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When are dynamic and static feedback equivalent?

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Abstract

An open question in Control Theory over commutative rings is: When does dynamic feedback equivalence imply (static) feedback equivalence? A second open problem is: Given a linear system $\Sigma = (A, B)$, when does there exist a matrix F such that $A + BF$ is invertible? In this paper we solve both questions, obtaining two characterizations of stable rings.

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1. Introduction

Throughout this paper R denotes a commutative ring with identity element and $\Sigma = (A, B)$ is an m-input, n-dimensional linear system over R (i.e. $A = (a_{ij})$ an $n \times n$ matrix and $B = (b_{ij})$ an $n \times m$ matrix with entries in R). Two linear

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systems $\Sigma = (A, B)$ and $\Sigma' = (A', B')$ are (static) feedback equivalent if there exist invertible matrices P and Q, and a feedback matrix F such that $B' = PBQ$ and $PA - A'P = B'F$.

The (static) feedback classification of linear dynamical systems over commutative rings is a classical and very difficult problem in Control Theory. The classification of matrices by similarity and the classification of matrices by equivalence are two particular cases of this problem, which has not been solved in general. In [4,6,7] and [9, Section 4] can be found solutions for certain special cases of the feedback classification problem. In [3] it is proved that, for linear systems over a large class of commutative rings (including the ring of integers), it is unlikely to obtain a complete system of invariants and a canonical form for the feedback class of a linear system, because the problem is "wild".

The objective of the feedback relation is to obtain a matrix F such that $P(A + BF)P^{-1}$ has some desired property. In some cases the difficulty of the static feedback classification is eluded by means of a technique called dynamic feedback, see [12, Section 6.2] for the classical case and [1, Section 3.4] and [2] for the case of linear systems over a commutative ring.

Two m-input, n-dimensional linear systems $\Sigma = (A, B)$ and $\Sigma' = (A', B')$ are dynamically feedback equivalent if $\Sigma(r)$ is statically feedback equivalent to $\Sigma'(r)$ for some positive integer r, where

$$\Sigma(r) = \left(\left(\begin{array}{c|c} 0_{r\times r} & 0 \\ 0 & A \end{array} \right), \left(\begin{array}{c|c} \mathrm{Id}_{r\times r} & 0 \\ 0 & B \end{array} \right) \right).$$

In [10] it is proved that the dynamic feedback equivalence over principal ideal domains is reduced to case $r = 1$.

Static feedback implies dynamic feedback, but the converse is not true in general. The open question is: When are dynamic and static feedback equivalent?

In Section 3 we solve this problem, in fact we characterize the class of commuttative rings in which the dynamic feedback equivalence implies the static feedback equivalence. The key for the proof of this characterization is given, in Section 2, by a solution of the following open problem: When there exists a matrix F such that $A + BF$ becomes invertible? In [5] this problem is related with the pole assignability property and with the standarization of generalized linear systems.

2. Feedback invertible matrices

Definition 1. Let $\Sigma = (A, B)$ be an m-input, n-dimensional linear system over R. The matrix A is feedback invertible modulo B if there exists an $m \times n$ matrix F such that $A + BF$ is invertible.

Let $\mathscr{U}_n(A|B)$ be the ideal of R generated by all the $n \times n$ minors of the block $n \times (n + m)$ matrix $(A|B)$, see [9, Section 1] for properties of determinantal ideals.

If A is feedback invertible modulo B then $\mathscr{U}_n(A|B) = R$ but the converse is not true in general (take, by example, $R = \mathbb{Z}$, $A = (5)$ and $B = (7)$). In the main result of this section, Theorem 5, we characterize the class of commutative rings in which the converse is true.

Lemma 2. *Let A_1 be an $n \times n$ matrix and B_1 be an $n \times m$ matrix. Consider the matrices*

$$A' = \left(\begin{array}{c|c} u & 0^{\mathrm{t}} \\ \hline \underline{*} & A_1 \end{array}\right) \quad \text{and} \quad B' = \left(\begin{array}{c} 0^{\mathrm{t}} \\ \hline B_1 \end{array}\right),$$

where u is a unit of R and $\underline{*}$ is an arbitrary column vector. Then

(i) $\mathscr{U}_n(A_1|B_1) = \mathscr{U}_{n+1}(A'|B')$.
(ii) *There exists F_1 such that $A_1 + B_1F_1$ is invertible if and only if there exists F' such that $A' + B'F'$ is invertible.*

Proof. Statement (i) is clear. Statement (ii) follows from the equality

$$\det(A' + B'F') = u \det(A_1 + B_1F_1)$$

for every matrix F' of the form $F' = (\underline{f}|F_1)$. $\square$

Lemma 3. *Let A and A' be two $n \times n$ matrices, B and B' be two $n \times m$ matrices, F' an $m \times n$ matrix and S an $(n + m) \times (n + m)$ matrix of the form*

$$S = \left(\begin{array}{c|c} U_{11} & 0 \\ \hline U_{21} & U_{22} \end{array}\right) \left(\begin{array}{c|c} U'_{11} & U'_{12} \\ \hline 0 & U'_{22} \end{array}\right),$$

where U_{11} and U'_{11} are $n \times n$ matrices. Suppose that the following statements hold:

(i) $(A|B)S = (A'|B')$.
(ii) $T = U_{11}U'_{11} + U_{11}U'_{12}F'$ *is invertible.*
(iii) $A' + B'F'$ *is invertible.*

Then there exists F such that $A + BF$ is invertible.

Proof. By (i) one has $A' = AU_{11}U'_{11} + BU_{21}U'_{11}$ and $B' = AU_{11}U'_{12} + B(U_{21}U'_{12} + U_{22}U'_{22})$. The result follows considering the matrix

$$F = \left(U_{21}U'_{11} + \left(U_{21}U'_{12} + U_{22}U'_{22}\right) F'\right) T^{-1}. \qquad \square$$

Definition 4. A commutative ring R is stable (or R have one in its stable range) if whenever the ideal generated by a and b is R then there exists c such that $a + bc$ is a unit.

Note that if R is stable and $a, b_1, b_2, \ldots, b_m$ generate R then there exist $c_1, c_2, \ldots, c_m$ of R such that $a + b_1c_1 + \cdots + b_mc_m$ is a unit. Semilocal rings, von Newmann regular rings and zero-dimensional rings are examples of stable rings. See [8] and [11, p. 53] for properties of stable rings.

Theorem 5. *Let R be a commutative ring. The following statements are equivalent*:

(i) *R is stable.*
(ii) *For every $n \times n$ matrix A and $n \times m$ matrix B, A is feedback invertible modulo B if and only if $\mathscr{U}_n(A|B) = R$.*

Proof. Assume (ii) and suppose that the ideal generated by a and b is R. Put $A = (a)$ and $B = (b)$. Then, by (ii), there exists c such that $a + bc$ is a unit.

Assume (i), we shall deduce (ii) by induction on n. For $n = 1$ and m arbitrary the result is clear. Suppose $n > 1$ and consider two matrices

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}$$

such that

$$\mathscr{U}_n(A|B) = R.$$

Hence the ideal of R generated by all elements of the first row of $(A|B)$ is R. Since R is stable, there exist elements $\lambda_{21}, \ldots, \lambda_{n1}, \mu_{11}, \ldots, \mu_{m1}$ of R such that

$$a_{11} + \lambda_{21}a_{12} + \cdots + \lambda_{n1}a_{1n} + \mu_{11}b_{11} + \cdots + \mu_{m1}b_{1m} = u$$

is a unit. Consider the invertible $n + m$ matrices

$$U = \left(\begin{array}{c|ccc||ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline \lambda_{21} & & & & & & \\ \vdots & & \mathrm{Id}_{n-1} & & & 0 & \\ \lambda_{n1} & & & & & & \\ \hline\hline \mu_{11} & & & & & & \\ \vdots & & 0 & & & \mathrm{Id}_m & \\ \mu_{m1} & & & & & & \end{array}\right) = \left(\begin{array}{c||c} U_{11} & 0 \\ \hline U_{21} & U_{22} \end{array}\right)$$

(where the double line denotes the partition on blocks) and

$$U' = \left(\begin{array}{c|ccc||ccc} 1 & -a_{12}u^{-1} & \cdots & -a_{1n}u^{-1} & -b_{11}u^{-1} & \cdots & -b_{1m}u^{-1} \\ \hline 0 & & & & & & \\ \vdots & & \mathrm{Id}_{n-1} & & & 0 & \\ 0 & & & & & & \\ \hline\hline 0 & & & & & & \\ \vdots & & 0 & & & \mathrm{Id}_m & \\ 0 & & & & & & \end{array}\right)$$

$$= \left(\begin{array}{c||c} U'_{11} & U'_{12} \\ \hline 0 & U'_{22} \end{array}\right).$$

Put $S = UU'$. Then

$$(A|B)S = (A'|B') = \left(\begin{array}{c|c||c} u & \underline{0}^{\mathrm{t}} & \underline{0}^{\mathrm{t}} \\ \hline \underline{*} & A_1 & B_1 \end{array}\right),$$

where A_1 is an $(n-1) \times (n-1)$ matrix and B_1 is an $(n-1) \times m$ matrix.

Since u is unit of R and S is an invertible matrix it follows

$$\mathscr{U}_{n-1}(A_1|B_1) = \mathscr{U}_n(A'|B') = \mathscr{U}_n(A|B) = R.$$

By induction hypothesis there exists F_1 such that $A_1 + B_1F_1$ is invertible.

Put $F' = \left(\underline{0} \mid F_1\right)$. Then, see Lemma 2, $A' + B'F'$ is invertible. Moreover the matrix

$$U_{11}U'_{11} + U_{11}U'_{12}F' = U_{11}(U'_{11} + U'_{12}F')$$

is invertible because, in the one hand U_{11} is invertible and in the other hand

$$\begin{aligned} U'_{11} + U'_{12}F' &= \left(\begin{array}{c|ccc} 1 & -a_{12}u^{-1} & \cdots & -a_{1n}u^{-1} \\ \hline 0 & & & \\ \vdots & & \mathrm{Id}_{n-1} & \\ 0 & & & \end{array}\right) \\ &\quad + \left(\begin{array}{ccc} -b_{11}u^{-1} & \cdots & -b_{1m}u^{-1} \\ \hline & 0_{(n-1)\times m} & \end{array}\right)\left(\underline{0} \mid F_1\right) \\ &= \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathrm{Id}_{n-1} & \\ 0 & & & \end{array}\right) + \left(\begin{array}{c|ccc} 0 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{array}\right) = \mathrm{Id}_n + N \end{aligned}$$

is invertible because N is a nilpotent matrix.

Finally (ii) is consequence of Lemma 3. □

3. Static vs. dynamic feedback

Let $\Sigma = (A, B)$ and $\Sigma' = (A', B')$ be two m-input, n-dimensional linear system over R. Recall that $\Sigma = (A, B)$ is (static) feedback equivalent to $\Sigma' = (A', B')$ if there exist invertible matrices P and Q, and a feedback matrix F such that

$$\begin{cases} PA - A'P = B'F, \\ B' = PBQ. \end{cases}$$

For a non-negative integer r we denote by $\Sigma(r)$ the $(m + r)$-input, $(n + r)$-dimensional linear system given by

$$\Sigma(r) = \left(\left(\begin{array}{c|c} 0_{r\times r} & 0 \\ \hline 0 & A \end{array} \right), \left(\begin{array}{c|c} \mathrm{Id}_{r\times r} & 0 \\ \hline 0 & B \end{array} \right) \right).$$

Definition 6. Σ is dynamically feedback equivalent to Σ' if there exists r such that $\Sigma(r)$ is statically feedback equivalent to $\Sigma'(r)$.

Theorem 7. *Let R be a commutative ring. Then the following statement are equivalent:*

(i) *R is stable.*
(ii) *Two linear systems are dynamically feedback equivalent if and only if they are statically feedback equivalent.*

Proof. Assume (i). Let $\Sigma = (A, B)$ and $\Sigma' = (A', B')$ be two m-input, n-dimensional linear system over R and suppose that $\Sigma(r)$ is statically feedback equivalent to $\Sigma'(r)$ for some r.

First we prove that we can suppose $B = B'$. Since $\Sigma(r)$ is statically feedback equivalent to $\Sigma'(r)$ then $\widetilde{B} = \left(\begin{array}{c|c} \mathrm{Id}_{r\times r} & 0 \\ \hline 0 & B \end{array} \right)$ and $\widetilde{B'} = \left(\begin{array}{c|c} \mathrm{Id}_{r\times r} & 0 \\ \hline 0 & B' \end{array} \right)$ are equivalent matrices. Therefore one has the following chain of isomorphisms

$$\mathrm{Coker}(B) \simeq \mathrm{Coker}(\widetilde{B}) \simeq \mathrm{Coker}(\widetilde{B'}) \simeq \mathrm{Coker}(B').$$

By [11, p. 149] it follows that B and B' are equivalent matrices because R is stable. Let P and Q be invertible matrices such that $PB'Q = B$. Then $\Sigma'' = (PA'P^{-1}, B)$ is feedback equivalent to $\Sigma' = (A', B')$ and hence we can suppose that $B = B'$.

Assume that

$$\Sigma(r) = \left(\widetilde{A} = \left(\begin{array}{c|c} 0_{r\times r} & 0 \\ \hline 0 & A \end{array} \right), \ \widetilde{B} = \left(\begin{array}{c|c} \mathrm{Id}_{r\times r} & 0 \\ \hline 0 & B \end{array} \right) \right)$$

and

$$\Sigma'(r) = \left(\widetilde{A'} = \left(\begin{array}{c|c} 0_{r\times r} & 0 \\ \hline 0 & A' \end{array} \right), \ \widetilde{B} = \left(\begin{array}{c|c} \mathrm{Id}_{r\times r} & 0 \\ \hline 0 & B \end{array} \right) \right)$$

are feedback equivalent. Then there exist an invertible $(r+n) \times (r+n)$ matrix $\widetilde{P} = \left(\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array} \right)$, where P_{22} is an $n \times n$ matrix, an invertible $(r+m) \times (r+m)$ matrix $\widetilde{Q} = \left(\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array} \right)$, where Q_{22} is an $m \times m$ matrix, and a $(r+m) \times (r+n)$ matrix $\widetilde{F} = \left(\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array} \right)$, where F_{22} is an $m \times n$ matrix such that

$$\widetilde{P}\widetilde{A} - \widetilde{A'}\widetilde{P} = \widetilde{B}\widetilde{F}, \tag{1}$$
$$\widetilde{P}\widetilde{B} = \widetilde{B}\widetilde{Q}. \tag{2}$$

By Eq. (1) one has

$$-A'P_{21} = BF_{21}, \tag{3}$$
$$P_{22}A - A'P_{22} = BF_{22}, \tag{4}$$

and by Eq. (2) one has

$$P_{11} = Q_{11}, \tag{5}$$
$$P_{12}B = Q_{12}, \tag{6}$$
$$P_{21} = BQ_{21}, \tag{7}$$
$$P_{22}B = BQ_{22}. \tag{8}$$

If P_{22} and Q_{22} are invertible then, by Eqs. (4) and (8), Σ and Σ' are feedback equivalent. Next we prove that, without loss of generality, we may suppose that P_{22} is invertible.

Since P is invertible it follows that $\mathscr{U}_n(P_{22}|P_{21}) = R$. As R is stable then, by Theorem 5, there exists an $r \times n$ matrix N such that $P_{22} + P_{21}N$ is invertible. Consider the invertible matrices

$$\widetilde{U} = \left(\begin{array}{c|c} \mathrm{Id}_{r\times r} & N \\ \hline 0 & \mathrm{Id}_{n\times n} \end{array}\right) \quad \text{and} \quad \widetilde{V} = \left(\begin{array}{c|c} \mathrm{Id}_{r\times r} & NB \\ \hline 0 & \mathrm{Id}_{m\times m} \end{array}\right).$$

Then, applying the above equations, one has

$$\widetilde{P}\widetilde{U}\widetilde{A} - \widetilde{A}'\widetilde{P}\widetilde{U} = \widetilde{B}\widetilde{H}$$

for a suitable matrix $\widetilde{H}$, and

$$\widetilde{P}\widetilde{U}\widetilde{B} = \widetilde{B}\widetilde{Q}\widetilde{V}.$$

Therefore $\Sigma(r)$ and $\Sigma'(r)$ are feedback equivalent via the matrices $\widetilde{P}\widetilde{U}$, $\widetilde{Q}\widetilde{V}$ and $\widetilde{H}$ where

$$\widetilde{P}\widetilde{U} = \left(\begin{array}{c|c} P_{11} & P_{11}N + P_{12} \\ \hline P_{21} & P_{21}N + P_{22} \end{array}\right),$$

with $P_{21}N + P_{22}$ invertible.

Suppose that P_{22} is invertible. By the formula of Schur [11, p. 30], one has

$$\det \widetilde{P} = \det(P_{11} - P_{12}P_{22}^{-1}P_{21}) \cdot \det P_{22}$$

and hence $P_{11} - P_{12}P_{22}^{-1}P_{21}$ is invertible because $\widetilde{P}$ is invertible. Now consider the invertible matrix

$$\widetilde{T} = \left(\begin{array}{c|c} \mathrm{Id}_{r\times r} & -P_{12}P_{22}^{-1}B \\ \hline 0 & \mathrm{Id}_{m\times m} \end{array}\right).$$

Then

$$\widetilde{T}\widetilde{Q} = \left(\begin{array}{c|c} Q_{11} - P_{12}P_{22}^{-1}BQ_{21} & Q_{12} - P_{12}P_{22}^{-1}BQ_{22} \\ \hline Q_{21} & Q_{22} \end{array}\right)$$

where, by Eqs. (6) and (8),

$$Q_{12} - P_{12}P_{22}^{-1}BQ_{22} = P_{12}(B - P_{22}^{-1}BQ_{22}) = 0.$$

Since $\det \widetilde{T}\widetilde{Q} = \det(Q_{11} - P_{12}P_{22}^{-1}BQ_{21}).\det Q_{22}$ it follows that Q_{22} is also invertible and with this it is proved that Statement (ii) holds.

Assume (ii). Let a and b be elements of R such that $1 = \lambda a + \mu b$ with λ and μ elements of R. Considering the matrices

$$\widetilde{P} = \begin{pmatrix} a & 0 & -\mu \\ 0 & 1 & 0 \\ b & 0 & \lambda \end{pmatrix}, \quad \widetilde{Q} = \begin{pmatrix} \lambda & \mu b \\ -1 & a \end{pmatrix} \quad \text{and} \quad \widetilde{F} = \begin{pmatrix} 0 & -\mu a & 0 \\ 0 & -\mu & 0 \end{pmatrix}$$

it is easy to prove that

$$\Sigma = \left(A = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ b \end{pmatrix} \right)$$

is 1-dynamically feedback equivalent to

$$\Sigma' = \left(A' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ B' = \begin{pmatrix} 0 \\ b \end{pmatrix} \right).$$

By (ii), Σ and Σ' are statically feedback equivalent. Hence there exist an invertible matrix $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$, a unit u of R and a matrix $F = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$ such that

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix}$$

and

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} u.$$

Then $p_{12}a = p_{12}b = 0$ and $p_{22}a - p_{11} = bk_1$. Therefore $p_{12} = 0$, because a and b generate R, and hence p_{11} and p_{22} are units, because P is invertible. Consequently $a - bp_{22}^{-1}k_1 = p_{22}^{-1}p_{11}$ is a unit and the proof is completed. □

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