The Poisson equation in axisymmetric domains with conical points
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Abstract

This paper analyzes the effects of conical points on the rotation axis of axisymmetric domains \( \hat{\Omega} \subset \mathbb{R}^3 \) on the regularity of the Fourier coefficients \( u_n \) \( (n \in \mathbb{Z}) \) of the solution \( \hat{u} \) of the Dirichlet problem for the Poisson equation

\[-\Delta \hat{u} = \hat{f} \text{ in } \hat{\Omega}.\]

The asymptotic behavior of the coefficients \( u_n \) near the conical points is carefully described and for \( \hat{f} \in L_2(\hat{\Omega}) \), it is proved that if the interior opening angle \( \theta_c \) at the conical point is greater than a certain critical angle \( \theta^* \), then the regularity of the coefficient \( u_0 \) will be lower than expected. Moreover, it is shown that conical points on the rotation axis of the axisymmetric domain do not affect the regularity of the coefficients \( u_n, \ n \neq 0 \). An approximation of the critical angle \( \theta^* \) is established numerically and a priori error estimate for the Fourier-finite-element solutions in the norm of \( W^1_2(\hat{\Omega}) \) is given.

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1. Introduction

The question on the behavior of solutions of elliptic boundary value problems near boundary singularities is particularly important if numerical techniques have to be employed to approximate the solution, since the regularity influences both the accuracy of the approximations and the rates of convergence in the error estimates. Moreover, a priori adaptive numerical schemes can be designed if the singularity functions are known. Thus, considerable work has been done on the singularity analysis of solutions of BVPs (cf. [2,4,5,9,11,12,15–23,25]). The theory of general elliptic boundary value problems

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in domains with conical points in the framework of weighted Sobolev spaces has been well developed (cf. [4,9,15–19,21,22]). In these works, the Mellin transformation is applied to a model problem connected with the boundary value problem.

From the point of view that the solution of elliptic BVPs in three-dimensional axisymmetric domains with nonaxisymmetric data can be approximated more economically by means of FFEM (cf. [3,5,7,13,24,26,28,29]), it is necessary for the effective application of the FFEM, to study the regularity of the Fourier coefficients of the solution, if the axisymmetric domain \( \Omega \) entails singularities. Although there is an extensive literature on the applications of the FFEM, much less work has been done on the mathematical framework, especially on regularity analyses of the solutions of the reduced problems in two-dimensional (2D) and the numerical implications. We refer here to [5,13,14,24,26,27] for some results.

In this paper, we consider the homogeneous Dirichlet problem for the Poisson equation \(-\Delta \hat{u} = \hat{f}\) in axisymmetric domains \( \hat{\Omega} \subset \mathbb{R}^3 \) with conical points on the rotation axis, and our primary concern is with the regularity of the Fourier coefficients \( u_n \ (n \in \mathbb{Z}) \) of the solution \( \hat{u} \). We use spaces originating from the usual \( L_2 \) Sobolev spaces and give a precise description of the asymptotic behavior of the coefficients \( u_n \) near the conical points and show that the singularity functions for \( u_n \) are related to the roots of some transcendental equations involving the Legendre functions. For \( \hat{f} \in L_2 (\hat{\Omega}) \), it is proved that conical points on the rotation axis of \( \hat{\Omega} \) affect only the regularity of the coefficient \( u_0 \) if the largest opening angle \( \theta_c \) at the conical points is greater than a certain critical angle \( \theta^* \), whereas the regularity of the coefficients \( u_n, n \neq 0 \) is not affected by the presence of conical points. An approximate value for \( \theta^* \) is derived numerically.

The results obtained are useful for the effective application of the FFEM for the solution of the BVP. For \( \theta_c > \theta^* \) FEM on graded meshes can be used to approximate \( u_0 \) and for \( u_n, n \neq 0 \), quasi-uniform mesh can be used. This is important since the coefficients \( u_n \) can be computed in parallel. It is proved that the error of the Fourier-finite-element approximation is of the order \( O (h + N^{-1}) \) if partial local mesh grading is used.

2. Analytical preliminaries

We consider the homogeneous Dirichlet problem for the Poisson equation, i.e.

\[-\Delta \hat{u} := - \sum_{i=1}^{3} \frac{\partial^2 \hat{u}}{\partial x_i^2} = \hat{f} \quad \text{in} \quad \hat{\Omega}, \quad \hat{f} \in L_2 (\hat{\Omega}), \quad \hat{u} = 0 \quad \text{on} \quad \hat{\Gamma}, \tag{2.1}\]

where \( \hat{\Omega} \subset \mathbb{R}^3 \) is a bounded domain with boundary \( \hat{\Gamma} := \partial \hat{\Omega}, \hat{\Gamma} \in C^{0,1} \cap PC^2 \). Let \((x_1, x_2, x_3)\) denote the Cartesian coordinates of the point \( x \in \mathbb{R}^3 \). Suppose that the domain \( \hat{\Omega} \) is axisymmetric with respect to the \( x_3 \)-axis, and that the set \( \hat{\Omega} \setminus \hat{\Gamma}_0 \) is generated by rotation of a plane meridian domain \( \Omega_a \) about the \( x_3 \)-axis, where \( \hat{\Gamma}_0 \) is the part of the \( x_3 \)-axis contained in \( \hat{\Omega} \). Let \( \partial \Omega_a \) denote the boundary of \( \Omega_a \) and let \( \hat{\Gamma}_a := \partial \Omega_a \setminus \hat{\Gamma}_0 \). Suppose that \( \hat{\Gamma}_a \in C^2 \) and that the angle \( \theta_c \) at one of the points of intersection of \( \hat{\Gamma}_0 \) and \( \hat{\Gamma}_a \) is not a right angle. If \( C_a = (0, z_c) \) denotes this point, then the rotation of \( C_a \) about the \( x_3 \)-axis yields a conical point \( \hat{C} \) on the boundary \( \hat{\Gamma} \) of \( \hat{\Omega} \), see Fig. 1. We have restricted ourselves for simplicity to domains with only one conical point and no edges, since several conical points could be treated similarly. For domains with axisymmetric edges, we refer, e.g., to [13,14].
Let $r, \varphi, z$ denote the cylindrical coordinate system, then the one-to-one mapping $x_1 = r \cos \varphi$, $x_2 = r \sin \varphi$, $x_3 = z$ transforms the sets $\hat{\Omega} \setminus \Gamma_0$ and $\hat{\Gamma}$ into the sets $\Omega := \Omega_a \times (-\pi, \pi)$ and $\Gamma := \Gamma_a \times (-\pi, \pi)$, respectively, in cylindrical system. Accordingly, for each function $\hat{v}(x), x \in \hat{\Omega} \setminus \Gamma_0$, some function $v$ is uniquely defined on $\Omega$ by

$$v(r, \varphi, z) := \hat{v}(r \cos \varphi, r \sin \varphi, z).$$  

Since the conical point $\hat{\mathcal{C}}$ on $\hat{\Gamma}$ affects the solution $\hat{u}$ of (2.1) only locally, we can analyse its influence on $\hat{u}$ and its Fourier coefficients by restricting $\hat{u}$ to a small neighbourhood $\hat{G}$ of $\hat{\mathcal{C}}$. Thus, we introduce in the $rz$-plane local polar coordinates $R, \theta$ with respect to the vertex at $C_a = (0, z_c)$ viz. $r = R \sin \theta$, $z - z_c = -R \cos \theta$, with $0 \leq \theta \leq \theta_c$, $0 \leq R \leq R_c < \infty$ ($R_c$ sufficiently small) and define the circular sector $\hat{G}_a \subset \hat{\Omega}_a$ by

$$\hat{G}_a := \{(r, z) \in \hat{\Omega}_a : 0 \leq R \leq R_c, 0 \leq \theta \leq \theta_c\}, \quad G_a := \hat{G}_a \setminus \partial G_a.$$  

Let $\hat{G}$ be the domain generated by the rotation of $G_a$ about the $x_3$-axis and $\partial \hat{G}$ its boundary, then the images of the sets $\hat{G} \setminus \Gamma_0$ and $\partial \hat{G} \setminus \Gamma_0$ in the ($r, z, \varphi$)-system are $G = G_a \times (-\pi, \pi]$ and $\partial G = \partial G_a \times (-\pi, \pi]$, respectively, where $\partial_0 G_a := \partial G_a \setminus \Gamma_0$. Let $W^j_{2} (\hat{\Omega})$ ($j = 0, 1, 2$) denote the usual Sobolev spaces, with $W^0_2 = L_2$. By (2.2) mappings $W^j_2 (\Omega \setminus \Gamma_0) \to X^j_{1/2} (\Omega)$ ($j = 0, 1, 2$) are defined, with $X^0_{1/2} (\Omega)$ from (2.4). Since $\Gamma_0$ is 1D, the spaces $W^j_2 (\hat{\Omega} \setminus \Gamma_0)$ and $W^j_2 (\hat{\Omega})$ can be identified. We have, see also [13,24]

$$L^2_2 (\Omega) := \left\{ v = v(r, \varphi, z) : \int_{\Omega} |v|^2 \, dr \, d\varphi \, dz < \infty, \, v \, 2\pi\text{-periodic w.r.t. } \varphi \right\},$$

$$X^0_{1/2} (\Omega) := \{ v = v(r, \varphi, z) : r^{1/2} v \in L^2_2 (\Omega) \},$$

$$X^1_{1/2} (\Omega) := \left\{ v \in X^0_{1/2} (\Omega) : \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \varphi}, \frac{1}{r} \frac{\partial v}{\partial \varphi} \in X^0_{1/2} (\Omega) \right\}.$$
\[ X_{1/2}^2(\Omega) := \left\{ v \in X_{1/2}^1(\Omega) : \frac{\partial^2 v}{\partial r^2}, \frac{\partial^2 v}{\partial z^2}, \frac{\partial^2 v}{\partial \varphi^2}, \frac{1}{r} \frac{\partial^2 v}{\partial z \partial r}, \frac{1}{r} \frac{\partial^2 v}{\partial \varphi \partial r}, \frac{1}{r^2} \frac{\partial^2 v}{\partial \varphi^2} \right\}, \]

\[ V_0(\Omega) := \{ v \in X_{1/2}^1(\Omega) : v|_\Gamma = 0 \}. \]

The norms \( \| \cdot \|_{X_{1/2}^2(\Omega)} \) of functions from (2.4) are generated from \( \| \cdot \|_{W_{2}^4(\Omega)} \) by transformation (2.2) and are given by

\[ \| v \|_{X_{1/2}^2(\Omega)} := \left\{ \int_{\Omega} |v|^2 r \, dr \, d\varphi \, dz \right\}^{1/2}, \]

\[ |v|^2_{X_{1/2}^2(\Omega)} := \left\{ \int_{\Omega} \left( \left| \frac{\partial v}{\partial r} \right|^2 + \left| \frac{\partial v}{\partial z} \right|^2 + \frac{1}{r} \left| \frac{\partial v}{\partial \varphi} \right|^2 \right) r \, dr \, d\varphi \, dz \right\}^{1/2}, \]

\[ |v|^2_{X_{1/2}^2(\Omega)} := \left\{ \int_{\Omega} \left( \left| \frac{\partial^2 v}{\partial r^2} \right|^2 + \left| \frac{\partial^2 v}{\partial z^2} \right|^2 + 2 \left| \frac{\partial^2 v}{\partial z \partial r} \right|^2 + 2 \frac{1}{r} \left| \frac{\partial^2 v}{\partial \varphi} \right|^2 \right) r \, dr \, d\varphi \, dz \right\}^{1/2}, \]

\[ \| v \|_{X_{1/2}^2(\Omega)} := \{ \| v \|^2_{X_{1/2}^2(\Omega)} + |v|^2_{X_{1/2}^2(\Omega)} \}^{1/2}, \]

\[ \| v \|_{V_0(\Omega)} := \| v \|^2_{X_{1/2}^2(\Omega)} \text{ for } v \in V_0(\Omega). \]

For \( f \in X_{1/2}^0(\Omega) \) the weak solution of (2.1) in cylindrical coordinates is obtained by looking for \( u \in V_0(\Omega) \) such that

\[ b(u, v) = f(v) \text{ for } v \in V_0(\Omega) \]

(2.6)

with

\[ b(u, v) := \int_{\Omega} \left( \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial u}{\partial \varphi} \frac{\partial v}{\partial \varphi} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right) r \, dr \, d\varphi \, dz, \quad f(v) := \int_{\Omega} f v r \, dr \, d\varphi \, dz, \]

(2.7)

where \( \bar{v} \) is the complex conjugate of \( v \). Clearly, the Lax/Milgram lemma infers that a unique solution \( u \in V_0(\Omega) \) of (2.6) exists. Solution may contain singularities due to the presence of conical points on the boundary of \( \Omega \), that is \( u \) may not belong to \( V_0(\Omega) \cap X_{1/2}^2(\Omega) \) as would be expected. For the analysis of the regularity of \( u \) we introduce a smooth cut-off function \( \eta = \eta(r, \varphi, z) = \tilde{\eta}(R) \in C^\infty[0, \infty) \) with \( 0 \leq \tilde{\eta} \leq 1 \) and

\[ \tilde{\eta} = \begin{cases} 1 & \text{for } 0 \leq R \leq R_c/3, \\ 0 & \text{for } R > 2R_c/3, \end{cases} \]
where $R$ and $R_c$ are from (2.3). The function $u_\eta := \eta u$ describes the solution $u$ of (2.6) in the neighbourhood $G$ of the conical point $C$. It can be shown, see for example [11,13,22], that the function $u_\eta$ is the unique solution of the variational problem: Find $u_\eta \in V_0(G)$ such that

$$b_\eta(u_\eta, v) = f_\eta(v) \quad \text{for } v \in V_0(G), \quad f_\eta \in X^1_{1/2}(G)$$

(2.8)

with

$$b_\eta(u_\eta, v) := \int_G \left( \frac{\partial u_\eta}{\partial r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial u_\eta}{\partial \varphi} \frac{\partial v}{\partial \varphi} + \frac{\partial u_\eta}{\partial z} \frac{\partial v}{\partial z} \right) r \, dr \, d\varphi \, dz,$$

$$f_\eta(v) := \int_G f_\eta \tilde{v} r \, dr \, d\varphi \, dz,$$

(2.9)

where $f_\eta$ depends on $\eta$, $f$, $u$ and its first derivatives. For simplicity, we omit in the sequel the subscript $\eta$. It would be clear from the context if we are referring to $G$ or $\Omega$.

**Remark 2.1.** The global regularity of the solution $u$ of (2.6) is determined by the regularity of the solution $u_\eta = \eta u$ of (2.8) and the function $(1 - \eta) u$. Owing to well-established results, see e.g., [9,11,16], we know that $\hat{f} \in L^2(\hat{\Omega})$ implies $\hat{u} \in W^2_2(\hat{N} \cap \hat{\Omega})$ for any open set $\hat{N}$ such that $\hat{N}$ does not contain any singular point of $\hat{\Omega}$.

### 3. Partial Fourier analysis

Functions $v(r, \varphi, z) \in X^{1/2}_{1/2}(\Omega) \ (j = 0,1,2)$ are $2\pi$-periodic with respect to the coordinate variable $\varphi \in (-\pi, \pi]$. Hence, using the orthogonal and complete system $\{e^{ik\varphi}\}_{k \in \mathbb{Z}} \ (i^2 = -1; \ Z := \{0, \pm 1, \pm 2, \ldots\})$ in $L^2(-\pi, \pi)$ these functions can be represented in Fourier series with Fourier coefficients defined on $\Omega_a$. For the analysis of functions defined on $\Omega_a$, it is natural to introduce weighted Sobolev spaces associated with the measure $r \, dr \, dz$. A more general study of such spaces can be found, e.g. in [6,22]. We have, see also [5,13,24]

$$L^2(\Omega_a) := \left\{ w = w(r, z) : \int_{\Omega_a} |w|^2 \, dr \, dz < \infty \right\},$$

$$L^2_{x}(\Omega_a) := \{ w = w(r, z) : r^2 w \in L^2(\Omega_a) \}, \ x \ \text{real,} \ X(\Omega_a) := L^2_{1/2}(\Omega_a),$$

$$W^1_{x}(\Omega_a) := \left\{ w \in L^2_{x}(\Omega_a) : \frac{\partial w}{\partial r}, \frac{\partial w}{\partial z} \in L^2_{x}(\Omega_a) \right\},$$

$$W^2_{x}(\Omega_a) := \left\{ w \in W^1_{x}(\Omega_a) : \frac{\partial^2 w}{\partial r^2}, \frac{\partial^2 w}{\partial z^2}, \frac{\partial^2 w}{\partial r \partial z} \in L^2_{x}(\Omega_a) \right\},$$

$$X^1_{1/2}(\Omega_a) := \{ w \in W^1_{1/2}(\Omega_a) : r^{-1} w \in L^2_{1/2}(\Omega_a) \},$$

$$V_0^1(\Omega_a) := \{ w \in W^1_{1/2}(\Omega_a) : w|_{\Gamma_a} = 0 \}.$$
The norms in these spaces are given by

\[ V_0^a(G_a) := \{ w \in W^{1,2}_1(G_a) : w|_{\partial G_a} = 0 \}, \]

\[ W_0^a(\Omega_a) := \{ w \in V_0^a(\Omega_a) : r^{-1} w \in L^{2,1/2}(\Omega_a) \}. \]  

(3.1)

The norms in these spaces are given by

\[ \| w \|_{L^2(\Omega_a)} = \left\{ \int_{\Omega_a} |w|^2 \, dr \, dz \right\}^{1/2}, \quad \| w \|_{L^{2,1/2}(\Omega_a)} = \left\{ \int_{\Omega_a} |r^2 w|^2 \, dr \, dz \right\}^{1/2}, \]

\[ \| w \|_{L^{1,2}(\Omega_a)} = \left\{ \| w \|^2_{L^2(\Omega_a)} + \left\| \frac{\partial w}{\partial r} \right\|_{L^{2,1/2}(\Omega_a)}^2 + \left\| \frac{\partial w}{\partial z} \right\|_{L^{2,1/2}(\Omega_a)}^2 \right\}^{1/2}, \]

\[ \| w \|_{L^{2,2}(\Omega_a)} = \left\{ \| w \|^2_{L^2(\Omega_a)} + \left\| \frac{\partial w}{\partial r} \right\|_{L^{2,1/2}(\Omega_a)}^2 + \left\| \frac{\partial w}{\partial z} \right\|_{L^{2,1/2}(\Omega_a)}^2 \right\}^{1/2}, \]

\[ \| w \|_{X^{1,2}_1(\Omega_a)} = \left\{ \frac{\| w \|^2}{r} \right\}^{1/2} \frac{\| \frac{\partial w}{\partial r} \|_{X(\Omega_a)}^2 + \| \frac{\partial w}{\partial z} \|_{X(\Omega_a)}^2}{X(\Omega_a)} \right\}^{1/2}, \]

\[ \| w \|_{v_0^a(\Omega_a)} := \| w \|_{W^{1,2}_1(\Omega_a)}, \quad \| w \|_{v_0^a(\Omega_a)} := \| w \|_{W^{1,2}_1(\Omega_a)}. \]  

(3.2)

Functions \( v \in X^{j}_{1/2}(\Omega) \) \((j = 0, 1, 2)\) can be characterized by their Fourier coefficients. We have the following lemma.

**Lemma 3.1 (Cf. Heinrich [13]).** For \( v \in X^{0}_{1/2}(\Omega) \), the Fourier coefficients \( v_k(r, z) \) defined by

\[ v_k(r, z) := \frac{1}{2\pi} \int_{\Omega} v(r, \varphi, z)e^{-ik\varphi} \, d\varphi, \quad k \in \mathbb{Z}, \]  

(3.3)

satisfy the relations

\[ v(r, \varphi, z) = \sum_{k \in \mathbb{Z}} v_k(r, z)e^{ik\varphi} \text{ (a.e. in } \Omega) \]  

(3.4)

and

\[ v_k \in X(\Omega_a), \quad \| v \|_{X^{0}_{1/2}(\Omega)}^2 = 2\pi \sum_{k \in \mathbb{Z}} \| v_k \|_{X(\Omega_a)} < \infty. \]  

(3.5)

If \( v \in X^{1}_{1/2}(\Omega) \), then relations (3.5) and

\[ |v|^2_{X^{1}_{1/2}(\Omega)} = 2\pi \sum_{k \in \mathbb{Z}} \left\{ \left\| \frac{\partial v_k}{\partial r} \right\|_{X(\Omega_a)}^2 + \left\| \frac{\partial v_k}{\partial z} \right\|_{X(\Omega_a)}^2 + k^2 \frac{v_k}{r} \right\} < \infty \]  

(3.6)
The solutions hold. If additionally \( v \in X^{1/2}_{1/2}(\Omega) \), then
\[
\|v\|^2_{X^{1/2}_{1/2}(\Omega)} = 2\pi \sum_{k \in \mathbb{Z}} \left\{ \frac{\partial^2 v_k}{\partial r^2} \right\}^2_{X(\Omega_0)} + \frac{\partial^2 v_k}{\partial z^2} \right\}^2_{X(\Omega_0)} + 2 \left\{ \frac{\partial^2 v_k}{\partial z \partial r} \right\}^2_{X(\Omega_0)} + 2k^2 \left\{ \frac{\partial v_k}{r} \right\}^2_{X(\Omega_0)} < \infty. \tag{3.7}
\]

Remark 3.2 (Cf. Heinrich [13]). If the function \( v(r, \varphi, z) \) and only some of its derivatives belong to \( X^{0}_{1/2}(\Omega) \), then corresponding completeness relationships hold. For example, let \( \partial^l v/\partial \varphi^l \in X^{0}_{1/2}(\Omega) \) \( (l = 0, 1, 2) \), then
\[
\sum_{l=0}^{2} \left\{ \frac{\partial^l v}{\partial \varphi^l} \right\}^2_{X^{0}_{1/2}(\Omega)} = 2\pi \sum_{k \in \mathbb{Z}} (1 + k^2 + k^4) \| v_k \|^2_{X(\Omega_0)} < \infty. \tag{3.8}
\]

By means of partial Fourier analysis with respect to the rotation angle \( \varphi \), the 3D BVPs (2.6) and (2.8) can be decomposed into a family of decoupled 2D BVPs on the meridian domain, see for example [13,5,24].

Lemma 3.3 (Cf. Heinrich [13]). For \( u, v \in X^{1}_{1/2}(\Omega), f \in X^{0}_{1/2}(\Omega) \), the functionals \( b(u, v) \) and \( f(v) \) from (2.6) can be represented by
\[
b(u, v) = 2\pi \sum_{k \in \mathbb{Z}} b_k(u_k, v_k), \quad f(v) = 2\pi \sum_{k \in \mathbb{Z}} f_k(v_k),
\]
\[
b_k(u_k, v_k) := \int_{\Omega_0} \left( \frac{\partial u_k}{\partial r} \frac{\partial v_k}{\partial r} + \frac{\partial u_k}{\partial z} \frac{\partial v_k}{\partial z} + k^2 \frac{u_k v_k}{r^2} \right) r \, dr \, dz,
\]
\[
f_k(v_k) := \int_{\Omega_0} f_k v_k r \, dr \, dz \tag{3.9}
\]
for \( k \in \mathbb{Z} \) and with \( u_k, v_k \) and \( f_k \) being the Fourier coefficients of \( u, v \) and \( f \), respectively. Moreover, the Fourier coefficients \( u_k \) of the solution \( u \) of (2.6) are unique solutions of the variational equations \( (\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}) \)
\[
k = 0 : \quad \text{find} \; u_0 \in V_{0}^{a}(\Omega_0) : b_0(u_0, w) = f_0(w) \; \; \; \text{for} \; \; w \in V_{0}^{a}(\Omega_0),
\]
\[
k \in \mathbb{Z}_0 : \quad \text{find} \; u_k \in W_{0}^{a}(\Omega_0) : b_k(u_k, w) = f_k(w) \; \; \; \text{for} \; \; w \in W_{0}^{a}(\Omega_0). \tag{3.10}
\]

The solutions \( u_k \) satisfy the following a priori estimates:
\[
\text{(a)} \; \| u_0 \|^2_{V_{0}^{a}(\Omega_0)} \leq C_1 \| f_0 \|^2_X,
\]
\[
\text{(b)} \; \| u_k \|^2_{W_{0}^{a}(\Omega_0)} \leq \left\{ \frac{\partial u_k}{\partial r} \right\}^2_{X} + \left\{ \frac{\partial u_k}{\partial z} \right\}^2_{X} + k^2 \frac{u_k}{r} \left\{ \frac{u_k}{r} \right\}^2_{X} \leq C_2 \frac{1}{r^2} \| f_k \|^2_X \; \; \; \text{for} \; \; k \in \mathbb{Z}_0,
\]
\[
\text{(c)} \; \| u_0 \|^2_{V_{0}^{a}(\Omega_0)} + \sum_{k \in \mathbb{Z}_0} k^2 \left\{ \frac{\partial u_k}{\partial r} \right\}^2_{X} + \left\{ \frac{\partial u_k}{\partial z} \right\}^2_{X} + k^2 \frac{u_k}{r} \left\{ \frac{u_k}{r} \right\}^2_{X} \leq C_3 \| f \|^2_{X^{1/2}_{1/2}(\Omega)} \tag{3.11}
\]
with some positive constants \( C_i \) \( (i = 1,2,3) \).
4. The solution of the 3D BVP near the conical point

For convenience we translate the vertex $G_a = (0, z_c)$ to the origin and consider the BVP (2.8), where $\tilde{G}$ is a neighbourhood of the origin. Our main concern in this section is to derive an analytic expression for the solution $u$ of (2.8). First, we notice that the solutions $u_k$ ($k \in \mathbb{Z}$) of the variational equations (3.10) with $\Omega_a$ replaced with $G_a$ are the generalized solutions of the following 2D BVPs:

$$\Delta_{r,z} u_k - \frac{k^2}{r^2} u_k = -f_k \quad \text{in } G_a, \quad (4.1)$$

$$u_k = 0 \quad \text{on } \partial_0 G_a, \quad |u_k(r, z)|_{r=0} < \infty \quad (4.2)$$

with $\Delta_{r,z} := \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$.

These problems can be written in terms of the local polar coordinates $R, \theta$ with respect to the origin $(0,0)$ as follows:

$$\frac{\partial^2 \tilde{u}_k}{\partial R^2} + 2 \frac{\partial \tilde{u}_k}{R \partial R} + \frac{1}{R^2} \frac{\partial^2 \tilde{u}_k}{\partial \theta^2} + \cot \theta \frac{\partial \tilde{u}_k}{\partial \theta} - \frac{k^2}{R^2 \sin^2 \theta} \tilde{u}_k = -\tilde{f}_k \quad \text{in } \tilde{G}_a. \quad (4.3)$$

$$\tilde{u}_k(R, \theta)|_{R=R_c} = 0, \quad |\tilde{u}_k(R, \theta)|_{R=0} < \infty, \quad k \in \mathbb{Z}, \quad (4.4)$$

$$\tilde{u}_k(R, \theta)|_{\theta=\theta_c} = 0, \quad |\tilde{u}_k(R, \theta)|_{\theta=0} < \infty, \quad k \in \mathbb{Z}, \quad (4.5)$$

where the functions $u_k$ and $\tilde{u}_k$ are related by $\tilde{u}_k(R, \theta) = u_k(R \sin \theta, -R \cos \theta)$.

We use the Fourier method to determine the solutions of (4.3)–(4.5). First, we consider the associated homogeneous equation to Eq. (4.3), i.e.,

$$\frac{\partial^2 \tilde{u}_k}{\partial R^2} + 2 \frac{\partial \tilde{u}_k}{R \partial R} + \frac{1}{R^2} \frac{\partial^2 \tilde{u}_k}{\partial \theta^2} + \cot \theta \frac{\partial \tilde{u}_k}{\partial \theta} - \frac{k^2}{R^2 \sin^2 \theta} \tilde{u}_k = 0, \quad k \in \mathbb{Z}, \quad (4.6)$$

Suppose that there is a solution of (4.6) having the multiplicative form

$$\tilde{u}_k(R, \theta) = Y_k(R) \Psi_k(\theta), \quad k \in \mathbb{Z}, \quad 0 < R < R_c, \quad 0 < \theta < \theta_c. \quad (4.7)$$

Substituting (4.7) in (4.6) we obtain the identity

$$\frac{R^2 Y_k''(R) + 2 R Y_k'(R)}{Y_k(R)} = \frac{\Psi_k''(\theta) + \cot \theta \Psi_k'(\theta) - k^2 \sin^2 \theta \Psi_k(\theta)}{\Psi_k(\theta)}. \quad (4.8)$$

Eq. (4.8) holds for all $0 < R < R_c$ and $0 < \theta < \theta_c$ such that $Y_k(R)$ and $\Psi_k(\theta)$ are different from zero. This leads to the equations

$$R^2 Y_k''(R) + 2 R Y_k'(R) - \lambda_k Y_k(R) = 0, \quad k \in \mathbb{Z}, \quad 0 < R < R_c, \quad (4.9)$$

$$\Psi_k''(\theta) + \cot \theta \Psi_k'(\theta) - k^2 \sin^2 \theta \Psi_k(\theta) + \lambda_k \Psi_k(\theta) = 0, \quad k \in \mathbb{Z}, \quad 0 < \theta < \theta_c, \quad (4.10)$$

where $\lambda_k$ ($k \in \mathbb{Z}$) are some complex numbers to be determined. We notice that Eq. (4.9) is the Euler differential equation. If the constant $\lambda_k$ is known, then the solution of (4.9) can be written out.
Set \( \lambda_k = v_k(v_k + 1) \) and \( t = \cos \theta, \) \( \cos \theta_c < t < 1 \). Then, from Eq. (4.10) and relations (4.5) we obtain

\[
(1 - t^2) \Psi_k''(t) - 2t \Psi_k'(t) + \left( v_k(v_k + 1) - \frac{k^2}{1 - t^2} \right) \Psi_k(t) = 0, \quad \cos \theta_c < t < 1,
\]

\[
\Psi_k(t)|_{t=\cos \theta_c} = 0, \quad |\Psi_k(t)|_{t=1} < \infty, \quad k \in \mathbb{Z}.
\]

We observe that Eq. (4.11) is the Legendre differential equation and its general solution is given by (cf. [1,10])

\[
\Psi_k(t) = C_1 P^k_{v_{k}}(t) + C_2 Q^k_{v_{k}}(t), \quad k \in \mathbb{Z},
\]

where \( P^k_{v_{k}} \) and \( Q^k_{v_{k}} \) are the associated Legendre functions of the first and second kind, respectively, of order \( v_{k} \) and degree \( k \) (cf. [1,10]). Since \( Q^k_{v_{k}}(t)|_{t=1} \) is unbounded and \( P^k_{v_{k}}(t) \) is bounded for \( \cos \theta_c \leq t \leq 1 \), it follows from (4.12) that in (4.13) \( C_2 \) must be zero and we are left with the solution (take \( C_1 = 1 \))

\[
\Psi_k(t) = P^k_{v_{k}}(t), \quad k \in \mathbb{Z}.
\]

The boundary condition at \( t = \cos \theta_c \) (cf. (4.12)) leads to the following deterministic expression for \( v_{k} \):

\[
P^k_{v_{k}}(\cos \theta_c) = 0, \quad k \in \mathbb{Z}.
\]

For each \( k \in \mathbb{Z} \), Eq. (4.15) has an increasing sequence of nonnegative simple real roots \( v_l^k, \ l = 1, 2, \ldots \), with no limit point and \( v_k^k \) and \( v_{-k}^k \) coincide (see Section 6). Let us denote by \( \{P^k_{v_{k}}(t)\}_{l=1}^{\infty} \) the corresponding system of eigenfunctions.

**Lemma 4.1.** *The system \( \{P^k_{v_{k}}(t)\}_{l=1}^{\infty} \) is orthogonal and complete in \( L_2(1, \cos \theta_c) \).*

**Proof.** To show the orthogonality property, we prove that

\[
\int_1^{\cos \theta_c} P^k_{v_{k}}(t) P^j_{v_{k}}(t) \, dt = 0 \quad \text{for} \ l \neq j.
\]

It follows from (4.11) the relations

\[
(1 - t^2) \frac{d^2 P^k_{v_{k}}(t)}{dt^2} - 2t \frac{dP^k_{v_{k}}(t)}{dt} + \left( v_k^k(v_k^k + 1) - \frac{k^2}{1 - t^2} \right) P^k_{v_{k}}(t) = 0,
\]

\[
(1 - t^2) \frac{d^2 P^k_{v_{k}}(t)}{dt^2} - 2t \frac{dP^k_{v_{k}}(t)}{dt} + \left( v_k^j(v_k^j + 1) - \frac{k^2}{1 - t^2} \right) P^k_{v_{k}}(t) = 0.
\]
Multiply the first equation in (4.17) by \( P^k_{v_k^j}(t) \), the second by \( P^k_{v_k^j}(t) \), subtract and integrate the result from 1 to \( \cos \theta_c \) to obtain

\[
\{v^j_k(v^j_k + 1) - v^j_k(v^j_k + 1)\} \int_1^{\cos \theta_c} P^k_{v_k^j}(t)P^k_{v_k^j}(t)\,dt
\]

\[
= \int_1^{\cos \theta_c} 2t \left[ \frac{dP^k_{v_k^j}(t)}{dt}P^k_{v_k^j}(t) - \frac{dP^k_{v_k^j}(t)}{dt}P^k_{v_k^j}(t) \right] dt
\]

\[
+ \int_1^{\cos \theta_c} (1 - t^2) \left[ \frac{d^2P^k_{v_k^j}(t)}{dt^2}P^k_{v_k^j}(t) - \frac{d^2P^k_{v_k^j}(t)}{dt^2}P^k_{v_k^j}(t) \right] dt.
\] (4.18)

Integration by part gives

\[
\int \frac{d^2P^k_{v_k^j}(t)}{dt^2} \{ (1 - t^2)P^k_{v_k^j}(t) \} \,dt
\]

\[
= (1 - t^2) \frac{dP^k_{v_k^j}(t)}{dt}P^k_{v_k^j}(t)
\]

\[
- \int (1 - t^2) \left[ \frac{dP^k_{v_k^j}(t)}{dt} \frac{dP^k_{v_k^j}(t)}{dt} \right] dt + \int 2t \frac{dP^k_{v_k^j}(t)}{dt}P^k_{v_k^j}(t)\,dt.
\] (4.19)

\[
\int \frac{d^2P^k_{v_k^j}(t)}{dt^2} \{ (1 - t^2)P^k_{v_k^j}(t) \} \,dt
\]

\[
= (1 - t^2) \frac{dP^k_{v_k^j}(t)}{dt}P^k_{v_k^j}(t)
\]

\[
- \int (1 - t^2) \left[ \frac{dP^k_{v_k^j}(t)}{dt} \frac{dP^k_{v_k^j}(t)}{dt} \right] dt + \int 2t \frac{dP^k_{v_k^j}(t)}{dt}P^k_{v_k^j}(t)\,dt.
\] (4.20)

Substituting relations (4.19) and (4.20) in (4.18) we obtain

\[
\{v^j_k(v^j_k + 1) - v^j_k(v^j_k + 1)\} \int_1^{\cos \theta_c} P^k_{v_k^j}(t)P^k_{v_k^j}(t)\,dt
\]

\[
= (1 - t^2) \left[ \frac{dP^k_{v_k^j}(t)}{dt}P^k_{v_k^j}(t) - \frac{dP^k_{v_k^j}(t)}{dt}P^k_{v_k^j}(t) \right] \bigg|_{t=1}
\] (4.21)

\[
= (1 - t^2) \left[ \frac{dP^k_{v_k^j}(t)}{dt}P^k_{v_k^j}(t) - \frac{dP^k_{v_k^j}(t)}{dt}P^k_{v_k^j}(t) \right] \bigg|_{t=1}
\] (4.22)

The right-hand side of Eq. (4.22) vanishes at \( t = \cos \theta_c \). Furthermore, due to the factor \((1 - t^2)\) and the fact that \( P^k_{v_k^j}(t) \) and \( dP^k_{v_k^j}(t)/dt \) are bounded at \( t = 1 \), the right-hand side of (4.22) also vanishes at \( t = 1 \). Finally, since \( v^j_k \neq v^j_i \), assertion (4.16) follows. □
The functions $\tilde{u}_k(R, \theta)$ and $\tilde{f}_k(R, \theta)$ from (4.3) can be represented in Fourier series with respect to the system $\{P_{v_k}^k(t)\}_{l=1}^\infty$. We have

$$\tilde{u}_k(R, \theta) = \sum_{l=1}^{\infty} u_{kl}(R) P_{v_k}^k(\cos \theta), \quad k \in \mathbb{Z},$$

$$\tilde{f}_k(R, \theta) = \sum_{l=1}^{\infty} f_{kl}(R) P_{v_k}^k(\cos \theta), \quad k \in \mathbb{Z}.$$  \label{eq:4.23}

The Fourier coefficients $u_{kl}$ and $f_{kl}$ are calculated from the relations

$$u_{kl}(R) = \frac{(\tilde{u}_k, P_{v_k}^k)}{(P_{v_k}^k, P_{v_k}^k)}, \quad f_{kl}(R) = \frac{(\tilde{f}_k, P_{v_k}^k)}{(P_{v_k}^k, P_{v_k}^k)},$$  \label{eq:4.24}

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $L_2$. Replacing in (4.3) the functions $\tilde{u}_k(R, \theta)$ and $\tilde{f}_k(R, \theta)$ by the corresponding Fourier series (4.23), we deduce by comparing coefficients that the $u_{kl}, k \in \mathbb{Z}; l \in \mathbb{N} = \{1, 2, \ldots\}$, from (4.24) are unique solutions of the boundary value problems

$$u_{kl}'' + 2R^{-1}u_{kl} - v_k^l(v_k^l + 1)R^{-2}u_{kl} = -f_{kl}, \quad k \in \mathbb{Z}; \quad l \in \mathbb{N}; \quad 0 < R < R_c,$$

$$|u_{kl}(0)| < \infty, \quad u_{kl}(R_c) = 0.$$  \label{eq:4.25}

The solution of problem (4.25), (4.26) can be obtained by the method of variation of parameters. We have

$$u_{kl}(R) = - \frac{R_c^{- (2v_k^l + 1)}}{2v_k^l + 1} \int_0^{R_c} f_{kl}(\tau)\tau^{v_k^l + 2} \, d\tau + \frac{R_c^{v_k^l}}{2v_k^l + 1} \int_R^{R_c} f_{kl}(\tau)\tau^{1-v_k^l} \, d\tau$$

$$+ \frac{R^{- (v_k^l + 1)}}{2v_k^l + 1} \int_0^R f_{kl}(\tau)\tau^{v_k^l + 2} \, d\tau.$$  \label{eq:4.27}

The above analysis can be summarized as follows:

**Theorem 4.1.** Let $\tilde{C}$ be a conical point of the axisymmetric domain $\hat{\Omega}$ situated at the origin and let $\hat{G} \subset \hat{\Omega}$ be a sufficiently small neighbourhood of $\tilde{C}$. Further, let $G_a$ denote the meridian plane of $\hat{G}$ and let $R, \theta$ be local polar coordinates with respect to the vertex $C_a$ of $G_a$. Then the solution $u$ of the variational problem (2.8) has the following representation:

$$u(r, \varphi, z) = \tilde{u}(R, \varphi, \theta) = \sum_{k \in \mathbb{Z}} \sum_{l=1}^{\infty} u_{kl}(R) P_{v_k}^k(\cos \theta) e^{ik\varphi},$$  \label{eq:4.28}

where $u_{kl}$ are given by (4.27) and

$$u_k(r, z) = \tilde{u}_k(R, \theta) = \sum_{l=1}^{\infty} u_{kl}(R) P_{v_k}^k(\cos \theta),$$  \label{eq:4.29}

are the solutions of the 2D variational equations (3.10) on $G_a$. 
5. The regularity of solutions of the 2D BVPs

In this section, we analyze the regularity of the solutions $u_k$ ($k \in \mathbb{Z}$) of the 2D BVPs (3.10). We assume for simplicity that the axisymmetric domain $\hat{\Omega}$ has only one conical point $\hat{C}$ situated at the origin and no edges. By the classical results (see, e.g. [13]) we know that for $f_k \in L_{2,1/2}(\Omega_a)$ and $\Omega_a$ sufficiently smooth, the solution $u_k$ of (3.10) belongs $W^{2,2}_{1/2}(\Omega_a)$ and the following a priori estimate is satisfied.

$$
\|u_k\|_{W^{2,2}_{1/2}(\Omega_a)} \leq C \|f_k\|_{L_{2,1/2}(\Omega_a)}.
$$

(5.1)

**Lemma 5.1.** For $\nu_l^k > 1/2$ ($\nu_l^k$ from (4.15)) the solutions $u_k$ ($k \in \mathbb{Z}$) of the variational equations (3.10) on $G_a$ have the property

$$
\left\{ \begin{array}{l}
u_k \in W^{2,2}_{1/2}(G_a) \\
\|u_k\|_{W^{2,2}_{1/2}(G_a)} \leq C \|f_k\|_{L_{2,1/2}(G_a)}.
\end{array} \right.
$$

(5.2)

**Proof.** In the following $C$ denotes a generic positive constant. It has different values at different points. The norm $\|u_k\|_{W^{2,2}_{1/2}(G_a)}$ of functions $u_k \in W^{2,2}_{1/2}(G_a)$ in terms of the local coordinates $R, \theta$ is given by

$$
\|u_k\|_{W^{2,2}_{1/2}(G_a)}^2 = \int_{\hat{G}_a} \left\{ \left| \tilde{u}_k \right|^2 + \left| \frac{\partial \tilde{u}_k}{\partial R} \right|^2 + \left| \frac{1}{R} \frac{\partial \tilde{u}_k}{\partial \theta} \right|^2 + \left| \frac{\partial^2 \tilde{u}_k}{\partial R^2} \right|^2 + \left| \frac{1}{R^2} \frac{\partial^2 \tilde{u}_k}{\partial \theta^2} \right|^2 + \left| \frac{1}{R} \frac{\partial \tilde{u}_k}{\partial \theta} \right|^2 \right\} R^2 \sin \theta \, dR \, d\theta.
$$

(5.3)

We need to show that each term on the right-hand side of (5.3) can be bounded by $\|f_k\|_{L_{2,1/2}(G_a)}$. Let us consider the term

$$
\int_{\hat{G}_a} \left| \frac{1}{R} \frac{\partial^2 \tilde{u}_k}{\partial R \partial \theta} - \frac{1}{R^2} \frac{\partial \tilde{u}_k}{\partial \theta} \right|^2 R^2 \sin \theta \, dR \, d\theta.
$$

Using representation (4.29) for $u_k(r, z) = \tilde{u}_k(R, \theta)$ with application of Levi’s and Fubini’s theorems we obtain

$$
\int_{\hat{G}_a} \left| \frac{1}{R} \frac{\partial^2 \tilde{u}_k}{\partial R \partial \theta} - \frac{1}{R^2} \frac{\partial \tilde{u}_k}{\partial \theta} \right|^2 R^2 \sin \theta \, dR \, d\theta \leq C \sum_{l=1}^{\infty} \int_{0}^{R_e} \left| \frac{1}{R} u_{kl} - \frac{1}{R^2} u_{kl} \right|^2 R^2 \, dR.
$$

(5.4)

From (4.27) we can write

$$
u_k(R) = -\frac{R^2(2\nu_l^k+1)}{2 \nu_l^k+1} \int_{0}^{R_e} f_{kl}(\tau) \tau^{\nu_l^k+2} \, d\tau + \frac{R^2\nu_l^k}{2 \nu_l^k+1} \int_{0}^{R_e} f_{kl}(\tau) \tau^{1-\nu_l^k} \, d\tau + \frac{R^2(2\nu_l^k+1)}{2 \nu_l^k+1} \int_{0}^{R_e} f_{kl}(\tau) \tau^{\nu_l^k+2} \, d\tau
$$

$$
= d_k^l R^{\nu_l^k} + F_{kl}(R) + G_{kl}(R).
$$

(5.5)
Considering the first term in (5.5) we have the estimate
\[
(v_k^l - 1)^2 (d_k^l)^2 \int_0^{R_c} R^{2(v_k^l - 1)} R = \frac{(v_k^l - 1)^2}{2v_k^l - 1} R_c^2 (d_k^l)^2 (d_k^l)^2 \leq C (d_k^l)^2
\] (5.6)
since \(v_k^l > \frac{1}{2}\). The constant factor \((d_k^l)^2\) can be bounded with the help of Cauchy inequality as follows:
\[
(d_k^l)^2 = \frac{R_c^{-2(2v_k^l + 1)}}{2v_k^l + 1} \left( \int_0^{R_c} f_{kl}(\tau) \tau^{v_k^l + 2} d\tau \right)^2 \leq C \int_0^{R_c} |f_{kl}(\tau)| \tau^2 d\tau \int_0^{R_c} \tau^{2v_k^l + 2} d\tau \leq C \int_0^{R_c} |f_{kl}(R)|^2 R^2 dR.
\] (5.7)
From the inequality
\[
\left| \int_R^{R_c} f(\tau) \tau^{1-v_k^l} d\tau \right| \leq \left( \int_R^{R_c} |f_{kl}(\tau)|^2 \tau^2 d\tau \right)^{1/2} \left( \int_R^{R_c} \tau^{-2v_k^l} d\tau \right)^{1/2} \leq C R^{1-2v_k^l/2} \left( \int_R^{R_c} |f_{kl}(\tau)|^2 \tau^2 d\tau \right)^{1/2}
\]
and integration by parts we obtain
\[
\int_0^{R_c} \left| \frac{1}{R} F_{kl} - \frac{1}{R^2} F_{kl} \right|^2 R^2 dR \leq C_1 \int_0^{R_c} R^{-1} \left| f_{kl}(\tau) \right|^2 \tau^2 d\tau + C_2 \int_0^{R_c} |f_{kl}(R)|^2 R^2 dR \leq C \int_0^{R_c} |f_{kl}(R)|^2 R^2 dR.
\] (5.8)
A similar argument leads to the inequality
\[
\int_0^{R_c} \left| \frac{1}{R} G_{kl} - \frac{1}{R^2} G_{kl} \right|^2 R^2 dR \leq C \int_0^{R_c} |f_{kl}(R)|^2 R^2 dR.
\] (5.9)
Finally, combining (5.4)–(5.9) and the completeness relation we obtain the inequality
\[
\int_{\tilde{G}_a} \left| \frac{1}{R} \frac{\partial^2 \tilde{u}_k}{\partial R \partial \theta} - \frac{1}{R^2} \frac{\partial \tilde{u}_k}{\partial \theta} \right|^2 R^2 \sin \theta dR d\theta \leq C \sum_{l=1}^{\infty} \| f_{kl} \|^2_{L_{2,1/2}(0,R_c)} \leq C \| f_k \|^2_{L_{2,1/2}(G_a)}.
\]
The other terms in (5.3) can be considered in the same way. □

**Lemma 5.2.** For \( k \in \mathbb{Z} \) let \( v_k^l \), \( l = 1, 2, \ldots \), be the roots of Eq. (4.15). Then
(a) for \( |k| \geq 1, v_k^l \geq 1/2 \) for any \( \theta \in (0, \pi) \), (b) for \( k = 0 \) and \( l \geq 2, v_k^l > 1/2 \) for any \( \theta \in (0, \pi) \).

**Proof.** See Section 6. □

It follows from Lemmas 5.1 and 5.2 that the solutions \( u_k \) of (3.10) for \( |k| \geq 1 \) belong to \( W_{1/2}^{2,2}(\Omega) \) independent of the opening angle \( \theta_c \) at \( C_a \). The solution \( u_0 \) may, depending on the opening angle \( \theta_c \),
exhibit singularity, i.e., may not belong to \( W_{1/2}^{2,2}(\Omega) \). For the structure of the singular function for the solution \( u_0 \) we have the following lemma.

**Lemma 5.3.** Suppose \( v_0^1 < \frac{1}{2} \). Then there exists a real constant \( \gamma_0^1 \) such that the solutions \( u_0 \) of the boundary value problem (3.10) for \( k = 0 \) can be split in the form

\[
\begin{align*}
  u_0(r, z) &= s_0(r, z) + w_0(r, z), & s_0(r, z) := \gamma_0^1 R_{v_0^1}^1 P_0^1(\cos \theta), \\
  w_0(r, z) &\in W_{1/2}^{2,2}(\Omega_a), & |\gamma_0^1| + \| w_0 \|_{W_{1/2}^{2,2}(\Omega_a)} &\leq C \| f_0 \|_{L_{2,1/2}(\Omega_a)}.
\end{align*}
\]

**Proof.** Using (4.29) we can write \( u_0 \) in the form

\[
\begin{align*}
  u_0(r, z) &= u_{01}(R) P_{v_0^1}(\cos \theta) + \sum_{l=2}^{\infty} u_{0l}(R) P_{v_0^l}(\cos \theta) = \bar{s}_0(r, z) + \bar{w}_0(r, z).
\end{align*}
\]

By Lemmas 5.1 and 5.2 the term \( \bar{w}_0 \) in relation (5.12) belongs to the space \( W_{1/2}^{2,2}(\Omega_a) \) and satisfies the inequality (5.1), so we need only investigate the term \( \bar{s}_0(r, z) \). We have

\[
\begin{align*}
  u_{01}(R) &= R_{v_0^1}^1 \int_0^R f_01(\tau) \left\{ \frac{1}{2v_0^1 + 1} \tau^{1-v_0^1} - \frac{1}{(2v_0^1 + 1)R_c(2v_0^1+1)} \tau^{v_0^1+2} \right\} d\tau \\
  &\quad + \left\{ -\frac{R_{v_0^1}^1}{2v_0^1} \int_0^R f_01(\tau) \tau^{1-v_0^1} d\tau + \frac{R_{v_0^1+1}^{-1}}{2v_0^1 + 1} \int_0^R f_01(\tau) \tau^{v_0^1+2} d\tau \right\} \\
  &= \gamma_0^1 R_{v_0^1}^1 + w_{01}(R).
\end{align*}
\]

Using the triangle and Cauchy inequalities we obtain the estimate

\[
\begin{align*}
  \| \gamma_0^1 \| &= \int_0^R f_01(\tau) \left\{ \frac{\tau^{1-v_0^1}}{2v_0^1 + 1} - \frac{\tau^{v_0^1+2}}{(2v_0^1 + 1)R_c(2v_0^1+1)} \right\} d\tau \\
  &\leq C \int_0^R f_01(\tau) \tau^{1-v_0^1} d\tau + C \int_0^R f_01(\tau) \tau^{v_0^1+2} d\tau \leq C \| f_01 \|_{L_{2,1/2}(0,R_c)}.
\end{align*}
\]

Obviously, the term \( T_0 := \gamma_0^1 R_{v_0^1}^1 P_0^1(\cos \theta) \) does not belong to \( W_{1/2}^{2,2}(G_a) \) for \( v_0^1 < \frac{1}{2} \) as we see from the estimate

\[
\begin{align*}
  \| T_0 \|_{W_{1/2}^{2,2}(G_a)}^2 &\geq \int_{G_a} \left| \frac{\partial^2 T_0}{\partial R^2} \right|^2 R^2 dR d\theta \geq C \int_0^R R^{2(v_0^1-1)} dR
\end{align*}
\]

and this integral converges only if \( 2(v_0^1 - 1) > -1 \), i.e., if \( v_0^1 > \frac{1}{2} \).
Similar arguments as in Lemma 5.1 lead to the relation \[ w_{01}(R) P_{v_0}^1(\cos \theta) \in W_{1/2}^{2,2}(G_a) \]. Thus \( u_0 \) can be written in the form

\[
\begin{align*}
   u_0(r, z) &= \gamma_0^1 R_0^1 P_{v_0}(\cos \theta) + \left\{ w_{01}(R) P_{v_0}^1(\cos \theta) + \sum_{l=2}^{\infty} u_{0l} P_{v_0}^l(\cos \theta) \right\} \\
   &= s_0(r, z) + w_0(r, z). \tag{5.14}
\end{align*}
\]

6. Numerical approximation of the singular exponent \( \nu \)

The roots of the equation \( P_{\nu}^k(\cos \theta) = 0 \) \((k \in \mathbb{Z})\) for a given \( \theta \in (0, \pi) \) can be determined by analytical means only in very few cases (cf. [1,10]). Thus, we need to consider numerical techniques for the approximation. From the relation

\[
P_{\nu}^k(x) = (-1)^k \frac{\Gamma(v-k+1)}{\Gamma(v+k+1)} P_{\nu}^k(x), \quad k \geq 1 \quad (x \text{ real})
\]

we see that the roots \( \nu_k^l \) and \( \nu_{-k}^l \) coincide and we can consider either positive or negative integers. We use the Mehler–Dirichlet formula

\[
P_{\nu}^k(\cos \theta) = \sqrt{\frac{2}{\pi}} \sin^{-k} \theta \int_0^\theta \frac{\cos(\nu + \frac{1}{2})t}{(\cos t - \cos \theta)^{(1/2)-k}} \, dt \tag{6.1}
\]

and seek values for \( v \) for which the integral with \( \theta = \theta_c \) vanishes. We note that the integrand is singular at \( t = \theta \). Using trigonometric identity we write the integrand in the form

\[
\frac{\cos(\nu + \frac{1}{2})t}{[-\sin((t+\theta)/2)]^{1/2-k}[2\sin((t-\theta)/2)]^{(1/2)-k}}
\]

and for \( t \) closed to \( \theta \) we use the approximation

\[
\frac{\cos(\nu + \frac{1}{2})t}{[-\sin((t+\theta)/2)](1/2)-k}(t-\theta)^{k-1/2}. \tag{6.3}
\]

For each \( \theta_c \in (0, \pi) \) and \( k=0, 1, 2, \ldots \), we approximate integral (6.1) using (6.2) or (6.3) if \( \theta_c - t \leq 0.001 \) by means of Gauss–Legendre quadrature formula, and then use a root-finding routine to determine \( v \). The results presented in Table 1 and Fig. 2 were obtained with the help of MATLAB 6 and MATHEMATICA software packages.\(^1\)

We observe that the roots \( \nu_k \) increase monotonically with increasing \( k \). For a fixed \( k \), the roots \( \nu_k \) decrease monotonically with increasing \( \theta \in (0, \pi) \), see Table 1. Also, we notice from the table that \( \nu_k > 0.5 \) for all \( |k| \geq 1 \) and all \( \theta \in (0, \pi) \). This implies that the functions \( R^k \nu_k^l P_{\nu_k}^l(\theta) \) (see Lemma 5.1) belong to \( W_{1/2}^{2,2}(G_a) \) for all \( |k| \geq 1 \) and \( \theta \in (0, \pi) \).

Thus, only the 0th Fourier coefficient can exhibit singular behaviour near the vertex. The critical angle for \( \nu_0 < 0.5 \) is approximately 132°, see Fig. 2.

\(^1\) MATLAB 6 and MATHEMATICA softwares.
Table 1
Solutions of Eq. (4.5) for $|k| = 0, 1, 2, 3$ and $l = 1, 2$

| $\theta$ | $k = 0$ | $|k| = 1$ | $|k| = 2$ | $|k| = 3$ |
|---------|--------|--------|--------|--------|
| 30      | 4.098731 | 6.831918 | 9.442626 | 12.00617 |
|         | 10.06059 | 12.77309 | 15.30654 | 17.70820 |
| 45      | 2.557918 | 4.403223 | 6.187597 | 7.953369 |
|         | 6.536980 | 8.356095 | 10.06619 | 11.69406 |
| 60      | 1.785291 | 3.194298 | 4.581784 | 5.970400 |
|         | 4.774713 | 6.150180 | 7.456466 | 8.707694 |
| 120     | 0.605268 | 1.424757 | 2.354219 | 3.383234 |
|         | 2.118863 | 2.864916 | 3.627150 | 4.395843 |
| 150     | 0.349228 | 1.117838 | 2.097829 | 3.271170 |
|         | 1.573440 | 2.240620 | 2.974657 | 8.068651 |
| 160     | 0.277411 | 1.057418 | 2.078944 | 3.183338 |
|         | 1.429770 | 2.102076 | 2.881017 | 8.068651 |
| 175     | 0.161142 | 1.008973 | 2.082583 | 3.095754 |
|         | 1.217214 | 1.969874 | 3.675072 | 7.292467 |

Fig. 2. Solutions of (4.15) for $k = 0$.

Table 1 shows the approximated values of the singular exponents $v_k^l$ for $k = 0, 1, 2, 3$ and $l = 1, 2$ for some special values of the opening angle $\theta_c$ at the vertex.

Fig. 2 shows the dependence of the exponent $v_0$ with respect to the opening angle $\theta_c$ at the vertex.
7. The Fourier-finite-element approximation

We consider in this section the approximation of the solution \( u \) of the variational equation (2.6) by means of the Fourier-finite-element method. First \( u \) is approximated by its truncated Fourier series \( u_N \) defined by

\[
 u_N = \sum_{|k| \leq N} u_k(r,z) e^{ik\phi}, \quad N > 0,
\]

where \( u_k \) \((k = 0, \pm 1, \ldots, \pm N)\) are the first \( 2N + 1 \) Fourier coefficients of \( u \), which are the unique solutions of the variational equations (3.10) for \(|k| \leq N\). The error \( u - u_N \) generated by replacing \( u \) by its truncated Fourier series can be estimated as follows:

**Theorem 7.1.** For \( f \in X^0_{1/2}(\Omega) \), let \( u \in V_0(\Omega) \) be the solution of the variational equation (2.6) and let \( u_N \) denote its Fourier series truncation defined by (7.1). Then there exists a constant \( C \) independent of \( N, u \) and \( f \), such that the following estimate holds:

\[
\| u - u_N \|_{V_0(\Omega)} \leq C N^{-1} \| f \|_{X^0_{1/2}(\Omega)}, \quad N > 0.
\]  

**Proof.** Taking into account the relation \( \| u - u_N \|_{V_0(\Omega)}^2 \leq C \| u - u_N \|_{V_0(\Omega)}^2 \) (use Friedrichs’ inequality), the completeness relations (3.5), (3.6) and the a priori estimate (3.11)(c), the following relations are obvious:

\[
\begin{align*}
\| u - u_N \|_{V_0(\Omega)}^2 &\leq C \sum_{|k| > N} \left\{ \left\| \frac{\partial u_k}{\partial r} \right\|_{X(\Omega_a)}^2 + \left\| \frac{\partial u_k}{\partial z} \right\|_{X(\Omega_a)}^2 + k^2 \left\| \frac{u_k}{r} \right\|_{X(\Omega_a)}^2 \right\} \\
&\leq C \frac{N^2}{N^2} \sum_{|k| > N} \left\{ \left\| \frac{\partial u_k}{\partial r} \right\|_{X(\Omega_a)}^2 + \left\| \frac{\partial u_k}{\partial z} \right\|_{X(\Omega_a)}^2 + k^2 \left\| \frac{u_k}{r} \right\|_{X(\Omega_a)}^2 \right\} \\
&\leq C N^{-2} \| f \|_{X^0_{1/2}(\Omega)}^2. \quad \Box
\end{align*}
\]  

The next step is to approximate the Fourier coefficients \( u_k \) in relation (7.1), which are solutions of the variational equations (3.10) for \(|k| \leq N\) by means of finite element method. We assume for simplicity that the domain \( \Omega_a \) is polygonally bounded. This is to avoid the technically involving proofs for error estimates which are needed for domains with curved boundaries, see e.g. [24]. Further assume that there are no re-entrant corners on the part \( \Gamma_a \) of the boundary, and that the angle \( \theta_c \) at the point \( C_a \) of intersection of \( \Gamma_a \) and \( \Gamma_0 \) is large enough to cause singularity on the solution \( u_0 \), i.e., the 3D problem has no edge singularities but conical singularities. For the treatment of problems with edge singularities see, e.g. [14].

For the finite element approximation, the domain \( \Omega_a \) is partitioned into a set of shape regular triangular elements \( \mathcal{T}_h = \{ T \} \) with mesh diameter \( 0 < h \leq h_0 \) \((h_0 \) sufficiently small), in such a way that the usual assumptions for conform finite element method (cf. [8]) are satisfied.

Since boundary singularities may reduce the rate of convergence of finite element approximations, we introduce local mesh grading near the vertex \( C_a \) with the help of a grading parameter \( 0 < \mu \leq 1 \), a grading function \( R_i \) and a step size \( h_i \) defined as follows (see [14] for details):

\[
 R_i := \frac{2}{3} R_0 (ih)^{1/\mu}, \quad i = 0, \ldots, n, \quad h_i := R_i - R_{i-1}, \quad i = 1, \ldots, n,
\]  

\[\text{(7.4)}\]
where \( n = [h^{-1}] \) is the integer part of \( h^{-1} \) and \( R_0 \) is taken from (2.3). The grading parameter \( \mu \) will be chosen according to the singular exponent \( v_0 \), see Lemma 5.3.

The triangulation \( \mathcal{T}_h = \{ T \} \) is further refined near the vertex \( C_a \) such that the following assumption is satisfied:

**Assumption 7.1.** The triangulation \( \mathcal{T}_h \) is graded around the vertex \( C_a \) by means of the parameter \( 0 < \mu \leq 1 \) such that \( h_T := \text{diam} \, T \) depends on the distance \( R_T \) of \( T \) from \( C_a \) in the following way:

\[
\begin{align*}
\varepsilon_1 h^{1/\mu} \leq h_T & \leq \varepsilon_1^{-1} h^{1/\mu} & \text{for } T \in \mathcal{T}_h : R_T = 0, \\
\varepsilon_2 h^{1-\mu} R_T \leq h_T \leq \varepsilon_2^{-1} h R_T^{1-\mu} & \text{for } T \in \mathcal{T}_h : 0 < R_T < R_0, \\
\varepsilon_3 h^{-\mu} & \leq h_T \leq \varepsilon_3^{-1} h & \text{for } T \in \mathcal{T}_h : R_T \geq R_0
\end{align*}
\]

(7.5)

with some constants \( 0 < \varepsilon_i \leq 1 \) \((i = 1, 2, 3)\).

For error analysis, we define the following subsets of \( \mathcal{T}_h \):

\[
B_0 := \{ T \in \mathcal{T}_h : R_T = 0 \}, \quad B_1 := \{ T \in \mathcal{T}_h : 0 < R_T < R_1 \}, \\
B_i := \{ T \in \mathcal{T}_h : R_{i-1} \leq R_T < R_i, \, i = 2, \ldots, n \}, \quad B_h := \bigcup_{i=0}^{n} B_i.
\]

(7.6)

Let \( n_i \) denote the total number of elements contained in the set \( B_i \) \((i = 1, \ldots, n)\), then the following relation can be verified (cf. [14]):

\[
n_i \leq Ci.
\]

(7.7)

The finite element spaces \( V_{0h}^a \) and \( W_{0h}^a \) are defined by

\[
\begin{align*}
V_{0h}^a &:= \{ v_h = v_h(r, z) : v_h \in C(\bar{\Omega}_a), \, v_h|_{T} \in P_1(T), \forall T \in \mathcal{T}_h, v_h = 0 \text{ on } \Gamma_a \}, \\
W_{0h}^a &:= \{ v_h = v_h(r, z) : v_h \in V_{0h}^a, v_h = 0 \text{ on } \Gamma_0 \}.
\end{align*}
\]

(7.8)

where \( P_1(T) \) denotes the space of all polynomials of degree \( \leq 1 \) on \( T \). For \( h \in (0, h_0] \) and \( N > 0 \), the Fourier-finite-element space \( V_{hN} \) is defined by

\[
V_{hN} := \left\{ v_{hN} = v_{hN}(r, \varphi, z) = \sum_{|k| \leq N} v_{kh}(r, z) e^{ik\varphi}, \, v_{0h} \in V_{0h}^a, \, v_{kh} \in W_{0h}^a, k \neq 0 \right\}.
\]

(7.9)

The relations \( V_{0h}^a \subset V_{0}^a(\Omega_a) \), \( W_{0h}^a \subset W_{0}^a(\Omega_a) \) and \( V_{hN} \subset V_{0}(\Omega) \) are easily verified.

The Fourier-finite-element approximation \( u_{hN} \) of the solution \( u \) of the 3D variational problem (2.6) is obtained by seeking for \( u_{hN} \in V_{hN} \) such that

\[
b(u_{hN}, v_{hN}) = f(v_{hN}) \quad \text{for } v_{hN} \in V_{hN}.
\]

(7.10)

\( b(\cdot, \cdot) \) and \( f(\cdot) \) are from (2.7). Cea’s lemma infers the existence of a unique solution \( u_{hN} \in V_{hN} \) of problem (7.10) that satisfies the a priori estimate

\[
\| u - u_{hN} \|_{V_0(\Omega)} \leq C \| u - u_{hN} \|_{V_0(\Omega)} \quad \text{for } u_{hN} \in V_{hN}.
\]

(7.11)
The solution $u_{hN}$ satisfies the relation

$$u_{hN} = \sum_{|k| \leq N} u_{kh}(r, z)e^{ik\phi},$$

(7.12)

where the Fourier coefficients $u_{kh}$ are the unique solutions of the following variational equations: Find $u_{kh} (|k| \leq N)$ such that

$$\begin{align*}
  u_{0h} &\in V_{0h}^a : b_0(u_{0h}, w_h) = f_0(w_h) \quad \text{for } w_h \in V_{0h}^a, \\
  u_{kh} &\in W_{0h}^a : b_k(u_{kh}, w_h) = f_k(w_h) \quad \text{for } w_h \in W_{0h}^a, \quad 1 \leq |k| \leq N,
\end{align*}$$

(7.13)

where $b_k(\cdot, \cdot)$ and $f(\cdot)$ are from (3.9).

For estimating the error $u - u_{hN}$ of the Fourier-finite-element approximation, we define a projection $r_{hN}u$ of the solution $u$ into the space $V_{hN}$ by

$$(r_{hN}u)(r, \phi, z) := \sum_{|k| \leq N} u_{kh}(r, z)e^{ik\phi} \quad \text{with} \quad \left\{ \begin{array}{ll}
  \Pi_h s_0 + \Pi_h w_0, & 1 \leq |k| \leq N, \\
  \Pi_h u_k, & 0 \leq |k| \leq N,
\end{array} \right.$$  \hspace{1cm} (7.14)

where $\Pi_h$ is the usual Lagrange’s interpolation operator. For the estimate of the error $u - \Pi_h u$, we have the following results.

**Lemma 7.1.** Let the triangulation $\mathcal{T}_h$ satisfy Assumption 7.1 with $0 < \mu \leq 1$. Then there exist constants $C$ independent of $h$ and $v$ such that the following inequalities hold:

$$\| v - \Pi_h v \|_{W^{1,2}_{1/2}(\Omega_a)} \leq C_h \| v \|_{W^{2,2}_{1/2}(\Omega_a)} \quad \text{for } v \in V_0^a(\Omega_a) \cap W^{2,2}_{1/2}(\Omega_a),$$

(7.15)

$$\| r^{-1}(v - \Pi_h v) \|_{X(\Omega_a)} \leq C_h \| v \|_{W^{2,2}_{1/2}(\Omega_a)} \quad \text{for } v \in W_0^a(\Omega_a) \cap W^{2,2}_{1/2}(\Omega_a).$$

(7.16)

**Proof.** The proof is the same as in [24, Proposition 6.1], and we omit it. \(\square\)

Since the singularity function $s_0(r, z) = \eta_0 R^\nu \cos \theta$ is continuous, the interpolation $\Pi_h s_0$ is well defined. For the error $s_0 - \Pi_h s_0$ we have the following results:

**Lemma 7.2.** Let the triangulation $\mathcal{T}_h$ satisfy Assumption 7.1 with $0 < \mu \leq 1$. Then there exist constant $C$ independent of $h$ and $s_0$ such that the following estimates hold:

$$\| s_0 - \Pi_h s_0 \|_{X(\Omega_a)} \leq C \| s_0 \|_{\gamma_0} h^{2\alpha} \quad \text{with} \quad \alpha = \begin{cases} 
  1 + \nu_0 & \text{if } \nu_0 < \mu \leq 1, \\
  2 & \text{if } 0 < \mu < \nu_0,
\end{cases}$$

(7.17)

$$| s_0 - \Pi_h s_0 |_{W^{1,2}_{1/2}(\Omega_a)} \leq C \| s_0 \|_{\gamma_0} \left\{ \begin{array}{ll}
  h^{2\nu_0/\mu} & \text{if } \nu_0 < \mu \leq 1, \\
  h^2 & \text{if } 0 < \mu < \nu_0.
\end{array} \right.$$  \hspace{1cm} (7.18)

**Proof.** We will prove relation (7.18). Relation (7.17) follows by analogy.

With $B_0, \ldots, B_n$ from (7.6) we can write

$$| s_0 - \Pi_h s_0 |_{W^{1,2}_{1/2}(\Omega_a)}^2 = \sum_{i=0}^n \sum_{T \in B_i} | s_0 - \Pi_h s_0 |_{W^{1,2}_{1/2}(T)}^2.$$

(7.19)
The seminorms $|v|_{W^{l,2}_i(T)}$ ($l = 1, 2$) in local polar co-ordinates $R, \theta$ is given by

$$|v|_{W^{1,2}_i(T)} := \int_T \left\{ \left| \frac{\partial v}{\partial R} \right|^2 + \frac{1}{R^2} \left| \frac{\partial v}{\partial \theta} \right|^2 \right\} R^2 \sin \theta \, dR \, d\theta,$$

$$|v|_{W^{2,2}_i(T)} := \int_T \left\{ \left| \frac{\partial^2 v}{\partial R^2} \right|^2 + 2 \left| \frac{1}{R^2} \frac{\partial^2 v}{\partial R \partial \theta} - \frac{1}{R^2} \frac{\partial v}{\partial \theta} \right|^2 + \left| \frac{1}{R} \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{R} \frac{\partial v}{\partial R} \right|^2 \right\} R^2 \sin \theta \, dR \, d\theta.$$  \hspace{1cm} (7.20)

From (7.19) we consider first triangles $T \in B_0$ which have as a vertex the corner point $C_a$. Then form

$$|s_0 - \Pi_h s_0|_{W^{1,2}_i(T)}^2 \leq 2 \left\{ |s_0|_{W^{1,2}_i(T)}^2 + |\Pi_h s_0|_{W^{1,2}_i(T)}^2 \right\}$$

we obtain

$$|s_0|_{W^{1,2}_i(T)} \leq C |\gamma_0|^2 \int_0^{h_T} R^{2\gamma_0} \, dR \leq C |\gamma_0|^2 h^{2\gamma_0/\mu},$$ \hspace{1cm} (7.22)

$$\left\{ |\Pi_h s_0|_{W^{1,2}_i(T)}^2 \right\} \leq C |\gamma_0|^2 h^{2\gamma_0/\mu}.$$ \hspace{1cm} (7.23)

The estimate (7.22) follows from relations (7.20) and (7.5). For the estimate of the seminorm $|\Pi_h s_0|_{W^{1,2}_i(T)}^2$ we use the linear interpolant $p_1(r, z)$ of $s_0$ and relation (7.5). Combining these two inequalities we obtain

$$\sum_{T \in B_0} |s_0 - \Pi_h s_0|_{W^{1,2}_i(T)}^2 \leq C |\gamma_0|^2 h^{2\gamma_0/\mu}.$$ \hspace{1cm} (7.24)

For triangles $T \in B_h \setminus B_0$ which do not have $C_a$ as a vertex, the seminorm $|\Pi_h s_0|_{W^{2,2}_i(T)}^2$ is bounded and the classical interpolation error estimate holds. Thus

$$\sum_{i=1}^n \sum_{T \in B_i} |s_0 - \Pi_h s_0|_{W^{1,2}_i(T)}^2 \leq C |\gamma_0|^2 \sum_{i=1}^n h_T^2 |s_0|_{W^{2,2}_i(T)}^2$$

$$\leq C |\gamma_0|^2 \sum_{i=1}^n i h_i^2 \int_{R_{i-1}}^R R^{2\gamma_0-2} \, dR \leq C |\gamma_0|^2 \left\{ h_1^{2\gamma_0} + \sum_{i=2}^n i h_i^2 \left[ \bar{R}_i - R_{i-1} \right] \right\}$$

$$\leq C |\gamma_0|^2 \left\{ h_1^{2\gamma_0} + \sum_{i=2}^n i h_i^3 R_i^{2\gamma_0-2} \right\} \leq C |\gamma_0|^2 h^{(2\gamma_0+1)/\mu} \sum_{i=1}^n i^{(2\gamma_0+1)/\mu - 3} \leq C |\gamma_0|^2 h^{2\gamma_0/\mu}.$$ \hspace{1cm} (7.25)

The inequalities in (7.25) are obtained basically by applying the relations (cf. [14])

$$h_T \leq C h_i, \quad n_i \leq C i, \quad R_{i-1} = R_{i-1}, \quad \bar{R}_i \leq R_i + C h_i,$$

$$\bar{R}_i \leq C h_i, \quad \bar{R}_i - R_{i-1} = C h_i, \quad R_i \leq C (i h)^{1/\mu}, \quad h_i \leq C h R_i^{1-\mu}.$$  \hspace{1cm} (7.25)

Assertion (7.18) follows from relations (7.24) and (7.25). \hspace{1cm} \Box
We can now give a bound for the error $u - u_{hN}$ of the Fourier-finite-element approximation.

**Theorem 7.2.** Let $u \in V_0(\Omega)$ be the solution of the variational problem (2.6) with $f \in X^0_{1/2}(\Omega)$ satisfying the additional condition $\partial f / \partial \phi \in X^0_{1/2}(\Omega)$. Let $u_{hN} \in V_{hN}$ be the Fourier-finite-element approximation defined according to (7.10). Suppose that the axisymmetric domain $\hat{\Omega}$ has only one conical point on the rotation axis that is situated at the origin and no reentrant edges. Let $T_h$ be a triangulation of the meridian plane $\hat{\Omega}$ that satisfies Assumption 7.1. Then there exists a constant $C$ independent of $h$, $N$ and $f$ such that

$$
\| u - u_{hN} \|_{V_0(\Omega)} \leq C(N^{-1} + h^2) \left( \| f \|_{X^0_{1/2}(\Omega)} + \left\| \frac{\partial f}{\partial \phi} \right\|_{X^0_{1/2}(\Omega)} \right)
$$

(7.26)

with

$$
\alpha = \begin{cases} 
  v_0 & \text{if } v_0 < \mu \leq 1, \\
  1 & \text{if } 0 < \mu < v_0.
\end{cases}
$$

**Proof.** It follows from the triangle inequality and relation (7.11) the estimate

$$
\| u - u_{hN} \|_{V_0(\Omega)} \leq C(\| u - u_N \|_{V_0(\Omega)} + \| u_N - r_{hN} u \|_{V_0(\Omega)}).
$$

(7.27)

We consider the second term on the right-hand side of (7.27) and use the completeness relations (3.5) and (3.6) to obtain

$$
\| u_N - r_{hN} u \|_{V_0(\Omega)}^2 = 2\pi \sum_{|k| \leq N} \left\{ \| u_k - u_{kh} \|_{W^{1,2}_1(\Omega_a)}^2 + k^2 \| u_k - u_{kh} \|_{L_{1/2}(\Omega_a)}^2 \right\}
$$

\( \leq C \left\{ \| s_0 - \Pi_h s_0 \|_{W^{1,2}_1(\Omega_a)}^2 + \| w_0 - \Pi_h w_0 \|_{W^{1,2}_1(\Omega_a)}^2 \right. \)

\[ + \sum_{1 \leq |k| \leq N} \left( \| u_k - \Pi_h u_k \|_{W^{1,2}_1(\Omega_a)}^2 + k^2 \| u_k - \Pi_h u_k \|_{L_{1/2}(\Omega_a)}^2 \right) \right\}

\( \leq C \left\{ \| s_0 \|_{W^{1,2}_1(\Omega_a)}^2 + h^2 \sum_{|k| \leq N} (1 + k^2) \| w_0 \|_{W^{1,2}_1(\Omega_a)}^2 \right\}

\( \leq C h^2 \left( \| f \|_{X^0_{1/2}(\Omega)}^2 + \left\| \frac{\partial f}{\partial \phi} \right\|_{X^0_{1/2}(\Omega)}^2 \right). \)  

(7.28)

The inequalities in (7.28) are obtained by exploiting the results of Lemmas 5.3, 7.1, 7.2, Remark 3.2 and inequalities (5.1), (5.11). Finally, assertion (7.26) follows from relations (7.2) and (7.28).

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References


