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# Twisted longitudinal index theorem for foliations and wrong way functoriality

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## Abstract

For a Lie groupoid  $\mathcal{G}$  with a twisting  $\sigma$  (a  $PU(H)$ -principal bundle over  $\mathcal{G}$ ), we use the (geometric) deformation quantization techniques supplied by Connes tangent groupoids to define an analytic index morphism

$$K^*(A^*\mathcal{G}, \pi^*\sigma) \xrightarrow{\text{Index}_{a,(\mathcal{G},\sigma)}} K^*(\mathcal{G}, \sigma)$$

in twisted K-theory. In the case the twisting is trivial we recover the analytic index morphism of the groupoid.

For a smooth foliated manifold with twistings on the holonomy groupoid we prove the twisted analog of the Connes–Skandalis longitudinal index theorem. When the foliation is given by fibers of a fibration, our index coincides with the one recently introduced by Mathai, Melrose, and Singer.

We construct the pushforward map in twisted K-theory associated to any smooth (generalized) map  $f: W \rightarrow M/F$  and a twisting  $\sigma$  on the holonomy groupoid  $M/F$ , next we use the longitudinal index theorem to prove the functoriality of this construction. We generalize in this way the wrong way functoriality results of Connes and Skandalis when the twisting is trivial and of Carey and Wang for manifolds.

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*Keywords:* Index theory; Foliations; Twisted K-theory; Tangent groupoids

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## 0. Introduction

This paper is part of a longstanding project where we aim to study and develop an index theory (à l’Atiyah–Singer and Connes–Skandalis) for foliations under the presence of twistings.

The Atiyah–Singer index theorem in [2], which states that for an elliptic pseudo-differential operator on a compact manifold, the analytical index is equal to the topological index, has been generalized in many other situations. In particular, Connes and Skandalis established an index theorem for longitudinal elliptic operators on foliated manifolds [15].

For a smooth foliated manifold  $(M, F)$ , there is a canonically defined  $C^*$ -algebra  $C^*(M, F) = C^*(\mathcal{G}_M)$  which is the  $C^*$ -algebra of the underlying holonomy groupoid  $\mathcal{G}_M = M/F$ . An elliptic pseudo-differential operator  $D$  along the leaves of the foliation defines an element

$$\mathbf{Index}_{a, M/F}(D) \in K_0(C^*(M, F)),$$

in the K-theory of the  $C^*$ -algebra. This is done by using pseudo-differential calculus for the holonomy groupoid  $\mathcal{G}_M$ . The element  $\mathbf{Index}_a(D)$  is called the analytical index of  $D$ , which depends only on the longitudinal symbol of  $D$ . Hence, one has the analytical index as a group homomorphism associated to the foliated manifold  $(M, F)$ ,

$$\mathbf{Index}_{a, M/F} : K^0(F^*) \longrightarrow K_0(C^*(M, F)).$$

The topological index for  $(M, F)$  is obtained by applying the Thom isomorphism and Bott periodicity in topological K-theory to an auxiliary embedding

$$\iota : M \longrightarrow \mathbb{R}^{2n}.$$

Let  $N$  be the total space of the normal bundle to the leaves. The Thom isomorphism implies that

$$K^0(F^*) \cong K^0(N).$$

There is an open neighborhood of the 0-section in  $N$  which is homeomorphic to an open transversal  $T$  of the induced foliation  $\tilde{F}$  on  $\tilde{M} = M \times \mathbb{R}^{2n}$ . For a suitable open neighborhood  $\Omega$  of  $T$  in  $M \times \mathbb{R}^{2n}$ , the  $C^*$ -algebra  $C^*(\Omega, \tilde{F}|_\Omega)$  is Morita equivalent to  $C_0(T)$  (the  $C^*$ -algebra of continuous functions on  $T$  vanishing at infinity). So one has

$$K^0(N) \cong K^0(T) \cong K_0(C^*(\Omega, \tilde{F}|_\Omega)) \longrightarrow K_0(C^*(\tilde{M}, \tilde{F}))$$

where the last map is induced by the inclusion  $C^*(\Omega, \tilde{F}|_\Omega) \subset C^*(\tilde{M}, \tilde{F})$ . The Bott periodicity gives rise to

$$K_0(C^*(\tilde{M}, \tilde{F})) = K_0(C^*(M, F) \otimes C_0(\mathbb{R}^{2n})) \cong K_0(C^*(M, F)).$$

This leads to the topological index for a foliated manifold  $(M, F)$ ,

$$\mathbf{Index}_{t, M/F} : K^0(F^*) \longrightarrow K_0(C^*(M, F)).$$

In [15], Connes and Skandalis showed that

$$\mathbf{Index}_{a, M/F} = \mathbf{Index}_{t, M/F}.$$

There is an equivalent definition (see [22,14,27]) of the analytical index for a foliated manifold  $(M, F)$  using the Connes tangent groupoid

$$\mathcal{G}_M^T := F \times \{0\} \sqcup \mathcal{G}_M \times (0, 1] \rightrightarrows M \times [0, 1]$$

of the holonomy groupoid  $\mathcal{G}_M$ . The definition works for any Lie groupoid  $\mathcal{G}$ . Let  $A\mathcal{G}$  be the associated Lie algebroid which can be viewed as a Lie groupoid given by the vector bundle structure, then one has the following exact sequence (see [22])

$$0 \longrightarrow C^*(\mathcal{G} \times (0, 1]) \longrightarrow C^*(\mathcal{G}^T) \xrightarrow{ev_0} C^*(A\mathcal{G}) \longrightarrow 0.$$

Monthubert and Pierrot in [27] show that

$$\mathbf{Index}_{a, M/F} = (ev_1)_* \circ (ev_0)_*^{-1} \circ \mathcal{F} := \mathbf{Index}_{a, \mathcal{G}_M} : K^0(F^*) \longrightarrow K_0(C^*(M, F))$$

where  $\mathcal{F} : K^0(F^*) = K^0(A^*\mathcal{G}) \cong K_0(C^*(A\mathcal{G}))$  is the isomorphism induced by the fiberwise Fourier isomorphism  $C^*(A\mathcal{G}) \cong C_0(A^*\mathcal{G})$  and the morphisms  $ev_i$  are the respective evaluations.

The analytic index morphism associated to a Lie groupoid arises naturally from a geometric construction, that of the Connes tangent groupoid [22,14,27,29]. This groupoid encodes the deformation of the Lie groupoid to its Lie algebroid. The use of deformation groupoids in index theory has proven to be very useful to prove index theorems, as well as to establish index theories in many different settings [22,14,16,1,8,21,9,10,34]. For example, even in the classic case of Atiyah–Singer theorem, Connes gives a very simple conceptual proof using the tangent groupoid (see [14], II.5).

There is also an equivalent definition of the topological index for a foliated manifold  $(M, F)$  (see [8]) by realizing the Thom inverse morphism of a complex vector bundle as the deformation index of some groupoid (the Thom groupoid, [16], Theorem 6.2). The resulting topological index is denoted by

$$\mathbf{Index}_{t, \mathcal{G}_M} : K^0(F^*) \longrightarrow K_0(C^*(M, F)).$$

One has the following equivalences between these four index morphisms:

$$\mathbf{Index}_{a, \mathcal{G}_M} = \mathbf{Index}_{a, M/F} = \mathbf{Index}_{t, M/F} = \mathbf{Index}_{t, \mathcal{G}_M}.$$

In this paper, we use these strict deformation quantization techniques to work out an index theory for foliation groupoids in the presence of a twisting which generalizes the index theory [35] for a smooth manifold with a twisting  $\sigma : M \rightarrow K(\mathbb{Z}, 3)$ , and Connes–Skandalis index theory when the twisting is trivial. The approach taken in this paper is of geometrical nature, we are going to define our indices and propose the corresponding index theorem by using deformation groupoid techniques. We will leave the discussion of the pseudodifferential calculus behind these for later works.

Following the construction of KK-elements developed in [22] (in particular Section 3) by Hilsum and Skandalis, we construct an analytic index morphism of any Lie groupoid  $\mathcal{G} \rightrightarrows M$  with a twisting given by a Hilsum–Skandalis’ generalized morphism  $\sigma : \mathcal{G} \dashrightarrow PU(H)$ ,

$$K^*(A\mathcal{G}, \pi^*\sigma) \xrightarrow{\mathbf{Index}_{a, (\mathcal{G}, \sigma)}} K^*(\mathcal{G}, \sigma).$$

This index  $\mathbf{Index}_{a, (\mathcal{G}, \sigma)}$  takes values in the twisted K-theory of  $(\mathcal{G}, \sigma)$  (see [33] for twisted K-theory for groupoids or Definition 2.8 below). When the twisting is trivial we recover the analytic index morphism of the groupoid defined by Monthubert and Pierrot in [27].

Equivalently a twisting is given by a  $PU(H)$ -principal bundle over the units of the groupoid  $\mathcal{G}$  together with a compatible  $\mathcal{G}$ -action. When the Lie groupoid represents a manifold, the twisted K-theory we are considering is the twisted K-theory for manifolds [3,23]. In general, our setting also includes, as particular cases, equivariant twisted K-theory and twisted K-theory for orbifolds [33].

For general Lie groupoids, we prove that our twisted index morphism satisfies the following three properties (Propositions 2.14, 2.15 and 2.17):

- (i) It is compatible with the Bott morphism, *i.e.*, the following diagram is commutative

$$\begin{CD} K^*(A^*\mathcal{G}, \pi^*\sigma_0) @>\mathbf{Index}_{a, (\mathcal{G}, \sigma)}>> K^*(\mathcal{G}, \sigma) \\ @V \text{Bott} VV @VV \text{Bott} V \\ K^*(A^*\mathcal{G} \times \mathbb{R}^2, \pi^*(\sigma \circ p)_0) @>\mathbf{Index}_{a, (\mathcal{G} \times \mathbb{R}^2, \sigma \circ p)}>> K^*(\mathcal{G} \times \mathbb{R}^2, \sigma \circ p) \end{CD}$$

where  $p : \mathcal{G} \times \mathbb{R}^2 \rightarrow \mathcal{G}$  is the projection,  $\sigma_0$  and  $(\sigma \circ p)_0$  are the induced twistings on the unit spaces of  $\mathcal{G}$  and  $\mathcal{G} \times \mathbb{R}^2$  respectively.

(ii) Let  $\mathcal{H} \xrightarrow{j} \mathcal{G}$  be an inclusion as an open subgroupoid. The following diagram is commutative:

$$\begin{CD}
 K^*(A^*\mathcal{H}, \pi^*(\sigma \circ j)_0) @>\text{Index}_{a,(\mathcal{H},\sigma \circ j)}>> K^*(\mathcal{H}, \sigma \circ j) \\
 @V j! VV @VV j! V \\
 K^*(A^*\mathcal{G}, \pi^*\sigma_0) @>\text{Index}_{a,(\mathcal{G},\sigma)}>> K^*(\mathcal{G}, \sigma)
 \end{CD}$$

where the vertical maps are induced from the inclusions by open subgroupoids.

(iii) Let  $N \rightarrow T$  be a real vector bundle. Consider the product groupoid  $N \times_T N \rightrightarrows N$ . This groupoid has a Lie algebroid  $N \oplus N \xrightarrow{\pi_N} N$ . The groupoid  $N \times_T N \rightrightarrows N$  is Morita equivalent to the identity groupoid  $T \rightrightarrows T$ . That is, there is an isomorphism in the Hilsum–Skandalis category

$$N \times_T N \xrightarrow{\mathcal{M}} T.$$

In the presence of a twisting  $\beta$  on the space  $T$  the twisted index morphism

$$K^*(N \oplus N, \pi^*\beta) \xrightarrow{\text{Index}_{a,(N \times_T N, \beta \circ \mathcal{M})}} K^*(N \times_T N, \beta \circ \mathcal{M}) \xrightarrow{\cong} K^*(T, \beta)$$

is the inverse (modulo a Morita equivalence and a Fourier isomorphism) of the Thom isomorphism (see [7,23]) in twisted K-theory  $K^*(T, \beta) \xrightarrow{T_\beta} K^*(N \oplus N, \pi_T^*\beta)$ .

These properties are the analogs to the axioms stated in [2] and will allow us to prove a twisted index theorem for foliations and in general we can then study twisted index theory for these objects. Indeed, for the case of regular foliations, we extend the topological index of Connes–Skandalis (Definition 3.2 below) to the twisted case and prove the following twisted longitudinal index theorem.

**Theorem 0.1.** *For a regular foliation  $(M, F)$  with a twisting  $\sigma : M/F \dashrightarrow PU(H)$  on the space of leaves, we have the following equality of morphisms:*

$$\text{Index}_{a,(M/F,\sigma)} = \text{Index}_{l,(M/F,\sigma)} : K^*(F, \pi^*\sigma) \longrightarrow K^*(M/F, \sigma).$$

In fact, any index morphism for foliations with twistings satisfying the three properties (i)–(iii) is equal to the twisted topological index by exactly the same proof. There is a second definition of the twisted analytic index using projective pseudo-differential operators along the leaves which also satisfies the three properties. We will return to this issue in a separate paper.

When the foliation is given by a fibration  $\pi : M \rightarrow B$  and the twisting is a torsion class in  $H^3(B, \mathbb{Z})$ , our index coincides with the one recently introduced in [25] by Mathai, Melrose, and Singer since the topological indices of both papers are the same. We remark that they also give the corresponding cohomological formula which was also recently proved by superconnection

methods by Benameur and Gorokhovsky [4]. We will leave the study of Chern–Connes type formulas in the twisted foliation context for further works.

In this paper, we also construct, for every smooth (generalized) map  $f : W \rightarrow M/F$  and a twisting  $\sigma$  on  $M/F$ , a pushforward morphism in twisted K-theory

$$f_! : K^{*+i}(W, f^*\sigma_0 + o_{TW \oplus f^*\nu_F}) \longrightarrow K^i(M/F, \sigma),$$

for an induced twisting on  $W$ . Here  $o_{TW \oplus f^*\nu_F}$  is the K-theoretical orientation twisting of the real vector bundle  $TW \oplus f^*\nu_F$  with  $\nu_F$  being the normal bundle to the foliation  $F$ . We then use the twisted longitudinal index theorem to prove the functoriality of this construction.

**Theorem 0.2.** *The pushforward morphism is functorial, that is, if we have a composition of smooth maps*

$$Z \xrightarrow{g} W \xrightarrow{f} M/F, \tag{0.1}$$

and a twisting  $\sigma : M/F \rightarrow PU(H)$  then  $(f \circ g)_! = f_! \circ g_!$

When  $\sigma$  is trivial and  $f : W \rightarrow M/F$  is K-oriented (that is  $TW \oplus \nu_F$  is K-oriented), our pushforward map in Definition 4.9 agrees with the one constructed in [15]. Also, when the foliation consists on a single leaf (hence a manifold) but the twisting is not necessarily trivial, we obtain otherwise the push-forward map defined in [7] by Carey and Wang.

We remark an important difference with respect to Connes–Skandalis approach in [15]. Indeed, we prove first, in a geometric way, the longitudinal index theorem and then we use it to prove the pushward functoriality. Recall that in [15] it is done conversely with analytic methods. Indeed, their fundamental technical result (as remarked by Connes and Skandalis), Lemma 4.7 in [15], is proved by means of the longitudinal pseudodifferential calculus and the KK-elements associated to it. With this result they prove the functoriality of the pushforward and then the longitudinal index theorem. In this paper, the twisted analog of the lemma mentioned above (Proposition 4.6) is proved using the longitudinal index theorem. The strict deformation quantization approach to index theory for foliations (without twistings) have already used in this sense in previous works by the first author in [8,9].

Some of the results presented in this paper were announced in [11]. There is however an important difference in the way of proving the pushforward functoriality. The arguments stated in that note for proving such a result were independent of the longitudinal index theorem. It is indeed possible to prove it directly by deforming some KK-elements, in the spirit of Theorem 6.2 in [16] or Theorem 2.17 below.

### 1. Twistings on Lie groupoids

In this section, we review the notion of twistings on Lie groupoids and discuss some examples which appear in this paper. Let us recall what a groupoid is:

**Definition 1.1.** A *groupoid* consists of the following data: two sets  $\mathcal{G}$  and  $\mathcal{G}^{(0)}$ , and maps

- (1)  $s, r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  called the source map and target map respectively,
- (2)  $m : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$  called the product map (where  $\mathcal{G}^{(2)} = \{(\gamma, \eta) \in \mathcal{G} \times \mathcal{G} : s(\gamma) = r(\eta)\}$ ),

together with two additional maps,  $u : \mathcal{G}^{(0)} \rightarrow \mathcal{G}$  (the unit map) and  $i : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  (the inverse map), such that, if we denote  $m(\gamma, \eta) = \gamma \cdot \eta$ ,  $u(x) = x$  and  $i(\gamma) = \gamma^{-1}$ , we have

- (i)  $r(\gamma \cdot \eta) = r(\gamma)$  and  $s(\gamma \cdot \eta) = s(\eta)$ .
- (ii)  $\gamma \cdot (\eta \cdot \delta) = (\gamma \cdot \eta) \cdot \delta$ ,  $\forall \gamma, \eta, \delta \in \mathcal{G}$  whenever this makes sense.
- (iii)  $\gamma \cdot x = \gamma$  and  $x \cdot \eta = \eta$ ,  $\forall \gamma, \eta \in \mathcal{G}$  with  $s(\gamma) = x$  and  $r(\eta) = x$ .
- (iv)  $\gamma \cdot \gamma^{-1} = u(r(\gamma))$  and  $\gamma^{-1} \cdot \gamma = u(s(\gamma))$ ,  $\forall \gamma \in \mathcal{G}$ .

For simplicity, we denote a groupoid by  $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ . A strict morphism  $f$  from a groupoid  $\mathcal{H} \rightrightarrows \mathcal{H}^{(0)}$  to a groupoid  $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$  is given by maps in

$$\begin{array}{ccc}
 \mathcal{H} & \xrightarrow{f} & \mathcal{G} \\
 \Downarrow & & \Downarrow \\
 \mathcal{H}^{(0)} & \xrightarrow{f} & \mathcal{G}^{(0)}
 \end{array}$$

which preserve the groupoid structure, i.e.,  $f$  commutes with the source, target, unit, inverse maps, and respects the groupoid product in the sense that  $f(h_1 \cdot h_2) = f(h_1) \cdot f(h_2)$  for any  $(h_1, h_2) \in \mathcal{H}^{(2)}$ .

In this paper we will only deal with Lie groupoids, that is, a groupoid in which  $\mathcal{G}$  and  $\mathcal{G}^{(0)}$  are smooth manifolds, and  $s, r, m, u$  are smooth maps (with  $s$  and  $r$  submersions, see [24,30]). For two subsets  $U$  and  $V$  of  $\mathcal{G}^{(0)}$ , we use the notation  $\mathcal{G}_U^V$  for the subset

$$\{\gamma \in \mathcal{G} : s(\gamma) \in U, r(\gamma) \in V\}.$$

Lie groupoids generalize the notion of Lie groups. For Lie groupoids there is also a notion of Lie algebroid playing the role of the Lie algebra in Lie theory.

**Definition 1.2** (*The Lie algebroid of a Lie groupoid*). Let  $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$  be a Lie groupoid. Denote by  $A\mathcal{G}$  the normal bundle associated to the inclusion  $\mathcal{G}^{(0)} \subset \mathcal{G}$ . We refer to this vector bundle  $\pi : A\mathcal{G} \rightarrow \mathcal{G}^{(0)}$  as the Lie algebroid of  $\mathcal{G}$ .

**Remark 1.3.** The Lie algebroid  $A\mathcal{G}$  of  $\mathcal{G}$  has a Lie algebroid structure. For see this it is usually more convenient to identify  $A\mathcal{G}$  with the restriction to the unit space of the vertical tangent bundle along the fiber of the source map  $s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ . This identification is not canonical, it corresponds indeed to a particular choice of splitting of the short exact sequence of vector bundles over  $\mathcal{G}^{(0)}$ .

$$0 \longrightarrow T\mathcal{G}^{(0)} \longrightarrow T_{\mathcal{G}^{(0)}}\mathcal{G} \longrightarrow A\mathcal{G} \longrightarrow 0.$$

The Lie algebroid structure consists of

$$(\pi : A\mathcal{G} \longrightarrow \mathcal{G}^{(0)}, dr : A\mathcal{G} \longrightarrow T\mathcal{G}^{(0)}, [ , ]_{A\mathcal{G}})$$

where the anchor map  $dr$  is the restriction of the differential of  $r$  to  $A\mathcal{G}$ , and the Lie bracket  $[ , ]_{A\mathcal{G}}$  on the space of section of  $A\mathcal{G}$  is given the usual Lie bracket of the right invariant vector fields, restricted to  $\mathcal{G}^{(0)}$ . We will not make use on this paper of this additional structure.

1.1. Generalized morphisms: the Hilsum–Skandalis category

Lie groupoids form a category with strict morphisms of groupoids. It is now classical in Lie groupoid’s theory that the right category to consider is the one in which Morita equivalences correspond precisely to isomorphisms. We review some basic definitions and properties of generalized morphisms between Lie groupoids, see [33], Section 2.1, or [22,28,26] for more detailed discussions.

We first recall the notions of groupoid actions and principal bundles over groupoids. Given a Lie groupoid  $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ , a right  $\mathcal{G}$ -bundle over a manifold  $M$  is a manifold  $P$  equipped with maps as in

$$\begin{array}{ccc}
 P & & \mathcal{G} \\
 \pi \downarrow & \searrow \epsilon & \downarrow r \downarrow s \\
 M & & \mathcal{G}^{(0)},
 \end{array}$$

together with a smooth right  $\mathcal{G}$ -action  $\mu : P \times_{(\epsilon,r)} \mathcal{G} \rightarrow P$ ,  $\mu(p, \gamma) = p\gamma$ , such that  $\pi(p\gamma) = \pi(p)$  and  $p(\gamma_1 \cdot \gamma_2) = (p\gamma_1)\gamma_2$  for any  $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$ . Here  $P \times_{(\epsilon,r)} \mathcal{G}$  denotes the fiber product of  $\epsilon : P \rightarrow \mathcal{G}^{(0)}$  and  $r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ . A left  $\mathcal{G}$ -bundle can be defined similarly. A  $\mathcal{G}$ -bundle  $P$  is called principal if

- (i)  $\pi$  is a surjective submersion, and
- (ii) the map  $P \times_{(\epsilon,r)} \mathcal{G} \rightarrow P \times_M P$ ,  $(p, \gamma) \mapsto (p, p\gamma)$  is a diffeomorphism.

A principal  $\mathcal{G}$ -bundle  $P$  over  $M$  is called locally trivial if  $P$  is isomorphic to a principal  $\mathcal{G}$ -bundle

$$\frac{\bigsqcup_i \Omega_i \times \mathcal{G}}{\{(x, j, \gamma) \sim (x, i, g_{ij}(x) \cdot \gamma)\}}$$

defined by a  $\mathcal{G}$ -valued 1-cocycle on  $M$ ,

$$\{g_{ij} : \Omega_i \cap \Omega_j \longrightarrow \mathcal{G} \mid g_{ij}(x) \cdot g_{jk}(x) = g_{ik}(x)\}$$

with respect to a cover  $\{\Omega_i\}$  of  $M$ .

Let  $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$  and  $\mathcal{H} \rightrightarrows \mathcal{H}^{(0)}$  be two Lie groupoids. A principal  $\mathcal{G}$ -bundle over  $\mathcal{H}$  is a right principal  $\mathcal{G}$ -bundle over  $\mathcal{H}^{(0)}$  which is also a left  $\mathcal{H}$ -bundle over  $\mathcal{G}^{(0)}$  such that the right  $\mathcal{G}$ -action and the left  $\mathcal{H}$ -action commute, formally denoted by

$$\begin{array}{ccc}
 \mathcal{H} & & \mathcal{G} \\
 \Downarrow & \swarrow P_f & \Downarrow \\
 \mathcal{H}^{(0)} & & \mathcal{G}^{(0)}.
 \end{array}$$



A  $\mathcal{G}$ -valued 1-cocycle on  $\mathcal{H}$  with respect to an indexed open covering  $\{\Omega_i\}_{i \in I}$  of  $\mathcal{H}^{(0)}$  is a collection of smooth maps

$$f_{ij} : \mathcal{H}_{\Omega_j}^{\Omega_i} \longrightarrow \mathcal{G},$$

satisfying the following cocycle condition:  $\forall \gamma \in \mathcal{H}_{ij}$  and  $\forall \gamma' \in \mathcal{H}_{jk}$  with  $s(\gamma) = r(\gamma')$ , we have

$$f_{ij}(\gamma)^{-1} = f_{ji}(\gamma^{-1}) \quad \text{and} \quad f_{ij}(\gamma) \cdot f_{jk}(\gamma') = f_{ik}(\gamma \cdot \gamma').$$

We will denote this data by  $f = \{(\Omega_i, f_{ij})\}_{i \in I}$ . It is equivalent to a strict morphism of groupoids

$$\begin{array}{ccc} \mathcal{H}_\Omega = \bigsqcup_{i,j} \mathcal{H}_{\Omega_j}^{\Omega_i} & \xrightarrow{f} & \mathcal{G} \\ \Downarrow & & \Downarrow \\ \bigsqcup_i \Omega_i & \longrightarrow & \mathcal{G}^{(0)}. \end{array}$$

Associated to a  $\mathcal{G}$ -valued 1-cocycle on  $\mathcal{H}$ , there is a canonical defined principal  $\mathcal{G}$ -bundle over  $\mathcal{H}$ . In fact, any principal  $\mathcal{G}$ -bundle over  $\mathcal{H}$  is locally trivial (cf. [26]).

We can now define generalized morphisms between two Lie groupoids.

**Definition 1.4 (Generalized morphism).** Let  $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$  and  $\mathcal{H} \rightrightarrows \mathcal{H}^{(0)}$  be two Lie groupoids. A right  $\mathcal{G}$ -principal bundle over  $\mathcal{H}$ , also called a generalized morphism (or Hilsum–Skandalis morphism),  $f : \mathcal{H} \dashrightarrow \mathcal{G}$ , is given by one of the three equivalent data:

- (1) A locally trivial right principal  $\mathcal{G}$ -bundle  $P_f$  over  $\mathcal{H}$ ,

$$\begin{array}{ccc} \mathcal{H} & & \mathcal{G} \\ & \swarrow P_f & \searrow \\ \mathcal{H}^{(0)} & & \mathcal{G}^{(0)}. \end{array}$$

- (2) A 1-cocycle  $f = \{(\Omega_i, f_{ij})\}_{i \in I}$  on  $\mathcal{H}$  with values in  $\mathcal{G}$ .
- (3) A strict morphism of groupoids

$$\begin{array}{ccc} \mathcal{H}_\Omega = \bigsqcup_{i,j} \mathcal{H}_{\Omega_j}^{\Omega_i} & \xrightarrow{f} & \mathcal{G} \\ \Downarrow & & \Downarrow \\ \bigsqcup_i \Omega_i & \longrightarrow & \mathcal{G}^{(0)}, \end{array}$$

for an open cover  $\Omega = \{\Omega_i\}$  of  $\mathcal{H}^{(0)}$ .

Two generalized morphisms  $f$  and  $g$  are called equivalent if the corresponding right  $\mathcal{G}$ -principal bundles  $P_f$  and  $P_g$  over  $\mathcal{H}$  are isomorphic.

As the name suggests, generalized morphism generalizes the notion of strict morphisms and can be composed. Indeed, if  $P$  and  $P'$  are generalized morphisms from  $\mathcal{H}$  to  $\mathcal{G}$  and from  $\mathcal{G}$  to  $\mathcal{L}$  respectively, then

$$P \times_{\mathcal{G}} P' := P \times_{\mathcal{G}^{(0)}} P' / (p, p') \sim (p \cdot \gamma, \gamma^{-1} \cdot p')$$

is a generalized morphism from  $\mathcal{H}$  to  $\mathcal{L}$ . Consider the category  $Grpd_{HS}$  with objects Lie groupoids and morphisms given by equivalent classes of generalized morphisms. There is a functor

$$Grpd \longrightarrow Grpd_{HS} \tag{1.1}$$

where  $Grpd$  is the strict category of groupoids. Then the composition is associative in  $Grpd_{HS}$ . Two groupoids are Morita equivalent if they are isomorphic in  $Grpd_{HS}$ . For example, given a Lie groupoid  $\mathcal{H} \rightrightarrows \mathcal{H}^{(0)}$  and an open covering  $\{\Omega_i\}_i$  of  $\mathcal{H}^{(0)}$ , the canonical strict morphism of groupoids  $\mathcal{H}_\Omega \rightarrow \mathcal{H}$  is a Morita equivalence.

1.2. *Twistings on Lie groupoids*

In this paper, we are only going to consider  $PU(H)$ -twistings on Lie groupoids where  $H$  is an infinite dimensional, complex and separable Hilbert space, and  $PU(H)$  is the projective unitary group  $PU(H)$  with the topology induced by the norm topology on  $U(H)$ .

**Definition 1.5.** A twisting  $\sigma$  on a Lie groupoid  $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$  is given by a generalized morphism

$$\sigma : \mathcal{G} \dashrightarrow PU(H).$$

Here  $PU(H)$  is viewed as a Lie groupoid with the unit space  $\{e\}$ . Two twistings  $\sigma$  and  $\sigma'$  are called equivalent if they are equivalent as generalized morphisms.

So a twisting on a Lie groupoid  $\mathcal{G}$  is a locally trivial right principal  $PU(H)$ -bundle over  $\mathcal{G}$ ,

$$\begin{array}{ccc}
 \mathcal{G} & P_\sigma & PU(H) \\
 \Downarrow & \swarrow & \Downarrow \\
 \mathcal{G}^{(0)} & & \{e\}.
 \end{array} \tag{1.2}$$

Equivalently, a twisting on  $\mathcal{G}$  is given by a  $PU(H)$ -valued 1-cocycle on  $\mathcal{G}$ ,

$$g_{ij} : \mathcal{G}_{\Omega_j}^{\Omega_i} \longrightarrow PU(H)$$

for an open cover  $\Omega = \{\Omega_i\}$  of  $\mathcal{G}^{(0)}$ . That is, a twisting  $\sigma$  on a Lie groupoid  $\mathcal{G}$  is given by a strict morphism of groupoids

$$\begin{array}{ccc}
 \mathcal{G}_\Omega = \bigsqcup_{i,j} \mathcal{G}_i^j & \longrightarrow & PU(H) \\
 \Downarrow & & \Downarrow \\
 \bigsqcup_i \Omega_i & \longrightarrow & \{e\}
 \end{array} \tag{1.3}$$

for an open cover  $\Omega = \{\Omega_i\}$  of  $\mathcal{G}^{(0)}$ .

**Remark 1.6.** The definition of generalized morphisms given in the last subsection was for two Lie groupoids. The group  $PU(H)$  it is not precisely a Lie group but it makes perfectly sense to speak of generalized morphisms from Lie groupoids to this infinite dimensional “Lie” groupoid following exactly the same definition. Indeed, it is obvious once one looks at equivalent datas (1.2) and (1.3) above (recall  $PU(H)$  is considered with the topology induced by the norm topology on  $U(H)$ ).

**Remark 1.7.** A twisting on a Lie groupoid  $\mathcal{G} \rightrightarrows M$  gives rise to a  $U(1)$ -central extension over the Morita equivalent groupoid  $\mathcal{G}_\Omega$  by pull-back the  $U(1)$ -central extension of  $PU(H)$

$$1 \longrightarrow U(1) \longrightarrow U(H) \longrightarrow PU(H) \longrightarrow 1.$$

We will not call a  $U(1)$ -central extension of a Morita equivalent groupoid of  $\mathcal{G}$  a twisting on  $\mathcal{G}$  as in [33]. This is due to the fact that the associated principal  $PU(H)$ -bundle might depend on the choice of Morita equivalence bibundles, even though the isomorphism class of principal  $PU(H)$ -bundle does not depend on the choice of Morita equivalence bibundles. It is important in applications to remember the  $PU(H)$ -bundle arising from a twisting, not just its isomorphism class.

**Example 1.8.** We give a list of various twistings on standard groupoids appearing in this paper.

(1) (Twisting on Lie groups.) Let  $G$  be a Lie group. Then

$$G \rightrightarrows \{e\}$$

is a Lie groupoid. A twisting on  $G$  is given by a projective unitary representation

$$G \longrightarrow PU(H).$$

(2) (Twisting on manifolds.) Let  $X$  be a  $C^\infty$ -manifold. We can consider the groupoid

$$X \rightrightarrows X$$

where every morphism is the identity over  $X$ . A twisting on  $X$  is given by a locally trivial principal  $PU(H)$ -bundle over  $X$ , or equivalently, a twisting on  $X$  is defined by a strict homomorphism

$$\begin{array}{ccc}
 X_\Omega = \bigsqcup_{i,j} \Omega_{ij} & \longrightarrow & PU(H) \\
 \Downarrow & & \Downarrow \\
 \bigsqcup_i \Omega_i & \longrightarrow & \{e\}
 \end{array}$$

with respect to an open cover  $\{\Omega_i\}$  of  $X$ , where  $\Omega_{ij} = \Omega_i \cap \Omega_j$ . Therefore, the restriction of a twisting  $\sigma$  on a Lie groupoid  $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$  to its unit  $\mathcal{G}^{(0)}$  defines a twisting  $\sigma_0$  on the manifold  $\mathcal{G}^{(0)}$ .

- (3) (Orientation twisting.) Let  $X$  be a smooth oriented manifold of dimension  $n$ . The tangent bundle  $TX \rightarrow X$  defines a natural generalized morphism

$$X \dashrightarrow SO(n).$$

Note that the fundamental representation of the  $\text{Spin}^c$  group  $\text{Spin}^c(n)$  gives rise to a commutative diagram of Lie group homomorphisms

$$\begin{array}{ccc}
 \text{Spin}^c(n) & \longrightarrow & U(\mathbb{C}^{2^n}) \\
 \downarrow & & \downarrow \\
 SO(n) & \longrightarrow & PU(\mathbb{C}^{2^n}).
 \end{array}$$

With a choice of inclusion  $\mathbb{C}^{2^n}$  into a Hilbert space  $H$ , we have a canonical twisting, called the orientation twisting of  $X$  associated to its tangent bundle, denoted by

$$o_{TX} : X \dashrightarrow PU(H).$$

- (4) (Pull-back twisting.) Given a twisting  $\sigma$  on  $\mathcal{G}$  and for any generalized homomorphism  $\phi : \mathcal{H} \rightarrow \mathcal{G}$ , there is a pull-back twisting

$$\phi^* \sigma : \mathcal{H} \dashrightarrow PU(H)$$

defined by the composition of  $\phi$  and  $\sigma$ . In particular, for a continuous map  $\phi : X \rightarrow Y$ , a twisting  $\sigma$  on  $Y$  gives a pull-back twisting  $\phi^* \sigma$  on  $X$ . The principal  $PU(H)$ -bundle over  $X$  defined by  $\phi^* \sigma$  is the pull-back of the principal  $PU(H)$ -bundle on  $Y$  associated to  $\sigma$ .

- (5) (Twisting on  $G$ -spaces.) Let  $G$  be a Lie group acting by diffeomorphisms in a manifold  $M$ . The transformation groupoid associated to this action is

$$M \rtimes G \rightrightarrows M.$$

As a set  $M \rtimes G = M \times G$ , and the maps are given by  $s(x, g) = x \cdot g$ ,  $r(x, g) = x$ , the product given by  $(x, g) \circ (x \cdot g, h) = (x, gh)$ , the unit is  $u(x) = (x, e)$  and with inverse  $(x, g)^{-1} = (x \cdot g, g^{-1})$ . The projection from  $M \rtimes G$  to  $G$  defines a strict homomorphism between groupoids  $M \rtimes G$  and  $G$ , then any projective representation  $G \rightarrow PU(H)$  defines a twisting on  $M \rtimes G$ , which is the pull-back twisting from a twisting on  $G$ .

(6) (Twisting on pair groupoid.) Let  $M$  be a  $C^\infty$ -manifold. We can consider the groupoid

$$M \times M \rightrightarrows M$$

with  $s(x, y) = y$ ,  $r(x, y) = x$  and the product given by  $(x, y) \circ (y, z) = (x, z)$ . Any locally trivial principal  $PU(H)$ -bundle  $P$  is trivial as the left action of  $M \times M \rightrightarrows M$  on  $P$  gives a canonical trivialization

$$P \cong M \times PU(H).$$

Hence, any twisting on the pair groupoid  $M \times M \rightrightarrows M$  is always trivial.

(7) (Twisting on fiber product groupoid.) Let  $N \xrightarrow{p} M$  be a submersion. We consider the fiber product  $N \times_M N := \{(n, n') \in N \times N : p(n) = p(n')\}$ , which is a manifold because  $p$  is a submersion. We can then take the groupoid

$$N \times_M N \rightrightarrows N$$

which is a subgroupoid of the pair groupoid  $N \times N \rightrightarrows N$ . Note that this groupoid is in fact Morita equivalent to the groupoid  $M \rightrightarrows M$ . A twisting on  $N \times_M N \rightrightarrows N$  is given by a pull-back twisting from a twisting on  $M$ .

(8) (Twisting on vector bundles.) Let  $\pi : E \rightarrow X$  be a vector bundle over a manifold  $X$ . We consider the groupoid

$$E \rightrightarrows X$$

with the source map  $s(\xi) = \pi(\xi)$ , the target map  $r(\xi) = \pi(\xi)$ , the product given by  $\xi \circ \eta = \xi + \eta$ . The unit is zero section and the inverse is the additive inverse at each fiber. With respect to a trivialization cover  $\{E|_{\Omega_i} \cong \Omega_i \times V\}$  of  $E$ , a twisting on the groupoid  $E \rightrightarrows X$  is given by a  $PU(H)$ -valued cocycle

$$g_{ij} : \Omega_{ij} \times V \longrightarrow PU(H)$$

satisfying  $g_{ij}(x, \xi) \cdot g_{jk}(x, \eta) = g_{ik}(x, \xi + \eta)$ . A twisting on the groupoid  $E \rightrightarrows X$  is a pull-back twisting  $\pi^* \sigma$  from a twisting  $\sigma$  on  $X$  if  $g_{ij} : \Omega_{ij} \times V \longrightarrow PU(H)$  is constant in  $V$ . Note that the pull-back twisting agrees with the pull-back twisting on  $E$  by  $\pi$  as a topological space.

In this paper, we will mainly deal with the holonomy groupoids associated to regular foliations. Let  $M$  be a manifold of dimension  $n$ . Let  $F$  be a subvector bundle of the tangent bundle  $TM$ . We say that  $F$  is integrable if  $C^\infty(F) := \{X \in C^\infty(M, TM) : \forall x \in M, X_x \in F_x\}$  is a Lie subalgebra of  $C^\infty(M, TM)$ . This induces a partition of  $M$  in embedded submanifolds (the leaves of the foliation), given by the solution of integrating  $F$ .

The holonomy groupoid of  $(M, F)$  is a Lie groupoid

$$\mathcal{G}_M \rightrightarrows M$$

with Lie algebroid  $A\mathcal{G} = F$  and minimal in the following sense: any Lie groupoid integrating the foliation, that is having  $F$  as Lie algebroid, contains an open subgroupoid which maps onto

the holonomy groupoid by a smooth morphism of Lie groupoids. The holonomy groupoid was constructed by Ehresmann [17] and Winkelkemper [36] (see also [6,19,30]).

**Definition 1.9** (*Twisting on the space of leaves of a foliation*). Let  $(M, F)$  be a regular foliation with holonomy groupoid  $\mathcal{G}_M$ . A twisting on the space of leaves is by definition a twisting on the holonomy groupoid  $\mathcal{G}_M$ . We will often use the notation

$$M/F \dashrightarrow PU(H)$$

for the correspondent generalized morphism.

Notice that by definition a twisting on the spaces of leaves is a twisting on the base  $M$  which admits a compatible action of the holonomy groupoid. It is however not enough to have a twisting on base which is leafwise constant, see for instance Remark 1.4(c) in [22].

### 1.3. Twistings on tangent groupoids

In this subsection, we review the notion of Connes’ tangent groupoids from deformation to the normal cone point of view, and discuss the induced twistings on tangent groupoids.

#### 1.3.1. Deformation to the normal cone

The tangent groupoid is a particular case of a geometric construction that we describe here.

Let  $M$  be a  $C^\infty$  manifold and  $X \subset M$  be a  $C^\infty$  submanifold. We denote by  $\mathcal{N}_X^M$  the normal bundle to  $X$  in  $M$ . We define the following set

$$\mathcal{D}_X^M := \mathcal{N}_X^M \times \mathbb{0} \sqcup M \times \mathbb{R}^*. \tag{1.4}$$

The purpose of this section is to recall how to define a  $C^\infty$ -structure in  $\mathcal{D}_X^M$ . This is more or less classical, for example it was extensively used in [22].

Let us first consider the case where  $M = \mathbb{R}^p \times \mathbb{R}^q$  and  $X = \mathbb{R}^p \times \{0\}$  (here we identify  $X$  canonically with  $\mathbb{R}^p$ ). We denote by  $q = n - p$  and by  $\mathcal{D}_p^n$  for  $\mathcal{D}_{\mathbb{R}^p}^{\mathbb{R}^n}$  as above. In this case we have that  $\mathcal{D}_p^n = \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}$  (as a set). Consider the bijection  $\psi : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} \rightarrow \mathcal{D}_p^n$  given by

$$\psi(x, \xi, t) = \begin{cases} (x, \xi, 0) & \text{if } t = 0, \\ (x, t\xi, t) & \text{if } t \neq 0, \end{cases} \tag{1.5}$$

whose inverse is given explicitly by

$$\psi^{-1}(x, \xi, t) = \begin{cases} (x, \xi, 0) & \text{if } t = 0, \\ (x, \frac{1}{t}\xi, t) & \text{if } t \neq 0. \end{cases}$$

We can consider the  $C^\infty$ -structure on  $\mathcal{D}_p^n$  induced by this bijection.

We pass now to the general case. A local chart  $(\mathcal{U}, \phi)$  in  $M$  is said to be an  $X$ -slice if:

- (1)  $\phi : \mathcal{U} \rightarrow U \subset \mathbb{R}^p \times \mathbb{R}^q$  is a diffeomorphism.
- (2) If  $V = U \cap (\mathbb{R}^p \times \{0\})$ , then  $\phi^{-1}(V) = \mathcal{U} \cap X$ , denoted by  $\mathcal{V}$ .

With this notation,  $\mathcal{D}_V^U \subset \mathcal{D}_p^n$  as an open subset. We may define a function

$$\tilde{\phi} : \mathcal{D}_{\mathcal{V}}^{\mathcal{U}} \longrightarrow \mathcal{D}_V^U \tag{1.6}$$

in the following way: For  $x \in \mathcal{V}$  we have  $\phi(x) \in \mathbb{R}^p \times \{0\}$ . If we write  $\phi(x) = (\phi_1(x), 0)$ , then

$$\phi_1 : \mathcal{V} \longrightarrow V \subset \mathbb{R}^p$$

is a diffeomorphism. We set  $\tilde{\phi}(v, \xi, 0) = (\phi_1(v), d_N \phi_v(\xi), 0)$  and  $\tilde{\phi}(u, t) = (\phi(u), t)$  for  $t \neq 0$ . Here  $d_N \phi_v : N_v \rightarrow \mathbb{R}^q$  is the normal component of the derivative  $d\phi_v$  for  $v \in \mathcal{V}$ . It is clear that  $\tilde{\phi}$  is also a bijection (in particular it induces a  $C^\infty$  structure on  $\mathcal{D}_{\mathcal{V}}^{\mathcal{U}}$ ). Now, let us consider an atlas  $\{(\mathcal{U}_\sigma, \phi_\sigma)\}_{\sigma \in \Delta}$  of  $M$  consisting of  $X$ -slices. Then the collection  $\{(\mathcal{D}_{\mathcal{V}_\sigma}^{\mathcal{U}_\sigma}, \tilde{\phi}_\sigma)\}_{\sigma \in \Delta}$  is a  $C^\infty$ -atlas of  $\mathcal{D}_X^M$  (Proposition 3.1 in [8]).

**Definition 1.10** (Deformation to the normal cone). Let  $X \subset M$  be as above. The set  $\mathcal{D}_X^M$  equipped with the  $C^\infty$  structure induced by the atlas of  $X$ -slices is called the deformation to the normal cone associated to the embedding  $X \subset M$ .

One important feature about the deformation to the normal cone is the functoriality. More explicitly, let  $f : (M, X) \rightarrow (M', X')$  be a  $C^\infty$  map  $f : M \rightarrow M'$  with  $f(X) \subset X'$ . Define  $\mathcal{D}(f) : \mathcal{D}_X^M \rightarrow \mathcal{D}_{X'}^{M'}$  by the following formulas:

- (1)  $\mathcal{D}(f)(m, t) = (f(m), t)$  for  $t \neq 0$ ,
- (2)  $\mathcal{D}(f)(x, \xi, 0) = (f(x), d_N f_x(\xi), 0)$ , where  $d_N f_x$  is by definition the map

$$(\mathcal{N}_X^M)_x \xrightarrow{d_N f_x} (\mathcal{N}_{X'}^{M'})_{f(x)}$$

induced by  $T_x M \xrightarrow{df_x} T_{f(x)} M'$ .

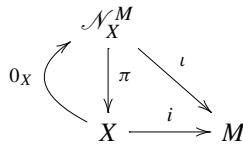
Then  $\mathcal{D}(f) : \mathcal{D}_X^M \rightarrow \mathcal{D}_{X'}^{M'}$  is a  $C^\infty$ -map (Proposition 3.4 in [8]). In the language of categories, the deformation to the normal cone construction defines a functor

$$\mathcal{D} : \mathcal{C}_2^\infty \longrightarrow \mathcal{C}^\infty, \tag{1.7}$$

where  $\mathcal{C}^\infty$  is the category of  $C^\infty$ -manifolds and  $\mathcal{C}_2^\infty$  is the category of pairs of  $C^\infty$ -manifolds.

**Proposition 1.11.** Given a twisting  $\sigma : M \rightarrow PU(H)$  as a principal  $PU(H)$ -bundle over  $M$ , there is a canonical twisting on  $\mathcal{D}_X^M$  which restricts to the pull-back twisting  $\pi^*(\sigma|_X)$  on  $\mathcal{N}_X^M$  by the map  $\pi : \mathcal{N}_X^M \rightarrow X$ .

**Proof.** Under the identification of the normal bundle  $\mathcal{N}_X^M$  with a tubular neighborhood of  $X$  in  $M$ , we have the following diagram of maps



where  $0_X$  is the zero section of the normal bundle. Note that  $\pi \circ 0_X = Id_X$ ,  $i = \iota \circ 0_X$ , and  $0_X \circ \pi$  is homotopic to the identity map on  $\mathcal{N}_X^M$ , which imply that

$$\pi^* \circ i^* \sigma = \pi^* \circ 0_X^* \circ \iota^* \sigma = (0_X \circ \pi)^* \circ \iota^* \sigma$$

is homotopic to  $\iota^* \sigma$ . Hence, the principal  $PU(H)$ -bundle over  $\mathcal{N}_X^M$  associated to  $\iota^* \sigma$  is isomorphic to the pull-back principal  $PU(H)$ -bundle on  $\mathcal{N}_X^M$  associated to  $\pi^*(\sigma|_X)$ .  $\square$

1.3.2. The tangent groupoid of a groupoid

**Definition 1.12** (Tangent groupoid). Let  $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$  be a Lie groupoid. The tangent groupoid associated to  $\mathcal{G}$  is the groupoid that has

$$\mathcal{D}_{\mathcal{G}^{(0)}}^{\mathcal{G}} = \mathcal{N}_{\mathcal{G}^{(0)}}^{\mathcal{G}} \times \{0\} \sqcup \mathcal{G} \times \mathbb{R}^*$$

as the set of arrows and  $\mathcal{G}^{(0)} \times \mathbb{R}$  as the units, with:

- (1)  $s^T(x, \eta, 0) = (x, 0)$  and  $r^T(x, \eta, 0) = (x, 0)$  at  $t = 0$ .
- (2)  $s^T(\gamma, t) = (s(\gamma), t)$  and  $r^T(\gamma, t) = (r(\gamma), t)$  at  $t \neq 0$ .
- (3) The product is given by  $m^T((x, \eta, 0), (x, \xi, 0)) = (x, \eta + \xi, 0)$  and  $m^T((\gamma, t), (\beta, t)) = (m(\gamma, \beta), t)$  if  $t \neq 0$  and if  $r(\beta) = s(\gamma)$ .
- (4) The unit map  $u^T : \mathcal{G}^{(0)} \rightarrow \mathcal{G}^T$  is given by  $u^T(x, 0) = (x, 0)$  and  $u^T(x, t) = (u(x), t)$  for  $t \neq 0$ .

We denote  $\mathcal{G}^T = \mathcal{D}_{\mathcal{G}^{(0)}}^{\mathcal{G}}$  and  $A\mathcal{G} = \mathcal{N}_{\mathcal{G}^{(0)}}^{\mathcal{G}}$  as a vector bundle over  $\mathcal{G}^{(0)}$ . Then we have a family of Lie groupoids parametrized by  $\mathbb{R}$ , which itself is a Lie groupoid

$$\mathcal{G}^T = A\mathcal{G} \times \{0\} \sqcup \mathcal{G} \times \mathbb{R}^* \rightrightarrows \mathcal{G}^{(0)} \times \mathbb{R}.$$

As a consequence of the functoriality of the deformation to the normal cone, one can show that the tangent groupoid is in fact a Lie groupoid compatible with the Lie groupoid structures of  $\mathcal{G}$  and  $A\mathcal{G}$  (considered as a Lie groupoid with its vector bundle structure). Indeed, it is immediate that if we identify in a canonical way  $\mathcal{D}_{\mathcal{G}^{(0)}}^{\mathcal{G}^{(2)}}$  with  $(\mathcal{G}^T)^{(2)}$ , then

$$m^T = \mathcal{D}(m), \quad s^T = \mathcal{D}(s), \quad r^T = \mathcal{D}(r), \quad u^T = \mathcal{D}(u)$$

where we are considering the following pair morphisms:

$$\begin{aligned} m &: (\mathcal{G}^{(2)}, \mathcal{G}^{(0)}) \longrightarrow (\mathcal{G}, \mathcal{G}^{(0)}), \\ s, r &: (\mathcal{G}, \mathcal{G}^{(0)}) \longrightarrow (\mathcal{G}^{(0)}, \mathcal{G}^{(0)}), \\ u &: (\mathcal{G}^{(0)}, \mathcal{G}^{(0)}) \longrightarrow (\mathcal{G}, \mathcal{G}^{(0)}). \end{aligned}$$



**Example 1.13.**

- (1) The tangent groupoid of a group: Let  $G$  be a Lie group. We consider it as a groupoid with one element,  $G \rightrightarrows \{e\}$ , as above. By definition  $AG = T_eG$ , *i.e.*, the Lie algebroid coincides with the Lie algebra  $\mathfrak{g}$  of the group  $G$ . Hence, the tangent groupoid is

$$G^T := \mathfrak{g} \times \{0\} \sqcup G \times \mathbb{R}^* \rightrightarrows G \times \mathbb{R}.$$

- (2) The tangent groupoid of a  $C^\infty$ -manifold: Let  $M$  be a  $C^\infty$ -manifold. We consider the associated pair groupoid  $M \times M \rightrightarrows M$ . In this case, the Lie algebroid can be identified with  $TM$  and the tangent groupoid take the following form

$$\mathcal{G}_M^T = TM \times \{0\} \sqcup M \times M \times (0, 1] \rightrightarrows M \times \mathbb{R}.$$

- (3) The Thom groupoid: Let  $N \xrightarrow{p} T$  be a vector bundle over a smooth manifold  $T$ . Consider the fiberwise pair groupoid over  $T$ ,

$$N \times_T N \rightrightarrows N. \tag{1.8}$$

The Lie algebroid is  $N \oplus N$  as a vector bundle over  $N$ , then the tangent groupoid, denoted by  $\mathcal{G}_{\text{Thom}}$ , takes the following form

$$\mathcal{G}_{\text{Thom}} = N \oplus N \times \{0\} \sqcup N \times_T N \times \mathbb{R}^* \rightrightarrows N \times \mathbb{R}.$$

See [16] for a motivation for the name of this groupoid.

- (4) Tangent groupoid of a holonomy groupoid: Let  $\mathcal{G}_{M/F}$  be the holonomy groupoid of a foliated manifold  $M/F$ . Then the tangent groupoid is given by

$$\mathcal{G}_{M/F}^T = F \times \{0\} \sqcup \mathcal{G}_{M/F} \rightrightarrows M \times \mathbb{R}.$$

The following proposition could be used alternatively to construct the twisted index morphism and to develop the twisted index theory. However in the sequel we will rather use the central extension approach because the proofs are simpler, and because we obtain naturally the twisted index theory as a factor of an  $S^1$ -equivariant index theory. It gives though a nice geometric idea of the strict deformation quantization methods.

**Proposition 1.14.** *Given a twisting  $\sigma$  on a Lie groupoid  $\mathcal{G}$ , there is a canonical twisting  $\sigma^T$  on its tangent groupoid  $\mathcal{G}^T$  such that*

$$\sigma^T|_{A\mathcal{G}} = \pi^*(\sigma|_{\mathcal{G}^{(0)}})$$

where  $\pi : A\mathcal{G} \rightarrow \mathcal{G}^{(0)}$  is the projection.

**Proof.** The twisting  $\sigma$  on  $\mathcal{G}$  is given by a  $PU(H)$ -principal bundle  $P_\sigma$  with a compatible left action of  $\mathcal{G}$ . By definition of the groupoid action, the units of the groupoid act as identities, hence we can consider the action as a  $C^\infty$ -morphism in the category of  $C^\infty$ -pairs:

$$(\mathcal{G} \times_M P_\sigma, M \times_M P_\sigma) \longrightarrow (P_\sigma, P_\sigma).$$

We can then apply the deformation to the normal cone functor to obtain the desired  $PU(H)$ -principal bundle with a compatible  $\mathcal{G}^T$ -action, which gives the desired twisting.  $\square$

*1.3.3. The tangent groupoid of a groupoid immersion*

We briefly discuss here the tangent groupoid of an immersion of groupoids which is called the normal groupoid in [22].

Consider an immersion of Lie groupoids  $\mathcal{G}_1 \xrightarrow{\varphi} \mathcal{G}_2$ . Let  $\mathcal{G}_1^N = \mathcal{N}_{\mathcal{G}_1}^{\mathcal{G}_2}$  be the total space of the normal bundle to  $\varphi$ , and  $(\mathcal{G}_1^{(0)})^N$  be the total space of the normal bundle to  $\varphi_0 : \mathcal{G}_1^{(0)} \rightarrow \mathcal{G}_2^{(0)}$ . The deformation to the normal bundle construction allows us to consider a  $C^\infty$  structure on

$$\mathcal{G}_\varphi := \mathcal{G}_1^N \times \{0\} \sqcup \mathcal{G}_2 \times \mathbb{R}^*,$$

such that  $\mathcal{G}_1^N \times \{0\}$  is a closed saturated submanifold and so  $\mathcal{G}_2 \times \mathbb{R}^*$  is an open submanifold.

**Remark 1.15.** Consider  $\mathcal{G}_1^N \rightrightarrows (\mathcal{G}_1^{(0)})^N$  with the following structure maps: The source map is the derivation in the normal direction  $d_{Ns} : \mathcal{G}_1^N \rightarrow (\mathcal{G}_1^{(0)})^N$  of the source map (seen as a pair of maps)  $s : (\mathcal{G}_2, \mathcal{G}_1) \rightarrow (\mathcal{G}_2^{(0)}, \mathcal{G}_1^{(0)})$  and similarly for the target map.

As remarked by Hilsaum and Skandalis (Remarks 3.1, 3.19 in [22]),  $\mathcal{G}_1^N$  may fail to inherit a Lie groupoid structure (see counterexample just before Section 4 in [22]). A sufficient condition is when  $(\mathcal{G}_1^{(0)})^N$  is a  $\mathcal{G}_1^N$ -vector bundle over  $\mathcal{G}_1^{(0)}$ . This is the case when  $\mathcal{G}_1^x \rightarrow \mathcal{G}_2^{\varphi(x)}$  is étale for every  $x \in \mathcal{G}_1^{(0)}$  (in particular if the groupoids are étale) or when one considers a manifold with two foliations  $F_1 \subset F_2$  and the induced immersion (again Remarks 3.1, 3.19 in [22]), but also for the case of the inclusion of extensions treated in (2.18) below. We will anyway describe explicitly the groupoid structure when needed.

The following results are an immediate consequence of the functoriality of the deformation to the normal cone construction.

**Proposition 1.16.** (See Hilsaum and Skandalis, Remarks 3.1, 3.19 in [22].) *Consider an immersion  $\mathcal{G}_1 \xrightarrow{\varphi} \mathcal{G}_2$  as above for which  $(\mathcal{G}_1)^N$  inherits a Lie groupoid structure (precedent remark). Let  $\mathcal{G}_{\varphi_0} := (\mathcal{G}_1^{(0)})^N \times \{0\} \sqcup \mathcal{G}_2^{(0)} \times \mathbb{R}^*$  be the deformation to the normal cone of the pair  $(\mathcal{G}_2^{(0)}, \mathcal{G}_1^{(0)})$ . The groupoid*

$$\mathcal{G}_\varphi \rightrightarrows \mathcal{G}_{\varphi_0} \tag{1.9}$$

*with structure maps compatible with the ones of the groupoids  $\mathcal{G}_2 \rightrightarrows \mathcal{G}_2^{(0)}$  and  $\mathcal{G}_1^N \rightrightarrows (\mathcal{G}_1^{(0)})^N$ , is a Lie groupoid with  $C^\infty$ -structures coming from the deformation to the normal cone.*

**Proposition 1.17.** *Given an immersion of Lie groupoids  $\mathcal{G}_1 \xrightarrow{\varphi} \mathcal{G}_2$  as above and a twisting  $\sigma$  on  $\mathcal{G}_2$ . There is a canonical twisting  $\sigma_\varphi$  on the Lie groupoid  $\mathcal{G}_\varphi \rightrightarrows \mathcal{G}_{\varphi_0}$ , extending the pull-back twisting on  $\mathcal{G}_2 \times \mathbb{R}^*$  from  $\sigma$ .*

## 2. Analytic index morphism for a twisted Lie groupoid

In this section, we first review the analytic index morphism for any Lie groupoid, and then develop an analytic index morphism for a Lie groupoid with a twisting.

### 2.1. The case of Lie groupoids

#### 2.1.1. The convolution $C^*$ -algebra of a Lie groupoid

We recall how to define the reduced and maximal  $C^*$ -algebras of a Lie groupoid  $\mathcal{G} \rightrightarrows M$ ,  $C_r^*(\mathcal{G})$  and  $C^*(\mathcal{G})$ .

We start with the reduced  $C^*$ -algebra. The basic elements of  $C_r^*(\mathcal{G})$  are smooth half densities with compact support on  $\mathcal{G}$  in  $C_c^\infty(\mathcal{G}, \Omega^{\frac{1}{2}})$  where  $\Omega^{\frac{1}{2}}$  is the real line bundle over  $\mathcal{G}$  given by  $s^*(\Omega_0^{\frac{1}{2}}) \otimes t^*(\Omega_0^{\frac{1}{2}})$  and  $\Omega^{\frac{1}{2}}$  denote the half density bundle associated to  $A\mathcal{G}$ . In particular,  $\Omega_\gamma^{\frac{1}{2}} = (\Omega_0)_{s(\gamma)}^{\frac{1}{2}} \otimes (\Omega_0)_{t(\gamma)}^{\frac{1}{2}}$ . The convolution product on  $C_c^\infty(\mathcal{G}, \Omega^{\frac{1}{2}})$  is given by

$$(f * g)(\gamma) = \int_{\gamma_1 \cdot \gamma_2 = \gamma} f(\gamma_1)g(\gamma_2),$$

where the integration is performed over the resulting 1-density. The  $*$ -operation is simply given by  $f^*(\gamma) = \overline{f(\gamma^{-1})}$ .

For every  $x \in M$ , there is a representation  $\pi_x$  of  $C_c^\infty(\mathcal{G}, \Omega^{\frac{1}{2}})$  on  $L^2(\mathcal{G}_x, \Omega^{\frac{1}{2}})$  given by

$$\pi_x(f)(\xi)(\gamma) = \int_{\gamma_1 \cdot \gamma_2 = \gamma} f(\gamma_1)\xi(\gamma_2).$$

The norm  $\|f\| := \sup_{x \in M} \pi_x(f)$  is well defined (see [14], Proposition 5.3, Chapter 2 or [31] for more details) and the reduced  $C^*$ -algebra  $C_r^*(\mathcal{G})$  is the  $C^*$ -completion with respect to this norm. For obtain the maximal, one has to take the completion with respect to the norm  $\|f\| := \sup_\pi \pi(f)$  over all involutive Hilbert representations of  $C_c^\infty(\mathcal{G}, \Omega^{\frac{1}{2}})$ .

The line bundle  $\Omega^{\frac{1}{2}}$  is trivial, even if there is not a canonical trivialization. We have therefore a non-canonical isomorphism between the sections  $C_c^\infty(\mathcal{G}, \Omega^{\frac{1}{2}})$  and the compactly supported functions  $C_c^\infty(\mathcal{G})$ . In the sequel we will often use the notation  $C_c^\infty(\mathcal{G})$  even if we are thinking on half densities.

#### 2.1.2. Analytic index morphism for Lie groupoids

A  $\mathcal{G}$ -pseudodifferential operator is a family of pseudodifferential operators  $\{P_x\}_{x \in \mathcal{G}(0)}$  acting in  $C_c^\infty(\mathcal{G}_x)$  such that if  $\gamma \in \mathcal{G}$  and

$$U_\gamma : C_c^\infty(\mathcal{G}_{s(\gamma)}) \longrightarrow C_c^\infty(\mathcal{G}_{r(\gamma)})$$

the induced operator, then we have the following compatibility condition

$$P_{r(\gamma)} \circ U_\gamma = U_\gamma \circ P_{s(\gamma)}.$$

We also admit, as usual, operators acting in sections of a complex vector bundle  $E \rightarrow \mathcal{G}^{(0)}$ . There is also a differentiability condition with respect to  $x$  that can be found in [29].

In this work we are going to work exclusively with uniformly supported operators, let us recall this notion. Let  $P = \{P_x, x \in \mathcal{G}^{(0)}\}$  be a  $\mathcal{G}$ -operator, we denote by  $k_x$  the Schwartz kernel of  $P_x$ . Let

$$\text{supp } P := \overline{\bigcup_x \text{supp } k_x}$$

and

$$\text{supp}_\mu P := \mu_1(\text{supp } P),$$

where  $\mu_1(g', g) = g'g^{-1}$ . We say that  $P$  is uniformly supported if  $\text{supp}_\mu P$  is compact.

We denote by  $\Psi^m(\mathcal{G}, E)$  the space of uniformly supported  $\mathcal{G}$ -operators of order  $m$ , acting on sections of a complex vector bundle  $E$ . We denote also

$$\Psi^\infty(\mathcal{G}, E) = \bigcup_m \Psi^m(\mathcal{G}, E) \quad \text{and} \quad \Psi^{-\infty}(\mathcal{G}, E) = \bigcap_m \Psi^m(\mathcal{G}, E).$$

The composition of two such operators is again of this kind (Lemma 3, [29]). In fact,  $\Psi^\infty(\mathcal{G}, E)$  is a filtered algebra (Theorem 1, [29]), *i.e.*,

$$\Psi^m(\mathcal{G}, E)\Psi^{m'}(\mathcal{G}, E) \subset \Psi^{m+m'}(\mathcal{G}, E).$$

In particular,  $\Psi^{-\infty}(\mathcal{G}, E)$  is a bilateral ideal.

**Remark 2.1.** The choice on the support can be justified by the fact that  $\Psi^{-\infty}(\mathcal{G}, E)$  is identified with  $C_c^\infty(\mathcal{G}, \text{End}(E))$ , thanks the Schwartz kernel theorem.

The notion of principal symbol extends also to this setting. Let us denote by  $\pi : A^*\mathcal{G} \rightarrow \mathcal{G}^{(0)}$  the projection. For  $P = \{P_x, x \in \mathcal{G}^{(0)}\} \in \Psi^m(\mathcal{G}, E, E)$ , the principal symbol of  $P_x$ ,  $\sigma_m(P_x)$ , is a  $C^\infty$  section of the vector bundle  $\text{Hom}(\pi_x^*r^*E, \pi_x^*r^*E)$  over  $T^*\mathcal{G}_x$  (where  $\pi_x : T^*\mathcal{G}_x \rightarrow \mathcal{G}_x$ ), such that at each fiber the morphism is homogeneous of degree  $m$  (see [2] for more details). There is a section  $\sigma_m(P)$  of  $\text{Hom}(\pi^*E, \pi^*E)$  over  $A^*\mathcal{G}$  such that

$$\sigma_m(P)(\xi) = \sigma_m(P_x)(\xi) \in \text{Hom}(E_x, E_x) \quad \text{if } \xi \in A_x^*\mathcal{G}. \tag{2.1}$$

This induces a unique surjective linear map

$$\sigma_m : \Psi^m(\mathcal{G}, E) \longrightarrow \mathcal{S}^m(A^*\mathcal{G}, \text{Hom}(E, E)), \tag{2.2}$$

with kernel  $\Psi^{m-1}(\mathcal{G}, E)$  (see for instance Proposition 2 in [29]) and where  $\mathcal{S}^m(A^*\mathcal{G}, \text{Hom}(E, E))$  denotes the sections of the vector bundle  $\text{Hom}(\pi^*E, \pi^*E)$  over  $A^*\mathcal{G}$ , homogeneous of degree  $m$  along each fiber.

**Definition 2.2.** Let  $P = \{P_x, x \in \mathcal{G}^{(0)}\}$  be a  $\mathcal{G}$ -pseudodifferential operator.  $P$  is elliptic if  $P_x$  is elliptic for each  $x$ . We denote by  $\text{Ell}(\mathcal{G})$  the set of elliptic  $\mathcal{G}$ -pseudodifferential operators.

The linear map (2.2) for elliptic  $\mathcal{G}$ -pseudodifferential operators defines a principal symbol map:

$$\text{Ell}(\mathcal{G}) \xrightarrow{[\sigma]} K^0(A^*\mathcal{G}). \tag{2.3}$$

Connes in [12] proved that if  $P = \{P_x, x \in \mathcal{G}^{(0)}\} \in \text{Ell}(\mathcal{G})$ , then it exists  $Q \in \Psi^{-m}(\mathcal{G}, E)$  such that

$$Id_E - PQ \in \Psi^{-\infty}(\mathcal{G}, E) \quad \text{and} \quad Id_E - QP \in \Psi^{-\infty}(\mathcal{G}, E),$$

where  $Id_E$  denotes the identity operator over  $E$ . In other words,  $P$  defines an element in  $K_0(C_c^\infty(\mathcal{G}))$ , when  $E$  is trivial, given by

$$\left[ T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T^{-1} \right] - \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0(\widetilde{C_c^\infty(\mathcal{G})}), \tag{2.4}$$

where 1 is the unit in  $\widetilde{C_c^\infty(\mathcal{G})}$  (the unitarization of  $C_c^\infty(\mathcal{G})$ ), and where  $T$  is given by

$$T = \begin{pmatrix} (1 - PQ)P + P & PQ - 1 \\ 1 - QP & Q \end{pmatrix}$$

with inverse

$$T^{-1} = \begin{pmatrix} Q & 1 - QP \\ PQ - 1 & (1 - PQ)P + P \end{pmatrix}.$$

If  $E$  is not trivial we obtain in the same way an element of  $K_0(C_c^\infty(\mathcal{G}, \text{Hom}(E, F))) \cong K_0(C_c^\infty(\mathcal{G}))$  since  $C_c^\infty(\mathcal{G}, \text{Hom}(E, F))$  is Morita equivalent to  $C_c^\infty(\mathcal{G})$ .

**Definition 2.3** ( $\mathcal{G}$ -index). Let  $P$  be an elliptic  $\mathcal{G}$ -pseudodifferential operator. We denote by

$$\mathbf{Index}_{\mathcal{G}}(P) \in K_0(C_c^\infty(\mathcal{G}))$$

the element defined by  $P$  as above, called the  $\mathcal{G}$ -index of  $P$ . The  $\mathcal{G}$ -index defines a correspondence

$$\text{Ell}(\mathcal{G}) \xrightarrow{\mathbf{Index}_{\mathcal{G}}} K_0(C_c^\infty(\mathcal{G})). \tag{2.5}$$

Consider the morphism

$$K_0(C_c^\infty(\mathcal{G})) \xrightarrow{j} K_0(C_r^*(\mathcal{G})) \tag{2.6}$$

induced by the inclusion  $C_c^\infty(\mathcal{G}) \subset C_r^*(\mathcal{G})$ , then the composition

$$\text{Ell}(\mathcal{G}) \xrightarrow{\mathbf{Index}_{\mathcal{G}}} K_0(C_c^\infty(\mathcal{G})) \xrightarrow{j} K_0(C_r^*(\mathcal{G}))$$

factors through the principal symbol class. In other words, we have the following commutative diagram:

$$\begin{array}{ccc}
 \text{Ell}(\mathcal{G}) & \xrightarrow{\text{Index}_{\mathcal{G}}} & K_0(C_c^\infty(\mathcal{G})) \\
 \downarrow [\sigma] & & \downarrow j \\
 K^0(A^*\mathcal{G}) & \xrightarrow{\text{Index}_a} & K_0(C_r^*(\mathcal{G})).
 \end{array}$$

The resulting morphism

$$K^0(A^*\mathcal{G}) \xrightarrow{\text{Index}_{a,\mathcal{G}}} K_0(C_r^*(\mathcal{G})) \tag{2.7}$$

is called the analytic index morphism of  $\mathcal{G}$ . In fact,  $\text{Index}_{a,\mathcal{G}}$  is the index morphism associated to the exact sequence of  $C^*$ -algebras (see [12,15,27,29])

$$0 \longrightarrow C_r^*(\mathcal{G}) \longrightarrow \overline{\Psi^0(\mathcal{G})} \xrightarrow{\sigma} C_0(S(A^*\mathcal{G})) \longrightarrow 0 \tag{2.8}$$

where  $\overline{\Psi^0(\mathcal{G})}$  is a certain  $C^*$ -completion of  $\Psi^0(\mathcal{G})$ ,  $S(A^*\mathcal{G})$  is the sphere bundle of  $A^*\mathcal{G}$  and  $\sigma$  is the extension of the principal symbol.

We remark that this index morphism  $\text{Index}_{a,\mathcal{G}}$  can also be constructed using the tangent groupoid and its  $C^*$ -algebra. We briefly recall this construction from [27].

Notice that the evaluation morphisms extend to the  $C^*$ -algebras as in [31]:

$$C^*(\mathcal{G}^T) \xrightarrow{ev_0} C^*(A\mathcal{G}) \quad \text{for } t = 0,$$

and

$$C^*(\mathcal{G}^T) \xrightarrow{ev_t} C^*(\mathcal{G}) \quad \text{for } t \neq 0.$$

Moreover, since  $\mathcal{G} \times (0, 1]$  is an open saturated subset of  $\mathcal{G}^T$  and  $A\mathcal{G}$  a saturated closed subset, we have the following exact sequence (see [31,22])

$$0 \longrightarrow C^*(\mathcal{G} \times (0, 1]) \longrightarrow C^*(\mathcal{G}^T) \xrightarrow{ev_0} C^*(A\mathcal{G}) \longrightarrow 0. \tag{2.9}$$

Now, the  $C^*$ -algebra  $C^*(\mathcal{G} \times (0, 1]) \cong C_0((0, 1], C^*(\mathcal{G}))$  is contractible. This implies that the groups  $K_i(C^*(\mathcal{G} \times (0, 1]))$  vanish, for  $i = 0, 1$ . Then, applying the K-theory functor to the exact sequence above, we obtain that

$$K_i(C^*(\mathcal{G}^T)) \xrightarrow{(ev_0)_*} K_i(C^*(A\mathcal{G}))$$

is an isomorphism, for  $i = 0, 1$ . In [27], Monthubert and Pierrot show that

$$\text{Index}_{a,\mathcal{G}} = (ev_1)_* \circ (ev_0)_*^{-1}, \tag{2.10}$$

modulo the Fourier isomorphism  $C^*(A\mathcal{G}) \cong C_0(A^*\mathcal{G})$ , when we consider the  $K_0$  evaluations (see also [22] and [29]). As usual we consider the index with values in the reduced  $C^*$ -algebra by taking the canonical morphism  $K_0(C^*(\mathcal{G})) \rightarrow K_0(C_r^*(\mathcal{G}))$ . Putting this in a commutative diagram, we have

$$\begin{array}{ccc}
 & K_0(C^*(\mathcal{G}^T)) & \\
 (ev_0)_* \swarrow & & \searrow (ev_1)_* \\
 K^0(A^*\mathcal{G}) & \xrightarrow{\text{Index}_{a,\mathcal{G}}} & K_0(C_r^*(\mathcal{G})).
 \end{array} \tag{2.11}$$

In summary, the algebra  $C^*(\mathcal{G}^T)$  is a strict deformation quantization of  $C_0(A^*\mathcal{G})$ , and the analytic index morphism of  $\mathcal{G}$  can be constructed by means of this deformation.

2.1.3. *Analytic indices morphisms for Lie groupoid immersions*

For Lie groupoid immersions, Hilsum and Skandalis in [22] shown that the same methods as above can be applied to define an index morphism (or even a KK-element) associated to an immersion of Lie groupoids  $\varphi : \mathcal{G}_1 \hookrightarrow \mathcal{G}_2$  for which  $(\mathcal{G}_1)^N$  inherits a Lie groupoid structure, and where we assume as in Section 3 of [22] that  $\mathcal{G}_1$  is amenable (so  $C^*(\mathcal{G}_1)$  is nuclear). Indeed, if we consider

$$\mathcal{G}_\varphi := \mathcal{G}_1^N \times \{0\} \sqcup \mathcal{G}_2 \times (0, 1]$$

and the short exact sequence

$$0 \longrightarrow C^*(\mathcal{G}_2 \times (0, 1]) \longrightarrow C^*(\mathcal{G}_\varphi) \xrightarrow{ev_0} C^*(\mathcal{G}_1^N) \longrightarrow 0, \tag{2.12}$$

we can define the index morphism

$$\text{Index}_\varphi : K_*(C_r^*(\mathcal{G}_1^N)) \longrightarrow K_*(C_r^*(\mathcal{G}_2)), \tag{2.13}$$

as the induced deformation morphism  $\text{Index}_\varphi := p_* \circ (ev_1)_* \circ (ev_0)_*^{-1}$ , where  $p : C^*(\mathcal{G}_2) \rightarrow C_r^*(\mathcal{G}_2)$  is the canonical morphism. Here we use the fact that  $\mathcal{G}_1^N$  is also amenable and so  $C_r^*(\mathcal{G}_1^N) = C^*(\mathcal{G}_1^N)$ .

**Remark 2.4.** (See Hilsum and Skandalis, Remark 3.17(2) in [22].) Even without the hypothesis on the amenability of  $\mathcal{G}_1$ , we still obtain in the same way an index morphism

$$\text{Index}_\varphi^{\max} : K_*(C^*(\mathcal{G}_1^N)) \longrightarrow K_*(C^*(\mathcal{G}_2))$$

between the K-theories of the maximal  $C^*$ -algebras.

## 2.2. The case of twisted groupoids

### 2.2.1. Twisted K-theory of a Lie groupoid with a twisting

Let  $(\mathcal{G}, \sigma)$  be a twisted groupoid. With respect to a covering  $\Omega = \{\Omega_i\}$  of  $\mathcal{G}^{(0)}$ , the twisting  $\sigma$  is given by a strict morphism of groupoids

$$\sigma : \mathcal{G}_\Omega \longrightarrow PU(H),$$

where  $\mathcal{G}_\Omega$  is the covering groupoid associated to  $\Omega$ . Consider the central extension of groups

$$S^1 \longrightarrow U(H) \longrightarrow PU(H),$$

we can pull it back to get an  $S^1$ -central extension of Lie groupoid  $R_\sigma$  over  $\mathcal{G}_\Omega$ ,

$$\begin{array}{ccc}
 S^1 & \longrightarrow & S^1 \\
 \downarrow & & \downarrow \\
 R_\sigma & \longrightarrow & U(H) \\
 \downarrow & & \downarrow \\
 \mathcal{G}_\Omega & \xrightarrow{\sigma} & PU(H).
 \end{array} \tag{2.14}$$

In particular,  $R_\sigma \rightrightarrows \bigsqcup_i \Omega_i$  is a Lie groupoid and  $R_\sigma \rightarrow \mathcal{G}_\Omega$  is an  $S^1$ -principal bundle.

We recall the definition of the convolution algebra and the  $C^*$ -algebra of a twisted Lie groupoid  $(\mathcal{G}, \sigma)$  [32,33]:

**Definition 2.5.** Let  $R_\sigma$  be the  $S^1$ -central extension of groupoids associated to a twisting  $\sigma$ . The convolution algebra of  $(\mathcal{G}, \sigma)$  is by definition the following sub-algebra of  $C_c^\infty(R_\sigma)$ :

$$C_c^\infty(\mathcal{G}, \sigma) = \{f \in C_c^\infty(R_\sigma) : f(\tilde{\gamma} \cdot \lambda) = \lambda^{-1} \cdot f(\tilde{\gamma}), \forall \tilde{\gamma} \in R_\sigma, \forall \lambda \in S^1\}. \tag{2.15}$$

The reduced (maximal resp.)  $C^*$ -algebra of  $(\mathcal{G}, \sigma)$ , denoted by  $C_r^*(\mathcal{G}, \sigma)$  ( $C^*(\mathcal{G}, \sigma)$  resp.), is the completion of  $C_c^\infty(\mathcal{G}, \sigma)$  in  $C_r^*(R_\sigma)$  ( $C^*(R_\sigma)$  resp.).

Let  $L_\sigma := R_\sigma \times_{S^1} \mathbb{C}$  be the complex line bundle over  $\mathcal{G}_\Omega$  which can be considered as a Fell bundle (using the groupoid structure of  $R_\sigma$ ) over  $\mathcal{G}_\Omega$ . In fact, the algebra of sections of this Fell bundle  $C_c^\infty(\mathcal{G}_\Omega, L_\sigma)$  is isomorphic to  $C_c^\infty(\mathcal{G}, \sigma)$ , and the same is true for the  $C^*$ -algebras,  $C^*(\mathcal{G}_\Omega, L_\sigma) \cong C^*(\mathcal{G}, \sigma)$  (see (23) in [33] for an explicit isomorphism).

**Remark 2.6.** (See [33].) Given the extension  $R_\sigma$  as above, the  $S^1$ -action on  $R_\sigma$  induces a  $\mathbb{Z}$ -grading in  $C_r^*(R_\sigma)$  (Proposition 3.2, [33]). More precisely, we have

$$C_r^*(R_\sigma) \cong \bigoplus_{n \in \mathbb{Z}} C_r^*(\mathcal{G}, \sigma^n) \tag{2.16}$$



where  $C_r^*(\mathcal{G}, \sigma^n)$  is the reduced  $C^*$ -algebra of the twisted groupoid  $(\mathcal{G}, \sigma^n)$  corresponding to the Fell bundle

$$L_\sigma^n = L_\sigma^{\otimes n} \longrightarrow \mathcal{G}_\Omega,$$

for all  $n \neq 0$ , and  $C_r^*(\mathcal{G}, \sigma^0) = C_r^*(\mathcal{G}_\Omega)$  by convention. Similar results hold for the maximal  $C^*$ -algebra.

**Remark 2.7.** If we take the twisting  $\sigma$  as the  $PU(H)$ -principal bundle over  $\mathcal{G}$  (as in Definition 1.4(1)), then the  $C^*$ -algebra  $C^*(R_\sigma)$  (maximal or reduced) is well defined up to a canonical strong Morita equivalence. Indeed, given  $\mathcal{G}_\Omega \xrightarrow{\sigma} PU(H)$ ,  $\mathcal{G}_{\Omega'} \xrightarrow{\sigma'} PU(H)$  two strict morphisms defining the same  $PU(H)$ -principal bundle over  $\mathcal{G}$ , they define two canonically Morita equivalent extensions  $R_\sigma$  and  $R_{\sigma'}$ . The induced strong Morita equivalence between  $C^*(R_\sigma)$  and  $C^*(R_{\sigma'})$  respects the grading of the precedent remark and hence it induces a strong Morita equivalence between  $C^*(\mathcal{G}, \sigma^n)$  and  $C^*(\mathcal{G}, (\sigma')^n)$  (see Proposition 3.3 in [33] or [32] for further details).

**Definition 2.8.** Following [33], we define the twisted K-theory of the twisted groupoid  $(\mathcal{G}, \sigma)$  by

$$K^i(\mathcal{G}, \sigma) := K_{-i}(C_r^*(\mathcal{G}, \sigma)). \tag{2.17}$$

In particular if  $\sigma$  is trivial we will be using the notation (unless specified otherwise)  $K^i(\mathcal{G})$  for the respective K-theory group of the reduced groupoid  $C^*$ -algebra.

**Remark 2.9.** By the Remark 2.7, the group  $K^i(\mathcal{G}, \sigma)$  is well defined, up to a canonical isomorphism coming from the respective Morita equivalences.

**Remark 2.10.** For the groupoid given by a manifold  $M \rightrightarrows M$ . A twisting on  $M$  can be given by a Dixmier–Douady class on  $H^3(M, \mathbb{Z})$ . In this event, the twisted K-theory, as we defined it, coincides with twisted K-theory defined in [3,23]. Indeed the  $C^*$ -algebra  $C^*(M, \sigma)$  is Morita equivalent to the continuous trace  $C^*$ -algebra defined by the correspondent Dixmier–Douady class (see for instance Theorem 1 in [18]).

2.2.2. *The Analytic index morphism of a twisted groupoid*

Let  $(\mathcal{G}, \sigma)$  be a twisted Lie groupoid with the induced central extension

$$R_\sigma \longrightarrow \mathcal{G}_\Omega$$

associated to an open cover  $\{\Omega_i\}$  of  $\mathcal{G}^{(0)} = M$ . Let  $(M, \sigma_0)$  be the twisted groupoid induced by restriction of  $\sigma$  to the unit space and let its central extension be

$$R_{\sigma_0} \longrightarrow M_\Omega.$$

Here  $M_\Omega$  denotes the covering groupoid

$$\bigsqcup_{i,j} \Omega_{ij} := \bigsqcup_{i,j} \Omega_i \cap \Omega_j \rightrightarrows \bigsqcup_i \Omega_i.$$

We can consider the inclusion of central extensions:

$$\begin{array}{ccc}
 R_{\sigma_0} & \hookrightarrow & R_{\sigma} \\
 \downarrow & & \downarrow \\
 M_{\Omega} & \hookrightarrow & \mathcal{G}_{\Omega}
 \end{array} \tag{2.18}$$

as an immersion of Lie groupoids  $\iota_{\sigma} : R_{\sigma_0} \rightarrow R_{\sigma}$ , which is  $S^1$ -equivariant and with  $R_{\sigma_0}$  amenable. We apply then to this immersion the tangent groupoid construction in Subsection 1.3.3

$$R_{\iota_{\sigma}} := R_{\sigma_0}^N \times \{0\} \sqcup R_{\sigma} \times (0, 1] \tag{2.19}$$

in order to define its index morphism as in (2.13):

$$\mathbf{Index}_{\iota_{\sigma}} : K_*(C_r^*(R_{\sigma_0}^N)) \longrightarrow K_*(C_r^*(R_{\sigma})). \tag{2.20}$$

Indeed,  $R_{\sigma_0}^N \rightrightarrows \bigsqcup_i \Omega_i$  inherits a Lie groupoid structure (see Remark 1.15). In fact,  $R_{\sigma}^N$  is also the  $S^1$ -central extension of some Lie groupoid. Let us look at this with more detail. Write  $\sigma = \{(\Omega_i, \sigma_{ij})\}$  as a  $PU(H)$ -valued cocycle, consider the inclusions  $\Omega_{ij} \subset \mathcal{G}_{\Omega_j}^{\Omega_i}$  and their normal bundles  $N_{ij}$ . We have a Lie groupoid

$$A_{\sigma} := \bigsqcup_{i,j} N_{ij} \rightrightarrows \bigsqcup_i \Omega_i,$$

where the source and target maps are the evident ones and where the product is induced from the derivation in the normal direction of the product

$$\mathcal{G}_{\Omega_j}^{\Omega_i} \times_{\Omega_j} \mathcal{G}_{\Omega_k}^{\Omega_j} \longrightarrow \mathcal{G}_{\Omega_k}^{\Omega_i}.$$

We have also a strict morphism of groupoids

$$A_{\sigma} \xrightarrow{\tilde{\sigma}_0} PU(H)$$

defined to be equal to  $\sigma_0$  in the base direction and constant in the normal vector fibers. The corresponding central extension can be described as follows. Consider the Lie algebroid  $A_{\mathcal{G}} \xrightarrow{\pi} M$  as a Lie groupoid  $A_{\mathcal{G}} \rightrightarrows M$  using its vector bundle structure. Then the central extension associated to  $(A_{\sigma}, \tilde{\sigma}_0)$  is given by the following pullback diagram

$$\begin{array}{ccc}
 R_{\sigma_0}^N & \longrightarrow & U(H) \\
 \downarrow & & \downarrow \\
 (A_{\mathcal{G}})_{\Omega} & \xrightarrow{\tilde{\sigma}_0} & PU(H) \\
 \Downarrow & & \Downarrow \\
 \bigsqcup_i \Omega_i & \longrightarrow & \{e\},
 \end{array} \tag{2.21}$$

via the identification  $N_{ij} \cong A\mathcal{G}|_{\Omega_{ij}}$ . In other words the subgroupoid  $R_{\sigma_0}^N$  is the  $S^1$ -central extension associated to the twisted groupoid  $(A\mathcal{G} \rightrightarrows M, \sigma_0 \circ \pi_A)$ , where  $\pi_A : A\mathcal{G} \rightarrow M$  is considered as a morphism of groupoids. This discussion is of course in agreement with Proposition 1.14.

Hence we can denote

$$R_{\sigma_0}^N = R_{\sigma_0 \circ \pi_A}, \tag{2.22}$$

that is,  $R_{\sigma_0}^N$  is the central extension groupoid associated to the twisted groupoid  $(A\mathcal{G} \rightrightarrows M, \sigma_0 \circ \pi_A)$ .

As in Remark 2.6, we have a decomposition

$$C_r^*(R_{\sigma_0 \circ \pi_A}) = \bigoplus_{n \in \mathbb{Z}} C_r^*(A\mathcal{G}, (\sigma_0 \circ \pi_A)^n).$$

Since, the inclusion of extensions (2.18) is  $S^1$ -equivariant, we have that the index morphism (2.20) respects the  $\mathbb{Z}$ -grading

$$\text{Index}_{t_\sigma} : \bigoplus_{n \in \mathbb{Z}} K^*(A\mathcal{G}, (\sigma_0 \circ \pi_A)^n) \longrightarrow \bigoplus_{n \in \mathbb{Z}} K^*(\mathcal{G}, \sigma^n). \tag{2.23}$$

Consider the projection  $A^*\mathcal{G} \xrightarrow{\pi_{A^*}} M$  and the pullback of  $\sigma_0$  to get the twisted groupoid

$$(A^*\mathcal{G} \rightrightarrows A^*\mathcal{G}, \sigma_0 \circ \pi_{A^*}).$$

Associated to this twisted groupoid we have a central extension given by the pull-back diagram

$$\begin{array}{ccccc}
 R_{\sigma_0 \circ \pi_{A^*}} & \longrightarrow & R_{\sigma_0} & \longrightarrow & U(H) \\
 \downarrow & & \downarrow & & \downarrow \\
 \bigsqcup A^*\mathcal{G}|_{\Omega_{ij}} & \xrightarrow{\pi_{A^*}} & \bigsqcup \Omega_{ij} & \xrightarrow{\sigma_0} & PU(H) \\
 \Downarrow & & \Downarrow & & \Downarrow \\
 \bigsqcup A^*\mathcal{G}|_{\Omega_i} & \longrightarrow & \bigsqcup \Omega_i & \longrightarrow & \{e\}.
 \end{array} \tag{2.24}$$

As in the untwisted case, we have the following proposition.

**Proposition 2.11.** *The fiberwise Fourier transform gives an isomorphism of  $C^*$ -algebras*

$$C_r^*(R_{\sigma_0 \circ \pi_A}) \cong C_r^*(R_{\sigma_0 \circ \pi_{A^*}}) \tag{2.25}$$

which preserves the grading under the  $S^1$ -action. In particular we have an isomorphism between the correspondent twisted  $K$ -theory groups

$$K^*(A\mathcal{G}, \sigma_0 \circ \pi_A) \cong K^*(A^*\mathcal{G}, \sigma_0 \circ \pi_{A^*}).$$

The right side is the twisted  $K$ -theory of the topological space  $A^*\mathcal{G}$  with the pull-ball twisting  $(\pi_{A^*})^*\sigma_0$ .

We will prove the proposition above in the more general case of vector bundles. Indeed, given a vector bundle  $E \xrightarrow{\pi_E} X$  and a twisting  $\beta$  on  $X$ , we can consider two twisted groupoids: the first is  $(E, \beta \circ \pi_E)$  where  $E \rightrightarrows X$  is considered as a groupoid and  $\pi$  as a groupoid morphism, and the second is  $(E^*, \beta \circ \pi_{E^*})$  where  $E^* \rightrightarrows E^*$  is the unit groupoid with the twisting on the topological space  $E^*$ .

**Proposition 2.12.** *The Fourier transform gives an isomorphism*

$$C_r^*(R_{\beta \circ \pi_E}) \cong C_r^*(R_{\beta \circ \pi_{E^*}})$$

which preserves the grading under the  $S^1$ -action. In particular we have an isomorphism of  $C^*$ -algebras

$$C_r^*(E, \beta \circ \pi_E) \cong C_r^*(E, \beta \circ \pi_{E^*}).$$

**Proof.** We will give the explicit isomorphism  $C_r^*(R_{\beta \circ \pi_E}) \xrightarrow{\mathcal{F}} C_r^*(R_{\beta \circ \pi_{E^*}})$ . It is defined at the level of  $C_c^\infty(R_{\beta \circ \pi_E})$  as follows: Let  $f \in C_c^\infty(R_{\beta \circ \pi_E})$ , and  $[(\eta_{ij}, u)] \in R_{\beta \circ \pi_{E^*}} = \bigsqcup_{i,j} (E^*|_{\Omega_{ij}} \times_{PU(H)} U(H))$  (see diagram (2.24) above). We let

$$\mathcal{F}(f)([(\eta_{ij}, u)]) = \int_{E_\pi(\eta_{ij})} e^{-i\eta_{ij}(X)} f([(X, u)]) dX. \tag{2.26}$$

For  $[(\eta_{ij}, u)] \in R_{\beta \circ \pi_E} = \bigsqcup_{i,j} (E^*|_{\Omega_{ij}} \times_{PU(H)} U(H))$  and  $X \in E_\pi(\eta_{ij})$ , we have

$$[(X, u)] \in R_{\beta \circ \pi_E} = \bigsqcup_{i,j} (E|_{\Omega_{ij}} \times_{PU(H)} U(H))$$

by the definition of the morphisms  $\beta \circ \pi_E$  (2.21) and  $\beta \circ \pi_{E^*}$  (2.24). It is also immediate that this function is  $S^1$ -equivariant.

Now, as in the untwisted case, the function  $\mathcal{F}(f)$  is not compactly supported. But it can be shown that it satisfies a Schwartz (rapid decaying) condition in the direction of the vector fibers of  $E$  (see for instance Proposition 4.5 in [8]).

Let us first assume  $E = X \times \mathbb{R}^q$ . We remark that we are not assuming triviality of the twisting  $\beta$  on  $X$ . In this case we have a very clear description of the extension groupoids: First for  $E \rightrightarrows X$ ,

$$R_{\beta \circ \pi_E} = R_\beta \times \mathbb{R}^q \rightrightarrows \bigsqcup_i \Omega_i,$$

where  $\mathbb{R}^q$  is considered as an additive group and  $R_\beta$  is the extension groupoid associated to  $(X, \beta)$ . Then for  $E^* \rightrightarrows E^*$ ,

$$R_{\beta \circ \pi_{E^*}} = R_\beta \times \mathbb{R}^q \rightrightarrows \bigsqcup_i \Omega_i \times \mathbb{R}^q,$$

where  $\mathbb{R}^q \rightrightarrows \mathbb{R}^q$  is considered as a unit groupoid ( $\mathbb{R}^q$  as a total space). Then, under this situation, it is immediate that

$$C_r^*(R_{\beta \circ \pi_E}) = C_r^*(R_\beta, (C_0(\mathbb{R}^q), *)) \quad \text{and} \quad C_r^*(R_{\beta \circ \pi_{E^*}}) = C_r^*(R_\beta, (C_0(\mathbb{R}^q), \cdot)),$$

where in the first  $C_0(\mathbb{R}^q)$  we have the convolution product and in the second the point-wise product. Since the product on  $C_r^*(R_\beta, (C_0(\mathbb{R}^q), *))$  is given by (at the  $C_c^\infty(R_\beta, (C_0(\mathbb{R}^q), *))$  level)

$$(f * g)(r) = f(r) * g(r),$$

we obtain, thanks to the continuity of the Fourier transform, that  $\mathcal{F}$  gives an isomorphism  $C_r^*(R_{\beta \circ \pi_E}) \cong C_r^*(R_{\beta \circ \pi_{E^*}})$ .

For a non-trivial vector bundle  $E$ , in order to see that  $\mathcal{F}$  gives the desired isomorphism from  $C_r^*(R_{\beta \circ \pi_E})$  to  $C_r^*(R_{\beta \circ \pi_{E^*}})$  we may use trivializing charts of the vector bundle  $E$  and consider the correspondent decomposition of  $C_r^*(R_{\beta \circ \pi_E})$  as in Eq. (15) in [8] (see Proposition 4.5 in [8] for more details).  $\square$

Now we define the index analytic morphism for the twisted groupoid  $(\mathcal{G}, \sigma)$  which agrees with the usual analytic index morphism for the groupoid  $\mathcal{G}$  when the twisting  $\sigma$  is trivial.

**Definition 2.13.** Let  $(\mathcal{G}, \sigma)$  be a twisted Lie groupoid. The analytic index morphism for  $(\mathcal{G}, \sigma)$ ,

$$\mathbf{Index}_{a, (\mathcal{G}, \sigma)} : K^i(A^*\mathcal{G}, \pi^*\sigma_0) \longrightarrow K^i(\mathcal{G}, \sigma) \tag{2.27}$$

is defined to be the composition of the Fourier isomorphism from Proposition 2.11 and the degree one term of the index morphism  $\mathbf{Index}_{t_\sigma}$  (2.23).

### 2.3. Properties of the twisted analytic index morphism

In this section we will establish three properties of twisted analytic index morphisms. These are the analogs to the axioms stated in [2] and will allow us to prove a twisted index theorem for foliations and in general we can then talk of a twisted index theory for these objects.

**Notation.** We recall for the benefit of the reader that  $K^*(\mathcal{G}, \sigma)$  denotes the K-theory group of the reduced twisted groupoid  $C^*$ -algebra  $K_{-*}(C_r^*(\mathcal{G}, \sigma))$  unless specified otherwise (Definition 2.8 above, see also [33]). In particular if  $\sigma$  is trivial we will be often using the notation  $K^*(\mathcal{G})$  for the respective K-theory group of the reduced groupoid  $C^*$ -algebra. Also, when the groupoid is a space, this notation is consistent with the notation for the twisted topological K-theory.



$$\begin{array}{ccc}
 K^*(A^*\mathcal{H}, \pi^*(\sigma \circ j)_0) & \xrightarrow{\text{Index}_{a,(\mathcal{H},\sigma \circ j)}} & K^*(\mathcal{H}, \sigma \circ j) \\
 \downarrow j! & & \downarrow j! \\
 K^*(A^*\mathcal{G}, \pi^*\sigma_0) & \xrightarrow{\text{Index}_{a,(\mathcal{G},\sigma)}} & K^*(\mathcal{G}, \sigma)
 \end{array}$$

where the vertical maps are induced from the inclusions by the open inclusion  $j$ .

**Proof.** We use again the notation of the proof of the proposition above. We observe this time that the deformation groupoid  $R_{l_{\sigma \circ j}}$  which gives the twisted index (2.23) for  $(\mathcal{H}, \sigma \circ j)$ , is an open subgroupoid of  $R_{l_\sigma}$ . The following diagram is obviously commutative

$$\begin{array}{ccccc}
 & & \text{Index}_{l_{\sigma \circ j}} & & \\
 & \swarrow & & \searrow & \\
 K^*(R_{\sigma_0 \circ j}^N) & \xleftarrow[\cong]{ev_0} & K^*(R_{l_{\sigma \circ j}}) & \xrightarrow{ev_1} & K^*(R_{\sigma \circ j}) \\
 \downarrow j! & & \downarrow j! & & \downarrow j! \\
 K^*(R_{\sigma_0}^N) & \xleftarrow[\cong]{ev_0} & K^*(R_{l_\sigma}) & \xrightarrow{ev_1} & K^*(R_\sigma) \\
 & \swarrow & \text{Index}_{l_\sigma} & \searrow & 
 \end{array} \tag{2.29}$$

The proposition easily follows from the diagram above and the fact that all the morphisms in (2.29) respect the  $\mathbb{Z}$ -grading under  $S^1$ -action.  $\square$

The last property (Proposition 2.17 below) involves in some way the compatibility of the twisted index morphism with the product in twisted K-theory. The idea is originated from Theorem 6.2 in [16] and the work of Connes on tangent groupoids (see [14]).

2.3.1. Thom inverse morphism in twisted K-theory

Let  $N \rightarrow X$  be a real vector bundle. Consider the fiber product groupoid  $N \times_X N \rightrightarrows N$  over  $X$  whose Lie algebroid is  $N \oplus N \xrightarrow{\pi_N} N$ . Note that the groupoid  $N \times_X N \rightrightarrows N$  is Morita equivalent to the unit groupoid  $X \rightrightarrows X$ , that is, there is an isomorphism in the Hilsum–Skandalis category

$$N \times_X N \xrightarrow{\mathcal{M}} X.$$

Debord, Lescure, and Nistor showed in [16] (Theorem 6.2) that, modulo Morita equivalence isomorphism, the analytic index morphism for the groupoid  $N \times_X N \rightrightarrows N$ :

$$K^*(N \oplus N^*) \xrightarrow{\text{Index}_{a,N \times_X N}} K^*(N \times_X N),$$

is the inverse of the Thom isomorphism for the vector bundle  $N \oplus N \xrightarrow{\pi_X} X$ . In other words the following diagram is commutative:

$$\begin{array}{ccc}
 K^*(N \oplus N^*) & \xrightarrow{=} & K^*(N \oplus N^*) \\
 \text{Index}_{a, N \times_X N} \downarrow & & \downarrow \text{Thom}^{-1} \\
 K^*(N \times_X N) & \xrightarrow{\mathcal{M}} & K^*(X).
 \end{array} \tag{2.30}$$

We will briefly recall the steps of the proof of the above assertion since we are going to follow similar steps in the twisted case. We recall that analytic indices for groupoids are realized by deformation groupoids. This point of view is essential for the proof of the above assertion in [16]. Indeed, they consider the tangent groupoid of  $N \times_X N$ , which they called the Thom groupoid, denoted by  $\mathcal{T}_N$ . Hence the above diagram takes the following form

$$\begin{array}{ccc}
 K^*(N \oplus N) & \xrightarrow[\text{Fourier}]{\cong} & K^*(N \oplus N^*) \\
 \text{ev}_0^{-1} \downarrow & & \downarrow \cong \text{Thom}^{-1} \\
 K^*(\mathcal{T}_N) & & \\
 \text{ev}_1 \downarrow & & \downarrow \\
 K^*(N \times_X N) & \xrightarrow[\mathcal{M}]{\cong} & K^*(X).
 \end{array} \tag{2.31}$$

They consider first the morphism  $T_0 := \text{Fourier} \circ \text{Thom} : K^*(X) \rightarrow K^*(N \oplus N)$ , the first pointed arrow below (in terms of KK-elements). Then, the main part of their proof is to construct the morphism  $T : K^*(X) \rightarrow K^*(\mathcal{T}_N)$  such that its corresponding evaluations make the following diagram commute.

$$\begin{array}{ccc}
 K^*(N \oplus N) & \xrightarrow[\text{Fourier}]{\cong} & K^*(N \oplus N^*) \\
 \text{ev}_0^{-1} \downarrow & \swarrow T_0 & \downarrow \cong \text{Thom}^{-1} \\
 K^*(\mathcal{T}_N) & & \\
 \text{ev}_1 \downarrow & \swarrow T & \downarrow \\
 K^*(N \times_X N) & \xrightarrow[\mathcal{M}]{\cong} & K^*(X).
 \end{array} \tag{2.32}$$

Let us now discuss the twisted case. Given a twisting  $\beta = \{(\Omega_i, \beta_{ij})\}$  on the space  $X$  with an open cover  $\Omega = \{\Omega_i\}$  of  $X$ . The bundle  $N \oplus N^*$  has a complex structure, hence we have the Thom isomorphism (see [7,23]) in twisted K-theory

$$\text{Thom}_\beta : K^i(X, \beta) \xrightarrow{\cong} K^i(N \oplus N^*, \pi_X^* \beta), \tag{2.33}$$

where  $N \oplus N^*$  is seen as a total space and  $\pi_X^* \beta$  is a twisting in the classic sense.



We will give now a KK-theoretic description of this isomorphism (modulo Fourier isomorphism). Consider the associated groupoid  $N \oplus N \rightrightarrows X$  (using the vector bundle structure) and the corresponding groupoid twisting  $\beta \circ \pi_X$ , where  $\pi_X : N \oplus N \rightarrow X$  is seen as a groupoid morphism. Consider the  $S^1$ -central extension associated to the twisted groupoid  $(N \oplus N, \beta \circ \pi_X)$ :

$$S^1 \longrightarrow R_{N \oplus N, \beta} \longrightarrow (N \oplus N)_\beta. \tag{2.34}$$

Hence,

$$(N \oplus N)_\beta = \bigsqcup_{i,j} (N \oplus N)_{ij} \rightrightarrows \bigsqcup_i \Omega_i \quad \text{and}$$

$$R_{N \oplus N, \beta} = \bigsqcup_{i,j} (N \oplus N)_{ij} \times_{PU(H)} U(H) \rightrightarrows \bigsqcup_i \Omega_i,$$

where the pullback is taken by the groupoid morphism  $(N \oplus N)_\beta \xrightarrow{\beta \circ \pi_X} PU(H)$ . Similarly, we have the  $S^1$ -central extension associated to  $(X, \beta)$ :

$$S^1 \longrightarrow R_{X, \beta} \longrightarrow X_\beta, \tag{2.35}$$

where  $X_\beta = \bigsqcup_{i,j} \Omega_{ij} \rightrightarrows \bigsqcup_i \Omega_i$  and  $R_{X, \beta} = \bigsqcup_{i,j} \Omega_{ij} \times_{PU(H)} U(H) \rightrightarrows \bigsqcup_i \Omega_i$ .

We are going to describe an element of  $KK^*(R_{X, \beta}, R_{N \oplus N, \beta})$  which realizes the twisted Thom isomorphism (2.33) modulo Fourier isomorphism. Let  $q : R_{N \oplus N, \beta} \rightarrow X$  be the composition of the obvious maps

$$R_{N \oplus N, \beta} \longrightarrow (N \oplus N)_\beta \longrightarrow \bigsqcup_i \Omega_i \longrightarrow X.$$

Let us now construct the  $(C^*(R_{X, \beta}) - C^*(R_{N \oplus N, \beta}))$ -Kasparov bimodule  $\mathcal{H}_0$ . It is the completion of  $C_c^\infty(R_{N \oplus N, \beta}, q^* \Lambda(N \oplus N) \otimes \Omega^{\frac{1}{2}})$  with respect to its  $C_c^\infty(R_{N \oplus N, \beta}, \Omega^{\frac{1}{2}}) \subset C^*(R_{N \oplus N, \beta})$  valued inner product:

$$\langle \xi, \eta \rangle(\gamma) = \int_{\gamma_1 \cdot \gamma_2 = \gamma} \langle \xi(\gamma_1), \eta(\gamma_2) \rangle_{N \oplus N}, \quad \text{for } \xi, \eta \in C_c^\infty(R_{N \oplus N, \beta}, q^* \Lambda(N \oplus N) \otimes \Omega^{\frac{1}{2}}),$$

where  $\langle \cdot, \cdot \rangle$  denotes the interior product on the (canonical) Hermitian bundle  $N \oplus N$ , and the integration is as usual over the resultant 1-density.

The right module structure is given by

$$(\xi f)(\gamma) = \int_{\gamma_1 \cdot \gamma_2 = \gamma} \xi(\gamma_1) f(\gamma_2),$$

for  $\xi \in C_c^\infty(R_{N \oplus N, \beta}, q^* \Lambda(N \oplus N) \otimes \Omega^{\frac{1}{2}})$  and  $f \in C_c^\infty(R_{N \oplus N, \beta}, \Omega^{\frac{1}{2}})$ .



**Proof.** Consider the deformation groupoid which realizes the twisted index morphism for  $(N \times_X N, \beta \circ \mathcal{M})$ :

$$R_{\beta^{tan}} = R_{N \oplus N, \pi^* \beta} \times \{0\} \sqcup R_{N \times_X N, \beta \circ \mathcal{M}} \times (0, 1].$$

That is, its deformation index gives the element  $[ev_0]^{-1} \otimes [ev_1] \in KK(R_{N \oplus N, \pi^* \beta}, R_{N \times_X N, \beta \circ \mathcal{M}})$  whose degree one induces the analytical index morphism **Index** $_{d, (N \times_X N, \beta \circ \mathcal{M})}$ , modulo the Fourier isomorphism.

We will define a KK-element in  $KK(R_{X, \beta}, R_{\beta^{tan}})$  in the following way: let  $\mathcal{H}$  be the  $C^*(R_{\beta^{tan}})$ -Hilbert module completion of  $C_c^\infty(R_{\beta^{tan}}, q^* \Lambda(N \oplus N))$  where  $q : R_{\beta^{tan}} \rightarrow X$  is the obvious projection (using the target map of  $R_{\beta^{tan}}$  for instance). The endomorphism of  $\mathcal{H}$  is given by

$$Fs(v, w, t) = \int_{(w', v') \in N_x \times N_x^*} e^{\frac{w-w'}{t} \cdot v'} C(v + iv')s(v, w') dw' \frac{dv'}{t^n}, \tag{2.38}$$

for  $t \neq 0$  (where  $n = \dim_{\text{vect}} N$ ) and by  $F_0$  for  $t = 0$ . We obtain an element  $[\mathcal{H}, F] \in KK(R_{X, \beta}, R_{\beta^{tan}})$ . Denote by  $T_{\beta^{tan}} : K(R_{X, \beta}) \rightarrow K(R_{\beta^{tan}})$  the induced morphism in K-theory. It fits by construction in the following commutative diagram:

$$\begin{array}{ccc}
 & & K(R_{N \oplus N, \beta}) \\
 & \nearrow T_\beta & \uparrow e_0 \\
 K(R_{X, \beta}) & \xrightarrow{T_{\beta^{tan}}} & K(R_{\beta^{tan}}) \\
 & \searrow e_1 \circ T_{\beta^{tan}} & \downarrow e_1 \\
 & & K(R_{N \times_X N, \beta}).
 \end{array} \tag{2.39}$$

To conclude the proof it is enough to show that  $\mathcal{M} \circ e_1 \circ T_{\beta^{tan}}$  gives the identity in  $K(R_{X, \beta})$ . Now, as a KK-element,  $T_{\beta^{tan}} \otimes e_1 \otimes \mathcal{M} \in KK(R_{X, \beta}, R_{X, \beta})$  can be represented by  $(H_{\Lambda E}, F_1)$  where

$$H_{\Lambda E} = \left( L^2 \left( \bigsqcup_{\{i \in I, x \in \Omega_i\}} N_x \times S^1, \Lambda E \right) \right)_{x \in \bigsqcup_j \Omega_j},$$

and the vector bundle is the pullback of  $\Lambda E$  by the canonical projection  $\bigsqcup_{\{i \in I, x \in \Omega_i\}} N_x \times S^1 \rightarrow \bigsqcup_j \Omega_j \rightarrow X$ . The operator  $F_1$  is as the operator  $F$  evaluated at  $t = 1$  but identified with a continuous family of Fredholm operators acting on  $L^2(\bigsqcup_{\{i \in I, x \in \Omega_i\}} N_x \times S^1, \Lambda E)$  by

$$F_1s(x, v) = \int_{(w', v') \in N_x \times N_x^*} e^{i(v-w') \cdot v'} C(v + iv')s(x, w') dw' dv'.$$

The Lemma 2.4 in [15] applies here and we have then that  $(H_{\Lambda E}, F_1)$  defines the unit element of  $KK(R_{X, \beta}, R_{X, \beta})$ .  $\square$

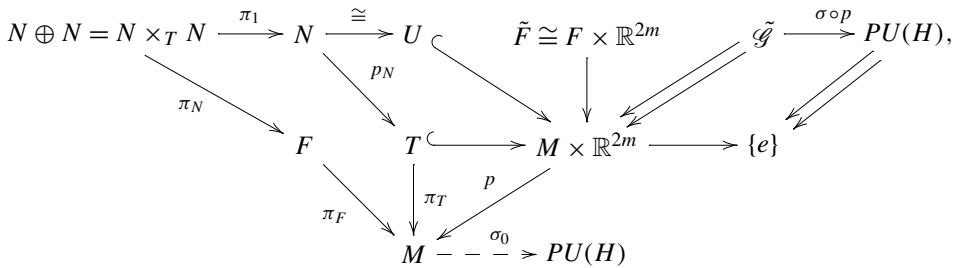
### 3. Longitudinal twisted index theorem for foliations

Consider the case of a regular foliation  $(M, F)$  with a twisting  $\sigma$  on its leaf space  $M/F$  in the sense of Definition 1.9, by using the holonomy groupoid  $\mathcal{G}$  of  $(M, F)$ . The induced twisting on  $M$  is denoted by  $\sigma_0$ . Recall that the Lie algebroid  $A\mathcal{G}$  is  $F \rightarrow M$ . In this section, we define the topological index for  $(M/F, \sigma)$  and show that it agrees with the twisted analytical index morphism for  $(M/F, \sigma)$ .

#### 3.1. Twisted topological index for foliation

Let  $i : M \hookrightarrow \mathbb{R}^{2m}$  be an embedding and  $T$  be the total space of the normal vector bundle  $\pi_T : T \rightarrow M$  to the foliation in  $\mathbb{R}^{2m}$  with  $T_x = (i_*(F_x))^\perp$  for  $x \in M$ . Consider the foliation  $M \times \mathbb{R}^{2m}$  given by the bundle  $\tilde{F} = F \times \mathbb{R}^{2m} \rightarrow M \times \mathbb{R}^{2m}$ . This foliation has  $\tilde{\mathcal{G}} = \mathcal{G} \times \mathbb{R}^{2m} \rightrightarrows M \times \mathbb{R}^{2m}$  as its holonomy groupoid equipped with the pull-back twisting, still denoted by  $\sigma$ . The map  $(x, \xi) \mapsto (x, i(x) + \xi)$  identifies an open neighborhood of the 0-section of  $T$  with an open transversal of  $(M \times \mathbb{R}^{2m}, \tilde{F})$ , that we still denote by  $T$  with the projection  $\pi_T : T \rightarrow M$ . Denote by  $N$  the total space of the normal vector bundle to the inclusion  $T \subset M \times \mathbb{R}^{2m}$ . We can identify  $N$  with a small open neighborhood  $U$  of  $T$  in  $M \times \mathbb{R}^{2m}$ .

We use the following diagram to streamline the above notations



where  $\pi_1$  denotes the projection to the first component, and  $\sigma_0$  denotes the restriction of the twisting  $\sigma$  on  $\mathcal{G}$  to the unit space  $M$ . As  $T$  is an open transversal to the foliation  $\tilde{F}$  on  $M \times \mathbb{R}^{2m}$ , its normal bundle  $N$  is the pull-back of  $\tilde{F} = \pi_T^* F$  to  $T$ . We can see that  $N \cong \pi_T^* F$  as vector bundles over  $T$ . Under the identification  $N$  and  $U$ ,  $N \times_T N = N \oplus N$  is the total space of  $\tilde{F}|_U$ . Hence, there is a canonically defined projection  $\pi_N$  making the above diagram commutative. The same arguments imply that the total space  $N \oplus N^*$  (as a vector bundle over  $T$ ) is diffeomorphic to the total space  $F^* \times \mathbb{R}^{2m}$  (as a vector bundle over  $M$ ).

**Lemma 3.1.**

- (1) *There exists an open neighborhood  $U$  of  $T$  in  $M \times \mathbb{R}^{2m}$  such that the holonomy groupoid of  $\tilde{F}|_U$  over  $U$  is strictly isomorphic to the groupoid  $N \times_T N \rightrightarrows N$  associated to the submersion  $N \rightarrow T$ . The latter groupoid  $N \times_T N \rightrightarrows N$  is Morita equivalent to the groupoid  $T \rightrightarrows T$ .*
- (2) *Let  $\mu : (N \times_T N \rightrightarrows N) \rightarrow (T \rightrightarrows T)$  be the Morita equivalence isomorphism of Lie groupoids,  $j : N \times_T N \cong \tilde{\mathcal{G}}|_U \hookrightarrow \mathcal{G} \times \mathbb{R}^{2m}$  be the inclusion as an open subgroupoid, and  $p : \mathcal{G} \times \mathbb{R}^{2m} \rightarrow \mathcal{G}$  be the projection. Then the twisting  $\sigma_0 \circ \pi_T \circ \mu$  agrees with the twisting  $\sigma \circ p \circ j$  over the groupoid  $N \times_T N \rightrightarrows N$ .*

**Proof.** The proof follows from the above commutative diagram and the fact that  $T$  is an open transversal to the foliation  $\tilde{F}$  on  $M \times \mathbb{R}^{2m}$ .  $\square$

This lemma enables us to define the twisted topological index of  $(M, F, \sigma)$  which agrees with the Connes–Skandalis’ longitudinal topological index of  $(M, F)$  when the twisting is trivial.

**Definition 3.2.** By the *twisted topological index* of  $(M, F, \sigma)$  we mean the morphism

$$\mathbf{Index}_{t,(M/F,\sigma)} : K^i(F^*, \sigma_0 \circ \pi_F) \longrightarrow K^i(M/F, \sigma) := K^i(\mathcal{G}, \sigma)$$

given by the composition of various isomorphisms in twisted K-theory of topological spaces, functoriality of open embedding and Bott isomorphism in twisted K-theory of groupoids

$$\begin{CD} K^i(F^*, \sigma_0 \circ \pi_F) @>{\cong \text{Bott}}>> K^i(F^* \times \mathbb{R}^{2m}, (\sigma \circ p)_0 \circ \pi_F) @>{\cong}>> K^i(N \oplus N^*, \sigma_0 \circ \pi_F \circ \pi_N) \\ @. @. @VV{\cong \text{Thom}^{-1}}V \\ K^i(N \times_T N, \sigma \circ p \circ j) @<{\cong}<< K^i(N \times_T N, \sigma_0 \circ \pi_T \circ \mu) @<{\cong \text{Morita}}<< K^i(T, \sigma_0 \circ \pi_T) \\ @VV{\cong}V @. @. @. \\ K^i(\mathcal{G}|_U, \sigma \circ p \circ j) @>{\cong j_!}>> K^i(\mathcal{G} \times \mathbb{R}^{2m}, \sigma \circ p) @>{\cong \text{Bott}^{-1}}>> K^i(\mathcal{G}, \sigma). \end{CD}$$

(3.1)

### 3.2. The twisted longitudinal index theorem

For a regular foliation  $(M, F)$  with a twisting  $\sigma$  the space of leaves, that is, a generalized morphism from the holonomy groupoid  $M/F$  to  $PU(H)$ , we have the twisted analytic index morphism (Definition 2.13)

$$\mathbf{Index}_{a,(M/F,\sigma)} : K^i(F^*, \sigma_0 \circ \pi_F) \longrightarrow K^i(M/F, \sigma).$$

The main theorem of the paper is to establish the equality between the twisted analytic index and the twisted topological index for  $(M/F, \sigma)$  which generalizes the longitudinal index theorem of Connes and Skandalis (see [15]) for the trivial twisting.

**Theorem 3.3.** For a regular foliation  $(M, F)$  with a twisting  $\sigma : M/F \dashrightarrow PU(H)$  we have the following equality of morphisms:

$$\mathbf{Index}_{a,(M/F,\sigma)} = \mathbf{Index}_{t,(M/F,\sigma)} : K^i(F^*, \sigma_0 \circ \pi_F) \longrightarrow K^i(M/F, \sigma). \tag{3.2}$$

**Proof.** We use the same notation and terminology of the definition of the twisted topological index for  $(M/F, \sigma)$ . Notice that the analytic twisted index morphism is compatible with the Bott isomorphism (Proposition 2.14)

$$\begin{CD}
 K^i(F^*, \sigma_0 \circ \pi_F) @>Bott>> K^i(F^* \times \mathbb{R}^{2m}, (\sigma \circ p)_0 \circ \pi_F) \\
 @V \text{Index}_{a,(\mathcal{G},\sigma)} VV @VV \text{Index}_{a,(\mathcal{G},\sigma \circ p)} V \\
 K^i(M/F, \sigma) @>Bott>> K^i(\mathcal{G} \times \mathbb{R}^{2m}, \sigma \circ p)
 \end{CD}$$

and open inclusions of subgroupoids (Proposition 2.15)

$$\begin{CD}
 K^i(A^*\tilde{\mathcal{G}}|_U, (\sigma \circ p \circ j)_0 \circ \pi) @>Bott>> K^i(F^* \times \mathbb{R}^{2m}, (\sigma \circ p)_0 \circ \pi_F) \\
 @V \text{Index}_{a,(\tilde{\mathcal{G}}|_U, \sigma \circ p)} VV @VV \text{Index}_{a,(\mathcal{G}, \sigma \circ p)} V \\
 K^i(\tilde{\mathcal{G}}|_U, \sigma \circ p \circ j) @>{j!}>> K^i(\mathcal{G} \times \mathbb{R}^{2m}, \sigma \circ p).
 \end{CD}$$

We only need to check that the analytic index morphism for  $(U, \tilde{F}|_U)$ ,

$$\text{Index}_{a,(\tilde{\mathcal{G}}|_U, \sigma \circ p \circ j)} : K^i(A^*\tilde{\mathcal{G}}|_U, (\sigma \circ p \circ j)_0 \circ \pi) \longrightarrow K^i(\tilde{\mathcal{G}}|_U, \sigma \circ p \circ j)$$

agrees with the composition of the following part in the definition of the twisted topological index for  $(M/F, \sigma)$ ,

$$K^i(N \oplus N^*, \sigma_0 \circ \pi_F \circ \pi_N) \xrightarrow[\cong]{Thom^{-1}} K^i(T, \sigma_0 \circ \pi_T) \xrightarrow[\cong]{Morita} K^i(\tilde{\mathcal{G}}|_U, \sigma \circ p \circ j).$$

Note that  $A^*\mathcal{G}|_U \cong N \oplus N^*$  as vector bundles over  $U \cong N$ . That is, the twisted analytic index morphism (modulo the Morita equivalence isomorphism) is the Thom inverse for twisted K-theory of Lie groupoids as in Proposition 2.17. Hence, we finish the proof of the twisted longitudinal index theorem.  $\square$

What we just proved in the last theorem is that any index morphism for foliations with twistings satisfying the three properties in Propositions 2.14, 2.15 and 2.17 is equal to the twisted topological index.

Indeed, if  $IND_{(\mathcal{H}, \beta)}$  is any such index morphism for twisted groupoids, the three mentioned properties imply that the following three square diagrams are commutative precisely by the same arguments as the proof above

$$\begin{CD}
 K^*(F^*, \sigma_0 \circ \pi_F) @>Bott>> K^*(F^* \times \mathbb{R}^{2m}, (\sigma \circ p)_0 \circ \pi_F) @<<{\cong}<< K^*(A^*\tilde{\mathcal{G}}|_U, A^*\tilde{\mathcal{G}}|_U, (\sigma \circ p \circ j)_0 \circ \pi) @<<{\cong}<< K^*(N \oplus N^*, \sigma_0 \circ \pi_F \circ \pi_N) \\
 @V \text{IND}_{(M/F, \sigma)} VV @V \text{IND}_{(M/F \times \mathbb{R}^{2m}, \sigma \circ p)} VV @V \text{IND}_{(\tilde{\mathcal{G}}|_U, \sigma \circ p \circ j)} VV @V \text{Thom}^{-1} VV \\
 K^*(M/F, \sigma) @>Bott>> K^*(M/F \times \mathbb{R}^{2m}, \sigma \circ p) @<<{j!}<< K^*(\tilde{\mathcal{G}}|_U, \sigma \circ p \circ j) @<<{\cong}<< K^*(T, \sigma_0 \circ \pi_T).
 \end{CD}$$

Hence, the big diagram is also commutative, and we obtain the equality  $IND_{(M/F, \sigma)} = \text{Index}_{t, (M/F, \sigma)}$ .

**Remark 3.4.**

- (1) There is a second definition of the twisted analytic index using projective pseudo-differential operators along the leaves which also satisfies the three properties in Propositions 2.14, 2.15 and 2.17. We will return to this issue in a separate paper.
- (2) The proof of the longitudinal index theorem we propose is not exactly that of Connes and Skandalis (see [15]) but more in the spirit of the index theorem proved in [8], Theorem 6.4, which is based on the fact that the Thom (inverse) isomorphism can be realized as the index of some deformation groupoid [16].

**4. Wrong way functoriality**

In this section we construct a push-forward map in twisted K-theory for any smooth oriented map from a manifold to  $M/F$ , the space of leaves of a foliation  $(M, F)$ . This construction generalizes Connes–Skandalis’ push-forward map in K-theory for any K-oriented map from a manifold to  $M/F$ . The main result will be the functoriality of these wrong way morphisms.

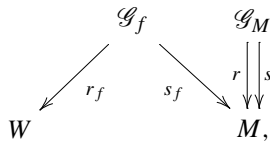
Throughout this section, we assume that with respect to an open cover  $\{U_\alpha\}$  of  $M$  and the foliation charts

$$k_\alpha : U_\alpha \cong \mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}^q,$$

the twisting  $\sigma$  on  $M/F$  is given by a  $PU(H)$ -valued 1-cocycle

$$\sigma_{\alpha\beta} : \mathcal{G}_{U_\alpha}^{U_\beta} \longrightarrow PU(H).$$

Recall that in [15] a smooth map  $f : W \rightarrow M/F$  is given by a principal right  $\mathcal{G}_M$ -bundle over  $W$ ,



and  $f : W \rightarrow M/F$  is called a submersion if the map  $s_f$  is a submersion. Equivalently, a smooth map  $f : W \rightarrow M/F$  is given by a  $\mathcal{G}_M$ -valued 1-cocycle  $(\Omega_i, f_{ij})$ ,

$$f_{ij} : \Omega_i \cap \Omega_j \longrightarrow \mathcal{G}_M, \tag{4.1}$$

with respect to an open covering  $\{\Omega_i\}$  of  $W$  such that

$$f_{ij}(x) \circ f_{jk}(x) = f_{ik}(x)$$

for any  $x \in \Omega_i \cap \Omega_j \cap \Omega_k$ . Note that  $f_{ii} : \Omega_i \rightarrow M$  due to the fact that  $f_{ii}(x) \circ f_{ii}(x) = f_{ii}(x)$  for any  $x \in \Omega_i$ . We always assume that for each  $\Omega_i$ ,

$$\overline{f_{ii}(\Omega_i)} \subset U_{\alpha(i)}$$

for a chosen foliation chart  $U_{\alpha(i)}$ . In particular, the pull-back twisting of  $\sigma$  by  $f$  on  $W$ , denoted by  $f^*\sigma$ , is the composition

$$f^*\sigma = \sigma_{\alpha(i)\alpha(j)} \circ f_{ij} : \Omega_i \cap \Omega_j \longrightarrow \mathcal{G}_{U_{\alpha(i)}}^{U_{\alpha(j)}} \longrightarrow PU(H).$$

Let  $\nu_F$  be the transverse bundle to the foliation  $F$ , that is,  $(\nu_F)_x = T_x M/F_x$ . One can check that the local vector bundles  $f_{ii}^*\nu_F$  can be glued together using  $f_{ij}$  to form a real vector bundle over  $W$ . We denote this vector bundle by  $f^*\nu_F$ . A map  $f : W \rightarrow M/F$  is called oriented if  $TW \oplus f^*\nu_F$  is an oriented vector bundle over  $W$ . The orientation twisting of  $TW \oplus f^*\nu_F$ ,

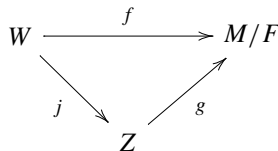
$$o_{TW \oplus f^*\nu_F} : W \longrightarrow PU(H)$$

is equivalent to a trivial twisting if and only if  $TW \oplus f^*\nu_F$  is  $K$ -orientable.

We are now going to construct the push-forward map, with a possible degree shift given by  $d(f) = \dim(W) + \text{rank}(\nu_F) \bmod 2$ ,

$$f_! : K^*(W, f^*\sigma + o_{TW \oplus f^*\nu_F}) \longrightarrow K^{*+d(f)}(M/F, \sigma) \tag{4.2}$$

associated to any smooth oriented map  $f : W \rightarrow M/F$  and a twisting  $\sigma : M/F \longrightarrow PU(H)$ , where  $f^*\sigma$  is the pull-back twisting on  $W$ , and  $o_{TW \oplus f^*\nu_F}$  is the orientation twisting of  $TW \oplus \nu_F$ . Let



be a factorization of  $f$  such that  $j$  is oriented and proper, and  $g$  is a submersion. Such a factorization can be found in [13,5] using the foliation microbundle associated to a Haefliger structure on  $W$ . According to [7], there is a push-forward map in twisted  $K$ -theory for any proper map  $j : W \rightarrow Z$ ,

$$j_! : K^*(W, f^*\sigma + o_{TW \oplus f^*\nu_F}) \longrightarrow K^*(Z, g^*\sigma + o_{TZ \oplus g^*\nu_F}).$$

Therefore, we only need to establish a push-forward map for any submersion  $g : Z \rightarrow M/F$ ,

$$g_! : K^*(Z, \hat{f}^*\sigma + o_{TZ \oplus g^*\nu_F}) \longrightarrow K^*(M/F, \sigma),$$

and show that  $f_! = g_! \circ j_!$  doesn't depend on the choice of the factorization  $f = g \circ j$ .

Let  $f : W \rightarrow M/F$  be a submersion, that is, with respect to an open cover  $\{\Omega_i\}$  of  $W$ , there is a  $\mathcal{G}_M$ -valued 1-cocycle  $\{\Omega_i, f_{ij}\}$ ,

$$f_{ij} : \Omega_i \cap \Omega_j \longrightarrow \mathcal{G}_M$$

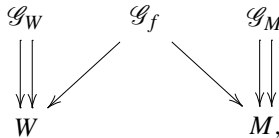


such that  $f_{ii} : \Omega_i \rightarrow M$  are transverse to the foliation  $F$ . Denote by  $F_W = f^*(F)$  the pull-back foliation on  $W$  given by the kernel of the homomorphism (independent of  $i$  and integrable)

$$\pi \circ df_{ii} : TW|_{\Omega_i} \longrightarrow TM \longrightarrow TM/F,$$

with the induced foliation charts given by  $k_{\alpha(i)} \circ f_{ii} : \Omega_i \rightarrow U_{\alpha(i)} \rightarrow \mathbb{R}^q$ . Note that the transverse bundle to the foliation  $F_W$ , denoted by  $\nu_{F_W}$ , is isomorphic to  $f^*\nu_F$ , the pull-back transverse bundle.

Denote by  $\mathcal{G}_W \rightrightarrows W$  the holonomy groupoid of  $(W, F_W)$ . In [15], Lemma 4.2, Connes and Skandalis give an explicit (left) action of  $\mathcal{G}_W$  on  $\mathcal{G}_f$ . In terms of Hilsum–Skandalis morphisms (Definition 1.1(i) in [22]) this means precisely that there is a generalized morphism  $\tilde{f} : \mathcal{G}_W \rightarrow \mathcal{G}_M$  given by the graph  $\mathcal{G}_f$  seen as a  $\mathcal{G}_M$ -principal bundle over  $\mathcal{G}_W$ ,



or equivalently that we can factorize the submersion  $f : W \rightarrow M/F$  as follows

$$\begin{array}{ccc} W & \xrightarrow{f} & M/F \\ & \searrow p_W & \nearrow \tilde{f} \\ & W/F_W & \end{array} \tag{4.3}$$

where  $p_W : W \rightarrow W/F_W$  is the natural submersion given by the inclusion of  $W$  into the unit space  $W/F_W$ . Denote by  $\tilde{f}^*\sigma$  be the pull-back twisting on  $W/F_W$ , then  $f^*\sigma$  is equivalent to the restriction of  $\tilde{f}^*\sigma$  to the unit space of  $\mathcal{G}_W$ . Note that twisting  $\sigma_{TW \oplus f^*\nu_F}$  is equivalent to the twisting  $\sigma_{\nu_{F_W}}$  as

$$TW \oplus f^*\nu_F \cong F_W \oplus \nu_{F_W} \oplus f^*\nu_F \cong F_W \oplus f^*(\nu_F \oplus \nu_F),$$

and  $\nu_F \oplus \nu_F$  has a canonical  $\text{Spin}^c$  structure. Hence, we have the following Thom isomorphism (cf. [7])

$$K^*(W, f^*\sigma + \sigma_{TW \oplus \nu_{F_W}}) \cong K^*(W, f^*\sigma + \sigma_{F_W}).$$

**Definition 4.1.** The push-forward map for the submersion  $p_W : W \rightarrow W/F_W$  is defined to be the composition of the Thom isomorphism in twisted K-theory and the twisted analytic index of  $(W/F_W, \tilde{f}^*\sigma)$ ,

$$(p_W)_! : K^*(W, f^*\sigma + \sigma_{F_W}) \xrightarrow{\cong} K^*(F_W^*, \pi_{F_W}^* \tilde{f}^*\sigma) \xrightarrow{\text{Index}_{a, (W/F_W, \tilde{f}^*\sigma)}} K^*(W/F_W, \tilde{f}^*\sigma),$$

with the degree shifted by the rank of  $F_W$ . Here  $\pi_F$  is the projection  $F_W^* \rightarrow W$ .

Now we are going to construct a push-forward map

$$\tilde{f}_! : K^*(W/F_W, \tilde{f}^* \sigma) \longrightarrow K^*(M/F, \sigma) \tag{4.4}$$

associated to the morphism  $\tilde{f} : W/F_W \rightarrow M/F$ .

**Remark 4.2.** Note that our index morphisms take values in the K-theory groups of reduced  $C^*$ -algebras (see (2.13)). However, to be precise in the construction of the morphism (4.4), we must in what follows, take everywhere the K-theory of the maximal  $C^*$ -algebras instead of the reduced ones (see again (2.13) and Remark 2.4). This wont change anything for those groupoids which are amenable (manifolds, vector bundles) for which we are going to use some results from [7]. Also, it is immediate that if the main result, Theorem 4.10, hold for the K-theory of the maximal  $C^*$ -algebras, then it holds for the K-theory of the reduced  $C^*$ -algebras by considering the canonical projection from the maximal to the reduced algebras. We decided to work with the maximal algebras to simplify the arguments of the proofs, but a direct proof at the reduced level is indeed possible by considering the construction of (4.4) as indicated in [11].

Note that a generalized morphism  $\tilde{f} : W/F_W \rightarrow M/F$  can be equivalently described (Definition–Proposition 1.1 in [22]) by a strict morphism of groupoids

$$\tilde{f}_T : (\mathcal{G}_W)_T \longrightarrow (\mathcal{G}_M)_T \tag{4.5}$$

between the étale groupoids obtained from the restriction to some complete faithful transversals. The way of relating the morphism  $\tilde{f}$  with  $\tilde{f}_T$  is by means of the Morita equivalence between an holonomy groupoid and the étale groupoid obtained by restriction to a transversal. In our case we have a commutative diagram

$$\begin{array}{ccc} \mathcal{G}_W & \xrightarrow{\tilde{f}} & \mathcal{G}_M \\ \mathcal{M}_W \uparrow \sim & & \sim \uparrow \mathcal{M}_M \\ (\mathcal{G}_W)_T & \xrightarrow{\tilde{f}_T} & (\mathcal{G}_M)_T, \end{array} \tag{4.6}$$

where  $\mathcal{M}$  stands for the corresponding Morita equivalence. We briefly recall the construction of  $\tilde{f}_T : (\mathcal{G}_W)_T \rightarrow (\mathcal{G}_M)_T$  (in detail can be found in [22], specially in Definition–Proposition 1.1 and Subsection 3.13).

Recall that the foliation  $F_W$  is constructed locally from the foliation chart

$$k_i : \Omega_i \xrightarrow{f_{ii}} U_{\alpha(i)} \xrightarrow{k_{\alpha(i)}} \mathbb{R}^q \tag{4.7}$$

where  $\{k_{\alpha(i)} : U_{\alpha(i)} \rightarrow \mathbb{R}^q\}$  is the foliation chart defining the foliation  $F$  on  $M$ . Let  $\mathcal{T}_i$  be a transversal of the foliation  $F_W$  on  $\Omega_i$  defined by choosing a right smooth inverse to  $k_i$ . For each foliation chart  $k_\alpha : U_\alpha \rightarrow \mathbb{R}^q$ , we can choose a smooth connected transversal

$$k_\alpha^{-1} : \mathbb{R}^q \longrightarrow \mathcal{T}_\alpha \subset U_\alpha,$$

in particular,  $k_\alpha \circ k_\alpha^{-1} = Id$ . We can assume that  $\{\mathcal{T}_\alpha\}$  are pairwise disjoint and  $f_{ii}$  defines an embedding

$$f_i : \mathcal{T}_i \longrightarrow \mathcal{T}_{\alpha(i)} \tag{4.8}$$

where  $\mathcal{T}_{\alpha(i)}$  is a transversal of the foliation  $F_M$  on  $U_{\alpha(i)}$ . Let  $\mathcal{T}_W = \bigsqcup_i \mathcal{T}_i$  and  $\mathcal{T}_M = \bigsqcup_i \mathcal{T}_{\alpha(i)}$ . We have then the restricted groupoids

$$(\mathcal{G}_W)_T := \bigsqcup_{i,j} (\mathcal{G}_W)_{\mathcal{T}_i}^{\mathcal{T}_j} \rightrightarrows \mathcal{T}_W \quad \text{and} \quad (\mathcal{G}_M)_T := \bigsqcup_{\alpha,\beta} (\mathcal{G}_M)_{\mathcal{T}_\alpha}^{\mathcal{T}_\beta} \rightrightarrows \mathcal{T}_M.$$

Recall that by definition  $\tilde{f}$  is given by a cocycle  $\{(\Omega_i, \tilde{f}_{ij})\}_i$  on  $\mathcal{G}_W$  with values on  $\mathcal{G}_M$ . The strict morphism of groupoids

$$\begin{array}{ccc} (\mathcal{G}_W)_T & \xrightarrow{\tilde{f}_T} & (\mathcal{G}_M)_T \\ \Downarrow & & \Downarrow \\ \mathcal{T}_W & \xrightarrow{\tilde{f}_T^{(0)}} & \mathcal{T}_M \end{array}$$

is defined as the restrictions of the  $\tilde{f}_{ij}$  to the  $(\mathcal{G}_W)_{\mathcal{T}_i}^{\mathcal{T}_j}$ , i.e.,

$$(\mathcal{G}_W)_{T_W} = \bigsqcup_{i,j} (\mathcal{G}_W)_{\mathcal{T}_i}^{\mathcal{T}_j} \xrightarrow{\bigsqcup_{i,j} \tilde{f}_{ij}} (\mathcal{G}_M)_{T_M} \subset \mathcal{G}_M.$$

Notice that the above map has its image in  $(\mathcal{G}_M)_T$  by construction, where we are canonically identifying  $(\mathcal{G}_M)_T$  as the sub-groupoid of  $\mathcal{G}_M$  given by

$$\{\gamma \in \mathcal{G}_M : s(\gamma) \in \mathcal{T}_{\alpha(i)}, r(\gamma) \in \mathcal{T}_{\alpha(j)}\}.$$

Remark that  $\mathcal{T}_W$  is a complete transversal and that, by enlarging  $\mathcal{T}_M$ , we can also assume  $T_M$  is complete. Even more, in our case, the embedding  $\tilde{f}_T^{(0)} : \mathcal{T}_W \hookrightarrow \mathcal{T}_M$  is étale (even we can assume it is a proper open inclusion, see (4.7), (4.8) or Remark 1.4 in [22] for more details). In particular  $\dim \mathcal{T}_W = \dim \mathcal{T}_M$ . The groupoids  $(\mathcal{G}_W)_T \rightrightarrows \mathcal{T}_W$  and  $(\mathcal{G}_M)_T \rightrightarrows \mathcal{T}_M$  are étale groupoids. Hence, the two arrow manifolds have the same dimension too:  $\dim(\mathcal{G}_W)_T = \dim \mathcal{T}_W = \dim \mathcal{T}_M = \dim(\mathcal{G}_M)_T$ .

**Lemma 4.3.** (See [20, p. 378], [15, Section IV], [22].) *The homomorphism  $\tilde{f}_T : (\mathcal{G}_W)_{T_W} \rightarrow (\mathcal{G}_M)_{T_M}$  is an injective étale map.*

Now, given a twisting  $\sigma : \mathcal{G}_M \dashrightarrow PU(H)$ , by taking the refinement of the foliation charts  $\{U_\alpha\}$  if necessary, we can construct as above a strict morphism of groupoids

$$\sigma_T : (\mathcal{G}_M)_T \longrightarrow PU(H),$$

which defines a twisting on  $(\mathcal{G}_M)_T$ . The  $C^*$ -algebras of two Morita equivalent extensions are also Morita equivalent [32,33]. Hence, we have

$$K^*(M/F, \sigma) \cong K^*((\mathcal{G}_M)_T, \sigma_T).$$

The twisting  $\sigma_T$  and  $\tilde{f}_T^* \sigma_T = \sigma_T \circ \tilde{f}_T$  induce the following injective immersions of groupoid extensions

$$\begin{array}{ccc}
 R_W & \longrightarrow & R_M \\
 \downarrow & & \downarrow \\
 (\mathcal{G}_W)_T & \xrightarrow{\tilde{f}_T} & (\mathcal{G}_M)_T.
 \end{array} \tag{4.9}$$

Taking the degree one index morphism associated to the immersion  $R_W \hookrightarrow R_M$  as in (2.20), we obtain a homomorphism

$$(\tilde{f}_T)! : K^*((\mathcal{G}_W)_T, \tilde{f}_T^* \sigma_T) \longrightarrow K^*((\mathcal{G}_M)_T, \sigma_T). \tag{4.10}$$

We have the following lemma.

**Lemma 4.4.** *With the notations above, the push-forward map*

$$\tilde{f}_! : K^*(W/F_W, \tilde{f}_T^* \sigma) \longrightarrow K^*(M/F, \sigma) \tag{4.11}$$

associated to  $\tilde{f} : W/F_W \rightarrow M/F$  given by the composition

$$K^*(W/F_W, \tilde{f}_T^* \sigma) \xrightarrow{\mathcal{M}_W} K^*((\mathcal{G}_W)_T, \tilde{f}_T^* \sigma_T) \xrightarrow{(\tilde{f}_T)!} K^*((\mathcal{G}_M)_T, \sigma_T) \xrightarrow{\mathcal{M}_M} K^*(M/F, \sigma),$$

where  $\mathcal{M}_W$  and  $\mathcal{M}_M$  denote the induced isomorphisms from the Morita equivalences (4.6), does not depend on the choices of complete transversals.

**Proof.** It is a direct adaptation of the proof of Lemma 3.14 in [22]. Just observe that we can also apply Remark 3.12 in [22] in our context since the morphism  $\tilde{f} : W/F_W \rightarrow M/F$  is étale.  $\square$

The following two propositions will allow us to use the factorization (4.3) and the precedent discussion to define the twisted pushforward map for any submersion.

**Proposition 4.5.** *Let  $f : W \rightarrow M/F$  be a smooth submersion and  $\sigma : M/F \dashrightarrow PU(H)$  be a twisting. Let  $X \xrightarrow{g} W/F_W$  be another submersion. Let  $\tilde{f} : W/F_W \rightarrow M/F$  and  $\tilde{g} : X/F_X \rightarrow W/F_W$  be the associated generalized morphisms. Then  $\tilde{f}_! \circ \tilde{g}_! = (\tilde{f} \circ \tilde{g})_!$ .*

**Proof.** By functoriality of the pullback foliation construction (for submersions) we have that  $F_X = g^* F_W = (\tilde{f} \circ \tilde{g})^* F_M$ . The commutative diagram of immersions of groupoids

$$\begin{array}{ccc}
 (\mathcal{G}_X)_T & \xrightarrow{\tilde{f}_T \circ \tilde{g}_T} & (\mathcal{G}_M)_T \\
 \searrow \tilde{g}_T & & \nearrow \tilde{f}_T \\
 & & (\mathcal{G}_W)_T
 \end{array}$$

induces the commutative diagram of immersions of central extensions

$$\begin{array}{ccc}
 R_X & \xrightarrow{\tilde{f}_T \circ \tilde{g}_T} & R_M \\
 \searrow \tilde{g}_T & & \nearrow \tilde{f}_T \\
 & & R_W.
 \end{array}$$

The functoriality of index morphism implies that the following commutative diagram:

$$\begin{array}{ccc}
 K^*((\mathcal{G}_X)_T, (\tilde{f}_T \circ \tilde{g}_T)^* \sigma_T) & \xrightarrow{(\tilde{f}_T \circ \tilde{g}_T)!} & K^*((\mathcal{G}_M)_T, \sigma_T) \\
 \searrow (\tilde{g}_T)! & & \nearrow (\tilde{f}_T)! \\
 & & K^*((\mathcal{G}_W)_T, (\tilde{f}_T)^* \sigma_T).
 \end{array}$$

Together with the functoriality of Morita equivalences, we complete the proof of  $\tilde{f}_! \circ \tilde{g}_! = (\tilde{f} \circ \tilde{g})_!$ .  $\square$

**Proposition 4.6.** Let  $(M/F, \sigma)$  be a twisted foliation,  $g : W \rightarrow M$  be a smooth submersion and  $F_W$  be the pull-back foliation on  $W$ . Denote  $f = p_M \circ g$ . Then

$$f_! = (p_M)_! \circ g_! : K^*(W, f^* \sigma + o_{F_W}) \longrightarrow K^*(M/F, \sigma)$$

with the degree shifted by the rank of  $F_W$  modulo 2.

**Proof.** Consider the commutative diagram

$$\begin{array}{ccc}
 W & \xrightarrow{p_W} & W/F_W \\
 \downarrow g & & \downarrow \tilde{f} \\
 M & \xrightarrow{p_M} & M/F_M.
 \end{array}$$





total spaces of vector bundles over  $T_W$  and  $T_M$ . We have the following commutative diagrams of maps:

$$\begin{array}{ccccc}
 F_W^* \oplus \mathbb{R}^{2m} & \xrightarrow{\cong} & N_W \oplus N_W^* & \xleftarrow{\iota} & T_W \\
 \downarrow dg \oplus h & & \downarrow & & \downarrow d_T g \\
 F_M^* \oplus \mathbb{R}^{2m} & \xrightarrow{\cong} & N_M \oplus N_M^* & \xleftarrow{\iota} & T_M
 \end{array}$$

where  $\iota_W$  and  $\iota_M$  are the zero sections of the vector bundles  $N_W \oplus N_W^* \rightarrow T_W$  and  $N_M \oplus N_M^* \rightarrow T_M$ . Hence, by the functoriality of Thom isomorphism in [7] and the fact that  $\iota_!$  (in twisted K-theory) coincides with the Thom isomorphism in twisted K-theory (see again [7]), we have the following commutative diagram:

$$\begin{array}{ccccc}
 K^*(F_W^* \oplus \mathbb{R}^{2m}, \pi_{F_W}^*(f^*\sigma)_0) & \xrightarrow{\cong} & K^*(N_W \oplus N_W^*, \pi_N^*(\pi_W^* f^* \sigma_0)) & \xrightarrow{Thom^{-1}} & K^*(T_W, \pi_W^* f^* \sigma_0) \\
 \downarrow (dg \oplus h)_! & & \downarrow & & \downarrow (d_T g)_! \\
 K^*(F_M^* \oplus \mathbb{R}^{2m}, \pi_{F_M}^* \sigma_0) & \xrightarrow{\cong} & K^*(N_M \oplus N_M^*, \pi_N^*(\pi_M^* \sigma_0)) & \xrightarrow{Thom^{-1}} & K^*(T_M, \pi_M^* \sigma_0).
 \end{array} \tag{4.16}$$

*Step 3.* Morita isomorphism is functorial. In the definition of the topological indices we can choose the neighborhoods  $U_W$  and  $U_M$  small enough such that the étale (generalized) morphism  $f \times id : \mathcal{G}_W \rightarrow \mathcal{G}_M$  restricts to an étale (generalized) morphism  $(f \times id)|_{U_W} : N_W \times_{T_W} N_W \rightarrow N_M \times_{T_M} N_M$  (remember  $U_W$  and  $U_M$  are chosen such that  $\mathcal{G}|_U = N \times_T N$ ). This last morphism can be equivalently described (Definition–Proposition 1.1 in [22] or our discussion after Definition 4.1) by a strict morphism of groupoids between the étale groupoids obtained from the restriction to some complete transversals. In this case, using the complete transversals  $T_W$  and  $T_M$  to the foliated manifolds  $(U_W, \widetilde{F}_W|_{U_W})$  and  $(U_M, \widetilde{F}_M|_{U_M})$  respectively, it is easily seen that  $(f \times id)|_{U_W}$  is given by  $d_T g : T_W \rightarrow T_M$  modulo the respective Morita equivalences  $N \times_T N \xrightarrow{\mathcal{M}} T$ .

By Lemma 4.4, the twisted K-theory morphism

$$(\tilde{f} \times id)|_{U_W} ! : K^*(\widetilde{\mathcal{G}}_W|_{U_W}, \tilde{f}^* \sigma \circ p_W \circ j_W) \longrightarrow K^*(\widetilde{\mathcal{G}}_M|_{U_M}, \sigma \circ p_M \circ j_M)$$

can be defined as the morphism  $(d_T g)_! : K^*(T_W, \pi_W^*(f^*\sigma)_0) \rightarrow K^*(T_M, \pi_M^*\sigma_0)$  via the induced morphisms by the Morita equivalences (independent of the choice of complete transversals). In other words, the following diagram is commutative:

$$\begin{array}{ccccc}
 K^*(T_W, \pi_W^*(f^*\sigma)_0) & \xrightarrow[\cong]{Morita} & K^*(N_W \times_{T_W} N_W, \pi_N^* \pi_W^*(f^*\sigma)_0) & \xrightarrow{\cong} & K^*(\widetilde{\mathcal{G}}_W|_{U_W}, \tilde{f}^* \sigma \circ p_W \circ j_W) \\
 \downarrow (d_T g)_! & & & & \downarrow ((f \times id)|_{U_W})_! \\
 K^*(T_M, \pi_M^* \sigma_0) & \xrightarrow[\cong]{Morita} & K^*(N_M \times_{T_M} N_M, \pi_N^* \pi_M^* \sigma_0) & \xrightarrow{\cong} & K^*(\widetilde{\mathcal{G}}_M|_{U_M}, \sigma \circ p_M \circ j_M).
 \end{array} \tag{4.17}$$



Step 4. Push-forward map for open embeddings in Lie groupoids is functorial.

The morphism  $\widetilde{f} \times id$  can be equivalently described by a strict morphism of groupoids between the étale groupoids obtained from the restriction to some complete transversals and it does not depend on the choice of these. It is obvious that these complete transversals can be chosen such that their restriction to the  $U$ 's give complete transversals for the restricted foliations. Hence, by Lemma 4.4 and Proposition 2.15, the following diagram is commutative

$$\begin{CD}
 K^*(\widetilde{\mathcal{G}}_W|_{U_W}, \widetilde{f}^*\sigma \circ p_W \circ j_W) @>j_!>> K^*(\widetilde{\mathcal{G}}_W, \widetilde{f}^*\sigma \circ p_W) \\
 @V{(\widetilde{f} \times id)|_{U_W}!}VV @VV{(\widetilde{f} \times id)!}V \\
 K^*(\widetilde{\mathcal{G}}_M|_{U_M}, \sigma \circ p_M \circ j_M) @>j_!>> K^*(\widetilde{\mathcal{G}}_M, \sigma \circ p_M).
 \end{CD} \tag{4.18}$$

Step 5. Bott isomorphism in twisted K-theory of Lie groupoids is functorial.

For the final diagram to be commutative it is enough to observe that if  $T$  is a complete transversal of  $(W, F)$  then  $T \times \mathbb{R}^m$  is a complete transversal of  $(W \times \mathbb{R}^m, F \times \{0\})$ . We apply Lemma 4.4 and Proposition 2.14 to obtain the commutative diagram

$$\begin{CD}
 K^*(\widetilde{\mathcal{G}}_W, \widetilde{f}^*\sigma \circ p_W) @>{Bott^{-1}}>> K^*(\mathcal{G}_W, \widetilde{f}^*\sigma) \\
 @V{(\widetilde{f} \times id)!}VV @VV{\widetilde{f}_!}V \\
 K^*(\widetilde{\mathcal{G}}_M, \sigma \circ p_M) @>{Bott^{-1}}>> K^*(\mathcal{G}_M, \sigma).
 \end{CD} \tag{4.19}$$

Putting together the commutative diagrams (4.15), (4.16), (4.17), (4.18) and (4.19), we establish the functoriality for the twisted topological index morphism. This completes the proof.  $\square$

We can now define the twisted pushforwards for smooth submersions.

**Definition 4.7.** The push-forward map  $f_! : K^*(W, f^*\sigma + o_{TW \oplus f^*v_F}) \rightarrow K^*(M/F_M, \sigma)$  for the submersion  $f : W \rightarrow M/F$  is defined to be the composition of the following maps

$$\begin{aligned}
 K^*(W, f^*\sigma + o_{TW \oplus f^*v_F}) &\xrightarrow{\cong} K^*(W, f^*\sigma + o_{F_W}) \xrightarrow{(p_W)!} K^*(W/F_W, \widetilde{f}^*\sigma) \\
 &\xrightarrow{\widetilde{f}_!} K^*(M/F_M, \sigma).
 \end{aligned} \tag{4.20}$$

We want now to give a definition for any smooth generalized morphism. We will need for that purpose the following result.

**Proposition 4.8.** *Let  $(M/F, \sigma)$  be a twisted foliation and  $f : W \rightarrow M/F$  be a smooth map. Assume  $f$  factors in two different ways:*

$$\begin{array}{ccc}
 & Z_1 & \\
 j_1 \nearrow & & \searrow g_1 \\
 W & \xrightarrow{f} & M/F \\
 j_2 \searrow & & \nearrow g_2 \\
 & Z_2 &
 \end{array} \tag{4.21}$$

where  $g_1$  and  $g_2$  are smooth submersions. Then  $(g_1)! \circ (j_1)! = (g_2)! \circ (j_2)!$ .

**Proof.** First, let us assume  $j_1$  and  $j_2$  are submersions. Consider the factorization of  $g_1 = \tilde{g}_1 \circ p_{Z_1}$ . Putting  $h_1 = p_{Z_1} \circ j_1$ , we have a commutative diagram

$$\begin{array}{ccccc}
 W & \xrightarrow{j_1} & Z_1 & & \\
 p_W \downarrow & \searrow h_1 & \swarrow p_{Z_1} & \downarrow g_1 & \\
 W/F_W & \xrightarrow{\tilde{h}_1} & Z_1/F_{Z_1} & \xrightarrow{\tilde{g}_1} & M/F
 \end{array}$$

By Propositions 4.6 and 4.5, we have  $(h_1)! = (\tilde{h}_1)! \circ (p_W)! = (p_{Z_1})! \circ (j_1)!$  and  $(\tilde{g}_1)! \circ (\tilde{h}_1)! = (\tilde{g}_1 \circ \tilde{h}_1)!$ . Hence,

$$(g_1)! \circ (j_1)! = (\tilde{g}_1)! \circ (p_{Z_1})! \circ (j_1)! = (\tilde{g}_1)! \circ (\tilde{h}_1)! \circ (p_W)! = (\tilde{g}_1 \circ \tilde{h}_1)! \circ (p_W)!$$

Similarly, the commutative diagram

$$\begin{array}{ccccc}
 W & \xrightarrow{j_1} & Z_2 & & \\
 p_W \downarrow & \searrow h_2 & \swarrow p_{Z_2} & \downarrow g_1 & \\
 W/F_W & \xrightarrow{\tilde{h}_2} & Z_2/F_{Z_2} & \xrightarrow{\tilde{g}_1} & M/F
 \end{array}$$

implies that

$$(g_2)! \circ (j_2)! = (\tilde{g}_2 \circ \tilde{h}_2)! \circ (p_W)!$$

Then  $(g_1)! \circ (j_1)! = (g_2)! \circ (j_2)!$  follows  $\tilde{g}_1 \circ \tilde{h}_1 = \tilde{g}_2 \circ \tilde{h}_2$ .

For the general case, recall that Connes and Skandalis construct in [15] the following commutative diagram:

$$\begin{array}{ccccc}
 & & Z_1 & & \\
 & j_1 \nearrow & \uparrow \pi_1 & \searrow g_1 & \\
 W & \xrightarrow{j} & Z & \xrightarrow{g} & M/F \\
 & j_2 \searrow & \downarrow \pi_2 & \nearrow g_2 & \\
 & & Z_2 & & 
 \end{array} \tag{4.22}$$

where  $Z$  is a smooth manifold,  $j$  is a smooth map and  $\pi_1, \pi_2$  and  $g$  are smooth submersions. The manifold  $Z$  corresponds to the fibered product of  $Z_1$  and  $Z_2$  over  $M/F$  (see also [5] for further discussions). The desire equality follows immediately from the wrong way functoriality in twisted K-theory for manifolds proved in [7] and the submersion case treated above.  $\square$

We can give the definition of the push-forward map for any smooth map  $f : W \rightarrow M/F$  where  $M/F$  is equipped with a twisting  $\sigma$ .

**Definition 4.9.** Let  $(M/F, \sigma)$  be a twisted foliation and  $f : W \rightarrow M/F$  be any smooth map. We define

$$f_! : K^*(W, f^*\sigma_0 + o_{TW \oplus f^*v_F}) \longrightarrow K^*(M/F, \sigma)$$

to be the composition  $g_! \circ j_!$  for any factorization  $f = g \circ j$  through  $g : Z \rightarrow M/F$  a smooth submersion. Here the push-forward map

$$j_! : K^*(W, f^*\sigma_0 + o_{TW \oplus f^*v_F}) \longrightarrow K^*(Z, g^*\sigma + o_{TZ \oplus g^*v_F})$$

is established in [7], and the push-forward map  $g_!$  for the submersion  $g$  is defined in Definition 4.7, with the possible shift on degree (see also Definition 4.1).

The main result of this section can be now stated.

**Theorem 4.10.** *The push-forward morphism is functorial, that means, if we have a composition of smooth maps*

$$X \xrightarrow{g} W \xrightarrow{f} M/F, \tag{4.23}$$

and a twisting  $\sigma : M/F \dashrightarrow PU(H)$ , then the following diagram commutes

$$\begin{array}{ccc}
 K^*(X, (f \circ g)^*\sigma + o_{TX \oplus (f \circ g)^*v_F}) & \xrightarrow{(f \circ g)_!} & K^*(M/F, \sigma) \\
 \searrow g_! & & \nearrow f_! \\
 & K^*(W, f^*\sigma + o_{TW \oplus f^*v_F}) & 
 \end{array}$$

**Proof.** Choose a factorization of  $f : W \rightarrow M/F$  as  $W \xrightarrow{j} Z \xrightarrow{q} M/F$  where  $q$  is a submersion, then  $f \circ g = (g \circ j) \circ q$  is a factorization of  $f \circ g$ . Then the claim follows Proposition 4.8 and the functorial property of the push-forward map in [7].  $\square$

We remark that when  $\sigma$  is trivial and  $f : W \rightarrow M/F$  is  $K$ -oriented (that is  $TW \oplus \nu_F$  is  $K$ -oriented), then our push-forward map in Definition 4.9 agrees with the one constructed in [15]. Also, when the foliation consists on a single leaf (hence a manifold) but the twisting is not necessarily trivial, we obtain otherwise the push-forward map defined in [7] by Carey and Wang.

## 5. Further developments

It is very natural now to use the wrong way functoriality studied in the last section to construct an assembly map adapted to our twisted situation. Indeed, it is possible to adapt to foliations the twisted geometric  $K$ -homology introduced in [35] by the second author.

As indicated in [11], we will obtain a twisted assembly map

$$\mu_\sigma : K_*^{geo}(M/F, \sigma + o_{\nu_F}) \longrightarrow K^*(M/F, \sigma). \quad (5.1)$$

There are two very interesting particular cases: the first is when  $\sigma$  is trivial, in this case we recover the assembly map considered by Connes and Skandalis:

$$\mu_\tau : K_*^{geo}(M/F, o_{\nu_F}) \longrightarrow K^*(M/F),$$

and the second is when  $\sigma = o_{\nu_F}$ , in this case we obtain the following assembly map:

$$\mu : K_*^{geo}(M/F) \longrightarrow K^*(M/F, o_{\nu_F}).$$

The situation here is more subtle than the untwisted case since we have not developed the appropriate pseudodifferential calculus and/or we have not discussed the construction of analytic elements from (twisted) correspondences. In particular the understanding of these subjects will lead us to prove and understand the bordism invariant of our twisted indices, and then, to understand how the assembly map fits into some kind of  $S^1$ -equivariant assembly map (we already saw that our indices are naturally factors of an  $S^1$ -equivariant index). We will discuss these topics in detail in a forthcoming paper.

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