On the interval Legendre polynomials

F. Patrício*, J.A. Ferreira, F. Oliveira

Department of Mathematics, University of Coimbra, Apartado 3008, Coimbra 3000, Portugal

Received 2 August 2001; received in revised form 15 September 2002

Abstract

In this paper, the extension to interval theory of the classical Legendre polynomials is considered. The so-called interval Legendre polynomials are introduced and some properties are studied. Based on these polynomials an interval minimum square approximation is introduced when continuous and discrete data are taken. © 2003 Elsevier Science B.V. All rights reserved.

1. Introduction

The role of Legendre polynomials on the approximation theory of real functions is well known. The aim of this paper is to study an interval version of those polynomials. We start by introducing some basic definitions. Using the recurrence relation of the definition of the classical Legendre polynomials we introduce an interval version of those functions—which we call interval Legendre polynomials—and some properties are studied. We introduce a minimum square approximation for continuous and discrete data based on those interval polynomials. Several examples showing the behaviour of the defined approximations are also presented in this paper. An application of the introduced theory to the computation of an estimate in the Hertzsprung–Russel diagram in Astrophysics is also included.

2. Basic definitions

In this section we introduce the basic definitions introduced by the authors in [1] and also some definitions known in the interval analysis theory.

* This work was supported by Centro de Matemática da Universidade de Coimbra.
* Corresponding author.
Definition 1. A real interval polynomial with degree $n$ is defined by

$$ P_n(x) = \sum_{j=0}^{n} A_j x^{n-j} $$

with $A_0 = 1$, $A_j = [a^{(1)}_j, a^{(2)}_j] \subset \mathbb{R}$, $j = 1, \ldots, n$.

According to (1) it is easy to see that $P_n(x)$ is a family of polynomials

$$ p_n(x) = \sum_{j=0}^{n} a_j x^{n-j} $$

with $a_0 = 1, a_j \in A_j, j = 1, \ldots, n$.

Using the definition of graph of a real function we introduce the graph of a real interval polynomial.

Definition 2. Let $P_n(x)$ be a real interval polynomial. The graph of $P_n(x)$ is denoted by $G(P_n)$ and is given by

$$ G(P_n) = \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 : \exists p_n(x) \in P_n(x), \tilde{y} = p_n(\tilde{x})\}. $$

In the next lemma we characterize the graph of a real interval polynomial $P_n(x)$ using the graphs of certain real polynomials. In order to do that, let $q_+(x)$, $r_+$, $q_-(x)$, $r_-$ be the following real polynomials

$$ q_+(x) = \sum_{j=0}^{n} q_{+,j} x^{n-j}, \quad r_+(x) = \sum_{j=0}^{n} r_{+,j} x^{n-j}, $$

$$ q_-(x) = \sum_{j=0}^{n} q_{-,j} x^{n-j}, \quad r_-(x) = \sum_{j=0}^{n} r_{-,j} x^{n-j}, $$

with

$$ q_{+,0} = r_{+,0} = q_{-,0} = r_{-,0} = 1, \quad q_{+,j} = a^{(2)}_j, \quad r_{+,j} = a^{(1)}_j, \quad j = 1, \ldots, n, $$

and

$$ q_{-,j} = \begin{cases} a^{(2)}_j, & \text{if } n-j \text{ is even} \\ a^{(1)}_j, & \text{if } n-j \text{ is odd} \end{cases}, \quad r_{-,j} = \begin{cases} a^{(1)}_j, & \text{if } n-j \text{ is even} \\ a^{(2)}_j, & \text{if } n-j \text{ is odd} \end{cases}. $$

Lemma 1. Let $P_n(x)$ be the real interval polynomial given by (1). The graph of $P_n$ is given by

$$ G(P_n) = \{(x, y) \in \mathbb{R}^2 : (r_+(x) \leq y \leq q_+(x) \text{ if } x \geq 0) \text{ or } (r_-(x) \leq y \leq q_-(x) \text{ if } x < 0)\}, $$

with $q_+(x), r_+(x)$ and $q_-(x), r_-(x)$ are given by (2).

Let us remember now the definition of the integral of an interval function [3,4].
Definition 3. Let $F(x)$ be a real interval function continuous in $[a, b]$ and $F(x) = [F_-(x), F_+(x)]$, $x \in [a, b]$. Then

$$\int_a^b F(x) \, dx = \left[ \int_a^b F_-(x) \, dx, \int_a^b F_+(x) \, dx \right].$$

Remark 1. If $F(x)$ is a real interval polynomial, that is, $F(x) = P_n(x)$ and $[a, b] \ni 0$ (respectively $[a, b] \ni 0$) then

$$\int_a^b P_n(x) \, dx = \left[ \int_a^b r_-(x) \, dx, \int_a^b q_-(x) \, dx \right],$$

(respectively $\int_a^b P_n(x) \, dx = [ \int_a^b r_-(x) \, dx, \int_a^b q_-(x) \, dx ]$).

Using definition (1) we introduce in the space of interval continuous functions defined in $[a, b]$ which we denote by $\mathbb{C}[a, b]$, the following “inner product” taking values in set of all real intervals $\mathbb{R}$.

Definition 4. Let $\langle \cdot, \cdot \rangle : \mathbb{C}[a, b] \times \mathbb{C}[a, b] \to \mathbb{R}$ be defined by

$$\langle F, G \rangle = \int_a^b F(x)G(x) \, dx, \quad F, G \in \mathbb{C}[a, b].$$

3. Interval Legendre polynomials

In this section we introduce an interval version of the Legendre polynomials.

Definition 5. Let us consider, for each natural number $k$, the family of interval polynomials defined by the following recursive formula

1. $L_{0,k}(x) = [1 - \frac{1}{k}, 1 + \frac{1}{k}]$,
2. $L_{1,k}(x) = [1 - \frac{1}{k}, 1 + \frac{1}{k}]x$,
3. for $n \in \mathbb{N}$,

$$L_{n+1,k}(x) = \frac{2n + 1}{n + 1}xL_{n,k}(x) - \frac{n}{n + 1} L_{n-1,k}(x).$$

For each $k \in \mathbb{N}$ and $n \in \mathbb{N}$, we call $L_{n,k}(x)$ interval Legendre polynomial.

Example 1. In Table 1 we present the interval Legendre polynomials for $n = 0, 1, 2, 3$. Their graphs are plotted in Figs. 1 and 2 for $k = 50$. 
Table 1
Interval Legendre polynomials of degree 0, 1, 2, 3

\[
L_{0,k}(x) = \left[1 - \frac{1}{k}, 1 + \frac{1}{k}\right]
\]

\[
L_{1,k}(x) = \left[1 - \frac{1}{k}, 1 + \frac{1}{k}\right] x
\]

\[
L_{2,k}(x) = \frac{3}{2} \left[1 - \frac{1}{k}, 1 + \frac{1}{k}\right] x^2 - \frac{1}{2} \left[1 - \frac{1}{k}, 1 + \frac{1}{k}\right]
\]

\[
L_{3,k}(x) = \frac{5}{2} \left[1 - \frac{1}{k}, 1 + \frac{1}{k}\right] x^3 - \frac{3}{2} \left[1 - \frac{1}{k}, 1 + \frac{1}{k}\right] x
\]

Fig. 1. Interval Legendre polynomials of degree zero and degree one.

Fig. 2. Interval Legendre polynomials of degree two and degree three.

In the next theorem we establish a relation between the interval Legendre polynomials and the well known Legendre polynomials \(\ell_n(x), n \in \mathbb{N}\).
Theorem 1. The interval Legendre polynomial $\mathbb{L}_{n,k}(x)$ is equal to the interval polynomial obtained from the Legendre polynomial $\ell_n(x)$ considering their coefficients multiplied by $[1 - \frac{1}{k}, 1 + \frac{1}{k}]$.

Proof. Follows from the definitions of $\mathbb{L}_{n,k}(x)$ and $\ell_n(x)$. $\square$

Theorem 2. The interval Legendre polynomials $\mathbb{L}_{n,k}(x)$, $n \in \mathbb{N}$, satisfy

1. If $n$ is even then the real polynomials $q_+, r_+, q_-$ and $r_-$ for $\mathbb{L}_{n,k}$ are given by

$$r_+(x) = q_+(x) = \sum_{j=0}^{n/2} a_j \left( 1 + \frac{(-1)^j}{k} \right) (-1)^j x^{2j},$$

$$r_-(x) = q_-(x) = \sum_{j=0}^{n/2} a_j \left( 1 + \frac{(-1)^{j+1}}{k} \right) (-1)^j x^{2j},$$

with

$$a_j = \frac{(n + 2j)!}{2^n (2j)! (\frac{n}{2} + j)! (\frac{n}{2} - j)!}.$$

2. If $n$ is odd then the real polynomials $q_+, r_+, q_-$ and $r_-$ for $\mathbb{L}_{n,k}$ are given by

$$r_+(x) = q_+(x) = \sum_{j=0}^{n-1 \over 2} a_j \left( 1 + \frac{(-1)^j}{k} \right) (-1)^{n-1 \over 2 - j} x^{2j+1},$$

$$q_-(x) = r_+(x) = \sum_{j=0}^{n-1 \over 2} a_j \left( 1 + \frac{(-1)^{j+1}}{k} \right) (-1)^{n-1 \over 2 - j} x^{2j+1},$$

with

$$a_j = \frac{(n + 1 + 2j)!}{2^n (2j - 1)! (\frac{n+1}{2} + j)! (\frac{n+1}{2} - j)!}.$$

3. For $n \neq m$, $\langle \mathbb{L}_{n,k}, \mathbb{L}_{m,k} \rangle \to 0$, $k \to +\infty$.

Proof. Using the definitions of $\mathbb{L}_{n,k}$, $q_+, r_+, q_-$ and $r_-$ we easily get 1 and 2.

Finally, the behaviour of $\mathbb{L}_{n,k}, \mathbb{L}_{n,k}$ when $k \to +\infty$, that is 3, is a consequence of the orthogonality of the Legendre polynomials $\ell_n$ in $[-1, 1]$. $\square$

Attending to the last result, we remark that the set $\{ \mathbb{L}_{n,k}, n \in \mathbb{N}_0 \}$ satisfies $\langle \mathbb{L}_n, \mathbb{L}_m \rangle \to 0$ when $k \to +\infty$. Attending to this fact we say that the set of the interval Legendre polynomials is asymptotically orthogonal.
4. Minimum square approximations

4.1. Continuous data

Let us consider now the space of interval Legendre polynomials \( \mathcal{L}_{n,k} \) of degree less or equal to \( n \)

\[
\mathcal{L}_{n,k} = \left\{ \sum_{j=0}^{n} a_j \mathcal{L}_{j,k}(x), a_j \in \mathbb{R} \right\}.
\]

**Definition 6.** Let \( F(x), x \in [-1, 1] \), be a given interval function and

\[
\text{proj}_{\mathcal{L}_{n,k}} F(x) = \sum_{j=0}^{n} \frac{\langle F, \mathcal{L}_{j,k} \rangle}{\langle \mathcal{L}_{j,k}, \mathcal{L}_{j,k} \rangle} \mathcal{L}_{j,k}(x),
\]

where \( \langle \cdot, \cdot \rangle \) is defined by (3). We call \( \text{proj}_{\mathcal{L}_{n,k}} F \) the asymptotically orthogonal projection of degree \( n \) of \( F \) into \( \mathcal{L}_{n,k} \).

**Example 2.** Let \( Y(x) = e^x, x \in [-1, 1] \). This function can be seen as a real interval function. In the following we compute \( \text{proj}_{\mathcal{L}_{2,k}} Y \). We have

\[
\text{proj}_{\mathcal{L}_{2,k}} Y(x) = \frac{\langle Y, \mathcal{L}_{0,k} \rangle}{\langle \mathcal{L}_{0,k}, \mathcal{L}_{0,k} \rangle} \mathcal{L}_{0,k}(x) + \frac{\langle Y, \mathcal{L}_{1,k} \rangle}{\langle \mathcal{L}_{1,k}, \mathcal{L}_{1,k} \rangle} \mathcal{L}_{1,k}(x) + \frac{\langle Y, \mathcal{L}_{2,k} \rangle}{\langle \mathcal{L}_{2,k}, \mathcal{L}_{2,k} \rangle} \mathcal{L}_{2,k}(x).
\]

We compute each term of \( \text{proj}_{\mathcal{L}_{2,k}} Y(x) \):

\[
\frac{\langle Y, \mathcal{L}_{0,k} \rangle}{\langle \mathcal{L}_{0,k}, \mathcal{L}_{0,k} \rangle} \mathcal{L}_{0,k}(x) = \frac{e^2 - 1}{2e} \left[ \frac{k - 1}{k+1} \right] \mathcal{L}_{0,k}(x)
\]

\[
\frac{\langle Y, \mathcal{L}_{1,k} \rangle}{\langle \mathcal{L}_{1,k}, \mathcal{L}_{1,k} \rangle} \mathcal{L}_{1,k}(x) = \frac{3}{2e} \left[ \frac{k(k-1)}{(k+1)^2} \right] \mathcal{L}_{1,k}(x)
\]

For the computation of \( \langle Y, \mathcal{L}_{2,k} \rangle/\langle \mathcal{L}_{2,k}, \mathcal{L}_{2,k} \rangle \mathcal{L}_{2,k}(x) \) we start by noting that

\[
\langle Y, \mathcal{L}_{2,k} \rangle = \frac{1}{2e} \left[ \frac{2}{k} (4 - e^2) + e^2 - 7, \frac{2}{k} (e^2 - 4) + e^2 - 7 \right].
\]
It is not a simple task to prove that
\[ \langle \mathbb{I}_{2,k}, \mathbb{I}_{2,k} \rangle = [a_k, b_k], \]
where
\[
\begin{align*}
a_k &= a \left( \frac{9}{10} a - b \right) + \frac{b^2}{2} + (b - a) \left( \frac{9}{10} (a\gamma_i^3 + b\gamma_s^5) + \frac{b - a}{2} (\gamma_s^3 - \gamma_i^3) - \frac{1}{2} (b\gamma_i + a\gamma_a) \right), \\
b_k &= b \left( \frac{9}{10} b - a \right) + \frac{a^2}{2} - \frac{2}{5\sqrt{3}} (a^2 - b^2),
\end{align*}
\]
and
\[
a = 1 - \frac{1}{k}, \quad b = \frac{1}{k}, \quad \gamma_i = \sqrt{\frac{k + 1}{3(k - 1)}}, \quad \gamma_s = \sqrt{\frac{k - 1}{3(k + 1)}},
\]
Then, attending that for \( k \geq 20, \frac{2}{k} (4 - e^2) + e^2 - 7 > 0, \) we obtain
\[
\frac{\langle Y, \mathbb{I}_{2} \rangle}{\langle \mathbb{I}_{2,k}, \mathbb{I}_{2,k} \rangle} \mathbb{I}_{2,k}(x) = \frac{1}{2e} \left[ \frac{1}{b_k} \left( \frac{2}{k} (4 - e^2) + e^2 - 7 \right) f_-(x), \frac{1}{a_k} \left( \frac{2}{k} (e^2 - 4) + e^2 - 7 \right) f_+(x) \right]
\]
where \( \mathbb{I}_{2,k}(x) = [f_-(x), f_+(x)] \).

In order to give an illustration of the behaviour of \( \text{proj}_{\mathcal{F}_{2,k}^x} Y(x) \) we only give an estimation for this interval function. We took \( \left[ \frac{2}{5}, \frac{2}{3} \right] \subset \langle \mathbb{I}_{2,k}, \mathbb{I}_{2,k} \rangle \) and we define \( F_k(x) \) by
\[
F_k(x) = [F_k_-(x), F_k_+(x)] \subset \text{proj}_{\mathcal{F}_{2,k}} Y(x), \quad x \in [-1, 1]
\]
with
\[
\begin{align*}
F_k_-(x) &= \frac{k - 1}{e(k + 1)^2} \left( \frac{e^2 - 1}{2} (k - 1) + 3k \left( \frac{1}{k} (1 - e) + 1 \right) x \right) \\
&\quad + \frac{5}{4e} \left( \frac{2}{k} (4 - e^2) + e^2 - 7 \right) f_-(x), \quad x \in [-1, 1], \\
F_k_+(x) &= \frac{k + 1}{e(k - 1)^2} \left( \frac{e^2 - 1}{2} (k + 1) + 3k \left( \frac{1}{k} (e - 1) + 1 \right) x \right) \\
&\quad + \frac{5}{4e} \left( \frac{2}{k} (e^2 - 4) + e^2 - 7 \right) f_+(x), \quad x \in [0, 1].
\end{align*}
\]
In Figs. 3 and 4 we plot the graphs of \( Y \) and \( F_k \) for different values of \( k \).

In the following we study the behaviour of \( d(F, \text{proj}_{\mathcal{F}_{2,k}}^x F) \) for a certain metric \( d(., .) \) defined for interval functions.

Let us consider \( \mathbb{C}[a,b] \). Let \( F, G \in \mathbb{C}[a,b] \), with \( F(x) = [f_-(x), f_+(x)] \), \( G(x) = [g_-(x), g_+(x)] \) for \( x \in [a,b] \). In order to define a metric on the last set we say that \( F = G \) if \( f_-(x) = g_-(x), g_+(x) = f_+(x) \) for \( x \in [a,b] \).

Let us define now a metric in \( \mathbb{C}[a,b] \).
Theorem 3. For $F, G \in \mathbb{C}[a, b]$ we define $d_{L^2}(F, G)$ by

$$d_{L^2}(F, G) = \max \{ \| f - g \|_{L^2[a, b]}, \| f^+ - g^+ \|_{L^2[a, b]} \}.$$ 

The last definition induces a metric in $\mathbb{C}[a, b]$.

Proof. In fact, it is easy to prove that

1. $d_{L^2}(F, G) = 0$ if and only if $F = G$ in $\mathbb{C}[a, b]$,
2. $d_{L^2}(F, G) \leq d_{L^2}(F, H) + d_{L^2}(H, G)$ for $F, G, H \in \mathbb{C}[a, b]$. □

Example 3. Let us consider the following interval function

$$\text{Exp}_\varepsilon(x) = [1 - \varepsilon, 1 + \varepsilon]e^x, \quad x \in [-1, 1], \quad \varepsilon \in (0, 1).$$

The minimum square approximation of first degree $\text{proj}_{\mathcal{L}_1} \text{Exp}_\varepsilon(x)$ is given by

$$\text{proj}_{\mathcal{L}_1} \text{Exp}_\varepsilon(x) = [A, B] + [C, D]x$$

with

$$A = \frac{(e^2 - 1)(1 - \varepsilon)(k - 1)^2}{2\varepsilon(k + 1)^2},$$
Fig. 5. The square of the distance of $F$ to $\text{proj}_{\psi_{1,5}} \text{Exp}_{0.5}$ as a function of $k$.

\[ B = \frac{(e^2 - 1)(1 + \varepsilon)(k + 1)^2}{2e(k - 1)^2}, \]
\[ C = 3(k - 1) \frac{(1 + \varepsilon)(k + 1) - e(1 + \varepsilon k)}{(k + 1)^2}, \quad D = 3(k - 1) \frac{(1 - \varepsilon)(k + 1) + e(1 + \varepsilon k)}{e(k - 1)^2}. \]

Let us look to the behaviour of $d_{L^2}(F, \text{proj}_{\psi_{1,5}} \text{Exp}_{0.5})^2$. For $k \geq 10$ we have
\[ d_{L^2}(F, \text{proj}_{\psi_{1,5}} \text{Exp}_{0.5}) = (1 + \varepsilon) \left( \frac{(1 + \varepsilon)(e^4 - 1)}{2e^2} - 2 \left( \frac{(B(e^2 - 1) + C(2 - e) + D)}{e} \right) \right) \]
\[ + B(2B + D - C) + \frac{D^2 + C^2}{3}. \]

In Fig. 5 we plot $d_{L^2}(F, \text{proj}_{\psi_{1,5}} \text{Exp}_{0.5})^2$.

Let us consider now $\text{proj}_{\psi_{2,5}} \text{Exp}_{\varepsilon}(x)$. We have
\[ \text{proj}_{\psi_{2,5}} \text{Exp}_{\varepsilon}(x) = \text{proj}_{\psi_{1,5}} \text{Exp}_{\varepsilon}(x) + \frac{\langle \text{Exp}_{\varepsilon}, \mathbb{I}_{2,k} \rangle}{\langle \mathbb{I}_{2,k}, \mathbb{I}_{2,k} \rangle} \mathbb{I}_{2,k}(x) \]
with
\[ \frac{\langle \text{Exp}_{\varepsilon}, \mathbb{I}_{2,k} \rangle}{\langle \mathbb{I}_{2,k}, \mathbb{I}_{2,k} \rangle} \mathbb{I}_{2,k}(x) = \begin{bmatrix} \text{Exp}_{\varepsilon,-} & \text{Exp}_{\varepsilon,+} \end{bmatrix} \begin{bmatrix} b_k \\ a_k \end{bmatrix} \mathbb{I}_{2,k}(x), \]
where $a_k$ and $b_k$ are defined by (4) and (5), and
\[ \text{Exp}_{\varepsilon,-} = (1 - \varepsilon) \left( \frac{3}{2} \left( 1 - \frac{1}{k} \right) \frac{e^2 - 5}{e} - \frac{1}{2} \left( 1 + \frac{1}{k} \right) \frac{e^2 - 1}{e} \right), \]
\[ \text{Exp}_{\varepsilon,+} = (1 + \varepsilon) \left( \frac{3}{2} \left( 1 + \frac{1}{k} \right) \frac{e^2 - 5}{e} - \frac{1}{2} \left( 1 - \frac{1}{k} \right) \frac{e^2 - 1}{e} \right). \]

Then
\[ d_{L^2}(F, \text{proj}_{\psi_{2,5}} \text{Exp}_{\varepsilon})^2 \]
\[ = \frac{1 + \varepsilon}{e} \left( \frac{(1 + \varepsilon)(e^4 - 1)}{2e^2} - 2 \left( (B - F)(e^2 - 1) + 2C + E(e^2 - 5) \right) \right) \]
Fig. 6. The square of the distance of $F$ to $\text{proj}_{\mathcal{L}_2} \text{Exp}_{0.5}$ as a function of $k$.

Fig. 7. The behaviour of $d_{\mathcal{L}_2}(F, \text{proj}_{\mathcal{L}_1} \text{Exp}_{0.5})^2$ and $d_{\mathcal{L}_2}(F, \text{proj}_{\mathcal{L}_2} \text{Exp}_{0.5})^2$.

\[ +2 \left( 2B \left( \frac{E}{3} - F \right) + (D - C) \left( \frac{E}{4} - \frac{F}{2} - (1 + \varepsilon) + \frac{B}{2} \right) \right) \]
\[ + 2 \left( B^2 + F^2 \right) + E \left( \frac{E}{5} - \frac{2F}{3} \right) + \frac{C^2 + D^2}{3}, \]

with
\[ E = \frac{3}{2ka_k} \text{Exp}_{\varepsilon+}(k + 1), \quad F = -\frac{1}{2ka_k} \text{Exp}_{\varepsilon+}(k - 1). \]

In Fig. 6 we plot $d_{\mathcal{L}_2}(F, \text{proj}_{\mathcal{L}_2} \text{Exp}_{0.5})^2$.

Finally in Fig. 7 we plot $d_{\mathcal{L}_2}(F, \text{proj}_{\mathcal{L}_1} \text{Exp}_{0.5})^2$ and $d_{\mathcal{L}_2}(F, \text{proj}_{\mathcal{L}_2} \text{Exp}_{0.5})^2$.

4.2. Discrete data

Let us consider a discrete set of data $\{(x_i, Y_i), i = 1, \ldots, n\}$, where $x_i \in \mathbb{R}$ and $Y_i = [y_i^{(1)}, y_i^{(2)}]$ is a compact real interval. In what follows we construct an interval function which “approximates” the discrete set given. We start by defining another set of discrete data $\{ (\tilde{x}_i, \tilde{y}_i), i = 1, \ldots, 2n \}$ with
\[ \hat{x}_1 = \hat{x}_2 = x_1, \hat{x}_3 = \hat{x}_4 = x_2, \ldots, \hat{x}_{2n-1} = \hat{x}_{2n} = x_n \quad \text{and} \quad \hat{y}_1 = y_1^{(1)}, \hat{y}_2 = y_1^{(2)}, \hat{y}_3 = y_2^{(1)}, \hat{y}_4 = y_2^{(2)}, \ldots, \hat{y}_{2n-1} = y_n^{(1)}, \hat{y}_{2n} = y_n^{(2)}.

For the last set we define \( \hat{\ell}(x) \) by

\[
\hat{\ell}(x) = \sum_{j=0}^{m} a_j^* \ell_j(x),
\]

where \( \ell_j(x) \) is the \( j \) Legendre polynomial and \( a_j^*, j = 0, \ldots, m, \) are the solutions of the minimization problem

\[
\min_{a_0, \ldots, a_m \in \mathbb{R}} \sum_{j=1}^{2n} (\hat{y}_j - \ell(x_j))^2 = \sum_{j=1}^{2n} (\hat{y}_j - \hat{\ell}(x_j))^2
\]

with \( \ell(x) = \sum_{j=0}^{m} a_j \ell_j(x), a_j \in \mathbb{R} \). That is \( \hat{\ell}(x) \) is the minimum square approximation for the set \( \{(\hat{x}_i, \hat{y}_i), i = 1, \ldots, 2n\} \) when Legendre polynomials are considered.

**Definition 7.** Let \( \hat{L}_k(x) \) be defined by

\[
\hat{L}_k(x) = \sum_{j=0}^{m} a_j^* \ell_{j,k}(x),
\]

where \( a_j^*, j = 0, \ldots, m, \) satisfy (6). We call \( \hat{L}_k(x) \) the interval Legendre minimum square approximation for the discrete set \( \{(x_i, Y_i), i = 1, \ldots, n\} \), where \( x_i \in \mathbb{R} \) and \( Y_i = [y_i^{(1)}, y_i^{(2)}] \).

In what follows we consider the performance of the defined interval Legendre minimum square approximation on the computation of an interval function which approximates a set of points.

**Example 4.** The theoretical models for certain stars enable to obtain the location of those stars using the so called Hertzsprung–Russel (HR) diagram. The location is obtained defining two curves which are the solutions of certain differential equations arising on the model. The Hipparcos mission has provided a very accurate data for a hundred disk stars of spectral types \( F \) to \( K \). In [2] those observations were analysed by means of the stellar models computed using the most recent input physics and the positions of the objects versus standard theoretical isochrones, corresponding to their chemical composition and age, were examined. Some discrepancy between the theoretical position and the position observed was detected.

Table 2 was taken from [2] for certain stars. The position of the star is \((\log T_{\text{eff}}, M_{\text{bol}})\) and \( \sigma_{M_{\text{bol}}} \) is the absolute error of the observed value \( M_{\text{bol}} \), that is, for the value \( T_{\text{eff}} \) the corresponding value \( M_{\text{bol}} \) belongs to the interval \([M_{\text{bol}} - \sigma_{M_{\text{bol}}}, M_{\text{bol}} + \sigma_{M_{\text{bol}}}]\). We use the minimum square Legendre polynomials on the computation of an estimate to the HR diagram.

The minimum Interval Legendre polynomial of first degree with coefficients presenting an exact decimal digit is

\[
\hat{L}_k(x) = 83.6 \left[ 1 - \frac{1}{k}, 1 + \frac{1}{k} \right] - 21.0 \left[ 1 - \frac{1}{k}, 1 + \frac{1}{k} \right] x.
\]

Its graph is plotted in Fig. 8 with \( k = 200 \).
The minimum interval Legendre polynomial of degree two with coefficients presenting an exact decimal digit is

\[ \hat{L}_k(x) = -887.3 \left[ 1 - \frac{1}{k}, 1 + \frac{1}{k} \right] + 501.6 \left[ 1 - \frac{1}{k}, 1 + \frac{1}{k} \right] x - 70.3 \left[ 1 - \frac{1}{k}, 1 + \frac{1}{k} \right] x^2, \]

and its graph is plotted in Fig. 9 for \( k = 10^4 \).
We remark that the behaviour of the minimum square interval Legendre polynomial of degree three with \( k \simeq 25 \times 10^4 \) is similar to the behaviour of the minimum square interval Legendre polynomial of degree two with \( k \simeq 10^4 \).

A piecewise version of the interval Legendre polynomials can be defined and more efficient curves containing all the observed values can be constructed.

References


