Journal of Pure and Applied Algebra 57 (1989) 5–24
North-Holland

HOMOTOPY COMMUTATIVE DIAGRAMS AND THEIR REALIZATIONS *

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Communicated by J.D. Stasheff
Received 11 September 1986
Revised 21 September 1987

In this paper we describe an obstruction theory for the problem of taking a commutative diagram in the homotopy category of topological spaces and lifting it to an actual commutative diagram of spaces. This directly generalizes the work of G. Cooke on extending a homotopy action of a group G to a topological action of G.

1. Introduction

1.1. *Summary.* In [2], Cooke asked when a homotopy action of a group G on a CW-complex X is equivalent, in an appropriate sense, to a topological action of G on some homotopically equivalent space. He converted this problem into a lifting problem, which then gave rise to a sequence of obstructions, whose vanishing insured the existence of the desired topological action. These obstructions were elements of the cohomology of G with local coefficients in the homotopy groups of the function space X^X.

As a group is just a category with one object in which all maps are invertible, one can consider the corresponding problem for homotopy actions of an arbitrary small topological category $\mathbf{D}$ on a set $\{X_D\}$ of CW-complexes, indexed by the objects $D \in \mathbf{D}$. The purpose of this paper is to generalize Cooke's results to this situation, i.e. to convert this problem into an equivalent lifting problem, which gives rise to obstructions, in the cohomology of the category $\mathbf{D}$ with local coefficients in the

* This research was in part supported by the National Science Foundation.

homotopy groups of the function complexes between the X_D's. There is also a relative version, in which one starts with a topological action of a subcategory $\mathbf{C}\subset\mathbf{D}$. And, of course, there are corresponding simplicial results.

1.2. *Further details.* We now give some more details for the case that $\mathbf{D}$ is an ordinary small discrete (as opposed to topological) category. First we formulate

(i) *The realization problem.* Given a homotopy commutative $\mathbf{D}$-diagram $\bar{Y}$ (i.e. a functor from $\mathbf{D}$ to the category of CW-complexes and homotopy classes of maps between them), the problem is whether $\bar{Y}$ has a realization. By this we mean a $\mathbf{D}$-diagram Y (i.e. a functor from $\mathbf{D}$ to the category of CW-complexes and continuous maps between them), together with a function z which assigns to every object $D\in\mathbf{D}$ a homotopy equivalence $zD: YD\to\bar{Y}D$ such that $(\bar{Y}d)[zD_1]=[zD_2][Yd]$ for every map $d: D_1\to D_2\in\mathbf{D}$ ($[-]$ denotes 'the homotopy class of $-$'). To attack this problem we define

(ii) *The space* real $\bar{Y}$ *of realizations of* $\bar{Y}$. This will be the classifying space of a certain category which has the realizations of $\bar{Y}$ as objects. *This space* real $\bar{Y}$ *is nonempty iff* $\bar{Y}$ *has a realization.* In order to get a more down to earth description of its homotopy type we consider

(iii) ∞*-homotopy commutative* $\mathbf{D}$*-diagrams.* An ∞-homotopy commutative $\mathbf{D}$-diagram Y is a function which assigns to every object $D\in\mathbf{D}$ a CW-complex YD, to every map $d: D_0\to D_1\in\mathbf{D}$ a continuous map $Yd: YD_0\to YD_1$, to every composable sequence of maps $d_1: D_0\to D_1,\dots,d_n: D_{n-1}\to D_n\in\mathbf{D}$ a homotopy between $Y(d_n\dots d_1)$ and $(Yd_n)\dots(Yd_1)$ and, in a certain precise manner, higher and higher such homotopies. There is an obvious map from the resulting 'space of ∞-homotopy commutative $\mathbf{D}$-diagrams involving a given set of CW-complexes' to the (discrete) 'space of homotopy commutative $\mathbf{D}$-diagrams involving the same CW-complexes'. Our key result then is that, given a homotopy commutative $\mathbf{D}$-diagram $\bar{Y}$, *the space* $\mathrm{hc}_\infty\bar{Y}$ *of 'the* ∞*-homotopy commutative* $\mathbf{D}$*-diagrams which lie over* $\bar{Y}$*' has the same (weak) homotopy type as the space* real $\bar{Y}$ *considered above.* Thus $\bar{Y}$ *has a realization iff* $\bar{Y}$ *can be lifted to an* ∞*-homotopy commutative* $\mathbf{D}$*-diagram.* To construct such a lifting, one first tries to lift $\bar{Y}$ to

(iv) *n-homotopy commutative* $\mathbf{D}$*-diagrams* ($n\geq 1$). One can obtain a definition of n-homotopy commutative $\mathbf{D}$-diagrams by 'truncating' the above-mentioned definition of ∞-homotopy commutative $\mathbf{D}$-diagrams. For instance, a 1-homotopy commutative $\mathbf{D}$-diagram Y is a function which assigns to every object $D\in\mathbf{D}$ a CW-complex YD and to every map $d_1: D_0\to D_1\in\mathbf{D}$ a continuous map $Yd_1: YD_0\to YD_1$ such that $Y(d_2d_1)$ is homotopic to $(Yd_2)(Yd_1)$ for every map $d_2: D_1\to D_2\in\mathbf{D}$. The space $\mathrm{hc}_\infty\bar{Y}$ then is an inverse limit as well as homotopy inverse limit of the resulting tower $\{\mathrm{hc}_n\bar{Y}\}_{n\geq 1}$ of 'spaces of n-homotopy commutative $\mathbf{D}$-diagrams over $\bar{Y}$' and it follows that $\bar{Y}$ *can be lifted to an* ∞*-homotopy commutative* $\mathbf{D}$*-diagram iff* $\bar{Y}$ *can be lifted to a 'compatible' sequence of n-homotopy commutative* $\mathbf{D}$*-diagrams.* It thus remains to find inductive algebraic conditions for the existence of such a 'compatible' sequence of liftings, which are usually called

(v) *Obstructions.* As $\mathrm{hc}_1 \bar{Y}$ is clearly contractible, one can always lift $\bar{Y}$ to a 1-homotopy commutative **D**-diagram. Whether $\bar{Y}$ can be lifted to a 2-homotopy commutative **D**-diagram turns out to be an (already algebraic) problem involving (fundamental) groupoids. Finally, given a lifting of $\bar{Y}$ to an n-homotopy commutative **D**-diagram ($n \geq 2$), the obstruction to the existence of a 'compatible' $(n+1)$-homotopy commutative **D**-diagram lies in the cohomology (in the sense of [11]) of the category **D** with local coefficients in the nth homotopy groups of the function spaces between the CW-complexes $\bar{Y}D$ ($D \in \mathbf{D}$).

The cohomology theory studied in [11] is defined in terms of maps from a free simplicial resolution of **D** into certain abelian Eilenberg–MacLane objects. However, in line with [12; II, §5] there is a more classical description of the groups involved. To be specific, suppose that $\bar{Y}$ has been lifted to an n-homotopy commutative **D**-diagram with $n \geq 2$. For any map $d: D_0 \to D_1$ in **D**, let $\hom(\bar{Y}D_0, \bar{Y}D_1)_{\bar{Y}d}$ denote the component of the function space $\hom(\bar{Y}D_0, \bar{Y}D_1)$ corresponding to the homotopy class $\bar{Y}d$. The 2-homotopy lifting of $\bar{Y}$ provides enough of a basepoint in $\hom(\bar{Y}D_0, \bar{Y}D_1)_{\bar{Y}d}$ to give an invariant determination of the nth homotopy group of this space; denote this nth homotopy group by $B_n(d)$. The construction $B_n(-)$ actually gives a functor from the twisted arrow category $\mathrm{Ar}^{\mathrm{t}}(\mathbf{D})$ to abelian groups; here $\mathrm{Ar}^{\mathrm{t}}(\mathbf{D})$ is the category in which an object is a map $d: D_0 \to D_1$ in **D** and a morphism from d to d' is a pair of maps $h_0: D_0' \to D_0$, $h_1: D_1 \to D_1'$ such that $h_1 d h_0 = d'$. Then it follows from (4.8) and [12] that the obstruction to lifting $\bar{Y}$ one step further to an $(n+1)$-homotopy commutative **D**-diagram lies in the group $\mathrm{Ext}^{n+2}(\mathbb{Z}, B_n)$ where $\mathrm{Ext}^*(-,-)$ is computed in the category of $\mathrm{Ar}^{\mathrm{t}}(\mathbf{D})$-diagrams of abelian groups and $\mathbb{Z}$ stands for the constant such diagram. As indicated in [12], this Ext can be interpreted as a type of Hochschild–Mitchell cohomology of **D**.

(vi) *A variation.* One might wonder why we did not require in (i) that the maps $zD: YD \to \bar{Y}D$ be homotopy classes of homotopy equivalences, as this would clearly have given rise to an equivalent realization problem. That is precisely what we did in [9]. It resulted, however, in a homotopically more complicated space of realizations and gave rise to obstructions in different and, unfortunately, less easily accessible cohomology groups. In fact, the complexity of the approach of [9] already becomes clear when one takes for **D** a category with only identity maps. In that case the present space of realizations is contractible, which is exactly what one would like it to be, while the space of realizations of [9] has the homotopy type of the product of the universal coverings of the classifying spaces of the spaces of self homotopy equivalences of the CW-complexes $\bar{Y}D$ ($D \in \mathbf{D}$).

(vii) *Example.* If **D** is a free category, that is, if the maps of **D** are freely generated by a set of basic maps $\{d_i: D_i \to D_i' \mid i \in I\}$, then any homotopy commutative **D**-diagram $\bar{Y}$ is realizable. Moreover, real $\bar{Y}$ is homotopic to the product (for $i \in I$) of the function space components $\hom(\bar{Y}D_i, \bar{Y}D_i')_{\bar{Y}d_i}$.

(viii) *Example.* Let **D** be the category of the free abelian monoid on two generators, so that **D** has one object D with endomorphism set freely generated by two maps $d_1, d_2: D \to D$ subject only to the commutation relation $d_1 d_2 = d_2 d_1$. It is possible to construct an explicit cofibrant resolution (1.4(iv)) of **D** with two zero-dimensional generators corresponding to d_1, d_2 and a single one-dimensional generator corresponding to the commutation relation. It then follows easily from Section 3 and [8] that any homotopy-commutative **D**-diagram $\bar{Y}$ can be realized.

(ix) *Example.* Let **D** be the 'retract category', i.e. let **D** have one object D and one non-identity map $d: D \to D$ such that $d^2 = d$. One can verify algebraically that the Hochschild–Mitchell cohomology of **D** ((1.2(v), [12]) vanishes above dimension 1. Consequently, a homotopy-commutative **D**-diagram $\bar{Y}$ can be realized if and only if a certain groupoid lifting problem can be solved. The relevant groupoid is the fundamental groupoid of the function space component $\hom(\bar{Y}D, \bar{Y}D)_{\bar{Y}d}$.

1.3. *Organization of the paper.* After fixing some notation and terminology (in 1.4), we state and prove our results first for *simplicial diagrams of simplicial sets* (the absolute case in Sections 2 and 3 and the relative case in Section 4) and then (in Section 5) use a *singular functor argument* to obtain the corresponding results for *topological diagrams of topological spaces.*

The proofs of our key result (2.4) and its relativization (4.4) make use of a *generalization of Quillen's theorem* B and of a *connection between simplicial categories and diagrams of simplicial sets*, which will be discussed in Sections 6 and 7 respectively.

1.4. *Notation, terminology, etc.* (i) *Simplicial sets.* We denote (see [1, Chapter VIII]) by **S** the category of *simplicial sets* (with its usual simplicial structure), by ho **S** its *homotopy category* (i.e. the category obtained from **S** by formally inverting the weak (homotopy) equivalences), by $\mathbf{S}^f \subset \mathbf{S}$ and ho $\mathbf{S}^f \subset$ ho **S** the full subcategories spanned by the fibrant objects (i.e. the simplicial sets which satisfy the extension condition) and by $\pi: \mathbf{S}^f \to \mathrm{ho}\, \mathbf{S}^f$ the *projection.* (The inclusion ho $\mathbf{S}^f \to$ ho **S** is an equivalence of categories). We also sometimes call a simplicial set a *space.*

(ii) *Simplicial categories.* A simplicial category is always assumed to have *the same objects in each dimension*; it thus is a category enriched over **S** [13, p. 181]. If **B** is a simplicial category, then we write $\mathbf{B}(B_1, B_2)$ or (if no confusion can arise) $\hom(B_1, B_2)$ for the simplicial hom-set between any two objects $B_1, B_2 \in \mathbf{B}$. Moreover, if $T: \mathbf{S} \to \mathbf{S}$ is a 'product preserving' functor, then $T\mathbf{B}$ will denote the simplicial category with the same objects as **B** and with simplicial hom-sets given by the formula $(T\mathbf{B})(B_1, B_2) = T(\mathbf{B}(B_1, B_2))$.

(iii) *Simplicial diagrams of simplicial sets.* Given a small simplicial category **D**, we denote by $\mathbf{S}^{\mathbf{D}}$ the *category of* **D**-*diagrams of simplicial sets*, i.e. the category which has as objects the functors $\mathbf{D} \to \mathbf{S}$ and as maps the natural transformations between them. *This category is* [6] *a closed simplicial model category* in which the simplicial structure is the obvious one and in which a map $X \to Y \in \mathbf{S}^{\mathbf{D}}$ is a weak equivalence or a fibration whenever, for every object $D \in \mathbf{D}$, the map $XD \to YD \in \mathbf{S}$ is a weak (homotopy) equivalence or a fibration.

(iv) *A category of small simplicial categories.* Given a small simplicial category

$\mathbf{D}$ with object set O, it is convenient to consider the category $\mathbf{S}O$-**Cat** which has as objects the small simplicial categories with this same object set O and which has as maps the functors between them which are the identity on O. *This category* $\mathbf{S}O$-**Cat** *admits* [3] *a closed simplicial model category structure* in which (cf. 7.1) the simplicial structure is the obvious one and in which a map $\mathbf{D}' \to \mathbf{D}'' \in \mathbf{S}O\text{-}\mathbf{Cat}$ is a weak equivalence or a fibration whenever, for every pair of elements $D_1, D_2 \in O$, the map $\mathbf{D}'(D_1, D_2) \to \mathbf{D}''(D_1, D_2) \in \mathbf{S}$ is a weak (homotopy) equivalence or a fibration. There are many *cofibrant resolutions* of $\mathbf{D}$, i.e. weak equivalences $\mathbf{D}' \to \mathbf{D} \in \mathbf{S}O\text{-}\mathbf{Cat}$ in which $\mathbf{D}'$ is cofibrant. A functorial and particularly convenient one is the *standard resolution* $p: F_*\mathbf{D} \to \mathbf{D} \in \mathbf{S}O\text{-}\mathbf{Cat}$ given by $(F_*\mathbf{D})_n = F^{n+1}\mathbf{D}_n$ $(n \geq 0)$, where [3] F is the functor which sends a small category $\mathbf{C}$ to the free category $F\mathbf{C}$ with the same objects as $\mathbf{C}$ and with as generators all the non-identity maps of $\mathbf{C}$, and $(F_*\mathbf{D})_n$ and $\mathbf{D}_n$ denote the 'categories in dimension n' of $F_*\mathbf{D}$ and $\mathbf{D}$ respectively.

Also useful are, given a functor $\bar{Y}: \mathbf{D} \to \text{ho}\,\mathbf{S}^{\text{f}}$, the objects $\mathbf{S}^{\#}$ and $\text{ho}\,\mathbf{S}^{\#}$ in $\mathbf{S}O$-**Cat** defined by the formulas $\mathbf{S}^{\#}(D_1, D_2) = \mathbf{S}(\bar{Y}D_1, \bar{Y}D_2)$ and $\text{ho}\,\mathbf{S}^{\#}(D_1, D_2) = \text{ho}\,\mathbf{S}(\bar{Y}D_1, \bar{Y}D_2)$. They come together with obvious functors $\mathbf{S}^{\#} \to \mathbf{S}$, $\text{ho}\,\mathbf{S}^{\#} \to \text{ho}\,\mathbf{S}$, $\pi: \mathbf{S}^{\#} \to \text{ho}\,\mathbf{S}^{\#}$ and $\mathbf{D} \to \text{ho}\,\mathbf{S}^{\#}$ of which the last one will also be denoted by $\bar{Y}$.

(v) *Nerves of categories.* If $\mathbf{C}$ is a small category, *we use the same symbol* $\mathbf{C}$ *for its nerve*, i.e. the simplicial set which has as n-simplicies $(n \geq 0)$ the composable sequences $C_0 \to \cdots \to C_n$ of maps in $\mathbf{C}$. We do the same if $\mathbf{C}$ is not necessarily small, as long as its nerve 'has homotopy meaning', i.e. is a *homotopically small* simplicial set in the sense of [5, §2].

(vi) *Topological spaces and categories.* The category of topological spaces will be denoted by $\mathbf{T}$. *No separation axioms* will be assumed, though our results remain valid if, for instance, all topological spaces are assumed to be compactly generated (but not necessarily Hausdorff) or singularly generated (i.e. the topology is the identification topology obtained from the geometric realization of the singular complex).

By a small topological category we mean a topological category with a *discrete set of objects*, i.e. a small category enriched over $\mathbf{T}$ [13, p. 181]. If $\mathbf{E}$ is such a small topological category, then we denote by $\mathbf{E}(E_1, E_2)$ the topological space of maps $E_1 \to E_2 \in \mathbf{E}$.

2. ∞-homotopy commutative diagrams and realizations of homotopy commutative diagrams

We start with stating and proving our main result (2.4), that *the 'space (i.e. simplicial set) of realizations' of a given homotopy commutative diagram is naturally weakly (homotopy) equivalent to its 'space of liftings to an ∞-homotopy commutative diagram'*. This implies (2.5) that *a homotopy commutative diagram has a realization iff it can be lifted to an ∞-homotopy commutative diagram.*

First the relevant definitions:

2.1. *Homotopy commutative diagrams.* Given (1.4(ii)) a small simplicial category $\mathbf{D}$, a *homotopy commutative* $\mathbf{D}$*-diagram* will be just a functor $\bar{Y}:\mathbf{D}\to \operatorname{ho}\mathbf{S}^{\mathrm{f}}$ (1.4(i)).

2.2. *Realizations of homotopy commutative diagrams.* Given a homotopy commutative $\mathbf{D}$-diagram $\bar{Y}$, the '*space of realizations of* $\bar{Y}$' will be the nerve (1.4(v)) of a category of realizations of $\bar{Y}$ and maps between them. In order that this space has the 'correct' (2.6) homotopy type, a *realization* of $\bar{Y}$ will be defined slightly more generally than in 1.2(i), as a $\mathbf{D}$-diagram $Y:\mathbf{D}\to\mathbf{S}^{\mathrm{f}}$ (1.4(i) and (iii)), together with a function z which assigns to every object $D\in\mathbf{D}$ a 'zigzag' of homotopy equivalences in $\mathbf{S}^{\mathrm{f}}$,

$$YD \xleftarrow{\;z'D\;} X'D \xrightarrow{\;z''D\;} \bar{Y}D,$$

such that the function $(\pi z'')(\pi z')^{-1}:\pi Y\to\bar{Y}$ (1.4(i)) is a natural equivalence. This clearly does not affect the realizability of $\bar{Y}$. Similarly a *map* $(Y_1,z_1)\to(Y_2,z_2)$ between two of these realizations will be a function which assigns to every object $D\in\mathbf{D}$ a commutative diagram

$$\begin{array}{ccccc}
Y_1D & \xleftarrow{\;z_1'D\;} & X_1'D & \xrightarrow{\;z_1''D\;} & \bar{Y}D\\
\Big\downarrow{\scriptstyle yD} & & \Big\downarrow{\scriptstyle x'D} & & \Big\downarrow{\scriptstyle \mathrm{id}}\\
Y_2D & \xleftarrow{\;z_2'D\;} & X_2'D & \xrightarrow{\;z_2''D\;} & \bar{Y}D
\end{array}$$

such that the function y is a map (and in fact a weak equivalence) $y:Y_1\to Y_2\in\mathbf{S}^{\mathbf{D}}$ (1.4(iii)). The *space of realizations* of $\bar{Y}$ thus is the (homotopically small (1.4(v)) simplicial set which has as vertices the realizations of $\bar{Y}$ and as n-simplices ($n\geq 1$) the composable sequences $(Y_0,z_0)\to\cdots\to(Y_n,z_n)$ of maps between them. It will be denoted by real $\bar{Y}$. Two realizations of $\bar{Y}$ will be called *equivalent* whenever they are in the same component of real $\bar{Y}$.

If $\mathbf{D}$ has only identity maps, one readily verifies that real $\bar{Y}$ is contractible.

2.3. ∞*-homotopy commutative diagrams.* Given a homotopy commutative $\mathbf{D}$-diagram $Y:\mathbf{D}\to\operatorname{ho}\mathbf{S}^{\mathrm{f}}$, an ∞*-homotopy commutative* $\mathbf{D}$*-diagram over* $\bar{Y}$ will (in the notation of 1.4(iv)) be a map $Y_\infty:F_*\mathbf{D}\to\mathbf{S}^{\#}\in\mathbf{S}O\text{-}\mathbf{Cat}$ such that the following diagram in $\mathbf{S}O$-$\mathbf{Cat}$ is commutative:

$$\begin{array}{ccc}
F_*\mathbf{D} & \xrightarrow{\;p\;} & \mathbf{D}\\
\Big\downarrow{\scriptstyle Y_\infty} & & \Big\downarrow{\scriptstyle \bar{Y}}\\
\mathbf{S}^{\#} & \xrightarrow{\;\pi\;} & \operatorname{ho}\mathbf{S}^{\#}
\end{array}$$

These ∞-homotopy commutative diagrams are just the vertices of a simplicial set $\hom_{\bar{Y}}(F_*\mathbf{D}, \mathbf{S}^\#)$ which has as n-simplices the commutative diagrams

$$\begin{array}{ccc} F_*\mathbf{D}\otimes\Delta[n] & \xrightarrow{\text{proj}} & \mathbf{D} \\ \downarrow & & \downarrow \bar{Y} \\ \mathbf{S}^\# & \xrightarrow{\pi} & \operatorname{ho}\mathbf{S}^\# \end{array}$$

(where $F_*\mathbf{D}\otimes\Delta[n]$ is as indicated in 1.4(iv)), which we therefore call the *space of ∞-homotopy commutative diagrams* over $\bar{Y}$ and denote also by $\operatorname{hc}_\infty \bar{Y}$. Two ∞-homotopy commutative diagrams over $\bar{Y}$ are called *equivalent* whenever they are in the same component of $\operatorname{hc}_\infty \bar{Y}$.

If $\mathbf{D}$ has only identity maps, then $F_*\mathbf{D}=\mathbf{D}$ and hence $\operatorname{hc}_\infty \bar{Y}\approx\Delta[0]$.

Our main result, which converts the realization problem into a lifting problem, now is

2.4. Theorem. *Let* $\mathbf{D}$ *be a small simplicial category and let* $\bar{Y}:\mathbf{D}\to\operatorname{ho}\mathbf{S}^{\mathrm{f}}$ *be a homotopy commutative* $\mathbf{D}$*-diagram. Then the spaces* real $\bar{Y}$ *and* $\operatorname{hc}_\infty \bar{Y}$ *are naturally weakly (homotopy) equivalent.*

2.5. Corollary. *A homotopy commutative* $\mathbf{D}$*-diagram has a realization iff there exists an ∞-homotopy commutative* $\mathbf{D}$*-diagram over* $\bar{Y}$*. Moreover the equivalence classes of the realizations of* $\bar{Y}$ *are in a natural* 1-1 *correspondence with the equivalence classes of the ∞-homotopy commutative* $\mathbf{D}$*-diagrams over* $\bar{Y}$*.*

The remainder of this section is devoted to a

Proof of Theorem 2.4. To get a hold on the homotopy type of real $\bar{Y}$, one notes first that real $\bar{Y}$ is a union of components of the 0-*realization space* $\mathrm{real}^0\,\bar{Y}$, which is defined as real $\bar{Y}$ (2.2), except that the function $(\pi z'')(\pi z')^{-1}$ is not required to be a natural transformation $\pi Y\to\bar{Y}$. This space $\mathrm{real}^0\,\bar{Y}$ is in turn closely related to the 0-*classification space* $c^0\bar{Y}$ of [7], which is the nerve of the category with as objects the functors $Y:\mathbf{D}\to\mathbf{S}^{\mathrm{f}}$ such that, for every object $D\in\mathbf{D}$, YD has the same homotopy type as $\bar{Y}D$, and as maps the natural transformations between such functors. In fact, if $\mathbf{D}^0\subset\mathbf{D}$ denotes the subcategory consisting of the objects (and their identity maps) only and $\bar{Y}|_{\mathbf{D}^0}$ is the restriction of $\bar{Y}$ to $\mathbf{D}^0$, then 6.8(ii) immediately implies

2.6. Lemma. *The obvious sequence*

$$\mathrm{real}^0\,\bar{Y}\to c^0\,\bar{Y}\to c^0(\bar{Y}|_{\mathbf{D}^0})$$

is a homotopy fibration sequence, i.e. $\operatorname{real}^0 \bar{Y}$ *is a homotopy fibre of the map* $c^0 \bar{Y} \to c^0(\bar{Y}|_{\mathbf{D}^0})$. $\square$

Similarly $\mathrm{hc}_\infty \bar{Y} = \hom_{\bar{Y}}(F_* \mathbf{D}, \mathbf{S}^{\#})$ is a union of components of the function complex $\hom(F_*\mathbf{D}, \mathbf{S}^{\#})$ and we will show (2.7) that there is a homotopy fibration sequence involving $\hom(F_*\mathbf{D}, \mathbf{S}^{\#})$ which is parallel to the above one (2.6) involving $\operatorname{real}^0 \bar{Y}$. Comparison of those two sequences then will lead to the conclusion that $\operatorname{real}^0 \bar{Y}$ is weakly homotopy equivalent to $\hom(F_*\mathbf{D}, \mathbf{S}^{\#})$ in such a manner that the components involved in real $\bar{Y}$ correspond to those involved in $\mathrm{hc}_\infty \bar{Y}$.

Let $d*$ be the division [9, 1.2(i)] of the terminal object $* \in \mathbf{S}O\text{-}\mathbf{Cat}$, i.e. $d*$ has as objects the finite sequences $(D_0, \ldots, D_k)$ $(k \geq 0)$ of elements of O (the object set of $\mathbf{D}$) and as maps the compositions of the 'deletions' and 'repetitions'. Given an object $\mathbf{A} \in \mathbf{S}O\text{-}\mathbf{Cat}$, let $V\mathbf{A}$, $V'\mathbf{A}$ and $V''\mathbf{A} \in \mathbf{S}^{d*}$ be the diagrams which assign to a sequence $(D_0, \ldots, D_k)$ the bar construction [7, §9]

$$B(\operatorname{haut} \mathbf{A}D_0, \mathbf{A}(D_0, D_1), \operatorname{haut} \mathbf{A}D_1, \ldots, \operatorname{haut} \mathbf{A}D_k)$$

where each $\operatorname{haut} \mathbf{A}D_i$ $(0 \leq i \leq k)$ is the maximum simplicial submonoid of $\mathbf{A}(D_i, D_i)$ which is grouplike (i.e. $\pi_0 \operatorname{haut} \mathbf{A}D_i$ is a group), the product of simplicial hom-sets

$$\mathbf{A}(D_0, D_1) \times \cdots \times \mathbf{A}(D_{k-1}, D_k)$$

and the product of classifying complexes [7, 1.3(v)]

$$B \operatorname{haut} \mathbf{A}D_0 \times \cdots \times B \operatorname{haut} \mathbf{A}D_k$$

respectively, let (see [7, 5.4]) $UA \to V'\mathbf{A} \in S^{d*}$ be a functorial cofibrant resolution (1.4(iii) and (iv)) and, for every diagram $Z \in \mathbf{S}^{d*}$, let $Z^{\mathrm{f}} \in \mathbf{S}^{d*}$ denote the weakly equivalent fibrant (1.4(iii)) diagram obtained by taking the singular complex of the geometric realization [1, Chapter VIII] of the simplicial sets involved. Proposition 7.6 and [5, 4.7] imply (by identifying function complexes in $\mathbf{S}O\text{-}\mathbf{Cat}$ and $\mathbf{S}^{d*}$ with suitable sets of components of function complexes in $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ and $\mathbf{S}^{\Delta^{\mathrm{op}}}$ respectively) that the function complexes computed in $\mathbf{S}O\text{-}\mathbf{Cat}$ are naturally weakly equivalent to the corresponding function complexes in $\mathbf{S}^{d*}$, as long as suitable cofibrant or fibrant models are chosen for the objects involved. In particular the composition

$$\hom(F_*\mathbf{D}, \mathbf{S}^{\#}) \xrightarrow{\approx} \hom(V'F_*\mathbf{D}, V'\mathbf{S}^{\#}) \longrightarrow \hom(UF_*\mathbf{D}, V'\mathbf{S}^{\#})$$

is a weak equivalence. Moreover

$$\hom(UF_*\mathbf{D}, (V''\mathbf{S}^{\#})^{\mathrm{f}}) \approx \hom(UF_*\mathbf{D}^0, (V\mathbf{S}^{\#})^{\mathrm{f}})$$

and it now follows from [7, 9.2(vii)] that

2.7. Lemma. *The obvious sequence*

$$\hom(F_*\mathbf{D}, \mathbf{S}^{\#}) \longrightarrow \hom(UF_*\mathbf{D}, (V\mathbf{S}^{\#})^{\mathrm{f}}) \longrightarrow \hom(UF_*\mathbf{D}^0, (V\mathbf{S}^{\#})^{\mathrm{f}})$$

is a homotopy fibration sequence. $\square$

The main classification result of [7, §5] essentially states that the right-hand maps in 2.6 and 2.7 are weakly equivalent and from this one now readily deduces that $\text{real}^0\,\bar{Y}$ and $\hom(F_*\mathbf{D},\mathbf{S})$ are weakly equivalent in such a manner that the components involved in $\text{real}\,\bar{Y}$ correspond to those involved in $\text{hc}_\infty\,\bar{Y}$.

3. *n*-homotopy commutative diagrams and obstructions to realizations

In view of 2.5 it remains to investigate the space $\text{hc}_\infty\,\bar{Y}$ of ∞-homotopy commutative diagrams over a given homotopy commutative diagram $\bar{Y}$. This space $\text{hc}_\infty\,\bar{Y}$ *is* (3.2) *an inverse limit as well as homotopy inverse limit of a tower* $\{\text{hc}_n\,\bar{Y}\}_{n>0}$ *of spaces of n-homotopy commutative diagrams over* $\bar{Y}$, and it follows (3.3) that $\bar{Y}$ *can be lifted to an* ∞*-homotopy commutative diagram* (*and hence* (2.5) *has a realization*) *iff it can be lifted to a sequence* $\{Y_n\}$ *of n-homotopy commutative diagrams which are compatible*, in the sense that each Y_n is in the same component of $\text{hc}_n\,\bar{Y}$ as the image of Y_{n+1}. As usual there is an accompanying sequence of obstructions (3.5).

We also (3.8) briefly compare our present treatment of the realization problem with that of [9].

3.1. *n-homotopy commutative diagrams.* A definition of *n-homotopy commutative* $\mathbf{D}$*-diagrams over* $\bar{Y}$ ($n>0$) can be obtained from Definition 2.3 by everywhere replacing the category $\mathbf{S}^\#$ by the category $\text{cosk}_n\,\mathbf{S}^\#$ (1.4(ii)), where $\text{cosk}_n:\mathbf{S}\to\mathbf{S}$ denotes the n-coskeleton functor, i.e. the right adjoint of the n-skeleton functor $\text{sk}_n:\mathbf{S}\to\mathbf{S}$. The resulting *space* $\hom_{\bar{Y}}(F_*\mathbf{D},\text{cosk}_n\,\mathbf{S}^\#)$ *of n-homotopy commutative* $\mathbf{D}$*-diagrams over* $\bar{Y}$ will be denoted by $\text{hc}_n\,\bar{Y}$ and its vertices are the n-homotopy commutative $\mathbf{D}$-diagrams over $\bar{Y}$.

This definition readily implies [11, 4.1]

3.2. Proposition. *The canonical map* [1, Chapter XI]

$$\text{hc}_\infty\,\bar{Y}=\varprojlim \text{hc}_n\,\bar{Y}\to \underset{\longleftarrow}{\text{holim}}\,\text{hc}_n\,\bar{Y}\in\mathbf{S}^{\text{f}}$$

is a homotopy equivalence. □

3.3. Corollary. *Let* $\{Y_n\}$ *be a compatible sequence of n-homotopy commutative* $\mathbf{D}$*-diagrams over* Y (i.e. each Y_n is in the same component of $\text{hc}_n\,\bar{Y}$ as the image of Y_{n+1}). *Then there exists an* ∞*-homotopy commutative* $\mathbf{D}$*-diagram* Y_∞ *such that, for every integer* $n>0$, *its image in* $\text{hc}_n\,\bar{Y}$ *is in the same component as* Y_n. □

3.4. Remark. As usual [11, 5.8], the component of such a Y_∞ is *not* uniquely determined by the Y_n. Given such a Y_∞, *the possible components are in* 1–1 *correspondence with the elements of the pointed set* $\varprojlim^1 \pi_1(\text{hc}_n\,\bar{Y};Y_n')$, where Y_n' ($n>0$) denotes the image of Y_∞ in $\text{hc}_n\,\bar{Y}$; the component containing Y_∞ corresponds to the base point.

There are, of course, inductive algebraic conditions for the existence of a compatible sequence $Y_1, \ldots, Y_n, \ldots$ of n-homotopy commutative diagrams, which one usually calls

3.5. *Obstructions.* Clearly $\mathrm{hc}_1 \bar{Y}$ *is contractible* and one can thus choose Y_1 to be any vertex of $\mathrm{hc}_1 \bar{Y}$. It is also not difficult to see that finding an (automatically) compatible $Y_2 \in \mathrm{hc}_2 \bar{Y}$ is an (already algebraic) problem involving *fundamental groupoids*. Assuming therefore that one already has obtained an n-homotopy commutative $\mathbf{D}$-diagram Y_n over $\bar{Y}$ $(n \geq 2)$, it remains to find an obstruction to the existence of a compatible $(n+1)$-homotopy commutative diagram.

To do this one applies the obstruction theory of [11, §5]. If (in the notation and terminology used there) one pulls the k-invariant $k^{n+1}\mathbf{S}^\#$ back along Y_n to a cycle $hY_n \in Z^{n+1}(F_*\mathbf{D}; \pi_n \mathbf{S}^\#)$ and denotes by $[hY_n]$ the image of hY_n under the composition of the canonical maps [11, §2]

$$Z^{n+1}(F_*\mathbf{D}; \pi_n \mathbf{S}^\#) \to H^{n+1}(F_*\mathbf{D}; \pi_n \mathbf{S}^\#) \to H^{n+1}(\mathbf{D}; p_*\pi_n \mathbf{S}^\#),$$

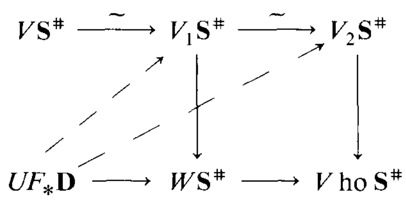
then one has [11, §5]

3.6. Proposition. *Let Y_n be an n-homotopy commutative* $\mathbf{D}$*-diagram over $\bar{Y}$ $(n \geq 2)$. Then there exists a compatible $(n+1)$-homotopy commutative* $\mathbf{D}$*-diagram iff $[hY_n] \in H^{n+1}(\mathbf{D}; p_*\pi_n \mathbf{S})$ is the zero element.* □

3.7. Remark. The results of Sections 2 and 3 are readily seen (1.4(iv)) to remain valid if one replaces everywhere the standard resolution (1.4(iv)) $p: F_*\mathbf{D} \to \mathbf{D}$ of $\mathbf{D}$ by any other cofibrant resolution $q: \mathbf{D}' \to \mathbf{D}$, i.e. the homotopy types of the simplicial sets $\hom_{\bar{Y}}(\mathbf{D}', \mathbf{S}^\#)$ and $\hom_{\bar{Y}}(\mathbf{D}', \mathrm{cosk}_n \mathbf{S}^\#)$ $(n > 0)$ as well as the resulting obstructions in $H^{n+1}(\mathbf{D}; q_*\pi_n \mathbf{S}^\#)$ (which is canonically isomorphic to $H^{n+1}(\mathbf{D}; p_*\pi_n \mathbf{S}^\#)$) are independent of the choice of such a cofibrant resolution. This is, however, not the case for the homotopy types of the simplicial sets $\hom_{\bar{Y}}(\mathrm{sk}_n \mathbf{D}', \mathbf{S}^\#)$ (where $\mathrm{sk}_n \mathbf{D}'$ denotes the subcategory of $\mathbf{D}'$ generated by the n-skeletons of the simplicial hom-sets), and that is why we did not define n-homotopy commutative $\mathbf{D}$-diagrams in terms of maps $\mathrm{sk}_n F_*\mathbf{D} \to \mathbf{S}^\# \in \mathbf{S}O\text{-}\mathbf{Cat}$, even though these are in a natural 1-1 correspondence with the maps $F_*\mathbf{D} \to \mathrm{cosk}_n \mathbf{S}^\# \in \mathbf{S}O\text{-}\mathbf{Cat}$.

We end with a

3.8. *Comparison with the realization results of* [9]. In the notation of the proof of Theorem 2.4, let $W\mathbf{S}^\#$ be the pullback of the diagram $V\,\mathrm{ho}\,\mathbf{S}^\# \to V''\,\mathrm{ho}\,\mathbf{S}^\# \leftarrow V''\,\mathbf{S}^\#$ and let $g: UF_*\mathbf{D} \to W\mathbf{S}^\# \in \mathbf{S}^{d*}$ be the obvious map determined by $\bar{Y}$. Then there is a commutative solid arrow diagram in $\mathbf{S}^{d*}$



in which the upper maps are weak equivalences and the vertical maps are fibrations (1.4(iii)), and it follows readily from 2.7 that $\bar{Y}$ has a realization iff there exists a broken arrow $UF_*\mathbf{D} \to V_1\mathbf{S}^\#$ such that the resulting diagram is commutative, while the results of [9] imply that $\bar{Y}$ has a realization iff there exists a broken arrow $UF_*\mathbf{D} \to V_2\mathbf{S}^\#$ with this property.

Application of the obstruction theory of [8] to the first of these lifting problems yields the same obstructions as above; the cohomology groups of $UF_*\mathbf{D}$ involved have coefficients in the homotopy groups of the products $\mathbf{S}(\bar{Y}D_0, \bar{Y}D_1) \times \cdots \times \mathbf{S}(\bar{Y}D_{k-1}, \bar{Y}D_k)$, and can be shown to be naturally isomorphic to the groups $H^{n+1}(\mathbf{D}; p_*\pi_n\mathbf{S}^\#)$ of 3.5 and 3.6. The second lifting problem, on the other hand, gives rise to obstructions which lie in the cohomology of $UF_*\mathbf{D}$ with coefficients in the less easily accessible homotopy groups of the bar constructions [7, §9]

$$B(\text{haut}\,\bar{Y}D_0, \mathbf{S}(\bar{Y}D_0, \bar{Y}D_1), \text{haut}\,\bar{Y}D_1, \ldots, \text{haut}\,\bar{Y}D_k)$$

which fit into long exact sequences involving the homotopy groups of the products $\mathbf{S}(\bar{Y}D_0, \bar{Y}D_1) \times \cdots \times \mathbf{S}(\bar{Y}D_{k-1}, \bar{Y}D_k)$ and $B\,\text{haut}\,\bar{Y}D_0 \times \cdots \times B\,\text{haut}\,\bar{Y}D_k$.

4. The relative case

Next we discuss a relative version of the results of Sections 2 and 3, i.e. the realization problem for

4.1. *Homotopy commutative* $(\mathbf{D}, \mathbf{C})$*-diagrams.* Let $\mathbf{D}$ be a small simplicial category and let $\mathbf{C} \subset \mathbf{D}$ be a subcategory which *contains all the objects.* A *homotopy commutative* $(\mathbf{D}, \mathbf{C})$*-diagram of simplicial sets* then will be just a commutative diagram of the form

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\text{incl}} & \mathbf{D} \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle \bar{Y}} \\ \mathbf{S}^{\mathrm{f}} & \xrightarrow{\pi} & \text{ho}\,\mathbf{S}^{\mathrm{f}} \end{array}$$

The corresponding relativizations of Definitions 2.2, 2.3 and 3.1 are:

4.2. *Realizations of homotopy commutative* $(\mathbf{D}, \mathbf{C})$*-diagrams.* A *realization* of a homotopy commutative $(\mathbf{D},\mathbf{C})$-diagram $(\bar{Y},X)$ consists of a $\mathbf{D}$-diagram $Y:\mathbf{D}\to\mathbf{S}^{\mathrm{f}}$, a $\mathbf{C}$-diagram $X':\mathbf{C}\to\mathbf{S}^{\mathrm{f}}$ and a 'zigzag' of weak equivalences in $\mathbf{S}^{\mathbf{C}}$

$$Y|_{\mathbf{C}} \xleftarrow{z'} X' \xrightarrow{z''} X$$

such that $(\pi z'')(\pi z')^{-1}:\pi Y\to\bar{Y}$ is a natural equivalence. A *map* $(Y_1,X_1',z_1',z_1'')\to(Y_2,X_2',z_2',z_2'')$ between two such realizations is a pair of weak equivalences $y:Y_1\to Y_2\in\mathbf{S}^{\mathbf{D}}$ and $x':X_1'\to X_2'\in\mathbf{S}^{\mathbf{C}}$ such that the diagram

$$\begin{array}{ccccc}
Y_1|_{\mathbf{C}} & \xleftarrow{z_1'} & X_1' & \xrightarrow{z_1''} & X \\
\downarrow{\scriptstyle y} & & \downarrow{\scriptstyle x'} & & \downarrow{\scriptstyle \mathrm{id}} \\
Y_2|_{\mathbf{C}} & \xleftarrow{z_2'} & X_2' & \xrightarrow{z_2''} & X
\end{array}$$

is commutative. The *space* $\mathrm{real}(\bar{Y},X)$ *of realizations of* $(\bar{Y},X)$ will be the nerve of the resulting category and two realizations of $(\bar{Y},X)$ are called *equivalent* whenever they are in the same component of $\mathrm{real}(\bar{Y},X)$.

4.3. *n-homotopy commutative* $(\mathbf{D},\mathbf{C})$*-diagrams* $(1\le n\le\infty)$. The inclusion $\mathbf{C}\to\mathbf{D}$ admits a functorial factorization

$$\mathbf{C} \longrightarrow F_*^{\mathbf{C}}\mathbf{D} \xrightarrow{p^{\mathbf{C}}} \mathbf{D}$$

in which $F_*^{\mathbf{C}}\mathbf{D}$ is the pushout of the diagram $\mathbf{C}\leftarrow F_*\mathbf{C}\to F_*\mathbf{D}$. Given a homotopy commutative $(\mathbf{D},\mathbf{C})$-diagram $(\bar{Y},X)$, an *n-homotopy commutative* $(\mathbf{D},\mathbf{C})$*-diagram over* $(\bar{Y},X)$ $(1\le n\le\infty)$ will be a factorization

$$\begin{array}{ccccc}
\mathbf{C} & \longrightarrow & F_*^{\mathbf{C}}\mathbf{D} & \xrightarrow{p^{\mathbf{C}}} & \mathbf{D} \\
\downarrow{\scriptstyle X} & & \downarrow{\scriptstyle Y} & & \downarrow{\scriptstyle \bar{Y}} \\
\mathbf{S}^{\mathrm{f}}=\mathrm{cosk}_\infty\,\mathbf{S}^\# & \longrightarrow & \mathrm{cosk}_n\,\mathbf{S}^\# & \longrightarrow & \mathrm{ho}\,\mathbf{S}^\#
\end{array}$$

of the diagram (4.1) representing $(\bar{Y},X)$, i.e. a vertex of the simplicial set

$$\mathrm{hc}_n(\bar{Y},X)=\hom_{\bar{Y}}^{X}(F_*^{\mathbf{C}}\mathbf{D},\mathrm{cosk}_n\,\mathbf{S}^\#)$$

which has as n-simplices the commutative diagrams

$$\begin{array}{ccccc}
\mathbf{C} & \xrightarrow{\mathrm{incl}} & F_*^{\mathbf{C}}\mathbf{D}\otimes^{\mathbf{C}}\Delta[n] & \xrightarrow{\mathrm{proj}} & \mathbf{D} \\
\downarrow{\scriptstyle X} & & \downarrow & & \downarrow{\scriptstyle \bar{Y}} \\
\mathbf{S}^\#=\mathrm{cosk}_\infty\,\mathbf{S}^\# & \longrightarrow & \mathrm{cosk}_n\,\mathbf{S}^\# & \longrightarrow & \mathrm{ho}\,\mathbf{S}^\#
\end{array}$$

where $F_*^{\mathbf{C}}\mathbf{D} \otimes^{\mathbf{C}} \Delta[n]$ denotes the pushout of the diagram $\mathbf{C} \leftarrow \mathbf{C} \otimes \Delta[n] \rightarrow F_*^{\mathbf{C}}\mathbf{D} \otimes \Delta[n]$. Two n-homotopy commutative $(\mathbf{D}, \mathbf{C})$-diagrams over $(\bar{Y}, X)$ are called *equivalent* whenever they lie in the same component of $\mathrm{hc}_n(\bar{Y}, X)$.

Theorem 2.4, Proposition 3.2 and their corollaries now become:

4.4. Theorem. *Let $(\bar{Y}, X)$ be a homotopy commutative $(\mathbf{D}, \mathbf{C})$-diagram. Then* $\mathrm{real}(\bar{Y}, X)$ *and* $\mathrm{hc}_\infty(\bar{Y}, X)$ *are naturally weakly (homotopy) equivalent.* □

4.5. Corollary. *$(\bar{Y}, X)$ has a realization iff there exists an ∞-homotopy commutative $(\mathbf{D}, \mathbf{C})$-diagram over $(\bar{Y}, X)$. Moreover, the equivalence classes of the realizations of $(\bar{Y}, X)$ are in a natural* 1-1 *correspondence with the equivalence classes of the ∞-homotopy commutative $(\mathbf{D}, \mathbf{C})$-diagrams over (Y, X).* □

4.6. Proposition. *The canonical map* [1, Chapter XI]

$$\mathrm{hc}_\infty(\bar{Y}, X) = \varprojlim \mathrm{hc}_n(\bar{Y}, X) \rightarrow \operatorname{holim}_{\longleftarrow} \mathrm{hc}_n(\bar{Y}, X) \in \mathbf{S}^{\mathrm{f}}$$

is a homotopy equivalence. □

4.7. Corollary. *Let $\{Y_n\}$ be a compatible sequence of n-homotopy commutative $(\mathbf{D}, \mathbf{C})$-diagrams over $(\bar{Y}, X)$ (i.e. each Y_n is in the same component of* $\mathrm{hc}_n(\bar{Y}, X)$ *as the image of Y_{n+1}). Then there exists an ∞-homotopy commutative $(\mathbf{D}, \mathbf{C})$-diagram Y_∞ such that, for every integer $n > 0$, its image in $\mathrm{hc}_n(\bar{Y}, X)$ is in the same component as Y_n.* □

Of course the discussion of 3.5 applies. In particular, an n-homotopy commutative $(\mathbf{D}, \mathbf{C})$-diagram Y_n $(n \geq 2)$ over $(\bar{Y}, X)$ gives rise to a cocycle $hY_n \in Z^{n+1}(F_*^{\mathbf{C}}\mathbf{D}, \mathbf{C}; \pi_n \mathbf{S}^\#)$ and hence to a cohomology class $[hY_n] \in H^{n+1}(\mathbf{D}, \mathbf{C}; p_*^{\mathbf{C}} \pi_n \mathbf{S}^\#)$ with the property:

4.8. Proposition. *Let Y_n be an n-homotopy commutative $(\mathbf{D}, \mathbf{C})$-diagram over $(\bar{Y}, X)$ $(n \geq 2)$. Then there exists a compatible $(n+1)$-homotopy commutative $(\mathbf{D}, \mathbf{C})$-diagram iff the element $[hY_n] \in H^{n+1}(\mathbf{D}, \mathbf{C}; p_*^{\mathbf{C}} \pi_n \mathbf{S}^\#)$ is the zero element.* □

4.9. Remark. If $\mathbf{C}$ consists of the objects of $\mathbf{D}$ and their identity maps only, then the above results reduce to the ones of Sections 2 and 3.

Propositions 4.6 and 4.8 are proved in the same manner as Propositions 3.2 and 3.6 and it thus remains to give a

Proof of Theorem 4.4. This is essentially the same as that of Theorem 2.4. One readily sees that Lemma 2.6 remains valid if $\mathrm{real}^0 \bar{Y}$ is replaced by its obvious relativization $\mathrm{real}^0(\bar{Y}, X)$ and $\mathbf{D}^0$ by $\mathbf{C}$. Similarly, Lemma 2.7 and the fact that $\mathbf{D}^0 = \mathbf{C}^0$

imply that Lemma 2.7 remains valid if $\hom(F_*\mathbf{D},\mathbf{S}^\#)$ is replaced by $\hom^X(F_*^{\mathbf{C}}\mathbf{D},\mathbf{S}^\#)$, the fibre of the restriction map $\hom(F_*\mathbf{D},\mathbf{S}^\#)\to\hom(F_*\mathbf{C},\mathbf{S}^\#)$, and again $\mathbf{D}^0$ by $\mathbf{C}$. □

5. The topological case

Finally we prove that the results of the preceding sections hold, not only for simplicial diagrams of simplicial sets, but also for topological diagrams of topological spaces. We do this by showing that the usual singular functor $\mathrm{Sin}:\mathbf{T}\to\mathbf{S}$ induces, for every small topological category $\mathbf{E}$ (with discrete object set), a singular functor $\mathrm{Sin}:\mathbf{T}^{\mathbf{E}}$ (from the category $\mathbf{T}^{\mathbf{E}}$ of $\mathbf{E}$-diagrams of topological spaces to the category $\mathbf{S}^{\mathrm{Sin}\,\mathbf{E}}$ of $\mathrm{Sin}\,\mathbf{E}$-diagrams of simplicial sets), *which is an equivalence of* homotopy theories in the strong sense of [10, §7]. This means that it induces, not only an equivalence between the homotopy categories of $\mathbf{T}^{\mathbf{E}}$ and $\mathbf{S}^{\mathrm{Sin}\,\mathbf{E}}$, but also weak (homotopy) equivalences between the function complexes.

We start with making precise what we mean by

5.1. *Topological diagrams of topological spaces.* Let (see 1.4(vi)) $\mathbf{T}$ be the category of topological spaces and $\mathbf{E}$ a small topological category. An $\mathbf{E}$-*diagram of topological spaces* then will be a function X which assigns

(i) to every object $E\in\mathbf{E}$ a space $XE\in\mathbf{T}$, and

(ii) to every pair of objects $E_1,E_2\in\mathbf{E}$ a map (1.4(vi))

$$X(E_1,E_2):\mathbf{E}(E_1,E_2)\times XE_1\to XE_2\in\mathbf{T}$$

subject to the obvious associativity and identity conditions. Similarly a *map* $x:X\to X'$ between two such $\mathbf{E}$-diagrams assigns to every object $E\in\mathbf{E}$ a map $xE:XE\to X'E\in\mathbf{T}$ such that, for every pair of objects $E_1,E_2\in\mathbf{E}$, the following diagram in $\mathbf{T}$ is commutative:

$$\begin{array}{ccc}
\mathbf{E}(E_1,E_2)\times XE_1 & \xrightarrow{X(E_1,E_2)} & XE_2 \\
\Big\downarrow{\scriptstyle \mathbf{E}(E_1,E_2)\times xE_1} & & \Big\downarrow{\scriptstyle xE_2} \\
\mathbf{E}(E_1,E_2)\times X'E_1 & \xrightarrow{X'(E_1,E_2)} & X'E_2
\end{array}$$

The resulting category will be denoted by $\mathbf{T}^{\mathbf{E}}$.

An obvious example of an $\mathbf{E}$-diagram is, for every object $E\in\mathbf{E}$, the diagram $\mathbf{E}(E,-)$ (1.4(vi)).

The usual simplicial structure on $\mathbf{T}$ gives rise to a simplicial structure on $\mathbf{T}^{\mathbf{E}}$ which is part of

5.2. A closed simplicial model category for $\mathbf{T}^{\mathbf{E}}$. *The category* $\mathbf{T}^{\mathbf{E}}$ *admits a closed* *simplicial model category structure in which the simplicial structure is the above one and in which a map $x: X \to X'$ is a weak equivalence or a fibration whenever, for every object $E \in \mathbf{E}$, the map $xE: XE \to X'E \in \mathbf{T}$ is a weak homotopy equivalence or a Serre fibration.*

Next we define

5.3. *The singular functor* $\operatorname{Sin}: \mathbf{T}^{\mathbf{E}} \to \mathbf{S}^{\operatorname{Sin} \mathbf{E}}$. This is the functor which assigns to an object $X \in \mathbf{T}^{\mathbf{E}}$ the object $\operatorname{Sin} X \in \mathbf{S}^{\operatorname{Sin} \mathbf{E}}$ (i.e. functor $\operatorname{Sin} \mathbf{E} \to \mathbf{S}$) such that $(\operatorname{Sin} X)E = \operatorname{Sin}(XE)$ for every object $E \in \mathbf{E}$, and such that, for every pair of objects $E_1, E_2 \in \mathbf{E}$, the map

$$\operatorname{Sin} \mathbf{E}(E_1, E_2) \to \mathbf{S}(\operatorname{Sin} XE_1, \operatorname{Sin} XE_2) \in \mathbf{S}$$

is the one which corresponds (under the usual adjointness) to the map

$$\operatorname{Sin} \mathbf{E}(E_1, E_2) \times \operatorname{Sin} XE_1 \to \operatorname{Sin} XE_2 \in \mathbf{S}$$

induced by the map $X(E_1, E_2)$.

As mentioned above this singular functor induces the desired

5.4. Equivalence of homotopy theories. *The functor* $\operatorname{Sin}: \mathbf{T}^{\mathbf{E}} \to \mathbf{S}^{\operatorname{Sin} \mathbf{E}}$ *is an equivalence of homotopy theories in the sense of* [10, §7]; *it preserves weak equivalences and induces a weak equivalence between the simplicial localizations of* $\mathbf{T}^{\mathbf{E}}$ *and* $\mathbf{S}^{\operatorname{Sin} \mathbf{E}}$.

It remains to give a

Proof of 5.2 and 5.4. These propositions follow readily from [6, 2.2 and 3.1] and the fact that

(i) for every pair of objects $X \in \mathbf{T}^{\mathbf{E}}$ and $E \in \mathbf{E}$, the simplicial hom-set $\mathbf{T}^{\mathbf{E}}(\mathbf{E}(E, -), X)$ is naturally isomorphic to the singular complex $\operatorname{Sin}(XE)$, and

(ii) the full simplicial subcategory of $\mathbf{T}^{\mathbf{E}}$ spanned by the objects $\mathbf{E}(E, -)$, for all $E \in \mathbf{E}$, is (in view of (i)) canonically isomorphic to the opposite of the small simplicial category $\operatorname{Sin} \mathbf{E}$. □

6. A generalization of Quillen's Theorem B

Given a functor $f: \mathbf{X} \to \mathbf{Y}$ and an object $Y \in \mathbf{Y}$, Quillen's Theorem B [15, p. 97] provides a sufficient condition in order that (the nerve of) the over category $f \downarrow Y$ (which has as objects the pairs (X, y) in which X is an object of $\mathbf{X}$ and y is a map $fX \to Y \in \mathbf{Y}$) is a homotopy fibre of (the nerve of) f. The aim of this section is to prove a similar theorem B_n in which the category $f \downarrow Y$ is replaced by the category

$f\downarrow_n Y$, which has as objects the pairs (X, y) in which X is an object of $\mathbf{X}$ and y is a 'zigzag'

$$fX = Y_n \cdots \rightarrow Y_2 \leftarrow Y_1 \rightarrow Y$$

in $\mathbf{Y}$ of length n. Of course there is also a dual version involving the category $f\uparrow_n Y$, which has as objects the pairs (X, y) in which X is an object of $\mathbf{X}$ and y is a zigzag

$$fX = Y_n \cdots \leftarrow Y_2 \rightarrow Y_1 \leftarrow Y$$

in $\mathbf{Y}$ of length n.

If $\mathbf{Y}$ is the category of weak equivalences between fibrant (resp. cofibrant) objects in a closed model category, then $f\downarrow_2 Y$ (resp. $f\uparrow_2 Y$) is a homotopy fibre of f (for any f), even though $f\downarrow Y$ (resp. $f\uparrow Y$) need *not* be so.

In order to simplify the formulation of Theorem B_n we first define

6.1. Property $\mathbf{B}_n$. Let n be an integer ≥ 1, let $f: \mathbf{X} \rightarrow \mathbf{Y}$ be a functor between small categories, let $Y \in \mathbf{Y}$ be an object, let $f\downarrow_n Y$ denote the category of which an object consists of an object $X \in \mathbf{X}$, together with a zigzag

$$fX = Y_n \cdots \rightarrow Y_2 \leftarrow Y_1 \rightarrow Y$$

in $\mathbf{Y}$ of length n, and of which a map consists of a map $x: X \rightarrow X' \in \mathbf{X}$ together with a commutative diagram

$$\begin{array}{ccccccc}
fX = Y_n \cdots & Y_2 & \longleftarrow & Y_1 & \longrightarrow & Y \\
\Big\downarrow fX & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \mathrm{id} \\
fX' = Y'_n \cdots & Y'_2 & \longleftarrow & Y'_1 & \longrightarrow & Y
\end{array}$$

and let $f\uparrow_n Y$ be defined similarly (see above). Then f will be said to have *property* B_n, if every map $Y' \rightarrow Y'' \in \mathbf{Y}$ induces (see 1.4(v)) a weak (homotopy) equivalence $f\downarrow_n Y' \rightarrow f\downarrow_n Y''$. This clearly implies that $f^{\mathrm{op}}: \mathbf{X}^{\mathrm{op}} \rightarrow \mathbf{Y}^{\mathrm{op}}$ has property B_n iff every map $Y' \rightarrow Y'' \in \mathbf{Y}$ induces a weak (homotopy) equivalence $f\uparrow_n Y'' \rightarrow f\uparrow_n Y'$.

Now we can state

6.2. Theorem $\mathbf{B}_n$. *If a functor $f: \mathbf{X} \rightarrow \mathbf{Y}$ (resp. $f^{\mathrm{op}}: \mathbf{X}^{\mathrm{op}} \rightarrow \mathbf{Y}^{\mathrm{op}}$) between small categories has property* B_n, *then, for every object $Y \in \mathbf{Y}$, (the nerve of) $f\downarrow_n Y$ (resp. $f\uparrow_n Y$) is a homotopy fibre of f over Y.*

Proof. For $n = 1$ this is Quillen's Theorem B [15, p. 97]. For $n > 1$ one has $f\downarrow_n Y = (f\uparrow_{n-1} \mathbf{Y})\downarrow Y$, where $f\uparrow_{n-1} \mathbf{Y}$ denotes the category of which an object consists of a pair of objects $X \in \mathbf{X}$, $Y_1 \in \mathbf{Y}$, together with a zigzag $fX = Y_n \cdots \rightarrow Y_2 \leftarrow Y_1$ in $\mathbf{Y}$ of length $n-1$, and the desired result therefore follows from Quillen's Theorem B and the fact that (the nerve of) the obvious functor $\mathbf{X} \rightarrow f\downarrow_{n-1} \mathbf{Y}$ is a weak (homotopy) equivalence. $\square$

Next we show that every functor $f: \mathbf{X} \to \mathbf{Y}$ between small categories has property B_n, if one assumes that $\mathbf{Y}$ has

6.3. Property C_n. Let $\mathbf{0}$ denote the category with only one object and its identity map. A small category $\mathbf{Y}$ then is said to have *property* C_n, if all functors $\mathbf{0} \to \mathbf{Y}$ have property B_n.

As just mentioned, the usefulness of this notion is due to

6.4. Theorem C_n. *If $f: \mathbf{X} \to \mathbf{Y}$ is a functor between small categories and $\mathbf{Y}$ (resp. $\mathbf{Y}^{\mathrm{op}}$) has property C_n, then f (resp. f^{op}) has property B_n.*

Proof. This follows immediately from the homotopy invariance of Grothendieck constructions [4, 9.6] and the fact that $f \downarrow_n Y$ is in an obvious manner a Grothendieck construction. □

A similar argument yields

6.5. Proposition. *If a small category $\mathbf{Y}$ has property C_n, then it also has property C_k for $k > n$.* □

6.6. *A slight generalization.* The above results remain valid for *not necessarily small categories* as long as all nerves of categories involved are *homotopically small* (1.4(v)).

We end with some examples.

6.7. Example. Every groupoid has property C_1. Conversely, if $\mathbf{Y}$ has property C_1 and if, for every two objects $Y_1, Y_2 \in \mathbf{Y}$, $\mathbf{Y}$ contains a map $Y_1 \to Y_2$ iff $\mathbf{Y}$ contains a map $Y_2 \to Y_1$, then $\mathbf{Y}$ is a groupoid.

6.8. Example. Let $\mathbf{C}$ be a closed model category [1, Chapter VIII], let $\mathbf{W} \subset \mathbf{C}$ be its category of weak equivalences, let $\mathbf{W}^{\mathrm{f}}$ (resp. $\mathbf{W}^{\mathrm{c}}$) be the full subcategory of $\mathbf{W}$ spanned by the fibrant (resp. cofibrant) objects and let $\mathbf{W}^{\mathrm{cf}} = \mathbf{W}^{\mathrm{c}} \cap \mathbf{W}^{\mathrm{f}}$. Then ([4, §8] and [5, §8])

(i) $\mathbf{W}^{\mathrm{op}}$ *has property* C_3, and

(ii) $\mathbf{W}^{\mathrm{f}}$, $\mathbf{W}^{\mathrm{cf}}$, $(\mathbf{W}^{\mathrm{c}})^{\mathrm{op}}$ *and* $(\mathbf{W}^{\mathrm{cf}})^{\mathrm{op}}$ *have property* C_2.

7. A connection between simplicial categories and diagrams of simplicial sets

The aim of this section is to show that *the homotopy theory of small simplicial categories* (1.4(ii)) with respect to (i.e. with as weak equivalences) the functors which are 1–1 and onto on objects and are weak (homotopy) equivalences on the simplicial hom-sets, *is equivalent to the homotopy theory of the diagrams of simplicial sets* indexed by the simplicial indexing category $\mathbf{\Delta}^{op}$, which are 'special' in the sense that they satisfy a slight variation on a condition of Segal [16, 1.5]. This will be done by establishing, more generally, that *the homotopy theory of the category* $\mathbf{Cat}^{\mathbf{\Delta}^{op}}$ (of $\mathbf{\Delta}^{op}$-diagrams of small categories), with respect to a slightly unusual notion of weak equivalence, *is equivalent to the homotopy theory of the category* $\mathbf{S}^{\mathbf{\Delta}^{op}}$ of all $\mathbf{\Delta}^{op}$-diagrams of simplicial sets.

We thus start with a brief discussion of

7.1. *The category* $\mathbf{Cat}^{\mathbf{\Delta}^{op}}$. As usual **Cat** will denote the category of *small categories* and $\mathbf{\Delta}$ the category of *finite ordered sets*. If, for every integer $n \geq 0$, $\mathbf{n} \in \mathbf{Cat}$ is the category which has as objects the integers $0, \ldots, n$ and which has exactly one map $i \to j$ whenever $i \leq j$, then $\mathbf{\Delta}$ can be identified with the full subcategory of **Cat** spanned by the $\mathbf{n}$ $(n \geq 0)$. The category $\mathbf{Cat}^{\mathbf{\Delta}^{op}}$ of *simplicial small categories* (*which has as objects the functors* $\mathbf{\Delta}^{op} \to \mathbf{Cat}$ and as maps the natural transformations between them) has an obvious simplicial structure. (If $\mathbf{C}$ is a simplicial small category and K is a simplicial set, then $\mathbf{C} \otimes K$ consists in dimension n of the sum in **Cat** (i.e. the disjoint union) of as many copies of the category $\mathbf{Cn}$ as there are n-simplices in K and the face and degeneracy operators in $\mathbf{C} \otimes K$ are the obvious ones. The simplicial set $\hom(\mathbf{C}, \mathbf{D})$ then has as its n-simplices the maps $\mathbf{C} \otimes \Delta[n] \to \mathbf{D} \in \mathbf{Cat}^{\mathbf{\Delta}^{op}}$.) One now readily verifies that the set of objects $\{\mathbf{n}\}_{n \geq 0}$ is a *set of orbits* for $\mathbf{Cat}^{\mathbf{\Delta}^{op}}$ in the sense of [6, 2.1]. Hence [6, 2.2]

7.2. Proposition. *The category* $\mathbf{Cat}^{\mathbf{\Delta}^{op}}$ *admits a closed simplicial model category structure in which the simplicial structure is the obvious one and in which a map* $Y \to Y' \in \mathbf{Cat}^{\mathbf{\Delta}^{op}}$ *is a weak equivalence or a fibration whenever, for every integer* $n \geq 0$, *the induced map of function complexes* $\hom(\mathbf{n}, Y) \to \hom(\mathbf{n}, Y') \in \mathbf{S}$ *is a weak (homotopy) equivalence or a fibration.* □

Moreover [6, 3.1]

7.3. Proposition. *The nerve or singular functor*

$$\hom(\mathbf{\Delta}, -) : \mathbf{Cat}^{\mathbf{\Delta}^{op}} \to \mathbf{S}^{\mathbf{\Delta}^{op}}$$

has a left adjoint, the realization functor

$$\mathbf{\Delta}\otimes : \mathbf{S}^{\mathbf{\Delta}^{op}} \to \mathbf{Cat}^{\mathbf{\Delta}^{op}}.$$

Furthermore

(i) *The functor* $\hom(\mathbf{\Delta}, -)$ *preserves fibrations and weak equivalences,*

(ii) *the functor* $\mathbf{\Delta}\otimes$ *preserves cofibrations and weak equivalences between cofibrant objects, and*

(iii) *for every cofibrant object* $X \in \mathbf{S}^{\mathbf{\Delta}^{op}}$ *and every fibrant object* $Y \in \mathbf{Cat}^{\mathbf{\Delta}^{op}}$, *a map* $\mathbf{\Delta} \otimes X \to Y \in \mathbf{Cat}^{\mathbf{\Delta}^{op}}$ *is a weak equivalence iff its adjoint* $X \to \hom(\mathbf{\Delta}, Y) \in \mathbf{S}^{\mathbf{\Delta}^{op}}$ *is so.* □

7.4. Corollary. *The functor* hom$(\mathbf{\Delta},-):\mathbf{Cat}^{\Delta^{\mathrm{op}}}\to\mathbf{S}^{\Delta^{\mathrm{op}}}$ *is an equivalence of homotopy theories in the sense of* [10, §7]; *it preserves weak equivalences and induces a weak equivalence between the simplicial localizations of* $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ *and* $\mathbf{S}^{\Delta^{\mathrm{op}}}$. □

Now we turn to

7.5. *The special case.* Let $\mathbf{S}_0^{\Delta^{\mathrm{op}}}\subset\mathbf{S}^{\Delta^{\mathrm{op}}}$ be the full subcategory spanned by the objects $X\in\mathbf{S}^{\Delta^{\mathrm{op}}}$ such that

(i) $X\mathbf{0}$ is discrete, and

(ii) for every integer $n\geq 2$, the natural map

$$X\mathbf{n}\to X\mathbf{1}\times_{X\mathbf{0}}\cdots\times_{X\mathbf{0}}X\mathbf{1}\in\mathbf{S}$$

induced by the (iterated) face maps $d_n\ldots\hat{d}_i\hat{d}_{i-1}\ldots d_0$ $(1\leq i\leq n)$, is a weak equivalence.

Similarly let $\mathbf{Cat}_0^{\Delta^{\mathrm{op}}}\subset\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ be the full subcategory spanned by the small simplicial categories (1.4(ii)), i.e. the simplicial small categories with discrete object sets. Then one has

7.6. Proposition. *The functor* hom$(\mathbf{\Delta},-):\mathbf{Cat}_0^{\Delta^{\mathrm{op}}}\to\mathbf{S}_0^{\Delta^{\mathrm{op}}}$ *is an equivalence of homotopy theories in the sense of* [10, §7]; *it preserves weak equivalences and induces a weak equivalence between the simplicial localizations of* $\mathbf{Cat}_0^{\Delta^{\mathrm{op}}}$ *and* $\mathbf{S}_0^{\Delta^{\mathrm{op}}}$.

Proof. Let $\overline{\mathbf{S}}_0^{\Delta^{\mathrm{op}}}\subset\mathbf{S}^{\Delta^{\mathrm{op}}}$ and $\overline{\mathbf{Cat}}_0^{\Delta^{\mathrm{op}}}\subset\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ be the full subcategories spanned by the objects which are weakly equivalent to objects in $\mathbf{S}_0^{\Delta^{\mathrm{op}}}$ and $\mathbf{Cat}_0^{\Delta^{\mathrm{op}}}$ respectively. Corollary 7.4 then readily implies that the functor hom$(\mathbf{\Delta},-):\overline{\mathbf{Cat}}_0^{\Delta^{\mathrm{op}}}\to\overline{\mathbf{S}}_0^{\Delta^{\mathrm{op}}}$ is an equivalence of homotopy theories and the desired result now follows from the fact that there are obvious functors

$$\overline{\mathbf{S}}_0^{\Delta^{\mathrm{op}}}\to\mathbf{S}_0^{\Delta^{\mathrm{op}}}\quad\text{and}\quad\overline{\mathbf{Cat}}_0^{\Delta^{\mathrm{op}}}\to\mathbf{Cat}_0^{\Delta^{\mathrm{op}}}$$

such that their compositions with the inclusion functors

$$\mathbf{S}_0^{\Delta^{\mathrm{op}}}\to\overline{\mathbf{S}}_0^{\Delta^{\mathrm{op}}}\quad\text{and}\quad\mathbf{Cat}_0^{\Delta^{\mathrm{op}}}\to\overline{\mathbf{Cat}}_0^{\Delta^{\mathrm{op}}}$$

are naturally weakly equivalent to the identity functors of $\overline{\mathbf{S}}_0^{\Delta^{\mathrm{op}}}$ and $\overline{\mathbf{Cat}}_0^{\Delta^{\mathrm{op}}}$. □

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