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Pointwise recurrent homeomorphisms with stable fixed points ☆

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Abstract

We prove that a pointwise recurrent, orientation preserving homeomorphism of the 2-sphere, which is different from the identity and whose fixed points are stable in the sense of Lyapunov must have exactly two fixed points. If moreover there are no periodic points, other than fixed, then every stable minimal set is connected and its complement has exactly two connected components. Finally, we study liftings of the restriction to the complement of the fixed point set to the universal covering space.

MSC: 54H20; 37B20

Keywords: Pointwise recurrent homeomorphism; Stable fixed point; 2-sphere

1. Introduction

A homeomorphism $f: X \to X$ of a compact metrizable space X is called *pointwise recurrent* if $x \in L^+(x) \cap L^-(x)$ for every $x \in X$, where

$$L^+(x) = \{y \in X : f^{n_k}(x) \to y \text{ for some } n_k \to +\infty\}$$

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is the positive limit set of x with respect to f and $L^{-}(x)$ is the positive limit set of x with respect to f^{-1}. A pointwise recurrent, orientation preserving homeomorphism of S^1 is topologically conjugate to a rotation. This is not true for pointwise recurrent, orientation preserving homeomorphisms of the 2-sphere S^2 and it is an interesting problem to seek for additional conditions which ensure topological conjugacy to a rotation. A first step towards a characterization of rotations modulo topological conjugacy in the class of pointwise recurrent, orientation preserving homeomorphisms of S^2 would be a theorem which guarantees the existence of only two fixed points.

A weakly almost periodic homeomorphism of a compact metrizable space is pointwise recurrent. It is proved in [6] that a weakly almost periodic, orientation preserving homeomorphism of S^2, different from the identity, has exactly two fixed points. In this note we generalize this result to the class of pointwise recurrent homeomorphisms with stable fixed points. More precisely, we prove that if $f: S^2 \to S^2$ is a pointwise recurrent, orientation preserving homeomorphism, different from the identity, and if every fixed point of f is stable, then f must have exactly two fixed points. A compact invariant set of f is stable, if it has a neighbourhood basis consisting of f-invariant, open sets. If f is weakly almost periodic, then every orbit closure of f is stable.

The idea of proof was inspired by the proof for weakly almost periodic homeomorphisms in [6], but is considerably simpler and shorter. This is due to the fact we prove first, that a stable fixed point of a pointwise recurrent, orientation preserving homeomorphism f of S^2 has a neighbourhood basis consisting of f-invariant topological open discs (see Corollary 3.2). This permits us to use the Brouwer Translation Theorem instead of the theory of prime ends, as it is done in [6].

Although pointwise recurrence is a property which is inherited by the iterates of a homeomorphism of a metric space, the stability of fixed points is not. It is clear however that if $f: S^2 \to S^2$ is a pointwise recurrent, orientation preserving homeomorphism, different from the identity, which has stable fixed points and has no periodic point, other than fixed, then f^n has the same properties for $n \neq 0$. As an application of the main theorem, we show that every stable minimal set of a homeomorphism in this class is connected and its complement in S^2 has exactly two connected components, which generalizes Theorem 6 in [6].

In the final section we are concerned with the problem of whether a lifting to the universal covering space $\mathbb{R}^2$ of the restriction of a pointwise recurrent, orientation preserving homeomorphism f of S^2, which is different from the identity and has stable fixed points, to the complement of the fixed point set, is topologically conjugate to translation. This is closely related to a conjecture made by Winkelnkemper in [9]. We give a partial affirmative answer in case f is a C^1 diffeomorphism near the fixed points, under the assumption that the infinitesimal rotation numbers at the fixed points are non-zero (see Theorem 4.2).

2. The fixed point set

Let $f: S^2 \to S^2$ be an orientation preserving homeomorphism. Then f has degree 1 and is homotopic to the identity. Therefore, the Lefschetz number of f coincides with the Euler characteristic of S^2, which is 2. From the Lefschetz Fixed Point Theorem we have that f has at least one fixed point, say x_0. If f is pointwise recurrent, it must have a second fixed point, from the Brouwer Translation Theorem (see [3]), because $S^2 \setminus \{x_0\}$ is f-invariant and homeomorphic to $\mathbb{R}^2$. We shall study the fixed point set of orientation preserving, pointwise recurrent homeomorphisms of S^2.

Lemma 2.1. *If $f : S^2 \to S^2$ is an orientation preserving, pointwise recurrent homeomorphism, different from the identity, then the fixed point set* $\mathrm{Fix}(f)$ *of f is not connected and no connected component of $S^2 \setminus \mathrm{Fix}(f)$ is topologically an open disc.*

Proof. Suppose that $\mathrm{Fix}(f)$ is connected. Then, $H_1(S^2 \setminus \mathrm{Fix}(f); \mathbb{Z}) = 0$, by Alexander Duality (see [7] or [8]), and therefore each connected component of $S^2 \setminus \mathrm{Fix}(f)$ is topologically an open disc. By a theorem of Brown and Kister (see [2]), $f(U) = U$ for every connected component U of $S^2 \setminus \mathrm{Fix}(f)$. Since f is not the identity, there exists such an U, and because f has no fixed point in U, every point of U must be wandering under f, by the Brouwer Translation Theorem. This cannot happen if f is pointwise recurrent. The same argument proves the second assertion also. $\square$

Let $K \subset \mathbb{R}^2$ be a continuum. The *acyclic hull* $A(K)$ of K is the union of K and the bounded connected components of $\mathbb{R}^2 \setminus K$. Obviously, $A(K)$ is compact, connected, $\partial A(K) \subset K$ and $\mathbb{R}^2 \setminus A(K)$ is the unbounded connected component of $\mathbb{R}^2 \setminus K$. Hence $A(K)$ is acyclic, meaning that it has the integral Alexander–Spanier cohomology of a point. If F is any other acyclic continuum containing K, then $\mathbb{R}^2 \setminus F$ is an unbounded, connected subset of $\mathbb{R}^2 \setminus K$. Therefore, $A(K) \subset F$. In other words, $A(K)$ is the smallest acyclic continuum containing K, and K is acyclic if and only if $A(K) = K$. If K, L are two disjoint continua in $\mathbb{R}^2$, then it is easy to see that either $A(K)$, $A(L)$ are disjoint or one contains the other.

Let D be a topological open disc and $K \subset D$ be a continuum. The *acyclic hull $A_D(K)$ of K in D* is the union of K and the connected components of $D \setminus K$ with compact closure in D. If $h : D \to \mathbb{R}^2$ is a homeomorphism, then $h(A_D(K)) = A(h(K))$.

The following theorem was essentially proved in [5].

Theorem 2.2 (M.W. Hirsch). *If $f : S^2 \to S^2$ is an orientation preserving, pointwise recurrent homeomorphism, different from the identity, then* $\mathrm{Fix}(f)$ *has at least two acyclic connected components.*

Proof. Since $\mathrm{Fix}(f)$ is not connected, by Lemma 2.1, it has at least two connected components, say K_1, K_2. There exists a simple closed curve C separating them (see [7]). Let D be the connected component of $S^2 \setminus C$ containing K_1. Let $\mathcal{C}$ be the set of connected components of $\mathrm{Fix}(f)$ contained in D. On $\mathcal{C}$ we consider the partial ordering $\preceq$ defined by $L \preceq K$ if and only if $L \subset A_D(K)$. Note that $L \preceq K$ and $L \neq K$ if and only if $A_D(L) \subset \mathrm{int}\, A_D(K)$. Let $\mathcal{A}$ be a maximal totally ordered subset of $\mathcal{C}$. The set $Z = \bigcap_{K \in \mathcal{A}} A_D(K)$ is nonempty and compact. If $z \in \partial Z$, there is a unique $L \in \mathcal{A}$ such that $z \in \partial A_D(L)$. If L' corresponds to another point $z' \in \partial Z$ and $L \preceq L'$, $L \neq L'$, then $z' \in A_D(L) \subset \mathrm{int}\, A_D(L')$, contradiction. This shows that there is a unique $L \in \mathcal{A}$ such that $\partial Z \subset L$ and L is a minimal element of $\mathcal{A}$. We shall prove that L is acyclic. If it is not, $S^2 \setminus L$ has a connected component U

whose closure is contained in D, and U is a topological open disc. From the theorem of Brown and Kister, the pointwise recurrence of f and the Brouwer Translation Theorem, f must have a fixed point in U. It follows that there is a connected component J of $\mathrm{Fix}(f)$ which is contained in U, since $J \cap \partial U \subset J \cap L = \emptyset$. This means that $J \in \mathcal{C}$ and $J \preceq L$, $J \neq L$, which contradicts the fact that L is a minimal element of $\mathcal{A}$. Similarly, the connected component of $S^2 \setminus C$ which contains K_2 contains an acyclic connected component of $\mathrm{Fix}(f)$. $\square$

We remark that each set $A_D(K)$ in the proof of Theorem 2.2 is f-invariant. To see this, it suffices to prove that every connected component W of $D \setminus K$ with compact closure in D is f-invariant. If V is a connected component of $W \setminus \mathrm{Fix}(f)$, then $\partial V \subset \partial W \cup \mathrm{Fix}(f) \subset K \cup \mathrm{Fix}(f) = \mathrm{Fix}(f)$, which implies that V is a connected component of $S^2 \setminus \mathrm{Fix}(f)$. By the theorem of Brown and Kister, V is f-invariant. Thus, every connected component of $W \setminus \mathrm{Fix}(f)$ is f-invariant, and hence also W.

3. Stability of fixed points

In this section we shall prove the main result of the paper, namely Theorem 3.4 below. We shall need the following.

Proposition 3.1. *Let $f : S^2 \to S^2$ be an orientation preserving, pointwise recurrent homeomorphism. Let $A \subset S^2$ be an f-invariant continuum and D be a topological closed disc containing A in its interior. There exists an f-invariant continuum K with the following properties*:

(i) $A \subset K \subset D$.
(ii) *Every simple closed curve in K bounds a topological disc in K.*
(iii) $K \cap \partial D \neq \emptyset$.

Proof. Let $B_0 = \operatorname{int} D$ and inductively let B_{n+1} be the connected component of $f(B_n) \cap B_0$ which contains A. Then, $\{B_n\colon n \in \mathbb{N}\}$ is a decreasing sequence of topological open discs. Let

$$K = \bigcap_{n=0}^{\infty} \overline{B}_n.$$

Assertion (i) is obvious, while (ii) follows from the Jordan–Schönflies Theorem. To see that K is f-invariant, note first that $f^{-1}(K) \subset K$. Since f is pointwise recurrent, for each $x \in K$ there are integers $n_k \to +\infty$ such that $f^{-n_k}(x) \to x$. It follows that

$$x \in \bigcap_{k=1}^{\infty} f^{-n_k}(K) = \bigcap_{n=0}^{\infty} f^{-n}(K).$$

This shows that $f^{-1}(K) = K$. To prove (iii), suppose by contradiction that $K \cap \partial D = \emptyset$. Then, $\overline{B}_{n_0+1} \cap \partial D = \emptyset$ for some integer $n_0 \geqslant 0$ and for any integer $j \geqslant 0$ we have

$$\partial B_{n_0+j} \subset \partial f(B_{n_0+j-1}) \cup \partial B_0 = \partial f(B_{n_0+j-1}) \cup \partial D,$$

and therefore $\partial B_{n_0+j} \subset \partial f(B_{n_0+j-1})$. Since both are simple closed curves, necessarily $\partial B_{n_0+j} = \partial f(B_{n_0+j-1})$ and $B_{n_0+j} = f(B_{n_0+j-1})$, because $B_{n_0+j} \subset f(B_{n_0+j-1})$. Hence $B_{n_0+j} = f^j(B_{n_0})$ for every $j \geqslant 0$. If now $f^j(\overline{B}_{n_0}) \cap \partial B_{n_0} = \emptyset$ for some $j \geqslant 0$, then

$$f^{j+n}(\overline{B}_{n_0}) = \overline{B}_{n_0+j+n} \subset \overline{B}_{n_0+j} = f^j(\overline{B}_{n_0}) \subset B_{n_0}.$$

If $x \in \partial B_{n_0}$, this means that $f^{n+j}(x) \in f^j(\overline{B}_{n_0})$ for every integer $n \geqslant 0$ and hence $L^+(x) \subset f^j(\overline{B}_{n_0})$, that is $L^+(x) \cap \partial B_{n_0} = \emptyset$, which contradicts the fact that $x \in L^+(x)$. Thus, $f^j(\overline{B}_{n_0}) \cap \partial B_{n_0} \neq \emptyset$ for every integer $j \geqslant 0$ and so $K \cap \partial B_{n_0} \neq \emptyset$. But $\partial B_{n_0} \subset \partial D \cup f(\partial D) \cup \cdots \cup f^{n_0}(\partial D)$ and consequently $K \cap f^n(\partial D) \neq \emptyset$ for some integer $0 \leqslant n \leqslant n_0$. It follows that $K \cap \partial D = f^{-n}(K) \cap \partial D \neq \emptyset$. $\square$

If $f : X \to X$ is a homeomorphism of a topological space X, a compact f-invariant set $A \subset X$ is called *f-stable* if for every open set $U \subset X$ with $A \subset U$ there exists an f-invariant open set $V \subset X$ such that $A \subset V \subset U$.

Recall that if $A \subset S^2$ is an acyclic continuum, then $S^2 \setminus A$ is a topological open disc, by Alexander Duality. It follows that for every open neighbourhood U of A there exists a topological closed disc D such that $A \subset \operatorname{int} D \subset D \subset U$.

Corollary 3.2. *Let $f : S^2 \to S^2$ be an orientation preserving, pointwise recurrent homeomorphism. If $A \subset S^2$ is an f-stable, f-invariant, acyclic continuum, then every neighbourhood of A contains an f-invariant open neighbourhood of A which is topologically an open disc.*

Proof. Let D be a closed disc containing A in its interior and let K be the f-invariant continuum provided by Proposition 3.1. We shall use the notation of the proof of Proposition 3.1. Since A is f-stable, there is an open disc W with $A \subset W$ and $f^n(W) \subset \operatorname{int} D$ for every $n \in \mathbb{Z}$. The set $V = \bigcup_{n\in\mathbb{Z}} f^n(W)$ is f-invariant, open, connected and $A \subset V \subset B_0$. Thus, $V = f(V) \subset f(B_0) \cap B_0$, and therefore $V \subset B_1$. Inductively now we see that $V \subset B_n$ for every integer $n \geqslant 0$, which means that $V \subset K$. In particular, K is a neighbourhood of A, and by property (ii) of K the connected component of $\operatorname{int} K$ which contains A is an f-invariant topological open disc contained in D. $\square$

In general the boundary of the invariant topological open disc of Corollary 3.2 will not be a simple closed curve. We shall now study pointwise recurrent, orientation preserving homeomorphisms of S^2 with stable fixed points.

Lemma 3.3. *Let $f : S^2 \to S^2$ be a pointwise recurrent, orientation preserving homeomorphism, different from the identity. If every fixed point of f is f-stable, there exists an f-invariant continuum, which contains no fixed point of f.*

Proof. Let A be an acyclic connected component of $\mathrm{Fix}(f)$. Then A is f-stable, by our assumption, and there exists a simple closed curve C, which is the boundary of a topological closed disc D such that $A \subset \operatorname{int} D$ and $C \cap \mathrm{Fix}(f) = \emptyset$. Let K be the f-invariant continuum given by Proposition 3.1. Recall that

$$K = \bigcap_{n=0}^{\infty} \overline{B_n},$$

where $B_0 = \operatorname{int} D$ and inductively B_{n+1} is the connected component of $f(B_n) \cap B_0$ which contains A. We shall prove that ∂K contains no fixed point of f. Suppose the contrary, and let $s \in \mathrm{Fix}(f) \cap \partial K$. Since C contains no fixed point, we see by induction that ∂B_n contains no fixed point of f, for every $n \geqslant 0$. Now s is the limit of a sequence $(y_k)_{k\in\mathbb{N}}$ such that $y_k \in \partial B_{n_k}$, for some $n_k \to +\infty$. Since $\partial B_{n_k} \subset C \cup f(C) \cup \cdots \cup f^{n_k}(C)$, for each $k \in \mathbb{N}$ there exists some $0 \leqslant m_k \leqslant n_k$ and some $x_k \in C$ such that $y_k = f^{m_k}(x_k)$. By our assumption, s has a neighbourhood basis consisting of open f-invariant sets. If V is such a neighbourhood, there exists $k_0 \in \mathbb{N}$ such that $x_k = f^{-m_k}(y_k) \in V$, for $k \geqslant k_0$. It follows that $x_k \to s$, and therefore $s \in C$. This contradiction shows that $\mathrm{Fix}(f) \cap \partial K = \emptyset$. As the proof of Corollary 3.2 shows, K is a neighbourhood of A and the connected component Δ of $\operatorname{int} K$ which contains A is an f-invariant topological open disc. Clearly, $\partial\Delta$ is an f-invariant continuum, which contains no fixed point of f. □

Theorem 3.4. *Let $f : S^2 \to S^2$ be a pointwise recurrent, orientation preserving homeomorphism, different from the identity. If every fixed point of f is f-stable, then f has exactly two fixed points.*

Proof. Let $A \subset S^2 \setminus \mathrm{Fix}(f)$ be an f-invariant continuum, given by Lemma 3.3. There is a finite number $U_1, \ldots, U_n$ of connected components of $S^2 \setminus A$ such that $\mathrm{Fix}(f) \cap U_j \neq \emptyset$ for $j = 1, \ldots, n$. Each one of them is f-invariant and

$$2 = \chi\big(S^2\big) = L(f) = i\big(f, S^2\big) = \sum_{j=1}^{n} i(f, U_j),$$

where i denotes the local fixed point index, L the Lefschetz number and χ the Euler characteristic. Since $i(f, U_j) = 1$, by [6, Lemma 2], we get $n = 2$. It suffices to prove that $\mathrm{Fix}(f) \cap U_j$ is a singleton, for $j = 1, 2$. Let U be the connected component of $U_1 \setminus \mathrm{Fix}(f)$ with $\partial U_1 \subset \partial U$. Then, $\partial U \setminus \partial U_1$ is a nonempty, closed subset of $\mathrm{Fix}(f)$.

First we shall prove that $\partial U \setminus \partial U_1$ is connected. Suppose that this is not the case. Then there exists a simple closed curve $C \subset U$ which separates $\partial U \setminus \partial U_1$. There exists also an arc $J \subset U$ with one endpoint in C and the other endpoint in ∂U_1. The continuum $Y = C \cup J \cup \partial U_1$ contains no fixed point of f. Since every point of $\mathrm{Fix}(f)$ is f-stable, there exists an f-invariant open set $W \subset U_1$ such that $\mathrm{Fix}(f) \cap U_1 \subset W$ and $Y \cap W = \emptyset$. If now

$$N = \overline{\bigcup_{k\in\mathbb{Z}} f^k(Y)},$$

then N is an f-invariant continuum in $\overline{U}_1$, which contains no fixed point of f. There is a finite number $V_1, \dots, V_m$ of connected components of $S^2 \setminus N$ such that $\mathrm{Fix}(f) \cap V_j \neq \emptyset$ and $V_j \subset U_1$, for $j = 1, \dots, m$. Since $i(f, V_j) = 1$, by [6, Lemma 2], we have

$$1 = i(f, U_1) = \sum_{j=1}^{m} i(f, V_j) = m.$$

This however contradicts the fact that C separates $\partial U \setminus \partial U_1$. Hence $\partial U \setminus \partial U_1$ must be connected.

It suffices to prove now that $\partial U \setminus \partial U_1$ is a singleton. Let $s \in \partial U \setminus \partial U_1$ and suppose that $\partial U \setminus \partial U_1$ is not a singleton. Since it is connected, as we previously showed, and s is f-stable, by assumption, there exists an f-invariant topological open disc D such that $\overline{D} \subset U_1$ and $\partial D \cap (\partial U \setminus \partial U_1) \neq \emptyset$, by Corollary 3.2. Thus, $\partial D \cup (\partial U \setminus \partial U_1)$ is an f-invariant continuum. Let S be a connected component of $S^2 \setminus (\partial D \cup (\partial U \setminus \partial U_1))$ which is contained in D and contains at least one point of U. Then $S \subset D \cap U \subset D \setminus \mathrm{Fix}(f)$. Actually, S is a connected component of $D \setminus \mathrm{Fix}(f)$, because $\partial S \subset \partial D \cup \mathrm{Fix}(f)$. Since D is f-invariant, it follows from the theorem of Brown and Kister that S is an f-invariant topological open disc, and $f|S$ is an orientation preserving, pointwise recurrent homeomorphism without fixed points. This contradicts the Brouwer Translation Theorem. □

Although pointwise recurrence is a property which is inherited by the iterates of a homeomorphism of a metric space (see [4, Theorem 7.04]), the property of the stability of fixed points is not. A simple example can be constructed as follows. Let $F : S^1 \times [0,1] \to S^1 \times [0,1]$ be the homeomorphism defined by $F(z,t) = (ze^{2\pi i t}, t)$. Then F is pointwise recurrent, orientation preserving and fixes the two boundary components pointwise. Identifying the boundary components to points, we get a pointwise recurrent, orientation preserving homeomorphism $f : S^2 \to S^2$ with stable fixed points, the north and the south pole, but for every integer $n > 1$ the iterate f^n does not have stable fixed points. It is however clear that if $f : S^2 \to S^2$ is a pointwise recurrent, orientation preserving homeomorphism, different from the identity, which has no periodic point, other than fixed, and every fixed point of f is f-stable, then f^n has the same properties for every $n \neq 0$. Theorem 3.4 can be applied to this class of homeomorphisms in order to prove the following proposition, which gives information about the topology of their minimal sets and generalizes [6, Theorem 6].

Proposition 3.5. *Let $f : S^2 \to S^2$ be a pointwise recurrent, orientation preserving homeomorphism, different from the identity, such that f has no periodic point, other than fixed, and every fixed point of f is stable. If $K \subset S^2 \setminus \mathrm{Fix}(f)$ is a stable, minimal set of f, then K is connected and $S^2 \setminus K$ has exactly two connected components.*

Proof. First we shall show that the two fixed points of f belong to different connected components of $S^2 \setminus K$. Suppose that they belong to the same. There exists then a topological closed disc $D \subset S^2 \setminus K$ such that $\mathrm{Fix}(f) \subset \operatorname{int} D$. Since f is orientation preserving and has no wandering point, we must have $f(\partial D) \cap \partial D \neq \emptyset$. The set

$$M = \overline{\bigcup_{n \in \mathbb{Z}} f^n(\partial D)}$$

is an f-invariant continuum and contains no fixed point of f, because the fixed points are assumed to be stable. Moreover, $K \cap M = \emptyset$, because K is assumed to be f-stable. Let U be a connected component of $S^2 \setminus M$ such that $K \cap U \neq \emptyset$. Then $\mathrm{Fix}(f) \cap U = \emptyset$ and there exists some $n > 0$ such that $f^n(U) = U$, since f is pointwise recurrent. Now f^n is also pointwise recurrent, orientation preserving and $f^n|U$ has no fixed point, by assumption, which contradicts the Brouwer Translation Theorem, as U is homeomorphic to $\mathbb{R}^2$. This shows that the two fixed points of f belong to different connected components of $S^2 \setminus K$. Let now V be a connected component of $S^2 \setminus K$ such that $V \cap \mathrm{Fix}(f) \neq \emptyset$. Let W be the connected component of $S^2 \setminus \overline{V}$, which contains the second fixed point of f. Then W is an f-invariant, topological open disc, and so ∂W is an f-invariant, continuum in K. Since K is f-minimal, we must have $K = \partial W$. This proves that K is connected. If $S^2 \setminus K$ had more than two connected components, then some connected component $Y \subset S^2 \setminus \mathrm{Fix}(f)$ of it would be invariant under some iterate of f, which contradicts the Brouwer Translation Theorem. □

4. Liftings of pointwise recurrent homeomorphisms with stable fixed points

Let $f : S^2 \to S^2$ be a pointwise recurrent, orientation preserving homeomorphism, different from the identity, which has stable fixed points. Let 0 and ∞ denote the two fixed points of f, by Theorem 3.4. By Corollary 3.2, there are f-invariant, topological open discs D_0 and D_∞, containing 0 and ∞, respectively, whose closures are disjoint. If f has no other periodic point than the two fixed points, then ∂D_0 separates S^2 into two topological open discs, each one containing a fixed point, because otherwise $S^2 \setminus \partial D_0$ would have a connected component containing no fixed point, which would be invariant under some iterate of f, and this contradicts the Brouwer Translation Theorem. Similarly for D_∞. Let A be the intersection of the connected component of $S^2 \setminus \partial D_0$, which contains ∞, with the connected component of $S^2 \setminus \partial D_\infty$, which contains 0. Then A is an f-invariant, open annulus. The prime end compactification $\hat{A}$ of A is homeomorphic to $S^1 \times [0, 1]$. Let $\hat{f} : \hat{A} \to \hat{A}$ be the extension of $f|A$. By a fixed point theorem of M. Barge and R.M. Gillette (see [1]), $\hat{f}$ has no fixed point on $\partial \hat{A}$, and therefore on $\hat{A}$. Obviously, $\hat{f}$ is orientation and boundary component preserving.

Let $\tilde{A}$ denote the universal covering space of $\hat{A}$, which is homeomorphic to the strip $\mathbb{R} \times [0, 1]$. The group of covering transformations is generated by the translation $T(x, y) = (x + 1, y)$.

Proposition 4.1. *If $\tilde{f} : \tilde{A} \to \tilde{A}$ is a lifting of $\hat{f}$, then $\tilde{f}$ is topologically conjugate to T.*

Proof. If $\tilde{f}$ is not topologically conjugate to T, then there exists in A a simple closed essential curve C such that $f(C) \cap C = \emptyset$, by a result of H.E. Winkelnkemper, which generalizes the Poincaré–Birkhoff Theorem (see [9]). Now C and $f(C)$ bound an open annulus $B \subset A$. Since f is an orientation preserving homeomorphism, and B has a common boundary component with $f(B)$, we have $f(B) \cap B = \emptyset$. Inductively, $f^n(B) \cap B = \emptyset$ for every $n \neq 0$. This implies that every point of B is wandering, which contradicts our assumption that f is pointwise recurrent. $\square$

The question now arises whether a lifting of $f|S^2 \setminus \mathrm{Fix}(f)$ to the universal covering space $\mathbb{R}^2$ is topologically conjugate to translation. This is closely related to a conjecture made by Winkelnkemper in [9]. In this case the compactification by prime ends does not work, since the prime end compactification of $S^2 \setminus \{0, \infty\}$ is again S^2 and $\hat{f}$ is f. Thus we need another method of compactification and extension to get a homeomorphism of a closed annulus. In case f is a C^1 diffeomorphism near the fixed points, we can obtain a homeomorphism of a closed annulus, if we blow up the fixed points. We shall briefly describe this procedure.

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 diffeomorphism such that 0 is the only fixed point of g. Let $h : S^1 \times (0, +\infty) \to \mathbb{R}^2 \setminus \{0\}$ be the homeomorphism $h(z, t) = tz$. Let $\hat{g} : S^1 \times [0, +\infty) \to S^1 \times [0, +\infty)$ be the map defined by $\hat{g}(z, t) = (h^{-1} \circ g \circ h)(z, t)$, if $t \neq 0$, and

$$\hat{g}(z, 0) = \left(\frac{Dg(0)z}{\|Dg(0)z\|}, 0 \right).$$

Since $Dg(0)$ is a linear isomorphism, $\hat{g}$ is a homeomorphism. If g is orientation preserving, then so is $\hat{g}$. The procedure of blowing up an isolated fixed point can be carried out for any homeomorphism of a smooth surface, which is a C^1 diffeomorphism of an open neighbourhood of the fixed point.

Recall now that if g is orientation preserving, then the *infinitesimal rotation number* of g at its fixed point 0 is the Poincaré rotation number of the circle homeomorphism $\bar{g} : S^1 \to S^1$ defined by

$$\bar{g}(z) = \frac{Dg(0)z}{\|Dg(0)z\|}.$$

A partial affirmative answer to the above question is given by the following.

Theorem 4.2. *Let $f : S^2 \to S^2$ be a pointwise recurrent, orientation preserving homeomorphism, different from the identity, with stable fixed points. We assume that f is a C^1 diffeomorphism in some open neighbourhoods of the fixed points. If the infinitesimal rotation numbers of f at the fixed points are both non-zero, then a lifting of $f|S^2 \setminus \mathrm{Fix}(f)$ to $\mathbb{R}^2$ is topologically conjugate to the translation T.*

Proof. Since f is a C^1 diffeomorphism in open neighbourhoods of the fixed points, we can blow up the two fixed points to get an orientation and boundary preserving homeomorphism $F : S^1 \times [0, 1] \to S^1 \times [0, 1]$ such that $F|S^1 \times (0, 1)$ is topologically conjugate to $f|S^2 \setminus \mathrm{Fix}(f)$. Since the infinitesimal rotation numbers at the fixed points of f coincide with the Poincaré rotation numbers of $F|S^1 \times \{0\}$ and $F|S^1 \times \{1\}$, if they are both non-zero, then F has no fixed point in $S^1 \times [0, 1]$. Thus, the method of proof of Proposition 4.1

works to show that a lifting $\widetilde{F}$ of F on the universal covering space $\mathbb{R} \times [0,1]$ is topologically conjugate to T on $\mathbb{R} \times [0,1]$. Hence the lifting $\widetilde{F}|\mathbb{R} \times (0,1)$ of $f|S^2 \setminus \mathrm{Fix}(f)$ is topologically conjugate to T on $\mathbb{R} \times (0,1)$. $\square$

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