Vassiliev invariants and knot polynomials

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Abstract

We give a criterion to detect whether the derivatives of knot polynomials at a point are Vassiliev invariants or not. As an application we show that for each nonnegative integer $n$, $J_K^{(n)}(a)$ is a Vassiliev invariant if and only if $a = 1$, where $J_K^{(n)}(a)$ is the $n$th derivative of the Jones polynomial $J_K(t)$ of a knot $K$ at $t = a$. Similarly we apply the criterion for the Conway, Alexander, $Q$-, HOMFLY and Kauffman polynomial.

Also we give two methods of constructing a polynomial invariant from a numerical Vassiliev invariant of degree $n$, by using a sequence of knots induced from a double dating tangle. These two new polynomial invariants are Vassiliev invariants of degree $\leq n$ and their values on a knot are also polynomials of degree $\leq n$.

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1. Introduction

In 1990, V.A. Vassiliev introduced the concept of a finite type invariant of knots, called Vassiliev invariants, by using singularity theory and algebraic topology [17]. These Vassiliev invariants provided us a unified framework in which to consider “quantum invariants” including knot polynomials. In particular, in 1993, J.S. Birman and X.-S. Lin showed that each coefficient in the Maclaurin series of the Jones, Kauffman, and HOMFLY polynomial, after a suitable change of variables, is a Vassiliev invariant [4]. From this...
viewpoint we find a criterion whether the derivatives of knot polynomials at some points are Vassiliev invariants or not.

In this paper, we find the points on which the derivatives of a knot polynomial can be Vassiliev invariants by considering a sequence of knots induced from a double dating tangle and we introduce two ways of constructing polynomial invariants from a Vassiliev invariant of degree \( n \). From each of these new polynomial invariants, we may get at most \( (n + 1) \) linearly independent numerical Vassiliev invariants.

We start with giving some preliminary materials. Throughout this paper all knots or links are assumed to be oriented unless otherwise stated and \( \mathbb{N}, \mathbb{R} \) denote the sets of nonnegative integers and the real numbers, respectively.

A knot or link invariant \( v \) taking values in an abelian group can be extended to a singular knot or link invariant by using the Vassiliev skein relation:

\[
v(K \times) = v(K_+) - v(K_-),
\]

where \( K_-, K_+ \) and \( K \times \) are singular knot or link diagrams which are identical except the indicated parts in Fig. 1.

A knot or link invariant \( v \) is called a Vassiliev invariant of degree \( n \) if \( n \) is the smallest nonnegative integer such that \( v \) vanishes on singular knots or links with more than \( n \) double points. A knot or link invariant \( v \) is called a Vassiliev invariant if \( v \) is a Vassiliev invariant of degree \( n \) for some nonnegative integer \( n \).

There are some analogies between Vassiliev invariants and polynomials. For example, in 1996, D. Bar-Natan showed that when a Vassiliev invariant of degree \( m \) is evaluated on a knot diagram having \( n \) crossings, the result is bounded by a constant times of \( n^m \) [3]. In [8], we defined a sequence of knots or links induced from a double dating tangle and showed that any Vassiliev invariant has a polynomial growth on this sequence.

**Definition 1.1.** Let \( J \) be a closed interval \( \{a, b\} \) and \( k \) a positive integer. Fix \( k \) points in the upper plane \( J^2 \times \{b\} \) of the cube \( J^3 \) and their corresponding \( k \) points in the lower plane \( J^2 \times \{a\} \) of the cube \( J^3 \). A \((k, k)\)-tangle is obtained by attaching, within \( J^3 \), to these \( 2k \) points \( k \) curves, none of which should intersect each other. A \((k, k)\)-tangle is said to be oriented if each of its \( k \) curves is oriented. Given two \((k, k)\)-tangles \( S \) and \( T \), roughly the tangle product \( ST \) is defined to be the tangle obtained by gluing the lower plane of the cube containing \( S \) to the upper plane of the cube containing \( T \). The closure \( \overline{T} \) of a tangle \( T \) is the unoriented knot or link obtained by attaching \( k \) parallel strands connecting the \( k \) points and their corresponding \( k \) points in the exterior of the cube containing \( T \). For oriented tangles \( S \) and \( T \), the oriented tangle \( ST \) is defined only when it respects the orientations of \( S \) and \( T \) and the closure \( \overline{S} \) has the orientation inherited from that of \( S \).

\[
K_-, K_+ K_\times
\]

Fig. 1.
Definition 1.2. For two \((k, k)\)-tangles \(S\) and \(T\), \(ST\) is the knot or link obtained by closing the \((k, k)\)-tangle \(ST\), depicted as in Fig. 2. In this case, we say that the \((k, k)\)-tangle \(S\) is the complementary tangle of \(T\), in the knot or link \(ST\). Note that \(ST = S\) when \(T\) is the trivial \((k, k)\)-tangle.

Definition 1.3. An oriented \((k, k)\)-tangle \(T\) is called a double dating tangle (DD-tangle for short) if there exist some ordered pairs of crossings of the form \((\ast)\) in Fig. 3, so that \(T\) becomes the trivial \((k, k)\)-tangle when we change all the crossings in the ordered pairs, where \(i\) and \(j\) in Fig. 3, denote components of the tangle. Note that a DD-tangle is always an oriented tangle.

Since any crossing in a \((1, 1)\)-tangle can be deformed to a pair of crossings as in Fig. 5, every \((1, 1)\)-tangle is a double dating tangle. Thus every knot is a closure of a double dating \((1, 1)\)-tangle. But there is a link \(L\) which is not the closure of any DD-tangle since the linking number of two components of \(L\) must be 0.

Definition 1.4. Given an oriented \((k, k)\)-tangle \(S\) and a double dating \((k, k)\)-tangle \(T\) such that the product \(ST\) is well-defined, we have a sequence of links \(\{L_i(S, T)\}_{i=0}^{\infty}\) obtained by...
setting $L_i(S, T) = ST^i$ where $T^i = TT \cdots T$ is the $i$-times self-product of $T$ and $T^0$ is the trivial $(k, k)$-tangle. We call $\{L_i(S, T)\}_{i=0}^\infty$ (or simply $\{L_i\}_{i=0}^\infty$) the sequence induced from the $(k, k)$-tangle $S$ and the double dating $(k, k)$-tangle $T$ or simply a sequence induced from the double dating tangle $T$.

In particular, if $S$ is a knot for a $(k, k)$-tangle $S$, then $L_i(S, T) = ST^i$ is a knot for each $i \in \mathbb{N}$ since $T^i$ can be trivialized by changing some crossings.

**Definition 1.5.** A DD$^m$-singular link is a singular link having $2m$ double points paired into $m$-ordered (by $+$, $-$ signs) pairs, so that each pair is of the form (**) in Fig. 6, where $i$ and $j$ denote components of the link.

Note that any numerical link invariant $v$ can be extended to a DD$^m$-singular link invariant by setting $v(L) = v(L^+) - v(L^-)$, where $L$, $L^+$ and $L^-$ are identical except the indicated parts in Fig. 7.
Definition 1.6. A link invariant \( v \) is called a \textit{double dating linking} (DD-linking for short) invariant of type \( m \) if it vanishes on all \( \text{DD}^{m+1} \)-singular links when we extend \( v \) to a \( \text{DD}^{m+1} \)-singular link invariant, and we say that \( v \) is a DD-linking invariant of finite type if it is a DD-linking invariant of type \( m \) for some \( m \in \mathbb{N} \).

In the case of knots, since the DD-linking knot invariant of degree \( n \) is exactly the knot invariant of degree \( n \) [1,8], we have

\[ v \mid \{ K_i \}_{i=0}^{\infty} \text{ is a polynomial function in } i \text{ of degree } \leq n. \]

Theorem 1.7 [8]. Let \( \{ L_i \}_{i=0}^{\infty} \) be a sequence of knots induced from a DD-tangle. Then any Vassiliev knot invariant \( v \) of degree \( n \) has a polynomial growth on \( \{ L_i \}_{i=0}^{\infty} \) of degree \( \leq n \).

Corollary 1.8 [8]. Let \( L \) and \( K \) be two knots. For each \( i \in \mathbb{N} \), let \( K_i = K \# L \# \cdots \# L \) be the connected sum of \( K \) to the \( i \)-times self-connected sum of \( L \). If \( v \) is a Vassiliev invariant of degree \( n \), then \( v \mid \{ K_i \}_{i=0}^{\infty} \) is a polynomial function in \( i \) of degree \( \leq n \).

In Section 2, we give a necessary condition for the derivatives of polynomials at a point to be Vassiliev invariants. As an application, we prove that the derivative \( J^{(n)}(a) \) of the Jones polynomial of a knot \( K \) at a point \( a \) is not a Vassiliev invariant if \( a \neq 1 \). Since the coefficient of \( x^n \) in the Maclaurin series of \( J_K(e^x) \) is a Vassiliev invariant of degree \( \leq n \) [4], the \( n \)th derivative \( J^{(n)}(1) \) at \( t = 1 \) is a Vassiliev invariant of degree \( \leq n \), where \( J_K(t) \) is the Jones polynomial of a knot \( K \). Hence we see that \( J^{(n)}(a) \) is a Vassiliev invariant if and only if \( a = 1 \), for each \( n \in \mathbb{N} \). For the Conway polynomial \( \nabla_K(z) \), the Alexander polynomial \( \Delta_K(t) \) and the \( Q \)-polynomial \( Q_K(x) \), we show that for each \( n \in \mathbb{N} \), \( \nabla^{(n)}_K(a) \) is a Vassiliev invariant if and only if \( a = 0 \), \( \Delta^{(n)}_K(a) \) is a Vassiliev invariant if and only if \( a = 1 \), and \( Q^{(n)}_K(a) \) is not a Vassiliev invariant if \( a \neq -2, 1 \).

In Section 3, from a Vassiliev invariant of degree \( n \), we introduce two new Vassiliev polynomial invariants \( \bar{v} \) and \( v^* \) of degrees \( \leq n \) by using double dating tangles.
2. The derivatives of knot polynomials and Vassiliev invariants

In this section, we deal with knot polynomials which are ambient isotopy invariant and multiplicative under the connected sum such as the Jones, Conway, Alexander, \(Q\)-, HOMFLY, and Kauffman polynomial and deal with their (partial) derivatives.

Though J.S. Birman and X.-S. Lin showed that, after a suitable change of variables, each coefficient of the Maclaurin series of the Jones, HOMFLY, and Kauffman polynomial is a Vassiliev invariant, it was shown that each coefficient of the Jones, Conway, Alexander, \(Q\)-, HOMFLY, and Kauffman polynomial is not a Vassiliev invariant [18,7,8].

We start with introducing knot polynomials defined via skein relations. See [11] for more details.

Definition 2.1. The HOMFLY polynomial \(P_L(a,z) \in \mathbb{Z}[a, a^{-1}, z, z^{-1}]\) of a knot or link \(L\) is a knot or link invariant defined by the following rules.

1. \(P_O(a,z) = 1\), where \(O\) is the unknot and \(L_+, L_-\) and \(L_0\) are identical except the indicated parts in Fig. 8.

The Jones polynomial \(J_L(t)\), the Conway polynomial \(\nabla_L(z)\), and the Alexander polynomial \(\Delta_L(t)\) of a knot or link \(L\) can be defined by \(J_L(t) = P_L(t, t^{1/2} - t^{-1/2})\), \(\nabla_L(z) = P_L(1, z)\) and \(\Delta_L(t) = P_L(1, t^{1/2} - t^{-1/2}),\) respectively.

By using the skein relation we can see that \(P_L(a,z)\) is multiplicative under the connected sum. I.e., \(P_{L_1 \# L_2}(a,z) = P_{L_1}(a,z)P_{L_2}(a,z)\) for all knots or links \(L_1\) and \(L_2\). So the Jones, Conway, and Alexander polynomials are also multiplicative under the connected sum.

Definition 2.2 [10]. For an unoriented knot or link diagram \(D\), there is a regular isotopy invariant \(\Lambda_D(a,x) \in \mathbb{Z}[a, a^{-1}, x, x^{-1}]\) satisfying the following rules.

1. \(\Lambda(O) = 1\), where \(O\) is the trivial knot diagram.
2. \(\Lambda(L_+) + \Lambda(L_-) = x(\Lambda(L_0) + \Lambda(L_\infty))\), where \(L_+, L_-\), \(L_0\) and \(L_\infty\) are identical except the indicated parts in Fig. 9.
3. \(\Lambda(\varphi^+) = a\Lambda(\varphi^0), \Lambda(\varphi^-) = a^{-1}\Lambda(\varphi^0)\), where \(\varphi^+\) and \(\varphi^-\) are identical except the indicated parts in Fig. 9.

\(F_L(a,x) = a^{-w(D)}\Lambda_D(a,x)\) is an ambient isotopy invariant of an unoriented knot or link \(L\) where \(D\) is a diagram of \(L\) and \(w(D)\) is the writhe of \(D\).
$F_L(a,x)$ is called the Kauffman polynomial of a knot or link $L$. Define the $Q$-polynomial $Q_L(x)$ of a knot or link $L$ by setting $Q_L(x) = F_L(1,x)$. Similarly as in the case of the HOMFLY polynomial, the Kauffman polynomial and the $Q$-polynomial are also multiplicative under the connected sum.

Throughout this section, knot polynomials are always assumed to be multiplicative under the connected sum and $K^i$ denotes the $i$-times self-connected sum of $K$ for a knot $K$.

We consider 1-variable knot polynomials first and then 2-variable knot polynomials.

**Lemma 2.3.** Let $f_K(x)$ be a knot polynomial of a knot $K$ such that $f_K(x)$ is infinitely differentiable in a neighborhood of a point $a$ and assume that $f_K(a) \neq 0$. Then there exists a polynomial $p(x)$ of degree $m$ such that $f_K^{(m)}(a) = (f_K(a))^i p(i)$ for $i > m$.

**Proof.** Fix $i$ and assume that $i > n$. Since $f_K(x) = (f_K(x))^i$, we see that

$$f_K^{(n)}(a) = \sum_{p_1 + \cdots + p_i = n} \frac{n!}{p_1! \cdots p_i!} f_K^{(p_1)}(a) \cdots f_K^{(p_i)}(a).$$

If $f_K(a) = 0$ then the statement holds. So we assume that $f_K(a) \neq 0$. Define a relation $\sim$ on

$$P = \{(p_1, \ldots, p_i) \mid p_1 + \cdots + p_i = n, \ p_j \in \mathbb{N} \text{ for } j = 1, \ldots, i\}$$

by $(p_1, \ldots, p_i) \sim (q_1, \ldots, q_i)$ if $(p_1, \ldots, p_i)$ is a rearrangement of $(q_1, \ldots, q_i)$.

Then $\sim$ is an equivalence relation on $P$ and the cardinality $|P/\sim|$ is independent of $i$ because $i > n$. Thus we can subdivide the summation for $f_K^{(n)}(a)$ in a finite number of sums, which is independent of $i$, according to the relation $\sim$. Let $(p_1, \ldots, p_i) \in P$ be fixed and let $A = [(p_1, \ldots, p_i)]$ denote the equivalence class of $(p_1, \ldots, p_i)$. Consider a subdivision $\sum_{(q_1, \ldots, q_i) \in A} \frac{n!}{q_1! \cdots q_i!} f_K^{(q_1)}(a) \cdots f_K^{(q_i)}(a)$ of the summation.

For each $(q_1, \ldots, q_i) \in A$,

$$\frac{n!}{q_1! \cdots q_i!} f_K^{(q_1)}(a) \cdots f_K^{(q_i)}(a) = \frac{n!}{p_1! \cdots p_i!} f_K^{(p_1)}(a) \cdots f_K^{(p_i)}(a) = c(f_K(a))^{i-l}$$

where $c$ is independent of $i$ and $l = |\{j \mid p_j \neq 0, \ j = 1, \ldots, i\}|$. Since the cardinality $|A|$ is a polynomial in $i$ of degree $l$ and $l$ is independent of $i$, we see that

$$\sum_{(q_1, \ldots, q_i) \in A} \frac{n!}{q_1! \cdots q_i!} f_K^{(q_1)}(a) \cdots f_K^{(q_i)}(a) = (f_K(a))^i q(i)$$
for some polynomial $q(i)$ in $i$ of degree $\leqslant l$. In particular if $(p_1, \ldots, p_l) = (1, \ldots, 1, 0, \ldots, 0)$ then we get $l = n$, which is maximal, and the degree of $q(i)$ is $n$ since $f_K^{(1)}(a) \neq 0$.

Thus when we add all of the subdivisions of the sum, we get a polynomial $p(i)$ of degree $n$ such that $f_K^{(n)}(a) = (f_K(a))' p(i)$.  

**Theorem 2.4.** Let $f_K(x)$ be a knot polynomial which is infinitely differentiable in a neighborhood of a point $a$ for all knot $K$. If there exists a knot $L$ such that $f_L(a) \neq 0, 1$ and $f_L^{(1)}(a) \neq 0$ then $f_K^{(n)}(a)$ is not a Vassiliev invariant for each $n \in \mathbb{N}$.

**Proof.** It follows immediately from Corollary 1.8 and Lemma 2.3.  

**Theorem 2.5.** For each $n \in \mathbb{N}$, we have

1. $J_K^{(n)}(a)$ is not a Vassiliev invariant if $a \neq 1$,
2. $\nabla^{(n)}(a)$ is not a Vassiliev invariant if $a \neq 0$,
3. $\Delta_K^{(n)}(a)$ is a Vassiliev invariant if and only if $a = 1$, and
4. $Q_L^{(n)}(a)$ is not a Vassiliev invariant if $a \neq -2, 1$.

**Proof.** In the following proof, the notations $3_1, 4_1$ will mean the trefoil knot and the figure eight knot from the Rolfsen’s knot table [14], respectively.

1. By direct calculations, $J_3(t) = t + t^3 - t^5$ and $J_{4_1}(t) = t^2 - t^{-1} + 1 - t + t^2$. Let $A = \{t \mid J_3(t) = 0, 1\}$ and $B = \{t \mid J_{4_1}(t) = 0, 1\}$. Then it is not hard to see that $A \cap B = \{1\}$. Thus if $a \neq 1$, by Theorem 2.4, $J_K^{(n)}(a)$ is not a Vassiliev invariant.

2. By direct calculations, $\nabla_{3_1}(z) = 1 + z^2$. Thus if $a \neq 0$ then $\nabla_{3_1}(a) \neq 0, 1$ and $\nabla_{4_1}^{(1)}(a) \neq 0$. By Theorem 2.4, $\nabla_K^{(n)}(a)$ is not a Vassiliev invariant if $a \neq 0$.

3. Since the coefficients of $z^n$ in $\nabla_K(z)$ are Vassiliev invariants [2], $\nabla_K^{(n)}(0)$ is a Vassiliev invariant. Since $\Delta_K(t) = \nabla_K(t^{1/2} - t^{1/2})$, we see that $\Delta_K^{(n)}(1)$ is a Vassiliev invariant for all nonnegative integer $n$. By direct calculations, $\Delta_{3_1}(t) = t^{-1} - 1 + t$ and $\Delta_{4_1}(t) = t^{-2} - t^{-1} + 1 - t + t^2$. Let $A = \{t \mid \Delta_{3_1}(t) = 0, 1\}$ and $B = \{t \mid \Delta_{4_1}(t) = 0, 1\}$. Then $A \cap B = \{1\}$. Again by applying Theorem 2.4, we have the desired result.

4. By direct calculations, $Q_3(x) = -3 + 2x + 2x^2$ and $Q_{4_1}(x) = -3 - 2x + 4x^2 + 2x^3$. Let $A = \{x \mid Q_3(x) = 0, 1\}$ and $B = \{x \mid Q_{4_1}(x) = 0, 1\}$. Then $A \cap B = \{-2, 1\}$. Thus, by Theorem 2.4, we see that $Q_K^{(n)}(a)$ is not a Vassiliev invariant if $a \neq -2, 1$.

**Corollary 2.6.** For each $n \in \mathbb{N}$, we have

1. $J_K^{(n)}(a)$ is a Vassiliev invariant if and only if $a = 1$, and
2. $\nabla_K^{(n)}(a)$ is a Vassiliev invariant if and only if $a = 0$.

**Proof.** (1) Since the coefficient of $x^n$ in the Taylor expansion of $J_K(e^x)$ is a Vassiliev invariant [4], $J_K^{(n)}(1)$ is a Vassiliev invariant. Thus if $a \neq 1$, by Theorem 2.5, $J_K^{(n)}(a)$ is not a Vassiliev invariant.
(2) Since the coefficient of $z^n$ in $\nabla_K(z) \in \mathbb{Z}[z]$ is a Vassiliev invariant [2], $\nabla_K^{(n)}(0)$ is a Vassiliev invariant. By Theorem 2.5, $\nabla_K^{(n)}(a)$ is not a Vassiliev invariant if $a \neq 0$. \hfill \Box

In [16], Trapp defined a twist sequence of knots to detect whether a knot invariant is a Vassiliev invariant or not.

**Definition 2.7** [16]. A sequence of singular knots $\{K_i\}_{i=0}^{\infty}$ is called a twist sequence if the knots $K_i$'s are identical outside a $(2, 2)$-tangle where consecutive knots $K_i$ and $K_{i+1}$ differ by a full twist which introduces two positive crossings.

**Theorem 2.8** [16]. If $v$ is a Vassiliev invariant of degree $n$ and $\{K_i\}_{i=0}^{\infty}$ is a twist sequence, then $v$ has a polynomial growth in $i$ of degree $\leq n$ on $\{K_i\}_{i=0}^{\infty}$.

In extending Theorem 2.5(4), we cannot apply Theorem 2.4 for the derivatives of the $Q$-polynomial at $x = -2$ or 1 because $Q_K(-2) = 1 = Q_K(1)$ for any knot $K$ [5]. But we can see that $Q_K^{(1)}(1)$ and $Q_K^{(2)}(1)$ are not Vassiliev invariants by using the R. Trapp's result.

**Proposition 2.9.** $Q_K^{(1)}(1)$ and $Q_K^{(2)}(1)$ are not Vassiliev invariants.

**Proof.** Let $T_i$ denote the $(2, 2i + 1)$-torus knot giving a twist sequence $\{T_i\}_{i=0}^{\infty}$ and $T_{i+1/2}$ the $(2, 2i)$-torus link having $2i$ half twists obtained from $T_i$ by splicing a single crossing. Then by applying the skein relation of the $Q$-polynomial we get

\[
\begin{align*}
Q_{T_{i+1}}(x) + Q_{T_i}(x) &= x(Q_{T_{i+1/2}}(x) + 1), \\
Q_{T_{i+1/2}}(x) + Q_{T_{i-1/2}}(x) &= x(Q_{T_i}(x) + 1), \\
Q_{T_i}(x) + Q_{T_{i-1/2}}(x) &= x(Q_{T_{i-1/2}}(x) + 1).
\end{align*}
\]

Thus we have $Q_{T_{i+1}}(x) + (2 - x^2)Q_{T_i}(x) + Q_{T_{i-1}}(x) = 2x + 1$. Since $Q_{T_1}(1) = 1$, for $n \geq 2$, by taking the $n$th derivatives of the identity at $x = 1$, we get

\[
Q_{T_{i+1}}^{(n)}(1) + Q_{T_i}^{(n)}(1) + Q_{T_{i-1}}^{(n)}(1) = 2n Q_{T_i}^{(n-1)}(1) + n(n-1)Q_{T_i}^{(n-2)}(1) + (2x + 1)^n|_{x=1}.
\]

And for $n = 1$, we get $Q_{T_{i+1}}^{(1)}(1) + Q_{T_i}^{(1)}(1) + Q_{T_{i-1}}^{(1)}(1) = 4$.

Since $Q_{T_1}(x) = -3 + 2x + 2x^2$, $Q_{T_1}^{(1)}(1) = 6$. Thus $Q_K^{(1)}(1)$ does not have a polynomial growth on $\{T_i\}_{i=0}^{\infty}$. By letting $n = 2$ in the above equation, we get $Q_{T_{i+1}}^{(2)}(1) + Q_{T_i}^{(2)}(1) + Q_{T_{i-1}}^{(2)}(1) = 4Q_{T_i}^{(1)}(1) + 2$. Since $Q_K^{(1)}(1)$ does not have a polynomial growth on $\{T_i\}_{i=0}^{\infty}$, neither does $Q_K^{(2)}(1)$ on $\{T_i\}_{i=0}^{\infty}$. Thus by applying Theorem 2.8, $Q_K^{(1)}(1)$ and $Q_K^{(2)}(1)$ are not Vassiliev invariants. \hfill \Box

**Remark 2.10.** Since $Q_K^{(1)}(-2) = J_K^{(2)}(1)$ [9], $Q_K^{(1)}(-2)$ is a Vassiliev invariant of degree $\leq 2$. It is still an open problem whether $Q_K^{(n)}(-2)$ is a Vassiliev invariant or not for $n \geq 2$ [15].
Now we will deal with 2-variable knot polynomials such as the HOMFLY polynomial $P_K(a, z) \in \mathbb{Z}[a, a^{-1}, z]$ and the Kauffman polynomial $F_K(a, x) \in \mathbb{Z}[a, a^{-1}, x]$. For a 2-variable Laurent polynomial $g(x, y)$ which is infinitely differentiable on a neighborhood of $(a, b)$, we denote $\frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial y^n} g(a, b)$ by $g^{(m,n)}(a, b)$ for all pairs $(m, n)$ of nonnegative integers.

**Lemma 2.11.** Let $g_K(x, y)$ be a knot polynomial of a knot $K$ which is infinitely differentiable on a neighborhood of $(a, b) \in \mathbb{R}^2$ such that both $g_1^{(1,0)}(a, b)$ and $g_1^{(0,1)}(a, b)$ are nonzero. Then there exists a polynomial $p(i)$ of degree $m + n$ such that $g^{(m,n)}_K(a, b) = (g_K(a, b))^i p(i)$ for $i > m + n$.

**Proof.** The proof is analogous to that of Lemma 2.3 and we omit it. 

**Theorem 2.12.** Let $g_K(x, y)$ be a 2-variable knot polynomial which is infinitely differentiable on a neighborhood of $(a, b)$ for all knot $K$. If there exists a knot $L$ such that $g_1^{(1,0)}(a, b) \neq 0$, $g_1^{(0,1)}(a, b) \neq 0$ and $g_L^{(1,0)}(a, b) \neq 0$ then $g_K^{(m,n)}(a, b)$ is not a Vassiliev invariant for all $m, n \in \mathbb{N}$.

**Proof.** It follows immediately from Corollary 1.8 and Lemma 2.7.

Whether a finite product of the derivatives of knot polynomials at some points is a Vassiliev invariant or not can be detected by using Lemmas 2.3, 2.11 and Corollary 1.8. For example, if there is a knot $L$ such that $J_L^{(1)}(a) \neq 0$, $Q_L^{(1)}(b) \neq 0$, $P_L^{(1,0)}(c, y) \neq 0$, $J_L(a)Q_L(b)P_L(c, y) \neq 0$, then $g_K^{(1,0)}(a, b)P_L^{(1,0)}(c, y)$ is not a Vassiliev invariant for all $k, l, m, n \in \mathbb{N}$.

More generally, we can use Lemmas 2.3, 2.11 and Corollary 1.8 to see whether a sum of products of the derivatives of knot polynomials at some points is a Vassiliev invariant or not.

We note that the coefficients in the Taylor expansion of a knot polynomial, after a change of variables, are closely related to the sum of derivatives of the knot polynomial at some points. As an example, for an infinitely differentiable function $f: \mathbb{R} \to \mathbb{R}$ with $f(0) \neq 0$ and $f(1)(0) \neq 0$, the coefficient of $x^2$ in $J_K \circ f(x)$ is the half of the sum $J_K^{(2)}(f(0))(f(1)(0))^2 + J_K^{(1)}(f(0))f'(0)J_K^{(1)}(f(0))$, and thus is not a Vassiliev invariant.

### 3. New polynomial invariants from Vassiliev invariants

In this section, a Vassiliev invariant $v$ always means a Vassiliev invariant taking values in a numerical number field $F$. We will give two ways of constructing new polynomial invariants from a given Vassiliev invariant by using difference sequences. The new polynomial invariants are also Vassiliev invariants and so we get various numerical Vassiliev invariants.

**Definition 3.1.** Let $\{a_i\}_{i=0}^\infty$ be a sequence of numbers. The $n$th difference sequence $\{\Delta^n a_i\}_{i=0}^\infty$ of $\{a_i\}_{i=0}^\infty$ is defined by setting $\Delta^0 a_i = a_i$ and $\Delta^{n+1} a_i = \Delta^n a_{i+1} - \Delta^n a_i$ inductively for $n \in \mathbb{N}$. 


Lemma 3.2. If $\Delta^{n+1}a_i = 0$ for all $i$ then there exists a unique polynomial $p(x)$ of degree $\leq n$ such that $a_i = p(i)$ for all $i \in \mathbb{N}$.

Proof. We use the Newton’s formula $a_i = \sum_{j=0}^{i} \binom{i}{j} \Delta^j a_0$ for all $i \in \mathbb{N}$ [6]. Let

$$\sum_{j=0}^{n} \frac{x(x-1) \cdots (x-j+1)}{j!} \Delta^j a_0 = p(x).$$

For $i \geq n$, $a_i = \sum_{j=0}^{n} \binom{i}{j} \Delta^j a_0 = p(i)$, since $\Delta^{n+1}a_0 = \Delta^{n+2}a_0 = \cdots = 0$. Assume that $i < n$. Then

$$a_i = \sum_{j=0}^{i} \binom{i}{j} \Delta^j a_0 = \sum_{j=0}^{i} \frac{i(i-1) \cdots (i-j+1)}{j!} \Delta^j a_0$$

$$= \sum_{j=0}^{n} \frac{x(x-1) \cdots (x-j+1)}{j!} \Delta^j a_0|_{x=i} = p(i),$$

since $\frac{i(i-1) \cdots (i-j+1)}{j!} = 0$ for $j > i$. To prove the uniqueness, suppose that there exist two polynomials $p(x)$ and $q(x)$ such that $a_i = p(i) = q(i)$ for all $i \in \mathbb{N}$. Then the equation $p(x) - q(x) = 0$ has infinitely many integral roots. This contradicts unless $p(x) = q(x)$. $\square$

By Theorem 1.7, we get the following:

Lemma 3.3. Let $\{K_i\}_{i=0}^{\infty}$ be a sequence of knots induced from a DD-tangle and let $v$ be a knot invariant of degree $n$. Then $\Delta^{n+1}v(K_i) = 0$.

From Lemmas 3.2 and 3.3, we get the following:

Corollary 3.4. Let $\{K_i\}_{i=0}^{\infty}$ be a sequence of knots induced from a DD-tangle and let $v$ be a knot invariant of degree $n$, then there exists a polynomial $p(x) \in \mathbb{F}[x]$ such that $v(K_i) = p(i)$ for all $i \in \mathbb{N}$.

Given a knot diagram $D$, let $A_1, \ldots, A_m$ be disjoint nonempty sets of crossings of $D$. For $\varepsilon_i = \pm (1 \leq i \leq m)$, let $D(A_{1}^{\varepsilon_1}, \ldots, A_{m}^{\varepsilon_m})$ denote the knot diagram obtained from $D$ by changing all the crossings in $A_i$ only if $\varepsilon_i = -$. When $A_i$ is a singleton set $\{x_i\}$, we will denote $A_i = x_i$ as usual.

Lemma 3.5. For a knot invariant $v$, the following are equivalent.

1. $v$ is a Vassiliev invariant of degree less than $n$.
2. For any knot diagram $D$ and $n$ chosen crossings $x_1, \ldots, x_n$ of $D$,

$$\sum_{\varepsilon_i = \pm} \varepsilon_1 \cdots \varepsilon_n v(D(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})) = 0.$$
(3) For any knot diagram $D$ and any collection $A_1, \ldots, A_n$ of disjoint nonempty sets of crossings of $D$, $\sum_{\varepsilon_i = \pm} \varepsilon_1 \cdots \varepsilon_n v(D(A_1^{\varepsilon_1}, \ldots, A_n^{\varepsilon_n})) = 0$.

**Proof.** (1)$\Leftrightarrow$(2). For crossings $x_1, \ldots, x_n$ of $D$, let $D(\mathcal{T}_1, \ldots, \mathcal{T}_n)$ denote the knot diagram obtained from $D$ by changing $x_i$ into a singular crossing $\mathcal{X}_i$ for each $i = 1, \ldots, n$. Then the equivalence follows from the identity that

$$\sum_{\varepsilon_i = \pm} \varepsilon_1 \cdots \varepsilon_n v(D(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})) = v(D(\mathcal{T}_1, \ldots, \mathcal{T}_n)).$$

(2)$\Leftrightarrow$(3). The implication (3) $\Rightarrow$ (2) is clear. We will prove the other implication by using the induction argument on $\max\{|A_i| \mid 1 \leq i \leq n\}$, where $|A_i|$ denotes the cardinality of the set $A_i$. If $\max\{|A_i| \mid 1 \leq i \leq n\} = 1$, then it is clear. Assume that $\sum_{\varepsilon_i = \pm} \varepsilon_1 \cdots \varepsilon_n v(D(A_1^{\varepsilon_1}, \ldots, A_n^{\varepsilon_n})) = 0$ for $\max\{|A_i| \mid 1 \leq i \leq n\} < l$. Suppose that $\max\{|A_i| \mid 1 \leq i \leq n\} = l$. Without the loss of generality we may assume that $|A_i| = l$ for all $i = 1, \ldots, j$, for some positive integer $j$.

Again we use induction argument on $j$. Assume that $j = 1$ and $A_1 = \{x_1, \ldots, x_j\}$. Partition $A_1$ into two disjoint subsets $A_{1,1} = \{x_1\}$ and $A_{1,2} = \{x_2, \ldots, x_j\}$. Then

$$\sum_{\varepsilon_i = \pm} \varepsilon_1 \cdots \varepsilon_n v(D(A_1^{\varepsilon_1}, \ldots, A_n^{\varepsilon_n}))$$

$$= \sum_{\varepsilon_2, \ldots, \varepsilon_n} \varepsilon_2 \cdots \varepsilon_n v(D(A_1^{\varepsilon_1}, A_{1,2}^{\varepsilon_2}, A_2^{\varepsilon_3}, \ldots, A_n^{\varepsilon_n}))$$

$$- \sum_{\varepsilon_2, \ldots, \varepsilon_n} \varepsilon_2 \cdots \varepsilon_n v(D(A_{1,1}^{\varepsilon_1}, A_2^{\varepsilon_2}, \ldots, A_n^{\varepsilon_n}))$$

$$= \left( \sum_{\varepsilon_2, \ldots, \varepsilon_n} \varepsilon_2 \cdots \varepsilon_n v(D(A_1^{\varepsilon_1}, A_{1,2}^{\varepsilon_2}, A_2^{\varepsilon_3}, \ldots, A_n^{\varepsilon_n})) \right)$$

$$- \sum_{\varepsilon_2, \ldots, \varepsilon_n} \varepsilon_2 \cdots \varepsilon_n v(D(A_{1,1}^{\varepsilon_1}, A_2^{\varepsilon_2}, \ldots, A_n^{\varepsilon_n}))$$

$$+ \left( \sum_{\varepsilon_2, \ldots, \varepsilon_n} \varepsilon_2 \cdots \varepsilon_n v(D(A_1^{\varepsilon_1}, A_{1,2}^{\varepsilon_2}, A_2^{\varepsilon_3}, \ldots, A_n^{\varepsilon_n})) \right)$$

$$- \sum_{\varepsilon_2, \ldots, \varepsilon_n} \varepsilon_2 \cdots \varepsilon_n v(D(A_{1,1}^{\varepsilon_1}, A_2^{\varepsilon_2}, \ldots, A_n^{\varepsilon_n}))$$

$$= 0$$

by the induction hypothesis.

Assume that it holds for $j = 1, \ldots, k - 1$. By applying the above argument repeatedly, we can conclude that the result holds for $j = k$ and this finishes the induction argument.  

Let $K$ and $L$ be two knots. Then there exist oriented $(1, 1)$-tangles $R$ and $S$ such that $K \sim L \oplus K$ and $L \oplus K$ such that $\left\{L \oplus K^i\right\}_{i=0}^{\infty}$ of knots induced from a DD-tangle $R$.

Let $U$ be an oriented $(k, k)$-tangle whose closure is a knot and let $T$ be a double dating $(k, k)$-tangle such that the product $UT$ is well-defined. Then for each $i \in \mathbb{N}$, we get a knot...
L_i = U T^{i}. Now K \# L_i can be obtained from U T^{i}, where U' is the complementary tangle of T_i in K \# U T^{i}. Thus \{K \# L_i\}_{i=0}^{\infty} is a sequence of knots induced from the DD-tangle T.

Let v be a Vassiliev invariant of degree n and fix a knot L, an oriented (k,k)-tangle U and a double dating (k,k)-tangle T as above. Then by Corollary 3.4, for each knot K there exist unique polynomials p_K(x) and q_K(x) in F[x] with degrees \leq n such that v(L \# K^i) = p_K(i) and v(K \# L_i) = q_K(i). We define two polynomial invariants \bar{v} and \bar{v}^* as follows.

Define \bar{v} : \{\text{knots}\} \rightarrow F[x] by \bar{v}(K) = p_K(x) and \bar{v}^* : \{\text{knots}\} \rightarrow F[x] by \bar{v}^*(K) = q_K(x). Then \bar{v}(K)|_{\varepsilon_{i}=j} = p_K(j) = v(L \# K^j) and \bar{v}^*(K)|_{\varepsilon_{i}=j} = q_K(j) = v(K \# L_j) for all j \in \mathbb{N}.

Now we want to show that \bar{v} and \bar{v}^* are Vassiliev invariants of degrees \leq n. Let D be a diagram of a knot K and let x_1, \ldots, x_{n+1} be any choice of crossings in D. Then by Lemma 3.5, for each positive integer j, we have

\[
\sum_{\varepsilon_{i}=\pm} \varepsilon_1 \cdots \varepsilon_{n+1} \bar{v}(D(x_1^{\varepsilon_1}, \ldots, x_{n+1}^{\varepsilon_{n+1}}))|_{\varepsilon_{i}=j} = \sum_{\varepsilon_{i}=\pm} \varepsilon_1 \cdots \varepsilon_{n+1} v(L \# D(x_1^{\varepsilon_1}, \ldots, x_{n+1}^{\varepsilon_{n+1}}))^{j} = \sum_{\varepsilon_{i}=\pm} \varepsilon_1 \cdots \varepsilon_{n+1} v(L \# D^{j}(A_1^{\varepsilon_1}, \ldots, A_{n+1}^{\varepsilon_{n+1}})) = 0,
\]

where D^j denotes the diagram of K^j corresponding to D for K, and A_j = \{the crossings in D^j corresponding to x_j in D\} and hence |A_j| = j for 1 \leq l \leq n + 1. And for each j \in \mathbb{N},

\[
\sum_{\varepsilon_{i}=\pm} \varepsilon_1 \cdots \varepsilon_{n+1} v^{*}(D(x_1^{\varepsilon_1}, \ldots, x_{n+1}^{\varepsilon_{n+1}}))|_{\varepsilon_{i}=j} = \sum_{\varepsilon_{i}=\pm} \varepsilon_1 \cdots \varepsilon_{n+1} v(L \# D(x_1^{\varepsilon_1}, \ldots, x_{n+1}^{\varepsilon_{n+1}}) \# L_j) = \sum_{\varepsilon_{i}=\pm} \varepsilon_1 \cdots \varepsilon_{n+1} v(D \# L_j(x_1^{\varepsilon_1}, \ldots, x_{n+1}^{\varepsilon_{n+1}})) = 0.
\]

Thus by Lemma 3.5, we see that \bar{v} and \bar{v}^* are Vassiliev invariants of degrees \leq n. Moreover if L is the unknot, \bar{v}(K)|_{\varepsilon_{i}=1} = v(K) and \bar{v} is a Vassiliev invariant of degree n. Similarly if L_j is the unknot for some j \in \mathbb{N}, \bar{v}^*(K)|_{\varepsilon_{i}=j} = v(K) and \bar{v}^* is a Vassiliev invariant of degree n. Thus we have proved the following

**Theorem 3.6.** Let v be a Vassiliev invariant of degree n taking values in a numerical field F.

1. For a fixed knot L, define \bar{v} : \{\text{knots}\} \rightarrow F[x] by \bar{v}(K) = p_K(x) where p_K(x) is the unique polynomial such that v(L \# K^i) = p_K(i) for all i \in \mathbb{N}. Then \bar{v} is a Vassiliev invariant of degree \leq n and the degree of x in \bar{v}(K) is \leq n. In particular if L is the unknot, \bar{v} is a Vassiliev invariant of degree n and \bar{v}(K)|_{\varepsilon_{i}=1} = v(K).

2. For a fixed sequence \{L_i\}_{i=0}^{\infty} of knots induced from a DD-tangle, define

\[
v^{*} : \{\text{knots}\} \rightarrow F[x]
\]
by $v^*(K) = q_K(x)$ where $q_K(x)$ is the unique polynomial such that $v(K \# L_i) = q_K(i)$ for all $i \in \mathbb{N}$. Then $v^*$ is a Vassiliev invariant of degree $\leq n$ and the degree of $x$ in $v^*(K)$ is $\leq n$. In particular if $L_j$ is the unknot for some $j \in \mathbb{N}$, then $v^*$ is a Vassiliev invariant of degree $n$ and $v^*(K)|_{x=j} = v(K)$.

Given a Vassiliev invariant $v$ of degree $n$, we may get at most $(n+1)$ linearly independent numerical Vassiliev invariants from the coefficients of the polynomial invariants $\bar{v}$ and $v^*$, respectively. Apply $\bar{-}$ and $^*$ operations repeatedly on the new Vassiliev invariants from the coefficients, then we may obtain various Vassiliev invariants.

Now we want to generalize the one variable knot polynomial invariants $\bar{v}$ and $v^*$ in Theorem 3.6 to two variable knot polynomial invariants $\bar{v}$ and $v^*$ with the same notation. Let $\{L_i\}_{i=0}^{\infty}$ and $\{S_j\}_{j=0}^{\infty}$ be two sequences of knots induced from DD-tangles $T$ and $S$, respectively. Let $v$ be a Vassiliev invariant of degree $n$ taking values in a numerical field $F$. By applying a similar argument as in the proof of Theorem 3.6, for each knot $K$ there exist unique 2-variable polynomials $p_K(x, y), q_K(x, y) \in F[x, y]$ such that $v(K' \# S_j) = p_K(i, j)$ and $v(K \# L_i \# S_j) = q_K(i, j)$ with degrees $\leq n$ of $x$ and $y$. In particular, if $\{S_j\}_{j=0}^{\infty}$ is a constant sequence, say $S_j = L$, then $p_K(i, j) = v(K' \# L) = v(L' \# K') = p_K(i)$ in Theorem 3.6. And if $\{S_j\}_{j=0}^{\infty}$ is a constant sequence of the trivial knot, then $q_K(i, j) = v(K \# L_i \# S_j) = v(K \# L_i) = q_K(i)$ in Theorem 3.6. By using the same argument in the proof of Theorem 3.6, we get the 2-variable polynomial knot invariants $\bar{v}$ and $v^*$ of degrees $\leq n$ which are Vassiliev invariants and hence

**Theorem 3.7.** Let $v$ be a Vassiliev invariant of degree $n$ taking values in a numerical field $F$.

1. For a fixed sequence $\{S_j\}_{j=0}^{\infty}$ of knots induced from a DD-tangle, define $\bar{v}: \{\text{knots}\} \rightarrow F[x, y]$ by $\bar{v}(K) = p_K(x, y)$ where $p_K(x, y)$ is the unique polynomial such that $v(K' \# S_j) = p_K(i, j)$ for all $i, j \in \mathbb{N}$. Then $\bar{v}$ is a Vassiliev invariant of degree $\leq n$ and the degrees of $x$ and $y$ in $v(\bar{v}(K))$ are $\leq n$. In particular if $S_j$ is the unknot for some $j \in \mathbb{N}$, then $\bar{v}$ is a Vassiliev invariant of degree $n$ and $v(\bar{v}(K)|_{(x,y)=(1,1)} = v(K)$.

2. For two fixed sequences $\{L_i\}_{i=0}^{\infty}$ and $\{S_j\}_{j=0}^{\infty}$ induced from DD-tangles, define $v^*: \{\text{knots}\} \rightarrow F[x, y]$ by $v^*(K) = q_K(x, y)$ where $q_K(x, y)$ is the unique polynomial such that $v(K \# L_i \# S_j) = q_K(i, j)$ for all $i, j \in \mathbb{N}$. Then $v^*$ is a Vassiliev invariant of degree $\leq n$ and the degrees of $x$ and $y$ in $v^*(K)$ are $\leq n$. In particular if $L_i$ and $S_j$ are the unknots for some $i, j \in \mathbb{N}$, $v^*$ is a Vassiliev invariant of degree $n$ and $v^*(K)|_{(x,y)=(i,j)} = v(K)$.

We note that for a Vassiliev invariant $v$ of degree $n$, since $\bar{v}(K)$ and $v^*(K)$ are polynomials of degrees $\leq n$ for any knot $K$, the polynomial invariants $\bar{v}$ and $v^*$ are completely determined by $\{\bar{v}(K)|_{x=i} \mid 0 \leq i \leq n\}$ and $\{v^*(K)|_{x=i} \mid 0 \leq i \leq n\}$, respectively, in 1-variable case and $\{\bar{v}(K)|_{(x,y)=(i,j)} \mid 0 \leq i, j \leq n\}$ and $\{v^*(K)|_{(x,y)=(i,j)} \mid 0 \leq i, j \leq n\}$, respectively, in 2-variable case.

Let $V_n$ be the space of Vassiliev invariants of degrees $\leq n$ and let $A_n \subset V_n$. For each nonnegative integer $j$, define $A^j_n$ as follows. Set $A^0_n = A_n$ and define inductively $A^j_n$ to be
the set of all Vassiliev invariants obtained from the coefficients of the new polynomial invariants \( \tilde{v} \) and \( v^* \) ranging over all \( v \in A_i^{-1} \), all knots \( L \) and all sequences \( \{ L_i \}_{i=0}^\infty \) induced from all DD-tangles in Theorem 3.6.

Define \( (A_n) = \bigcup_{j=0}^\infty A_n^j \). We ask ourselves the following

**Question.** Find a minimal finite subset \( A_0 \) of \( V_n \) such that \( \text{span}(A_n) = V_n \).

**Example.** Let \( v(K) = J^K(7)(1) \) be the seventh derivative of the Jones polynomial of a knot \( K \) at 1. Then \( v \) is a Vassiliev invariant of degree \( \leq 7 \) [4]. Take \( L = \) the unknot and \( L_i = T_i \) in Theorem 3.6, where \( T \) is the trefoil knot.

Assume that \( i \geq 7 \). Since the Jones polynomial is multiplicative under the connected sum, \( J^K(t)_{L_i} = J^K(t)_L J^K(t)_{L_i} \). Therefore,

\[
J^K(7)(1)_{L_i} = \sum_{p+p_1+\cdots+p_n=7} \frac{7!}{p!p_1!\cdots p_n!} J^K(p)_L J^K(p_1)_L \ldots J^K(p_n)_L.
\]

Similarly

\[
J^K(7)(1)_{L_i} = \sum_{p+p_1+\cdots+p_n=7} \frac{7!}{p!p_1!\cdots p_n!} J^K(q)_L J^K(q_1)_L \ldots J^K(q_n)_L.
\]

Define a relation \( \sim \) on \( P = \{(p_1, \ldots, p_i) \mid p + p_1 + \cdots + p_i = 7, \ p_j \in \mathbb{N} \} \) for \( j = 1, \ldots, i \) by \( (p_1, \ldots, p_i) \sim (q_1, \ldots, q_i) \) if \( p = q \) and \( (p_1, \ldots, p_i) \) is a rearrangement of \( (q_1, \ldots, q_i) \).

Then \( \sim \) is an equivalence relation on \( P \) and the cardinality \( |P/\sim| \) is independent of \( i \). Since

\[
\frac{7!}{p!p_1!\cdots p_n!} J^K(p)_L J^K(p_1)_L \ldots J^K(p_n)_L = \frac{7!}{q!q_1!\cdots q_n!} J^K(q)_L J^K(q_1)_L \ldots J^K(q_n)_L
\]

for \( (p, p_1, \ldots, p_i) \sim (q, q_1, \ldots, q_i) \), we subdivide the summation for \( J^K(7)(1) \) according to the relation \( \sim \). Since \( J^K(1) = 1 \) and \( J^K(1) = 0 \) for any knot \( K \), we only consider the equivalence classes on \( P \) which does not include 1. By calculating the cardinality of the nonvanishing equivalence classes, we have

\[
\begin{align*}
[0, 7, 0, \ldots, 0] &= i, & \ [0, 5, 2, 0, \ldots, 0] &= i(i-1), \\
[0, 2, 2, 3, 0, \ldots, 0] &= i(i-1)(i-2)/2, & \ [0, 3, 4, 0, \ldots, 0] &= i(i-1), \\
[2, 2, 3, 0, \ldots, 0] &= i(i-1), & \ [2, 5, 0, \ldots, 0] &= i, \\
[3, 2, 2, 0, \ldots, 0] &= i(i-1)/2, & \ [3, 4, 0, \ldots, 0] &= i, \\
[4, 3, 0, \ldots, 0] &= i, & \ [5, 2, 0, \ldots, 0] &= i, & \ [7, 0, 0, \ldots, 0] &= 1.
\end{align*}
\]

Since \( J^K(t) = 1 \) we get

\[
J^K(7)(1)_{L_i} = iJ^K(7)(1) + 21i(i-1)J^K(5)(1)J^K(2)(1)
+ 105i(i-1)(i-2)J^K(2)(1)J^K(2)(1)J^K(3)(1)
+ 35i(i-1)J^K(3)(1)J^K(3)(1).
\]
Similarly we get
\[
J_{\mathcal{L}^i}(7) = -68040(i-1)(i-2) + 15120i(i-1) + 22680i(i-1)J_{K}^{(2)}(1) \\
+ 3780i(i-1)J_{K}^{(3)}(1) - 840iJ_{K}^{(3)}(1) - 630iJ_{K}^{(4)}(1) - 72iJ_{K}^{(5)}(1) \\
+ J_{K}^{(7)}(1).
\]

By substituting \(i\) with \(x\) in the above equations, we get
\[
\bar{v}(K) = xJ_{K}^{(7)}(1) + 21x(x-1)J_{K}^{(5)}(1)J_{K}^{(2)}(1) \\
+ 105x(x-1)(x-2)J_{K}^{(3)}(1)J_{K}^{(2)}(1)J_{K}^{(3)}(1) \\
+ 35x(x-1)J_{K}^{(3)}(1)J_{K}^{(4)}(1) \\
\text{and}
\]
\[
v^*(K) = -68040x(x-1)(x-2) + 15120x(x-1) + 22680x(x-1)J_{K}^{(2)}(1) \\
+ 3780x(x-1)J_{K}^{(3)}(1) - 840xJ_{K}^{(3)}(1) - 630xJ_{K}^{(4)}(1) - 72xJ_{K}^{(5)}(1) \\
+ J_{K}^{(7)}(1).
\]

If \(v_n\) and \(v_m\) are Vassiliev invariants of degrees \(n\) and \(m\), respectively, then the product \(v_nv_m\) is a Vassiliev invariant of degree \(\leq n+m\) [2,19]. Since \(J_{K}^{(3)}(1)\) is a Vassiliev invariant of degree \(\leq n\) [4], each coefficient of \(\bar{v}(K)\) and \(v^*(K)\) is a Vassiliev invariant.

**Remark 3.8.** Since \(\bar{v}\) and \(v^*\) are polynomial invariants which are Vassiliev invariants, each coefficient of \(\bar{v}\) and \(v^*\) is also a Vassiliev invariant.

**References**