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Formulas for the Unramified Brauer Group of a Principal Homogeneous Space of a Linear Algebraic Group

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For a smooth compactification V of a principal homogeneous space E under a connected linear algebraic group G defined over a field k of characteristic zero, we present two formulas expressing $\mathrm{Br}\, V/\mathrm{Br}\, k$ in terms of G. © 2000 Academic Press

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INTRODUCTION

Let G be a connected linear algebraic group over a field k of characteristic zero, let E be a principal homogeneous space (torsor) under G, and let V be a smooth complete variety over k containing E as a dense open subset. Since the Brauer group $\mathrm{Br}\, V = H^2_{ét}(V, \mathbb{G}_m)$ is a birational invariant, it does not depend on the choice of V but only depends on G and E; it is denoted $\mathrm{Br}_{nr}(k(E)/k)$ and is called the unramified Brauer group of E. In this paper we give two formulas for $\mathrm{Br}\, V/\mathrm{Br}\, k$ in terms of G.

Formulas for $\mathrm{Br}\, V/\mathrm{Br}\, k$ were first given by Voskresenskiĭ [V] and Sansuc [S] in the case when k is a number field. A generalization to an arbitrary ground field was presented in [CTK]. In all these three papers, G is of some special type: either a torus, or a semisimple group, or a group admitting a finite cover of the type $G_0 \times S$ where G_0 is a semisimple simply connected group and S is a quasi-trivial torus.

In this paper we compute $\mathrm{Br}\, V/\mathrm{Br}\, k$ for any connected k-group G. The paper is based on results of [CTK]. We use the method of z-extensions developed by Kottwitz [K2, K3].

We now briefly describe our results. Let G^{u} denote the unipotent radical of G. Set $G^{\mathrm{red}} = G/G^{\mathrm{u}}$; it is a reductive group. Let G^{ss} denote the derived group of G^{red}; it is semisimple. Set $G^{\mathrm{tor}} = G^{\mathrm{red}}/G^{\mathrm{ss}}$; it is a torus. Let G^{sc} denote the universal covering of G^{ss}; it is simply connected. Consider the composite map

$$\rho\colon G^{\mathrm{sc}} \to G^{\mathrm{ss}} \to G^{\mathrm{red}}.$$

Let $T \subset G^{\mathrm{red}}$ be a maximal torus. Set $T^{\mathrm{sc}} = \rho^{-1}(T) \subset G^{\mathrm{sc}}$. Denote $L^{-1} = \mathbf{X}^*(T)$, $L^0 = \mathbf{X}^*(T^{\mathrm{sc}})$, where $\mathbf{X}^*(\cdot) = \mathrm{Hom}_{\bar{k}}(\cdot, \mathbb{G}_m)$ stands for the character group, and $\bar{k}$ denotes an algebraic closure of k. Consider the complex

$$L^\bullet = (0 \to L^{-1} \to L^0 \to 0) = (0 \to \mathbf{X}^*(T) \to \mathbf{X}^*(T^{\mathrm{sc}}) \to 0).$$

The Galois group $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$ acts on $L^\bullet$. Let $\mathbb{H}^i(\mathfrak{g}, L^\bullet)$ denote the hypercohomology group of $\mathfrak{g}$ with coefficients in the complex $L^\bullet$. Set

$$\text{Ш}^i_\omega(\mathfrak{g}, L^\bullet) = \ker\left[\mathbb{H}^i(\mathfrak{g}, L^\bullet) \to \prod_\gamma \mathbb{H}^i(\gamma, L^\bullet)\right],$$

where γ runs over all closed procyclic subgroups of $\mathfrak{g}$. Let $\overline{V} = V \times_k \bar{k}$, and denote by $\mathrm{Pic}\, \overline{V}$ the Picard group; it is a $\mathfrak{g}$-module. Our main results are the following theorem and corollary.

THEOREM A. *With the above assumptions and notation*,

$$H^1(k, \mathrm{Pic}\, \overline{V}) = \text{Ш}^1_\omega(\mathfrak{g}, L^\bullet).$$

COROLLARY B. *There is an injection*

$$\mathrm{Br}\, V/\,\mathrm{Br}\, k \hookrightarrow Ш^1_\omega(\mathfrak{g}, L^\bullet)$$

which is an isomorphism provided that either $V(k) \neq \emptyset$, *or* $H^3(k, \bar{k}^\times) = 0$.

Note that Corollary B, which, in general, gives an estimate for $\mathrm{Br}\, V/\,\mathrm{Br}\, k$ in terms of the group only depending on G, in many cases gives the precise value of this invariant. Namely, this is the case for $E = G$ (when $V(k) \neq \emptyset$), or for k local or global (when $H^3(k, \bar{k}^\times) = 0$).

Let now $Z(\hat{G})$ denote the center of a connected Langlands dual group for a connected reductive group G, cf. [K2, 1.5]. It is a $\mathbf{C}$-group of multiplicative type. It turns out that $Z(\hat{G}) = \ker[L^{-1} \otimes \mathbf{C}^\times \to L^0 \otimes \mathbf{C}^\times]$.

PROPOSITION C.

$$Ш^1_\omega(\mathfrak{g}, L^\bullet) = Ш^1_\omega(\mathfrak{g}, Z(\hat{G})).$$

Thus

$$H^1(k, \mathrm{Pic}\, \overline{V}) = Ш^1_\omega(\mathfrak{g}, Z(\hat{G}^{\mathrm{red}})).$$

We obtain a new case of the following Kottwitz principle [K2]: an invariant of reductive groups which is trivial for semisimple simply connected groups can be computed from the Galois module $Z(\hat{G})$.

Note that although the above statements and their proofs presented below are purely algebraic, we heavily rely upon a result of [CTK] containing a deep arithmetic ingredient (Chebotarev's density theorem). It would be interesting to find a purely algebraic proof of Theorem A.

The structure of the paper is as follows. In Sect. 1 we collect required information on linear algebraic groups, Brauer groups, Galois cohomology and hypercohomology and prove Proposition C (Proposition 1.3.2). In Sect. 2 we state and prove our main results (Theorem A = Theorems 2.1 and 2.4, and Corollary B = Corollary 2.2). In Sect. 3 we present some comments and remarks relating our results to previously known ones, and give an example of computation.

NOTATION AND CONVENTIONS

Throughout the paper, k denotes a field of characteristic zero, $\bar{k}$ is a fixed algebraic closure of k, $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$ is the absolute Galois group of k, $k^\times$ denotes the multiplicative group of k. An algebraic k-torus T is called quasi-trivial if it is a direct product of tori of the form $R_{K/k}\mathbb{G}_m$ where K/k is a finite extension and $R_{K/k}$ stands for Weil's restriction of scalars. We denote by $\mathbf{X}^*(G)$ the group of characters of a linear algebraic group G

and by $\mathbf{X}_*(T) = \mathrm{Hom}_{\bar{k}}(\mathbb{G}_m, T)$ the group of cocharacters of a torus T; one can view $\mathbf{X}^*(G)$ and $\mathbf{X}_*(T)$ as $\mathfrak{g}$-modules. For a torus T, $\mathbf{X}^*(T)$ is a $\mathbf{Z}$-free $\mathfrak{g}$-module of finite rank; if T is quasi-trivial, $\mathbf{X}^*(T)$ is a permutation module (i.e., it has a $\mathbf{Z}$-basis permuted by $\mathfrak{g}$). If M is a Galois module, we denote by $H^i(k, M)$ (or by $H^i(\mathfrak{g}, M)$) the ith Galois cohomology group. For a smooth projective k-variety X we denote $\overline{X} = X \times_k \bar{k}$. Set Pic $X = H^1_{ét}(X, \mathbb{G}_m)$, Br $X = H^2_{ét}(X, \mathbb{G}_m)$, these are the Picard group and the Brauer group of X, respectively. Other notation is explained in the Introduction.

1. PRELIMINARIES

1.1. *Linear Algebraic Groups*

1.1.1. Let G be a connected linear algebraic group over a field k of characteristic zero. By Chevalley's theorem [C] the k-variety G is rational, i.e., $\bar{k}$-birationally equivalent to an affine space.

Every unipotent k-group is k-biregular to an affine space and hence k-rational.

Every quasi-trivial k-torus Z is k-rational, and by Hilbert 90 $H^1(K, Z) = 1$ for every extension K of k.

1.1.2. Levi Decomposition. For any connected linear algebraic group G over a field k of characteristic zero, there is an isomorphism $G^{\mathrm{u}} \rtimes G^{\mathrm{red}} \xrightarrow{\sim} G$ (Levi decomposition) which gives rise to a k-biregular morphism of varieties

$$G^{\mathrm{u}} \times G^{\mathrm{red}} \xrightarrow{\sim} G.$$

1.1.3. z-Extensions. A z-extension of a connected reductive k-group G is an epimorphism of reductive groups $\alpha\colon H \to G$ with kernel Z, such that H^{ss} is simply connected and Z is central and is a quasi-trivial k-torus. The notion of z-extension was introduced by Langlands [L]. We say that a z-extension $\alpha_1\colon H_1 \to G$ dominates a z-extension $\alpha_2\colon H_2 \to G$ if there exists a homomorphism $\phi\colon H_1 \to H_2$ such that $\alpha_2 = \phi \circ \alpha_1$.

LEMMA 1.1.4 (Kottwitz). (1) *For every connected reductive k-group G and a cohomology class $\xi \in H^1(k, G)$ there exists a z-extension $\alpha\colon H \to G$ such that $\xi \in \mathrm{im}[\alpha_*\colon H^1(k, H) \to H^1(k, G)]$.*

(2) *For every two z-extensions $\alpha_1\colon H_1 \to G$ and $\alpha_2\colon H_2 \to G$ of G there exists a z-extension $\alpha_3\colon H_3 \to G$ that dominates both α_1 and α_2.*

(3) *Let $G_1 \to G_2$ be a homomorphism, and let $H_i \to G_i$ $(i = 1, 2)$ be z-extensions. Then there exists a commutative diagram*

$$\begin{array}{ccccc} H_1 & \longleftarrow & H_3 & \longrightarrow & H_2 \\ \downarrow & & \downarrow & & \downarrow \\ G_1 & \xleftarrow{\ id\ } & G_1 & \longrightarrow & G_2 \end{array}$$

in which $H_3 \to G_1$ *is a* z*-extension.*

Proof. (1) For a proof of existence of some z-extension of G see [MS, Proposition 3.1]. The existence of a z-extension such that ξ lifts to $H^1(k, H)$ is proved in [K3] in the proof of Theorem 1.2, p. 369.

(2) See [K1, Lemma 1.1(2)].

(3) See [K2, Lemma 2.4.4]. ∎

1.2. *Birational Invariants*

1.2.1. Permutation Modules. A permutation $\mathfrak{g}$-module P can be written as a direct sum of induced modules $\mathbf{Z}[\mathfrak{g}/\mathfrak{h}]$, where $\mathfrak{h}$ is a closed subgroup of finite index in $\mathfrak{g}$. By Shapiro's lemma, $H^1(\mathfrak{g}, \mathbf{Z}[\mathfrak{g}/\mathfrak{h}]) = H^1(\mathfrak{h}, \mathbf{Z}) = 0$, hence $H^1(\mathfrak{g}, P) = 0$. Moreover, $H^1(\gamma, P) = 0$ for any closed subgroup $\gamma \subset \mathfrak{g}$.

We also have $\text{III}^2_\omega(\mathfrak{g}, P) = 0$ (cf. [S, (1.9.1)] for the case where k is a number field). Indeed, it suffices to prove this for an induced module $M = \mathbf{Z}[\mathfrak{g}/\mathfrak{h}]$. We have $H^2(\mathfrak{g}, M) = H^2(\mathfrak{h}, \mathbf{Z}) = \mathrm{Hom}(\mathfrak{h}, \mathbf{Q}/\mathbf{Z})$. Since any continuous homomorphism $\mathfrak{h} \to \mathbf{Q}/\mathbf{Z}$ factors through a finite quotient of $\mathfrak{g}$, we may assume that $\mathfrak{h}$ and $\mathfrak{g}$ are finite. Since any non–trivial homomorphism $\mathfrak{h} \to \mathbf{Q}/\mathbf{Z}$ is non–trivial on some cyclic subgroup of $\mathfrak{h}$, we conclude that $\text{III}^2_\omega(\mathfrak{g}, M) = 0$.

1.2.2. Smooth Compactifications. By Hironaka [H], any smooth affine k-variety X can be embedded into a smooth complete k-variety $V(X)$ containing X as an open subset. Indeed, one has to map X biregularly onto a closed subscheme of an affine space, embed it into the projective space, take the projective closure, and resolve its singularities. We call $V(X)$ a smooth k-compactification of X. If V_1 and V_2 are two smooth k-compactifications of X, then there exists an isomorphism of $\mathfrak{g}$-modules $\mathrm{Pic}\, \overline{V}_1 \oplus P_1 \cong \mathrm{Pic}\, \overline{V}_2 \oplus P_2$, where P_1 and P_2 are permutation $\mathfrak{g}$-modules (cf. [V, Theorem 1]). By 1.2.1, this gives an isomorphism $H^1(k, \mathrm{Pic}\, \overline{V}_1) \to H^1(k, \mathrm{Pic}\, \overline{V}_2)$, and the construction in [V] shows that this isomorphism is canonical. This also shows that $H^1(k, \mathrm{Pic}\, \overline{V}(X))$ is a birational invariant of X.

Moreover, the group $H^1(k, \mathrm{Pic}\, \overline{V}(X))$ is functorial in X. Indeed, let $f\colon X_1 \to X_2$ be a k-morphism of smooth integral k-varieties. We wish to extend f to a k-morphism $f'\colon V_1 \to V_2$, where V_i is a suitable smooth compactification of X_i, $i = 1, 2$. We are very grateful to Colliot–Thélène for communicating to us the following construction. Let U denote the graph of f in $X_1 \times_k X_2$. Choose smooth compactifications W_i of X_i, $i = 1, 2$. Let W be the closure of U in $W_1 \times_k W_2$. Then U is a smooth open subvariety of W. By Hironaka [H] there exists a proper morphism (desingularization) $\pi\colon V_1 \to W$ such that V_1 is smooth and the restriction $\pi^{-1}(U) \to U$ is an isomorphism. Clearly V_1 is a smooth compactification of X_1. Set $V_2 = W_2$ and define $f'\colon V_1 \to V_2$ to be the composite map $V_1 \to W \to V_2$ where the second arrow is the restriction of the canonical projection $W_1 \times W_2 \to W_2 = V_2$. The map f' induces a homomorphism $H^1(k, \mathrm{Pic}\, \overline{V}_1) \to H^1(k, \mathrm{Pic}\, \overline{V}_2)$, as required.

We prove the following property of the functor $H^1(k, \mathrm{Pic}\, \overline{V}(X))$: if Z is a k-rational variety, then $H^1(k, \mathrm{Pic}\, \overline{V}(X \times_k Z)) \cong H^1(k, \mathrm{Pic}\, \overline{V}(X))$. Indeed, let V_X, V_Z be smooth compactifications of X, Z, respectively. One can then take $V_X \times V_Z$ as a smooth compactification of $X \times Z$. The variety $\overline{V}_Z$ is rational, hence by [CTS1, Lemme 11, p. 188], the canonical homomorphism $\mathrm{Pic}\, \overline{V}_X \oplus \mathrm{Pic}\, \overline{V}_Z \to \mathrm{Pic}(\overline{V}_X \times \overline{V}_Z)$ is an isomorphism. Since Z is k-rational, $H^1(k, \mathrm{Pic}\, \overline{V}_Z) = 0$. Thus $H^1(k, \mathrm{Pic}\, \overline{V}(X \times Z)) \cong H^1(k, \mathrm{Pic}\, \overline{V}(X))$, as required. This isomorphism is induced by the canonical projection $\mathrm{pr}_X\colon X \times Z \to X$.

1.2.3. Brauer Group. For a geometrically integral smooth projective k-variety X we have an exact sequence

$$\mathrm{Br}\, k \to \ker[\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X}] \to H^1(k, \mathrm{Pic}\, \overline{X}) \to H^3(k, \bar{k}^\times);$$

if X has a k-point, we have an exact sequence

$$0 \to \mathrm{Br}\, k \to \ker[\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X}] \to H^1(k, \mathrm{Pic}\, \overline{X}) \to 0$$

(cf. [CTS2, 1.5.0]). If X is k-rational, this gives an isomorphism $\mathrm{Br}\, k \xrightarrow{\sim} \mathrm{Br}\, X$; if X is a smooth k-compactification of a G-torsor E with $X(k) \neq \emptyset$, this gives an isomorphism

$$\mathrm{Br}\, X/\, \mathrm{Br}\, k \cong H^1(k, \mathrm{Pic}\, \overline{X}),$$

because by 1.1.1 $\overline{X}$ is rational, and since $\overline{X}$ is projective and rational, $\mathrm{Br}\, \overline{X} = 0$.

1.3. *Hypercohomology*

1.3.1. Let $M^\bullet = (0 \to M^{-1} \to M^0 \to 0)$ be a short complex of $\mathfrak{g}$-modules. We often shorten notation to $(M^{-1} \to M^0)$. We define the hypercohomology $\mathbb{H}^i(\mathfrak{g}, M^\bullet)$ as the cohomology H^i of the ordinary chain complex corresponding to the double complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^0 & \longrightarrow & C^1(\mathfrak{g}, M^0) & \longrightarrow & C^2(\mathfrak{g}, M^0) & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & M^{-1} & \longrightarrow & C^1(\mathfrak{g}, M^{-1}) & \longrightarrow & C^2(\mathfrak{g}, M^{-1}) & \longrightarrow & \dots \end{array}$$

where C^i is the usual group of non–homogeneous continuous i-cochains and the bidegree of M^{-1} is $(-1, 0)$.

For a subgroup $\gamma \subset \mathfrak{g}$ one can define the restriction map $\mathbb{H}^i(\mathfrak{g}, M^\bullet) \to \mathbb{H}^i(\gamma, M^\bullet)$ and define

$$\text{III}^i_\omega(\mathfrak{g}, M^\bullet) = \ker\left[\mathbb{H}^i(\mathfrak{g}, M^\bullet) \to \prod_\gamma \mathbb{H}^i(\gamma, M^\bullet)\right],$$

where γ runs over all closed procyclic subgroups of $\mathfrak{g}$.

1.3.2. Let T be a maximal k-torus in a connected reductive k-group G, and let $Z(\hat{G})$ denote the center of a connected Langlands dual group for G (see Introduction). Denote $L^{-1} = \mathbf{X}^*(T)$, $L^0 = \mathbf{X}^*(T^{\text{sc}})$, $L^\bullet = (L^{-1} \to L^0)$.

PROPOSITION. *With the above notation*

$$\text{III}^1_\omega(\mathfrak{g}, L^\bullet) = \text{III}^1_\omega(\mathfrak{g}, Z(\hat{G})).$$

1.3.3. We first prove the following lemma.

LEMMA. *There is a short exact sequence of* $\mathbf{C}$*-groups*

$$1 \to Z(\hat{G}) \to L^{-1} \otimes \mathbf{C}^\times \to L^0 \otimes \mathbf{C}^\times \to 1.$$

Proof. Set $\pi_1(G) = \mathbf{X}_*(T)/\rho(\mathbf{X}_*(T^{\text{sc}}))$, where $\mathbf{X}_*(\cdot) = \text{Hom}_{\bar{k}}(\mathbb{G}_m, \cdot)$ is the cocharacter group. By [B, 1.2] the Galois module $\pi_1(G)$ does not depend on the choice of $T \subset G$. We call $\pi_1(G)$ the algebraic fundamental group of G. By [B, 1.10] we have $\pi_1(G) = \text{Hom}(Z(\hat{G}), \mathbf{C}^\times)$. Hence $Z(\hat{G}) = \text{Hom}(\pi_1(G), \mathbf{C}^\times)$.

By the definition of $\pi_1(G)$, there is an exact sequence

$$0 \to \mathbf{X}_*(T^{\text{sc}}) \to \mathbf{X}_*(T) \to \pi_1(G) \to 0.$$

We thus obtain an exact sequence

$$1 \to \text{Hom}(\pi_1(G), \mathbf{C}^\times) \to \text{Hom}(\mathbf{X}_*(T), \mathbf{C}^\times) \to \text{Hom}(\mathbf{X}_*(T^{\text{sc}}), \mathbf{C}^\times) \to 1,$$

or

$$1 \to Z(\hat{G}) \to L^{-1} \otimes \mathbf{C}^\times \to L^0 \otimes \mathbf{C}^\times \to 1. \quad \blacksquare$$

1.3.4. Proof of Proposition 1.3.2. From Lemma 1.3.3 we obtain a quasi-isomorphism of complexes $(Z(\hat{G}) \to 1) \to L^\bullet \otimes \mathbf{C}^\times$. Hence

$$\mathbb{H}^0(\mathfrak{g}, L^\bullet \otimes \mathbf{C}^\times) = \mathbb{H}^0(\mathfrak{g}, Z(\hat{G}) \to 1) = H^1(\mathfrak{g}, Z(\hat{G})). \tag{1.3.4.1}$$

On the other hand, the short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{C} \to \mathbf{C}^\times \to 1$$

induces a short exact sequence of complexes

$$0 \to L^\bullet \to L^\bullet \otimes \mathbf{C} \to L^\bullet \otimes \mathbf{C}^\times \to 1$$

and a hypercohomology exact sequence

$$\mathbb{H}^0(\mathfrak{g}, L^\bullet \otimes \mathbf{C}) \to \mathbb{H}^0(\mathfrak{g}, L^\bullet \otimes \mathbf{C}^\times) \to \mathbb{H}^1(\mathfrak{g}, L^\bullet) \to \mathbb{H}^1(\mathfrak{g}, L^\bullet \otimes \mathbf{C}). \tag{1.3.4.2}$$

We prove that $\mathbb{H}^0(\mathfrak{g}, L^\bullet \otimes \mathbf{C}) = 0$ and $\mathbb{H}^1(\mathfrak{g}, L^\bullet \otimes \mathbf{C}) = 0$. Let $T^{\mathrm{ss}} = T \cap G^{\mathrm{ss}}$. Then $\mathbf{X}^*(T^{\mathrm{ss}})$ is a subgroup of finite index of $\mathbf{X}^*(T^{\mathrm{sc}})$, and so $\mathbf{X}^*(T^{\mathrm{sc}}) \otimes \mathbf{C} = \mathbf{X}^*(T^{\mathrm{ss}}) \otimes \mathbf{C}$. We see that

$$L^\bullet \otimes \mathbf{C} = (\mathbf{X}^*(T) \otimes \mathbf{C} \to \mathbf{X}^*(T^{\mathrm{ss}}) \otimes \mathbf{C}).$$

It follows that $L^\bullet \otimes \mathbf{C}$ is quasi-isomorphic to $(\mathbf{X}^*(G^{\mathrm{tor}}) \otimes \mathbf{C} \to 0)$. Hence

$$\mathbb{H}^0(\mathfrak{g}, L^\bullet \otimes \mathbf{C}) = \mathbb{H}^0(\mathfrak{g}, (\mathbf{X}^*(G^{\mathrm{tor}}) \otimes \mathbf{C} \to 0)) = H^1(\mathfrak{g}, \mathbf{X}^*(G^{\mathrm{tor}}) \otimes \mathbf{C}).$$

But $H^1(\mathfrak{g}, \mathbf{X}^*(G^{\mathrm{tor}}) \otimes \mathbf{C}) = 0$ because $\mathbf{X}^*(G^{\mathrm{tor}}) \otimes \mathbf{C}$ is a uniquely divisible group. Thus $\mathbb{H}^0(\mathfrak{g}, L^\bullet \otimes \mathbf{C}) = 0$. Similarly $\mathbb{H}^1(\mathfrak{g}, L^\bullet \otimes \mathbf{C}) = 0$. From the exact sequence (1.3.4.2) we then obtain

$$\mathbb{H}^0(\mathfrak{g}, L^\bullet \otimes \mathbf{C}^\times) = \mathbb{H}^1(\mathfrak{g}, L^\bullet).$$

We see from (1.3.4.1) that

$$H^1(\mathfrak{g}, Z(\hat{G})) = \mathbb{H}^1(\mathfrak{g}, L^\bullet).$$

Similarly, $H^1(\gamma, Z(\hat{G})) = \mathbb{H}^1(\gamma, L^\bullet)$ for every closed subgroup $\gamma \subset \mathfrak{g}$. We conclude that

$$Ш^1_\omega(\mathfrak{g}, L^\bullet) = Ш^1_\omega(\mathfrak{g}, Z(\hat{G})). \quad \blacksquare$$

1.3.5. Remark. Proposition 1.3.2 shows that $Ш^1_\omega(\mathfrak{g}, L^\bullet)$ does not depend on the choice of T. Indeed, it only depends on the algebraic fundamental group $\pi_1(G)$ which does not depend on T ([B, 1.2]).

2. MAIN RESULTS

THEOREM 2.1. *Let k be a field of characteristic zero, $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$, G a connected linear algebraic k-group, V a smooth k-compactification of G, $T \subset G$ a maximal k-torus, $L^\bullet = (\mathbf{X}^*(T) \to \mathbf{X}^*(T^{\mathrm{sc}}))$. Then the group $H^1(k, \mathrm{Pic}\,\overline{V})$ is canonically isomorphic to $Ш^1_\omega(\mathfrak{g}, L^\bullet)$, and this isomorphism is functorial in G.*

COROLLARY 2.2. *With the notation of Theorem 2.1, there is an injection*

$$\mathrm{Br}\,V/\,\mathrm{Br}\,k \hookrightarrow Ш^1_\omega(\mathfrak{g}, L^\bullet)$$

which is an isomorphism provided that either $V(k) \neq \emptyset$, or $H^3(k, \bar{k}^\times) = 0$.

Proof. By 1.1.1 G is rational, hence V is projective and rational, and $\mathrm{Br}\,\overline{V} = 0$; see 1.2.3. The corollary now follows from 1.2.3 and Theorem 2.1. ∎

2.3. Proof of Theorem 2.1. We first assume that G is reductive and G^{ss} is simply connected. We use Voskresenskiĭ's exact sequence

$$0 \to \mathbf{X}^*(G) \to P \to \mathrm{Pic}\,\overline{V} \to \mathrm{Pic}\,\overline{G} \to 0 \tag{2.3.1}$$

which is valid for any connected k-group G (cf. [V], see also [S, 9.0.0]). Here P is a permutation $\mathfrak{g}$-module. The exact sequence of $\bar{k}$-groups

$$1 \to \overline{G}^{\mathrm{ss}} \to \overline{G} \to \overline{G}^{\mathrm{tor}} \to 1$$

induces the exact sequence ([S, (6.11.4)])

$$0 \to \mathbf{X}^*(G^{\mathrm{tor}}) \to \mathbf{X}^*(G) \to \mathbf{X}^*(G^{\mathrm{ss}}) \to \mathrm{Pic}\,\overline{G}^{\mathrm{tor}} \to \mathrm{Pic}\,\overline{G} \to \mathrm{Pic}\,\overline{G}^{\mathrm{ss}} \to 0.$$

Since $\overline{G}^{\mathrm{ss}}$ is simply connected, we have $\mathrm{Pic}\,\overline{G}^{\mathrm{ss}} = 0$; since G^{ss} is semisimple, we have $\mathbf{X}^*(G^{\mathrm{ss}}) = 0$; since G^{tor} is a torus, we have $\mathrm{Pic}\,\overline{G}^{\mathrm{tor}} = 0$, cf. [S, 6.9]. We conclude that $\mathrm{Pic}\,\overline{G} = 0$ and $\mathbf{X}^*(G) = \mathbf{X}^*(G^{\mathrm{tor}})$. The exact sequence (2.3.1) is thus reduced to

$$0 \to \mathbf{X}^*(G^{\mathrm{tor}}) \to P \to \mathrm{Pic}\,\overline{V} \to 0. \tag{2.3.2}$$

We can now use the following fundamental property of the Picard group $\mathrm{Pic}\,\overline{V}$ of a smooth compactification of a principal homogeneous space of a connected linear group proved in [CTK, Proposition 3.2]: $H^1(\gamma, \mathrm{Pic}\,\overline{V}) = 0$

for all closed procyclic subgroups $\gamma \subset \mathfrak{g}$. From the exact sequence (2.3.2) we obtain a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & Ш^1_\omega(\mathfrak{g}, \operatorname{Pic} \overline{V}) & \longrightarrow & Ш^2_\omega(\mathfrak{g}, \mathbf{X}^*(G^{\mathrm{tor}})) & \longrightarrow & Ш^2_\omega(\mathfrak{g}, P) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1(\mathfrak{g}, \operatorname{Pic} \overline{V}) & \longrightarrow & H^2(\mathfrak{g}, \mathbf{X}^*(G^{\mathrm{tor}})) & \longrightarrow & H^2(\mathfrak{g}, P) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_\gamma H^1(\gamma, \operatorname{Pic} \overline{V}) & \longrightarrow & \prod_\gamma H^2(\gamma, \mathbf{X}^*(G^{\mathrm{tor}})) & \longrightarrow & \prod_\gamma H^2(\gamma, P)
\end{array}$$

in which the middle and bottom rows are exact. The term $Ш^2_\omega(\mathfrak{g}, P)$ is zero because P is a permutation $\mathfrak{g}$-module. By diagram chasing one can easily prove that the top row is also exact. We thus obtain

$$H^1(\mathfrak{g}, \operatorname{Pic} \overline{V}) = Ш^2_\omega(\mathfrak{g}, \mathbf{X}^*(G^{\mathrm{tor}})) \tag{2.3.3}$$

(see also [CTS2, Proposition 9.5(ii)]).

We now prove that $Ш^2_\omega(\mathfrak{g}, \mathbf{X}^*(G^{\mathrm{tor}})) \cong Ш^1_\omega(\mathfrak{g}, L^\bullet)$. Since G^{ss} is simply connected, we have an exact sequence of tori

$$1 \to T^{\mathrm{sc}} \to T \to G^{\mathrm{tor}} \to 1$$

and the dual exact sequence of character groups

$$0 \to \mathbf{X}^*(G^{\mathrm{tor}}) \to \mathbf{X}^*(T) \to \mathbf{X}^*(T^{\mathrm{sc}}) \to 0.$$

It induces a morphism of complexes

$$(\mathbf{X}^*(G^{\mathrm{tor}}) \to 0) \to L^\bullet$$

which is a quasi-isomorphism. Thus

$$H^{i+1}(\gamma, \mathbf{X}^*(G^{\mathrm{tor}})) = \mathbb{H}^i(\gamma, L^\bullet)$$

for every natural i and every closed subgroup $\gamma \subseteq \mathfrak{g}$. We conclude that

$$Ш^2_\omega(\mathfrak{g}, \mathbf{X}^*(G^{\mathrm{tor}})) \cong Ш^1_\omega(\mathfrak{g}, L^\bullet).$$

Thus $H^1(\mathfrak{g}, \operatorname{Pic} \overline{V}) \cong Ш^1_\omega(\mathfrak{g}, L^\bullet)$, and this isomorphism is functorial in G. The theorem is proved for reductive G with G^{ss} simply connected.

Let now G be an arbitrary connected reductive k-group. Let $H \xrightarrow{\alpha} G$ be a z-extension with kernel Z. Let V_G be a a smooth compactification of G and let V_H be a smooth compactification of H. We have a homomorphism

$$\alpha_*\colon H^1(k, \operatorname{Pic} \overline{V}_H) \to H^1(k, \operatorname{Pic} \overline{V}_G).$$

We prove that α_* is an isomorphism, hence

$$H^1(k, \operatorname{Pic} \overline{V}_H) \cong H^1(k, \operatorname{Pic} \overline{V}_G). \tag{2.3.4}$$

Since Z is a quasi-trivial torus, the map α admits a rational k-section $s\colon G \to H$. Indeed, the obstruction to the existence of such a section lies in $H^1(k(G), Z) = 0$. The rational section s gives rise to a biregular k-isomorphism $i\colon U_H \to U_G \times Z$, where $U_H \subset H$ and $U_G \subset G$ are open k-subvarieties, U_G is an open subvariety on which s is defined, and $U_H = s(U_G) \cdot Z$. The projections are defined as follows: $\mathrm{pr}_{U_G}(h) = \alpha(h)$, $\mathrm{pr}_Z(h) = h \cdot s(\alpha(h))^{-1}$, where $h \in U_H$. Since Z is a quasi-trivial torus, it is k-rational, and by 1.2.2 we obtain a canonical isomorphism $H^1(k, \operatorname{Pic} \overline{V}(U_H)) \xrightarrow{\sim} H^1(k, \operatorname{Pic} \overline{V}(U_G))$ induced by the projection $\mathrm{pr}_{U_G}\colon U_H \to U_G$. This gives us (2.3.4), and we see that (2.3.4) is the canonical isomorphism induced by $\alpha\colon H \to G$. It does not depend on s.

Let $T_G \subset G$ be a maximal torus and set $L_G^\bullet = (\mathbf{X}^*(T_G) \to \mathbf{X}^*(T_G^{\mathrm{sc}}))$. Set $T_H = \alpha^{-1}(T_G) \subset H$, $L_H^\bullet = (\mathbf{X}^*(T_H) \to \mathbf{X}^*(T_H^{\mathrm{sc}}))$. We prove that

$$\text{Ш}^1_\omega(k, L_G^\bullet) \cong \text{Ш}^1_\omega(k, L_H^\bullet). \tag{2.3.5}$$

We note that the z-extension $H \to G$ induces an exact sequence of complexes

$$1 \to L_G^\bullet \to L_H^\bullet \to (\mathbf{X}^*(Z) \to 1) \to 1$$

which leads to the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \| \\
0 = \text{Ш}^1_\omega(\mathfrak{g}, \mathbf{X}^*(Z)) & \longrightarrow & \text{Ш}^1_\omega(\mathfrak{g}, L_G^\bullet) & \longrightarrow & \text{Ш}^1_\omega(\mathfrak{g}, L_H^\bullet) & \longrightarrow & \text{Ш}^2_\omega(\mathfrak{g}, \mathbf{X}^*(Z)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 = H^1(\mathfrak{g}, \mathbf{X}^*(Z)) & \longrightarrow & \mathbb{H}^1(\mathfrak{g}, L_G^\bullet) & \longrightarrow & \mathbb{H}^1(\mathfrak{g}, L_H^\bullet) & \longrightarrow & H^2(\mathfrak{g}, \mathbf{X}^*(Z)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 = \prod_\gamma H^1(\gamma, \mathbf{X}^*(Z)) & \longrightarrow & \prod_\gamma \mathbb{H}^1(\gamma, L_G^\bullet) & \longrightarrow & \prod_\gamma \mathbb{H}^1(\gamma, L_H^\bullet) & \longrightarrow & \prod_\gamma H^2(\gamma, \mathbf{X}^*(Z))
\end{array}$$

In this diagram, the middle and bottom rows are exact, and the terms in the left column are all zero since $\mathbf{X}^*(Z)$ is a permutation $\mathfrak{g}$-module. The group $\text{Ш}^2_\omega(\mathfrak{g}, \mathbf{X}^*(Z))$ is zero for the same reason (see 1.2.1). By diagram chasing one can easily prove that the top row is exact. We have proved isomorphism (2.3.5).

We have already proved that $H^1(k, \operatorname{Pic} \overline{V}_H) \cong \text{Ш}^1_\omega(k, L_H^\bullet)$, because H^{ss} is simply connected. Together with the isomorphisms (2.3.4) and (2.3.5) we obtain an isomorphism

$$H^1(k, \operatorname{Pic} \overline{V}_G) \cong \text{Ш}^1_\omega(k, L_G^\bullet). \tag{2.3.6}$$

Using Lemma 1.1.4(2), one can easily prove that isomorphism (2.3.6) does not depend on the choice of a z-extension $H \to G$. Using Lemma 1.1.4(3), one can easily check that isomorphism (2.3.6) is functorial in G. This establishes the theorem for reductive k-groups.

Let now G be an arbitrary connected k-group. Let V_G be a smooth compactification of G and let $V_{G^{\text{red}}}$ be a smooth compactification of G^{red}. By 1.1.2, there is an isomorphism of k-varieties $G \cong G^{\text{red}} \times G^{\text{u}}$ where G^{u} is k-rational; by 1.2.2, we then obtain

$$H^1(\mathfrak{g}, \operatorname{Pic} \overline{V}_G) \cong H^1(\mathfrak{g}, \operatorname{Pic} \overline{V}_{G^{\text{red}}}). \tag{2.3.7}$$

Let the complex $L^\bullet$ be defined as in the Introduction, i.e., in terms of G^{red}. Since G^{red} is reductive, we have already proved that $H^1(\mathfrak{g}, \operatorname{Pic} \overline{V}_{G^{\text{red}}}) \cong$ Ш$^1_\omega(\mathfrak{g}, L^\bullet)$. Together with (2.3.7) we obtain

$$H^1(\mathfrak{g}, \operatorname{Pic} \overline{V}_G) \cong \text{Ш}^1_\omega(\mathfrak{g}, L^\bullet).$$

This isomorphism is functorial in G. ∎

THEOREM 2.4. *Let k be a field of characteristic zero*, $\mathfrak{g} = \operatorname{Gal}(\bar{k}/k)$, *$G$ a connected linear algebraic k-group*, *E a G-torsor*, *V a smooth compactification of E*, *$T \subset G$ a maximal k-torus*, $L^\bullet = (\mathbf{X}^*(T) \to \mathbf{X}^*(T^{\text{sc}}))$. *Then the group $H^1(k, \operatorname{Pic} \overline{V})$ is isomorphic to* Ш$^1_\omega(\mathfrak{g}, L^\bullet)$.

2.5. Proof of Theorem 2.4. First assume that G is reductive and G^{ss} is simply connected. If $E = G$, Theorem 2.4 coincides with Theorem 2.1. If E has no rational points, one just has to make use of the device of passage to the generic point (see [CTS3, Appendix 2B, pp. 462–463]; cf. [CTK, proof of Theorem 4.1]). We reproduce here this argument adapted to our setting. Let $K = k(E)$, $L = \bar{k}(E)$, and let M be an algebraic closure of K containing L. We have $\operatorname{Gal}(L/K) = \operatorname{Gal}(\bar{k}/k) = \mathfrak{g}$. Let $\mathfrak{g}_1 = \operatorname{Gal}(M/K)$, $\mathfrak{h} = \operatorname{Gal}(M/L)$: $\mathfrak{h}$ is a normal subgroup of $\mathfrak{g}_1$, and $\mathfrak{g} = \mathfrak{g}_1/\mathfrak{h}$. Since $\overline{V}$ is a proper smooth rational variety, the natural inclusions of free abelian groups of finite rank

$$\operatorname{Pic}(V \times_k \bar{k}) \hookrightarrow \operatorname{Pic}(V \times_k L) \hookrightarrow \operatorname{Pic}(V \times_k M)$$

are in fact equalities. Denote this abelian group by $\operatorname{Pic} \overline{V}$. The group $\mathfrak{h}$ acts trivially on $\operatorname{Pic}(V \times_k M)$. We write down the restriction–inflation exact sequence for the extensions $M/L/K$:

$$0 \to H^1(\mathfrak{g}, \operatorname{Pic} \overline{V}) \overset{\text{inf}}{\to} H^1(\mathfrak{g}_1, \operatorname{Pic} \overline{V}) \overset{\text{res}}{\to} H^1(\mathfrak{h}, \operatorname{Pic} \overline{V}) = 0.$$

This gives an isomorphism

$$H^1(\operatorname{Gal}(\bar{k}/k), \operatorname{Pic}(V \times_k \bar{k})) \cong H^1(\operatorname{Gal}(M/K), \operatorname{Pic}(V \times_k M)). \tag{2.5.1}$$

Recall that G is reductive and G^{ss} is simply connected. The K-variety $V \times_k K$ is a smooth compactification of the torsor $E \times_k K$ which has a K-point and is hence isomorphic to $G \times_k K$. Formula (2.3.3) and isomorphism (2.5.1) then show that there is an isomorphism

$$H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{Pic}(V \times_k \bar{k})) \cong \text{Ш}^2_\omega(\mathrm{Gal}(M/K), \mathbf{X}^*(G^{\mathrm{tor}})). \tag{2.5.2}$$

Since $\mathfrak{h} = \mathrm{Gal}(M/L)$ acts trivially on $\mathbf{X}^*(G^{\mathrm{tor}})$, and $\mathbf{X}^*(G^{\mathrm{tor}})$ is a torsion–free module, we can use the equality $H^1(\mathfrak{h}, \mathbf{X}^*(G^{\mathrm{tor}})) = 0$ and write down the restriction–inflation exact sequence for H^2

$$0 \to H^2(\mathfrak{g}, \mathbf{X}^*(G^{\mathrm{tor}})) \overset{\mathrm{inf}}{\to} H^2(\mathfrak{g}_1, \mathbf{X}^*(G^{\mathrm{tor}})) \overset{\mathrm{res}}{\to} H^2(\mathfrak{h}, \mathbf{X}^*(G^{\mathrm{tor}})).$$

Since $\text{Ш}^2_\omega(\mathfrak{h}, \mathbf{X}^*(G^{\mathrm{tor}})) = 0$, we obtain

$$\text{Ш}^2_\omega(\mathrm{Gal}(\bar{k}/k), \mathbf{X}^*(G^{\mathrm{tor}})) \cong \text{Ш}^2_\omega(\mathrm{Gal}(M/K), \mathbf{X}^*(G^{\mathrm{tor}})). \tag{2.5.3}$$

Putting together isomorphisms (2.5.2) and (2.5.3), we obtain $H^1(\mathfrak{g}, \mathrm{Pic}\, \overline{V}) \cong \text{Ш}^2_\omega(\mathfrak{g}, \mathbf{X}^*(G^{\mathrm{tor}}))$. Since G^{ss} is simply connected, we have $\text{Ш}^2_\omega(\mathfrak{g}, \mathbf{X}^*(G^{\mathrm{tor}})) \cong \text{Ш}^1_\omega(\mathfrak{g}, L^\bullet)$. Thus $H^1(\mathfrak{g}, \mathrm{Pic}\, \overline{V}) \cong \text{Ш}^1_\omega(\mathfrak{g}, L^\bullet)$, and we obtain the theorem for G reductive with G^{ss} simply connected.

Let now G be an arbitrary reductive group and E_G a principal homogeneous space of G. By Lemma 1.1.4(1), there exists a z-extension α: $H \to G$ with kernel Z such that the class $\mathrm{Cl}(E_G) \in H^1(k, G)$ is the image of some $\mathrm{Cl}(E_H) \in H^1(k, H)$ where E_H is a principal homogeneous space of H. The cohomology map α_*: $H^1(k, H) \to H^1(k, G)$ is represented by the map $E_H \mapsto E_H/Z$. We may therefore assume that $E_G = E_H/Z$. The canonical projection $E_H \to E_G = E_H/Z$ admits a k-rational section, because Z is a quasi-trivial torus. Indeed, the obstruction to the existence of such a section lies in $H^1(k(E_G), Z)$, and this cohomology group is zero by Hilbert 90. Therefore we have a birational isomorphism f: $E_G \times Z \to E_H$ given by $f(x, z) = s(x) \cdot z$ where $x \in E_G$ and $z \in Z$. The quasi-trivial torus Z is k-rational. By 1.2.2 the birational isomorphism f gives an isomorphism

$$H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{Pic}(V_G \times_k \bar{k})) \cong H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{Pic}(V_H \times_k \bar{k})),$$

where V_G (resp. V_H) stands for a smooth compactification of E_G (resp. E_H). Since H^{ss} is simply connected, by the preceding part of the proof we have

$$H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{Pic}(V_H \times_k \bar{k})) \cong \text{Ш}^2_\omega(\mathrm{Gal}(\bar{k}/k), \mathbf{X}^*(H^{\mathrm{tor}})).$$

As shown in the proof of Theorem 2.1, $\text{Ш}^2_\omega(\mathrm{Gal}(\bar{k}/k), \mathbf{X}^*(H^{\mathrm{tor}}))$ is isomorphic to $\text{Ш}^1_\omega(\mathfrak{g}, L^\bullet)$. Thus $H^1(\mathfrak{g}, \mathrm{Pic}\, V_G) \cong \text{Ш}^1_\omega(\mathfrak{g}, L^\bullet)$. This proves the theorem for any reductive group.

Let now G be an arbitrary (not necessarily reductive) connected k-group. The canonical homomorphism r: $G \to G^{\mathrm{red}}$ induces a bijection of Galois cohomology pointed sets r_*: $H^1(k, G) \to H^1(k, G^{\mathrm{red}})$, cf. [S, 1.13]. The map r_* is represented by the map of torsors $E \mapsto E/G^{\mathrm{u}}$, where E is a torsor under G and E/G^{u} is a torsor under $G/G^{\mathrm{u}} = G^{\mathrm{red}}$. We wish to prove that

$$H^1(k, \operatorname{Pic} \overline{V}_E) \cong H^1(k, \operatorname{Pic} \overline{V}_{E/G^{\mathrm{u}}}),$$

where V_E and $V_{E/G^{\mathrm{u}}}$ are smooth compactifications of E and E/G^{u}, respectively. Since our functor $\text{Ш}^1_\omega(\mathfrak{g}, L^\bullet)$ is, by definition, the same for G and for G^{red}, this will prove the theorem.

We fix a Levi decomposition $G^{\mathrm{u}} \rtimes G^{\mathrm{red}} \xrightarrow{\sim} G$. It defines a natural homomorphism φ: $G^{\mathrm{red}} \to G$ and a map φ_*: $H^1(k, G^{\mathrm{red}}) \to H^1(k, G)$, inverse to r_*. We want to describe φ_* in terms of torsors.

Let X be a torsor under G^{red}. Set $Y = X \times G^{\mathrm{u}}$. We define a right action of $G = G^{\mathrm{red}} \ltimes G^{\mathrm{u}}$ on Y by

$$(x, v) \cdot (g, u) = (x \cdot g, v^g \cdot u),$$

where $x \in X$, $v, u \in G^{\mathrm{u}}$, $g \in G^{\mathrm{red}}$, and v^g refers to the right action of G^{red} on G^{u} defined by the Levi decomposition. One can easily check that this is a well–defined action and that the map $X \mapsto Y$ represents φ_*.

Since $Y = X \times G^{\mathrm{u}}$ and G^{u} is k-rational, by 1.2.2 we have $H^1(k, \operatorname{Pic} \overline{V}(Y)) \cong H^1(k, \operatorname{Pic} \overline{V}(X))$. Since φ_* is inverse to r_*, any torsor E of G is isomorphic to Y for $X = E/G^{\mathrm{u}}$. We obtain $H^1(k, \operatorname{Pic} \overline{V}_E) \cong H^1(k, \operatorname{Pic} \overline{V}_{E/G^{\mathrm{u}}})$. This proves the theorem. ∎

3. COMMENTS AND REMARKS

3.1. If $G = T$ is a torus, we have $\text{Ш}^1_\omega(\mathfrak{g}, L^\bullet) = \text{Ш}^2_\omega(\mathfrak{g}, \mathbf{X}^*(T))$, and the formula of Theorem 2.1 reduces to

$$H^1(k, \operatorname{Pic} \overline{V}) \cong \text{Ш}^2_\omega(k, \mathbf{X}^*(T));$$

cf. [S, Proposition. 9.8], in the number field case, [CTS2] for $E = T$ over an arbitrary field, and [CTK] in general.

3.2. If G is a semisimple group, we have $\text{Ш}^1_\omega(\mathfrak{g}, L^\bullet) = \text{Ш}^1_\omega(\mathfrak{g}, \mathbf{X}^*(B))$ where $B = \ker[G^{\mathrm{sc}} \to G]$ is the fundamental group of G, and the formula of Theorem 2.1 reduces to

$$H^1(k, \operatorname{Pic} \overline{V}) \cong \text{Ш}^1_\omega(k, \mathbf{X}^*(B)); \tag{3.2.1}$$

cf. [S, 9.6], in the number field case and [CTK] in general.

3.3. Let now G be a reductive group admitting a special covering μ: $G_0 \times S \to G$ with kernel B, where G_0 is a simply connected group, S is a quasi-trivial torus, and B is a finite group. We show that in this case Theorem 2.1 reduces to formula (3.2.1), cf. [S, Proposition 9.8], in the number field case and [CTK] in general. Recall that T is a maximal torus of G, $L^\bullet = (\mathbf{X}^*(T) \to \mathbf{X}^*(T^{\mathrm{sc}}))$. We have to prove that

$$\text{Ш}^1_\omega(k, \mathbf{X}^*(B)) = \text{Ш}^1_\omega(\mathfrak{g}, L^\bullet).$$

Write $\mu^{-1}(T) = T_0 \times S$, where T_0 is a maximal torus of G_0. We have $G^{\mathrm{sc}} = G_0$, $T^{\mathrm{sc}} = T_0$. Set $L_1^\bullet = (\mathbf{X}^*(T) \to \mathbf{X}^*(T_0) \times \mathbf{X}^*(S))$. Consider an exact sequence

$$1 \to B \to T_0 \times S \to T \to 1.$$

We see that the complexes $(B \to 1)$ and $(T_0 \times S \to T)$ are quasi-isomorphic. Hence the complexes $(0 \to \mathbf{X}^*(B))$ and $L_1^\bullet$ are also quasi-isomorphic, and therefore

$$\text{Ш}^1_\omega(k, \mathbf{X}^*(B)) = \text{Ш}^1_\omega(\mathfrak{g}, L_1^\bullet).$$

We now consider a short exact sequence of complexes

$$0 \to (0 \to \mathbf{X}^*(S)) \to L_1^\bullet \to L^\bullet \to 0.$$

It induces the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ш}^1_\omega(\mathfrak{g}, L_1^\bullet) & \longrightarrow & \text{Ш}^1_\omega(\mathfrak{g}, L^\bullet) & \longrightarrow & \text{Ш}^2_\omega(\mathfrak{g}, \mathbf{X}^*(S)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 = H^1(\mathfrak{g}, \mathbf{X}^*(S)) & \longrightarrow & \mathbb{H}^1(\mathfrak{g}, L_1^\bullet) & \longrightarrow & \mathbb{H}^1(\mathfrak{g}, L^\bullet) & \longrightarrow & H^2(\mathfrak{g}, \mathbf{X}^*(S)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 = \prod_\gamma H^1(\gamma, \mathbf{X}^*(S)) & \longrightarrow & \prod_\gamma \mathbb{H}^1(\gamma, L_1^\bullet) & \longrightarrow & \prod_\gamma H^1(\gamma, L^\bullet) & \longrightarrow & \prod_\gamma H^2(\gamma, \mathbf{X}^*(S))
\end{array}$$

The middle and bottom rows are exact. Since S is a quasi-trivial torus, we have $\text{Ш}^2_\omega(\mathfrak{g}, \mathbf{X}^*(S)) = 0$. By diagram chasing, one can show that the top row of the diagram is also exact. Thus

$$\text{Ш}^1_\omega(k, \mathbf{X}^*(B)) = \text{Ш}^1_\omega(\mathfrak{g}, L_1^\bullet) = \text{Ш}^1_\omega(\mathfrak{g}, L^\bullet),$$

as required. ∎

3.4. EXAMPLE. Here we consider an example of a reductive k-group G which is not a torus and does not admit a special covering as in 3.3. Let V be a smooth compactification of G. We use Corollary B to compute $\operatorname{Br} V/\operatorname{Br} k$.

We take $k = \mathbf{Q}$. Let $L = \mathbf{Q}(\sqrt{a}, \sqrt{b})$ be a biquadratic extension of $\mathbf{Q}$, it is a Galois extension with group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Consider the composite homomorphism

$$\varphi\colon R_{L/\mathbf{Q}}\,\mathrm{GL}_{2,L} \xrightarrow{R_{L/\mathbf{Q}}\det} R_{L/\mathbf{Q}}\mathbb{G}_{m,L} \xrightarrow{N_{L/\mathbf{Q}}} \mathbb{G}_{m,\mathbf{Q}},$$

where det denotes the determinant homomorphism and $N_{L/\mathbf{Q}}$ denotes the norm homomorphism. Set $G_1 = \ker\varphi$. Let $\mu_{2,L}$ denote the center of $\mathrm{SL}_{2,L}$. We set $G = G_1/R_{L/\mathbf{Q}}\mu_{2,L}$.

We have $G^{\mathrm{sc}} = R_{L/\mathbf{Q}}\,\mathrm{SL}_{2,L}$, $G^{\mathrm{ss}} = R_{L/\mathbf{Q}}\,\mathrm{PSL}_{2,L}$, $G^{\mathrm{tor}} = R^1_{L/\mathbf{Q}}\mathbb{G}_{m,L}$, where by definition

$$R^1_{L/\mathbf{Q}}\mathbb{G}_{m,L} = \ker[N_{L/\mathbf{Q}}\colon R_{L/\mathbf{Q}}\mathbb{G}_{m,L} \to \mathbb{G}_{m,\mathbf{Q}}].$$

Notice that G^{tor} is an anisotropic $\mathbf{Q}$-torus (i.e., G^{tor} has no non–trivial character defined over $\mathbf{Q}$). It follows that G does not admit special covering as in 3.3, and one thus cannot compute $\mathrm{Br}\,V/\,\mathrm{Br}\,k$ with the help of formulas given in [CTK].

We choose a maximal torus $T \subset G$ such that $T^{\mathrm{sc}} \cong R_{L/\mathbf{Q}}\mathbb{G}_{m,L}$. Set $T^{\mathrm{ss}} = T \cap G^{\mathrm{ss}}$. Then

$$T^{\mathrm{ss}} \cong T^{\mathrm{sc}}/R_{L/\mathbf{Q}}\mu_{2,L} \cong R_{L/\mathbf{Q}}(\mathbb{G}_{m,L}/\mu_{2,L}) \cong R_{L/\mathbf{Q}}\mathbb{G}_{m,L}.$$

We have $H^3(\mathbf{Q}, \mathbb{G}_m) = 0$. By Corollary B, we have $\mathrm{Br}\,V/\,\mathrm{Br}\,k \cong Ш^1_\omega(\mathfrak{g}, L^\bullet)$ where $L^\bullet = (L^{-1} \to L^0)$, $L^{-1} = \mathbf{X}^*(T)$, $L^0 = \mathbf{X}^*(T^{\mathrm{sc}})$. Here we compute explicitly $Ш^1_\omega(\mathfrak{g}, L^\bullet)$. We use the fact that $\mathbf{X}^*(T^{\mathrm{sc}})$ and $\mathbf{X}^*(T^{\mathrm{ss}})$ are permutation modules because T^{sc} and T^{ss} are quasi-trivial tori.

From the exact sequence of complexes

$$0 \to (0 \to L^0) \to (L^{-1} \to L^0) \to (L^{-1} \to 0) \to 0$$

we obtain an exact sequence

$$H^1(\mathfrak{g}, L^0) \to \mathbb{H}^1(\mathfrak{g}, L^{-1} \to L^0) \to H^2(\mathfrak{g}, L^{-1}) \to H^2(\mathfrak{g}, L^0).$$

Since $L^0 = \mathbf{X}^*(T^{\mathrm{sc}})$ is a permutation module, by 1.2.1 we have $H^1(\mathfrak{g}, L^0) = 0$, $H^1(\gamma, L^0) = 0$ for all closed subgroups $\gamma \subset \mathfrak{g}$, $Ш^1_\omega(\mathfrak{g}, L^0) = 0$, $Ш^2_\omega(\mathfrak{g}, L^0) = 0$. We write a commutative diagram similar to those of 2.3 and 3.3. By diagram chasing we can prove that

$$Ш^2_\omega(\mathfrak{g}, L^{-1} \to L^0) = Ш^2_\omega(\mathfrak{g}, L^{-1}).$$

We write exact sequences

$$1 \to T^{\mathrm{ss}} \to T \to G^{\mathrm{tor}} \to 1,$$

$$0 \to \mathbf{X}^*(G^{\mathrm{tor}}) \to \mathbf{X}^*(T) \to \mathbf{X}^*(T^{\mathrm{ss}}) \to 0,$$

$$H^1(\mathfrak{g}, \mathbf{X}^*(T^{\mathrm{ss}})) \to H^2(\mathfrak{g}, \mathbf{X}^*(G^{\mathrm{tor}})) \to H^2(\mathfrak{g}, \mathbf{X}^*(T)) \to H^2(\mathfrak{g}, \mathbf{X}^*(T^{\mathrm{ss}})).$$

Since $\mathbf{X}^*(T^{ss})$ is a permutation module, by 1.2.1 we have $H^1(\mathfrak{g}, \mathbf{X}^*(T^{ss})) = 0$, $H^1(\gamma, \mathbf{X}^*(T^{ss})) = 0$ for all $\gamma \subset \mathfrak{g}$, $Ш^1_\omega(\mathfrak{g}, \mathbf{X}^*(T^{ss})) = 0$, $Ш^2_\omega(\mathfrak{g}, \mathbf{X}^*(T^{ss})) = 0$. Once again we write a commutative diagram similar to those of 2.3 and 3.3. By diagram chasing we can prove that

$$Ш^2_\omega(\mathfrak{g}, \mathbf{X}^*(G^{tor})) = Ш^2_\omega(\mathfrak{g}, \mathbf{X}^*(T)).$$

Recall that since G^{tor} is a torus, we have

$$Ш^2_\omega(\mathfrak{g}, \mathbf{X}^*(G^{tor})) \cong H^1(\mathbf{Q}, \mathrm{Pic}\ \overline{W}),$$

where W is a smooth compactification of G^{tor} (cf. 3.1). Now $G^{tor} = R^1_{L/\mathbf{Q}}\mathbb{G}_{m,L}$ where $L/\mathbf{Q}$ is a biquadratic extension. Since $H^1(\mathbf{Q}, \mathrm{Pic}\ \overline{W}) \cong \mathbf{Z}/2\mathbf{Z}$ (see, for example, [V, Corollary of Theorem 7]), we have

$$Ш^1_\omega(\mathfrak{g}, L^{-1} \to L^0) = Ш^2_\omega(\mathfrak{g}, L^{-1}) = Ш^2_\omega(\mathfrak{g}, \mathbf{X}^*(G^{tor})) = \mathbf{Z}/2\mathbf{Z}.$$

By Corollary B, we conclude that $\mathrm{Br}\ V/\mathrm{Br}\ k = \mathbf{Z}/2\mathbf{Z}$.

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