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On matrices which have signed null-spaces ☆

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Abstract

We denote by $\mathcal{Q}(A)$ the set of all matrices with the same sign pattern as A. A matrix A has *signed null-space* provided there exists a set $\mathcal{S}$ of sign patterns such that the set of sign patterns of vectors in the null-space of $\widetilde{A}$ is $\mathcal{S}$, for each $\widetilde{A} \in \mathcal{Q}(A)$. We show that if A is an m by n matrix with no duplicate columns up to multiplication by -1 and A has signed null-space, then $n \leqslant 3m - 2$. We also classify the set of matrices satisfying the equality.

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1. Introduction

The *sign* of a real number a is defined by

$$\operatorname{sign}(a) = \begin{cases} -1 & \text{if } a < 0, \\ 0 & \text{if } a = 0, \\ 1 & \text{if } a > 0. \end{cases}$$

A *sign pattern* is a $(0, 1, -1)$-matrix. The *sign pattern of a matrix A* is the matrix obtained from A by replacing each entry by its sign. We denote by $\mathcal{Q}(A)$ the set of

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all matrices with the same sign pattern as A. The *zero pattern* of a matrix A is the (0, 1)-matrix obtained from A by replacing each nonzero entry by 1.

Let A be an m by n matrix and b an m by 1 vector. The linear system $Ax = b$ has *signed solutions* provided there exists a collection $\mathscr{S}$ of n by 1 sign patterns such that the set of sign patterns of the solutions to $\widetilde{A}x = \tilde{b}$ is $\mathscr{S}$, for each $\widetilde{A} \in \mathscr{Q}(A)$ and $\tilde{b} \in \mathscr{Q}(b)$. This notion generalizes that of a sign-solvable linear system (see [1] and references therein). The linear system, $Ax = b$, is *sign-solvable* provided each linear system $\widetilde{A}x = \tilde{b}$ ($\widetilde{A} \in \mathscr{Q}(A)$, $\tilde{b} \in \mathscr{Q}(b)$) has a solution and all solutions have the same sign pattern. Thus, $Ax = b$ is sign-solvable if and only if $Ax = b$ has signed solutions and the set $\mathscr{S}$ has cardinality 1.

The matrix A has *signed null-space* provided $Ax = 0$ has signed solutions. Thus, A has signed null-space if and only if there exists a set $\mathscr{S}$ of sign patterns such that the set of sign patterns of vectors in the null-space of $\widetilde{A}$ is $\mathscr{S}$, for each $\widetilde{A} \in \mathscr{Q}(A)$. An *L-matrix* is a matrix, A, with the property that each matrix in $\mathscr{Q}(A)$ has linearly independent rows. A square L-matrix is a *sign-nonsingular*, or SNS-matrix for short. A *totally L-matrix* is an $m \times n$ matrix such that each $m \times m$ submatrix is an SNS-matrix. It is known that totally L-matrices are matrices with signed null-spaces [3]. Hence matrices with signed null-spaces generalize totally L-matrices.

A vector is *mixed* if it has a positive entry and a negative entry. A matrix is *row-mixed* if each of its rows is mixed. A *signing* is a nonzero, diagonal $(0, 1, -1)$-matrix. A signing is *strict* if each of its diagonal entries is nonzero. A matrix B is *strictly row-mixable* provided there exists a strict signing D such that BD is row mixed.

In this paper, we show that if A is an m by n matrix with no duplicate columns up to multiplication by -1 and it has signed null-space, then $n \leqslant 3m - 2$. Equality holds if and only if there exist permutation matrices P and Q such that the zero pattern of PAQ is in $\mathscr{M}_m$ (for definition see Section 2).

We use the following standard notations throughout the paper. If k is a positive integer, then $\langle k \rangle$ denotes the set $\{1, 2, \ldots, k\}$. Let A be an $m \times n$ matrix. If α is a subset of $\{1, 2, \ldots, m\}$ and β is a subset of $\{1, 2, \ldots, n\}$, then $A[\alpha|\beta]$ denotes the submatrix of A determined by the rows whose indices are in α and the columns whose indices are in β. We sometimes use $A[*|\beta]$ instead of $A[\langle m \rangle|\beta]$. The submatrix complementary to $A[\alpha|\beta]$ is denoted by $A(\alpha|\beta)$. In particular, $A(-|\beta)$ denotes the submatrix obtained from A by deleting the columns whose indices are in β. We write $\mathrm{diag}(d_1, d_2, \ldots, d_n)$ for the n by n diagonal matrix whose (i, i)-entry is d_i. Let $J_{m,n}$ denote the m by n matrix all of whose entries are 1 and let e_i denote the column vector all of whose entries are 0 except for the ith entry which is 1.

2. Matrices with signed null-spaces

We make use of the following property of matrices with signed null-spaces.

Theorem A ([2,3]). *If a strictly row-mixable matrix A has signed null-space, then there exist matrices B and C (possibly with no rows), and nonzero vectors b and c such that B and C are strictly row-mixable matrices with signed null-spaces,*

$$\begin{bmatrix} B \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ C \end{bmatrix}$$

have signed null-spaces, and up to permutation of rows and columns

$$A = \begin{bmatrix} B & O \\ b & c \\ O & C \end{bmatrix}.$$

The converse also holds.

Let A be an m by n $(0, 1, -1)$-matrix. The matrix B is *conformally contractible* to A provided there exists an index k such that the rows and columns of B can be permuted so that B has the form

$$\left[\begin{array}{c|c|c} A[\langle m\rangle|\langle n\rangle\backslash\{k\}] & x & y \\ \hline 0 \quad \cdots \quad 0 & 1 & -1 \end{array}\right],$$

where $x = [x_1, \ldots, x_m]^{\mathrm{T}}$ and $y = [y_1, \ldots, y_m]^{\mathrm{T}}$ are $(0, 1, -1)$ vectors such that $x_i y_i \geqslant 0$ for $i = 1, 2, \ldots, m$, and the sign pattern of $x + y$ is the kth column of A.

Let B be conformally contractible to A. It is known that A has signed null-space if and only if B has signed null-space, and a strictly row-mixable m by n matrix A has signed null-space if and only if A has term rank m and has signed mth compound [3].

All matrices we consider from now on are assumed to be $(0, 1, -1)$ matrices.

Lemma 1. *Let J be the* 2 *by* 3 *matrix obtained from an m by m* + 1 *matrix A by a sequence of conformal contractions. If the zero pattern of J is* $J_{2,3}$, *then A does not have signed null-space.*

Proof. Let $A = A_m, A_{m-1}, \ldots, A_2 = J$ such that A_{i+1} is conformally contractible to A_i for all $i = 2, \ldots, m-1$. If A has signed null-space, then $A_{m-1}, \ldots, A_2 = J$ have signed null-space. Since J has no zero entry, J does not have signed null-space. This is impossible. □

Corollary 2. *Let an m by n matrix A have a k by k* + 1 *submatrix B whose complementary submatrix in A has term rank* $m - k$. *If there is a matrix* B^* *obtained from B by replacing some nonzero entries with* 0*'s (if necessary) such that* $J_{2,3}$ *is the zero pattern of the matrix obtained from* B^* *by a sequence of conformal contractions, then A does not have signed null-space.*

Proof. By Lemma 1, B^* does not have signed null-space. Hence B^* contains a k by k submatrix which is not SNS-matrix with term rank k. Thus A contains an m by

m submatrix which is not SNS-matrix with term rank m. This implies that A does not have signed null-space. □

Let $\mathscr{M}_m$ be the set of m by $3m-2$ (0, 1)-matrices defined inductively as follows: $\mathscr{M}_1 = \{[1]\}$. Let $S \in \mathscr{M}_m$. Then S is of the form

$$\left[\begin{array}{ccc|ccc} & S_{m-1} & & & C & \\ \hline 0 & \cdots & 0 & 1 & 1 & 1 \end{array}\right], \tag{1}$$

where $S_{m-1} \in \mathscr{M}_{m-1}$ and all rows but a row which is $(1, 1, 0)$ in C are zero.

Proposition 3. *For any $S \in \mathscr{M}_m$, there exists a matrix with signed null-space whose zero pattern is S.*

Proof. We prove it by induction on m. It is clear for $m = 1$. Let $m > 1$ and let S be of the form in (1). By induction, there is a matrix B with signed null-space whose zero pattern is equal to S_{m-1}. Let A be the m by $3m-2$ matrix of the form

$$\left[\begin{array}{ccc|ccc} & B & & & C & \\ \hline 0 & \cdots & 0 & 1 & -1 & 1 \end{array}\right]. \tag{2}$$

Then the zero pattern of A is equal to S. Since S_{m-1} contains a submatrix which is the identity matrix I_{m-1} of order $m-1$ and B has signed null-space, the matrix $A^* = A(m|3m-3, 3m-2)$ has signed null-space. Since $A(-|3m-2)$ is conformally contractible to A^*, $A(-|3m-2)$ and hence A has signed null-space. □

Let A be a matrix with signed null-space. A is a *maximal* matrix with signed null-space if any matrix obtained from A by replacing a zero entry by a nonzero entry does not have signed null-space. Let E_{ij} denote the matrix all of whose entries are 0 except for the (i, j) entry which is 1. Let A be a strictly row-mixable m by n matrix with signed null-space. Then A is a maximal matrix with signed null-space if and only if there is an m by m submatrix B of A such that $B \pm E_{ij}$ has term rank m but $B \pm E_{ij}$ is not an SNS-matrix for any (i, j) with $a_{ij} = 0$.

Proposition 4. *Let A have signed null-space. If the zero pattern of A is in $\mathscr{M}_m$, then A is a maximal matrix with signed null-space.*

Proof. Let A be an m by $3m-2$ matrix with signed null-space and let its zero pattern be S_m in $\mathscr{M}_m$. Without loss of generality, we may assume that $A = [a_{ij}]$ is of the form in (2) where B is a matrix with signed null-space whose zero pattern is in $\mathscr{M}_{m-1}$. Let $Z_1 = \{(i, j) \in \langle m-1\rangle \times \{3m-4, 3m-3, 3m-2\} | a_{ij} = 0\}$ and $Z_2 = \{(m, j) | j \in \langle 3m-5\rangle\}$. Since B is a maximal matrix with signed null-space by induction, it is sufficient to show that $S_m + E_{ij}$ is not zero pattern of a matrix with signed null-space for any $(i, j) \in Z_1 \cup Z_2$.

Let the kth row of C be $(1, 1, 0)$ and the other rows of C be zero. If $(i, j) = (k, 3m-2)$, then $S_m + E_{ij}$ has $J_{2,3}$ as a submatrix such that its complementary submatrix has term rank $m-2$. Hence $A \pm E_{ij}$ does not have signed null-space by Corollary 2. Let $(i,j) \in Z_1$ with $j \in \{3m-4, 3m-3\}$. Then there exist distinct $p_1, p_2, \ldots, p_t$, and distinct $q_1, q_2, \ldots, q_t$ such that $a_{p_1,q_1}, a_{p_2,q_1}, \ldots, a_{p_t,q_t}$ are nonzero where $p_1 = 1$, $p_t = i$ and $q_t = j$. Similarly, there exist distinct $i_1, i_2, \ldots, i_s$, and distinct $j_1, j_2, \ldots, j_{s+2}$ such that $a_{i_1,j_1}, a_{i_2,j_1}, \ldots, a_{i_s,j_{s-1}}, a_{i_s,j_s}, a_{i_s,j_{s+1}}, a_{i_s,j_{s+2}}$ are nonzero where $i_1 = 1, i_s = k, j_{s+1} = 3m-4$ and $j_{s+2} = 3m-3$. Choosing some entries from these entries, we obtain a matrix which is conformally contractible to a matrix whose zero pattern is $J_{2,3}$. By Corollary 2, $S_m + E_{ij}$ is not zero pattern of a matrix with signed null-space. Analogously we can show that $S_m + E_{ij}$ is not zero pattern of a matrix with signed null-space for $i \neq k$ and $j = 3m-2$, or $(i,j) \in Z_2$. $\square$

Proposition 5. *Let A be a strictly row-mixable m by n matrix of the form*

$$\left[\begin{array}{c|c|c} & 0 & \\ B & \vdots & O \\ & 0 & \\ \hline b & 1 & c \\ \hline & 0 & \\ O & \vdots & C \\ & 0 & \end{array}\right].$$

If

$$M = \begin{bmatrix} B & O \\ b & 1 \end{bmatrix} \quad \textit{and} \quad N = \begin{bmatrix} 1 & c \\ O & C \end{bmatrix}$$

have signed null-spaces, then A has signed null-space. Moreover, if the zero patterns of M and N are in $\mathscr{M}_k$ and $\mathscr{M}_{m-k+1}$ by permuting rows and columns, respectively, then the zero pattern of A is in $\mathscr{M}_m$ by permuting rows and columns.

Proof. Clearly B has signed null-space whether b is zero or not. Analogously C has signed null-space. Hence A has signed null-space by Theorem A. Let S_M and S_N be the zero patterns of M and N, respectively. Then there exist permutation matrices P_1, P_2, Q_1 and Q_2 such that $P_1 S_M Q_1 \in \mathscr{M}_k$ and $P_2 S_N Q_2 \in \mathscr{M}_{m-k+1}$. Let the first row of S_N have moved to the pth row of $P_2 S_N Q_2$. Then there exist distinct $i_1 = 1, i_2, \ldots, i_q = p$ and distinct $j_1 = 1, j_2, \ldots, j_{3q-2}$ such that $L = P_2 S_N Q_2[i_1, i_2, \ldots, i_q | j_1, j_2, \ldots, j_{3q-2}]$ is of the form

$$\begin{bmatrix} 1 & 1 & 1 & & & & & & & \\ & 1 & 1 & 1 & 1 & 1 & & & & \\ & & & & 1 & 1 & 1 & & O & \\ & & O & & & \ddots & \ddots & & & \\ & & & & & & & & 1 & 1 & \\ & & & & & & & & 1 & 1 & 1 \end{bmatrix}.$$

Let L' be the matrix obtained from L by permuting rows and columns in reverse order. Then $L = L'$ and hence there exist permutation matrices P, Q such that $PS_NQ \in \mathscr{M}_{m-k+1}$ where its first row corresponds to the first row of N. This implies that the zero pattern of A is in $\mathscr{M}_m$ by permuting rows and columns. □

Proposition 6. *Let A be a strictly row-mixable m by n matrix with no duplicate columns up to multiplication by* -1. *If A has signed null-space, then* $n \leqslant 3m - 2$. *Equality holds if and only if there exist permutation matrices P and Q such that the zero pattern of PAQ is in* $\mathscr{M}_m$.

Proof. We prove it by induction on m. For $m = 2$, there is nothing to prove. Without loss of generality, we may assume that $m \geqslant 3$ and A can be rearranged as

$$A = \begin{bmatrix} B & O \\ b & c \\ O & C \end{bmatrix}, \tag{3}$$

where matrices B and C (possibly with no rows) are strictly row-mixable matrices which have signed null-spaces, and vectors b and c are nonzero. Also

$$\begin{bmatrix} B \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ C \end{bmatrix}$$

have signed null-spaces. Let

$$A[\alpha|\beta] = \begin{bmatrix} B \\ b \end{bmatrix} \quad \text{and} \quad A[\gamma|\delta] = \begin{bmatrix} c \\ C \end{bmatrix}$$

such that $|\alpha| = k$, $|\beta| = s$, $|\gamma| = l$ and $|\delta| = t$. Then $k + l - 1 = m$ and $s + t = n$.

Let $k > 1$ and $l > 1$. If $A[\alpha|\beta]$ has one of the unit vectors $\pm e_k$ as a column, then we can assume that $A[\alpha|\beta]$ is of the form

$$\begin{bmatrix} B' & O \\ b' & 1 \end{bmatrix}.$$

If $b' = \mathbf{0}$, then B' is a strictly row-mixable matrix with no duplicate columns up to multiplication by -1. By induction, we have $s - 1 \leqslant 3(k - 1) - 2$. Hence $s < 3k - 2$. If $b' \neq \mathbf{0}$, then $A[\alpha|\beta]$ is a strictly row-mixable matrix with no duplicate columns up to multiplication by -1. Hence $s \leqslant 3k - 2$. Let $C' = A[\gamma|\{s\} \cup \delta]$. Then we have $t + 1 \leqslant 3l - 2$ since C' satisfies the conditions of hypothesis. This implies that $3m - 2 - n = 3(k + l - 1) - 2 - (s + t) = (3k - s - 2) + (3l - t - 3) \geqslant 0$. Hence $n \leqslant 3m - 2$. Similarly, in the case that $A[\gamma|\delta]$ has one of the unit vectors $\pm e_1$ as columns we can show that $n \leqslant 3m - 2$. Assume that $A[\alpha|\beta]$ and $A[\gamma|\delta]$ do not have the unit vectors $\pm e_k$ and $\pm e_1$ as a column respectively. Since b is nonzero, the k by $s + 1$ matrix B^* obtained from $A[\alpha|\beta]$ by adding e_k as the last column is a strictly row-mixable matrix with no duplicate columns up to multiplication by -1. Also B^* has signed null-space. Hence $s + 1 \leqslant 3k - 2$.

Similarly we also have $t+1 \leqslant 3l-2$. Hence $3m-2 = 3(k+l-1)-2 = (3k-3)+(3l-3)+1 \geqslant s+t+1 > s+t = n$.

Let $k=1$. Then $s=1$ since columns of A are distinct. Hence we may assume that A is the form of

$$\begin{bmatrix} 1 & c \\ O & C \end{bmatrix}.$$

If C has no duplicate columns up to multiplication by -1, then $t \leqslant 3(m-1)-2$ by induction. Hence $n = t+1 \leqslant 3m-4 < 3m-2$. Let C have duplicate columns up to multiplication by -1. That is, let the columns 1, 2 of C be a pair of identical columns up to multiplication by -1. Let D be a strict signing such that $M = CD = [m_{ij}]$ is row-mixed and let $(m_{i_1 1}, \ldots, m_{i_p 1}) = (m_{i_1 2}, \ldots, m_{i_p 2})$ or $(m_{i_1 1}, \ldots, m_{i_p 1}) = -(m_{i_1 2}, \ldots, m_{i_p 2})$ for some positive integer p and $m_{i1} = m_{i,2} = 0$ for all $i \in \langle l-1 \rangle \backslash \{i_1, i_2, \ldots, i_p\}$. Since C has signed null-space, M has no mixed cycles and hence the columns 1 and 2 of M must be identical or $p=1$. If $p \geqslant 2$, then the matrix M' obtained from M by multiplying column 1 by -1 has a mixed cycle which is impossible. Hence $p=1$. This implies that the number of the same ones as the first column of C up to multiplication by -1 is at most 3. Thus without loss of generality, we may assume that the zero pattern of A is of the form

$$\left[\begin{array}{c|ccc|ccc|ccc|c}
1 & u & \cdots & u & v & \cdots & v & w & \cdots & w & 0 \text{ or } 1\\ \hline
 & x & & & & & & & & & \\
 & & \ddots & & & & & & & & \\
 & & & x & & & & & & & \\
 & & & & v & & & & & & \\
 & & & & & \ddots & & & & & S\\
 & & & & & & v & & & & \\
 & & & & & & & v & & & \\
 & & & & & & & & \ddots & & \\
 & & & & & & & & & v & \\ \hline
 & & & & & & & & & & T
\end{array}\right],$$

where $u = (1,1,0)$, $v = (1,1)$, $w = (1,0)$ and $x = (1,1,1)$, and the unspecified entries are zero. Let ϵ be the set of indices of columns corresponding to

$$\begin{bmatrix} S \\ T \end{bmatrix}$$

in A. Then we may also assume that $A[\gamma\backslash\{1\}|\epsilon]$ has no duplicate columns up to multiplication by -1 and the columns are also different from the ones of $A(1|\epsilon)$ up to multiplication by -1. If $A[\gamma\backslash\{1\}|\epsilon]$ is vacuous, we are done. Let only T be vacuous. Notice that every column of S has at least two nonzero entries. Any row of S has at most one nonzero entry. For, suppose that a row of S has two nonzero entries. Since the columns of $A[\gamma\backslash\{1\}|\epsilon]$ are distinct up to multiplication by -1, we may assume that there exists one of submatrices of A whose zero patterns are

$$\begin{bmatrix} 1 & 1 & * & * \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 1 & * & * \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & * \\ 0 & 0 & 1 & * & 1 \end{bmatrix},$$

where * is 0 or 1. By Corollary 2, A does not have signed null-space. This is a contradiction. Next, suppose that a row r of $A[\gamma\backslash\{1\}|\langle n\rangle]$ has four nonzero entries. Since each row of S has at most one nonzero entry and each column of S has at least two nonzero entries, we have a submatrix of A whose zero pattern is

$$\begin{bmatrix} 1 & 1 & 1 & * \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which is also impossible by Corollary 2. Hence every row of $A[\gamma\backslash\{1\}|\langle n\rangle]$ has at most three nonzero entries. Thus we have $n \leqslant 3m-2$. Let T be nonvacuous. Notice that the submatrix of A corresponding to T is a strictly row-mixable matrix with signed null-space. Let ϵ' be the set of indices of nonzero columns in T and let $T[*|\epsilon'] = T'$. Then we may assume that $T = [O \;\; T']$. Let γ_1 and γ_2 be the set of indices of rows corresponding to the rows of S and T respectively. Notice that $A(\gamma_2|\epsilon')$ has at most $3(|\gamma_1|+1)-2$ columns by the similar method we have shown in the case that only T is vacuous. If the submatrix A' of A corresponding to T' has no duplicate columns up to multiplication by -1, then $n \leqslant 3(|\gamma_1|+1)-2+3|\gamma_2|-2 = 3(|\gamma_1|+|\gamma_2|+1)-4 = 3m-4 < 3m-2$ by induction. Hence we have the result. Suppose that A' has duplicate columns up to multiplication by -1. It is easy to show that such columns of A' have exactly one nonzero entry. We want to show that the number of such duplicate columns is at most 3. Suppose that there are four duplicate columns in A' up to multiplication by -1. We may assume that the zero pattern of the submatrix consisting of such duplicate columns of A' is of the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ & O & & \end{bmatrix}.$$

Since $A[\gamma\backslash\{1\}|\epsilon']$ has no duplicate columns up to multiplication by -1, we may assume that $A[\gamma\backslash\{1\}|\epsilon']$ must have a submatrix whose zero pattern is of the form

$$\begin{bmatrix} 1 & * & * \\ * & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

where $*$ is 0 or 1. Hence we can have a submatrix N of A whose zero pattern is

$$\begin{bmatrix} 1 & 1 & * & * & * \\ 1 & 0 & 1 & * & * \\ 0 & 1 & * & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 1 & * & * & * \\ 1 & 0 & 0 & 1 & * & * \\ 0 & 1 & 0 & * & 1 & * \\ 0 & 0 & 1 & * & * & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

where $*$ is 0 or 1. By Corollary 2, A does not have signed null-space. This is a contradiction. Thus we can assume that T' is of the form

$$\begin{bmatrix} T_1' & T_2' \\ O & T_3' \end{bmatrix},$$

where T_1' is a block diagonal matrix whose diagonal blocks are (1 1) or (1 1 1), and

$$\begin{bmatrix} T_2' \\ T_3' \end{bmatrix}$$

has no duplicate columns up to multiplication by -1. Continuing this process, we can assume that T is of the form

$$\begin{bmatrix} T_1 & & * \\ O & \ddots & \\ & & T_q \end{bmatrix},$$

where $T_i = [O \;\; T_i']$ for $i = 1, 2, \ldots, q$ and T_i' are block diagonal matrices whose diagonal blocks are either (1 1) or (1 1 1) for $i = 1, 2, \ldots, q-1$. Let λ_i be the set of indices of rows in T_i. Let ϵ_i and δ_i be the set of indices of nonzero columns and zero columns in T_i respectively. It is easy to show that each row of $A[\lambda_i|\epsilon_i \cup \delta_{i+1}]$ has at most three nonzero entries for $i = 1, 2, \ldots, q-1$ by the similar method we have shown above. Hence $A(\lambda_q|\epsilon_q)$ has at most $3(m - |\lambda_q|) - 2$ columns. If the submatrix A_q' of A corresponding to T_q' has no duplicate columns up to multiplication by -1, then $|\epsilon_q| \leqslant 3|\lambda_q| - 2$ by induction. Hence $n \leqslant 3(m - |\lambda_q|) - 2 + |\epsilon_q| \leqslant 3m - 4 < 3m - 2$ and we have the result. If A_q' has duplicate columns up to multiplication by -1, we may assume that T_q' is of the form $[T_q'' \;\; T_q''']$ where T_q'' is a block diagonal matrix whose diagonal blocks are (1 1 1) or (1 1). As we have shown in the case that T is vacuous, every row of T_q' has exactly three nonzero entries. Thus we have the result. Similarly we have the same result for $l = 1$.

Let A be an m by n matrix such that the zero pattern of PAQ is in $\mathscr{M}_m$ for some permutation matrices P and Q. Clearly we have $n = 3m - 2$. Conversely, assume that A is an m by n matrix of the form in (3) with $n = 3m - 2$. If $k > 1$ and $l > 1$, then $s \leqslant 3k - 2$ and $t \leqslant 3l - 2$. Since $n = 3m - 2$, $s = 3k - 2$, $t = 3l - 3$

or $s = 3k - 3, t = 3l - 2$. Let $s = 3k - 2$ and $t = 3l - 3$. By induction, there exist permutation matrices P_1 and Q_1 such that the zero pattern of $P_1\left[\begin{smallmatrix} B \\ b \end{smallmatrix}\right]Q_1$ is in $\mathscr{M}_k$. Hence A has e_k or $-e_k$ as a column. Thus the submatrix C' of A obtained from $\left[\begin{smallmatrix} c \\ C \end{smallmatrix}\right]$ by adding e_1 or $-e_1$ according as A has e_k or $-e_k$ as the first column is a l by $t + 1$ matrix and $t + 1 = 3l - 2$. By induction, there exist permutation matrices P_2 and Q_2 such that the zero pattern of $P_2C'Q_2$ is in $\mathscr{M}_l$. By Proposition 5, there exist permutation matrices P and Q such that the zero pattern of PAQ is in $\mathscr{M}_m$. Analogously we have the result for $s = 3k - 3, t = 3l - 2$.

Next let $k = 1$. Since $n = 3m - 2$, we may assume that T'_q is a block diagonal matrix whose diagonal blocks are (1 1 1). Then $A(m|n - 2, n - 1, n)$ is a strictly row-mixable $m - 1$ by $n - 3$ matrix with no duplicate columns up to multiplication by -1 and it has signed null-space. Hence the zero pattern of $P(A(m|n - 2, n - 1, n))Q$ is in $\mathscr{M}_{m-1}$ for some permutation matrices P and Q by induction. If $(P \oplus 1)A(Q \oplus I_3)[\langle m\rangle|i, j]$ has a submatrix whose the zero pattern is of the form

$$\begin{bmatrix} 1 & * \\ * & 1 \\ 1 & 1 \end{bmatrix},$$

where $n - 2 \leqslant i, j \leqslant n$, then we have a submatrix E of $(P \oplus 1)A(Q \oplus I_3)$ whose zero pattern is of the form

$$\begin{bmatrix} 1 & 1 & & & & & & & & 1 & \\ 1 & 1 & 1 & 1 & & & & & & & \\ & & & & \ddots & & & * & & & \\ & & O & & & 1 & 1 & & & & \\ & & & & & 1 & 1 & 1 & 1 & & \\ & & & & & & & 1 & 1 & & 1 \\ & & & & & & & & & 1 & 1 \end{bmatrix}.$$

It is impossible by Corollary 2. Thus we may assume that the zero pattern of $(P \oplus 1)A(Q \oplus I_3)[\langle m\rangle|n - 2, n - 1, n]$ is of the form

$$\begin{bmatrix} & O & \\ 1 & 1 & 0 \\ & O & \\ 1 & 1 & 1 \end{bmatrix}.$$

Hence $(P \oplus 1)A(Q \oplus I_3) \in \mathscr{M}_m$. □

Corollary 7. *Let A be an m by n matrix with no duplicate columns up to multiplication by -1. If A has signed null-space, then $n \leqslant 3m - 2$.*

Proof. The result comes from Proposition 5, and Lemma 6 in [3]. □

Let A be an m by n matrix whose last row is x and let B be an r by s matrix whose first row is y. Let $A \diamond B$ be such that

$$A \diamond B = \left[\begin{array}{c|c} A[\langle m-1\rangle|\langle n\rangle] & O \\ \hline x & y \\ \hline O & B[\langle r\rangle\backslash 1|\langle s\rangle] \end{array}\right].$$

Proposition 8. *Let m be a positive integer with $m \geqslant 2$. Then there exists a strictly row-mixable $m \times n$ matrix with no duplicate columns up to multiplication by -1 which has signed null-space for any n with $m+1 \leqslant n \leqslant 3m-2$.*

Proof. Let

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

Then $C = A \diamond \overbrace{B \diamond \cdots \diamond B}^{m-2}$ is an m by $3m-2$ row-mixed matrix with signed null-space whose zero pattern is in $\mathscr{M}_m$ by Proposition 5. Let

$$D = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \diamond \overbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix} \diamond \cdots \diamond \begin{bmatrix} -1 \\ 1 \end{bmatrix}}^{m-3} \diamond \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}.$$

Then D is an m by $m+1$ row-mixed submatrix of C which has signed null-space. Hence any submatrix of C containing D is a row-mixed submatrix of C which has signed null-space. Thus we have the result. $\square$

References

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