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Centralizing Mappings of Operator Algebras

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1. INTRODUCTION

If A is an algebra, a mapping $\phi: A \to A$ is called centralizing if $[\phi(x), x] \in Z_A$ for all $x \in A$ where $[x, y] = xy - yx$ and Z_A is the centre of A. In this paper we prove theorems for certain centralizing mappings of C^*-algebras and von Neumann algebras which are related to theorems of Posner [6], Mayne [4], and Herstein [3] for prime or simple rings. Namely, we show that if $d: A \to A$ is a derivation on a C^*-algebra A with $[p(d)(x), x] \in Z_A$ for all $x \in A$, where $p(t)$ is a complex polynomial, then $p(d)(x) = 0$ for all $x \in A$. Moreover, if A is a von Neumann algebra and $p(d)(x) = 0$ $\forall x \in A$ there exists $z \in Z_A$ such that $p(a - z) = 0$ where $d(x) = [a, x]$, $a \in A$. In the case that $\phi: A \to A$ is a centralizing $*$-automorphism of a von Neumann algebra, then $A = A_1 \oplus A_2$ where $\phi|_{A_1}$ is the identity on A_1, and A_2 is abelian.

Although C^*-algebras are semi-prime and have many special algebraic properties they are not, in general, prime. In fact, a von Neumann algebra is prime if and only if it is a factor (i.e. its centre consists of scalar multiples of the identity). The presence of central projections (self-adjoint idempotents) in von Neumann algebras means that phenomena of an "either $\cdots$ or" nature in prime rings can occur simultaneously but on complementary summands in the von Neumann case. This is the situation with regard to Mayne's theorem which states that for prime rings a centralizing automorphism is the identity or the ring is commutative.

2. NOTATION AND PRELIMINARIES

We denote by $\mathscr{L}(H)$ the ring of all linear operators $T: H \to H$, H a complex Hilbert space with inner product $(\cdot, \cdot)$, which are bounded in the uniform norm $\| T \| = \sup_{\|h\| \leqslant 1} \| Th \|$, $h \in H$.

In this norm $\mathscr{L}(H)$ is a Banach $*$-algebra with identity operator I and $*$ operation defined by $(Th, k) = (h, T^*k)$ $\forall h, k \in H$. An operator $T \in \mathscr{L}(H)$ is

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positive if $T = S^*S$. A B^*-algebra is a Banach *-algebra A, with complex field, such that $\| xx^* \| = \| x \|^2 \ \forall x \in A$. Any B^*-algebra is isometrically *-isomorphic to a uniformly closed *-algebra of bounded linear operators on a complex Hilbert space. Such operator algebras are called C^*-algebras.

In addition to the uniform topology on $\mathscr{L}(H)$ we shall be concerned with the weakest topology making the linear functionals $T \to \sum_{n=1}^{\infty}(Th_n, k_n)$ continuous for all sequences $\{h_n\}, \{k_n\}$ for which $\sum_{n=1}^{\infty} \| h_n \|, \sum_{n=1}^{\infty} \| k_n \| < \infty$. Multiplication is continuous in each variable separably and the *-operation is continuous in this topology which is called the ultra-weak ($=$ uw) topology. A vòn Neumann or W^*-algebra is an ultra-weakly closed subalgebra of $\mathscr{L}(H)$ which contains I. It is a fact that if M is a von Neumann algebra then M is the smallest ultra-weakly closed linear subspace of $\mathscr{L}(H)$ containing $\{p \in M \mid p = p^2 = p^*\}$. Such operators p are called projections. If $p \in M$ is a projection, the core of p, denoted by $\mathbf{p}$, is defined to be $\mathrm{LUB}\{s \in Z_M \mid s = s^* \leqslant p\}$. $\mathbf{p}$ always exists, is a projection, and is in M if $p \in M$. The support of p, denoted by $\bar{p}$, is defined to be the least central projection in M which is larger than p. Two projections $p, q \in M$ are parallel if $\bar{p}\bar{q} = 0$ and this occurs iff $pxq = 0 \ \forall x \in M$. If p is a projection, $Mp = \{p \times p \mid x \in M\}$. If $\mathscr{I}$ is an ultra-weakly closed two sided ideal in a von Neumann algebra M then $\mathscr{I} = Mc$ where c is a central projection.

If A is a C^*-algebra then A is isometrically *-isomorphic to $\mathscr{U}(A)$ its universal representation. Moreover $U(A)^{-uw}$ is W^*-isomorphic to A^{**} where A^* is the (norm) dual of A. Automorphisms are understood to be *-automorphisms but derivations do not necessarily preserve the *-operation. If $\phi: A \to A$ is a *-automorphism of a von Neumann algebra then it is ultra-weakly bi-continuous. If $d: A \to A$ is a derivation of a C^*-algebra A then d is norm continuous. If A is a simple C^*-algebra or a von Neumann algebra and d a derivation on A then $d(x) = [a, x]$ for some $a \in A$. Such derivations are called inner. If A is a C^*-algebra and $\mathscr{I}$ a closed 2-sided ideal in A then $\mathscr{I}^2 = \{\sum_{i=1}^{n} a_i b_i \mid a_i, b_i \in \mathscr{I}\}$ is uniformly dense in $\mathscr{I}$. In particular if d is a derivation and $\mathscr{I}$ an ideal then $d(\mathscr{I}) \subseteq \mathscr{I}$ by the uniform continuity of d.

For an excellent account of these and other properties of operator algebras we refer the reader to [1] or [8].

3. Centralizing Derivations

Let $p(t) = c_0 + c_1 t + \cdots + c_n t^n$ be a complex polynomial of degree $n > 0$, so $c_n \neq 0$. Let A be an algebra with identity e and for each $a \in A$ let ad a be the derivation $(\mathrm{ad}\, a)(x) = [a, x]$, $x \in A$.

Lemma 1. (i)

$$p(\mathrm{ad}\, a)(x) = p(a)x + \sum_{k=1}^{n} u_k(a)\, xa^k$$

where $u_k(a) = (-1)^k p^{(k)}(a)/k!$ *and* $p(\text{ad } a)(x) = xp(-a) + \sum_{k=1}^{n} a^k x v_k(a)$ *where* $v_k(a) = p^{(k)}(-a)/k!$.

(ii) *If* $[p(\text{ad } a)(x), x] = 0\ \forall x$ *then* $[p(\text{ad } a)(x), y] + [p(\text{ad } a)(y), x] = 0$ $\forall x, y \in A$, *and* $(\text{ad } a)(p(\text{ad } a)(x)) = 0\ \forall x$.

(iii) *If* A *is a* C^**-algebra,* $[p(\text{ad } a)(x), x] \in Z_A$ *implies* $p(\text{ad } a)(x) = 0$.

(iv) *If* A *is a* C^**-algebra and* $d\colon A \to A$ *is any derivation then* $p(d)(x) = 0$ $\forall x \in A$ *implies* $c_0 = 0$.

Proof. (i) Follows by expansion of $p(\text{ad } a)(x) = \sum_{i=0}^{n} c_i (\text{ad } a)^i(x)$.

(ii) Replace x by $x + y$ and use the linearity of ad a. In the relation $[p(\text{ad } a)(x), y] + [p(\text{ad } a)(y), x] = 0$, set $y = a$ and get $0 = [p(\text{ad } a)(x), a] = -(\text{ad } a)(p(\text{ad } a)(x))$.

(iii) The assertion follows from [7: Theorem 1.3.1].

(iv) By [8: 4.1.2], $d|_{Z_A} = 0$, so that if $z \neq 0$, $z \in Z_A$, we have $0 = p(d)(z) = c_0 z$. ∎

LEMMA 2 [6, Theorem 1]. *If* A *is a prime ring and* d_1, d_2 *derivations on* A *with* $(d_1 d_2)(x) = d_1(d_2(x))$ *a derivation, then one of* d_1 *or* d_2 *is zero.*

COROLLARY. *If* A *is a factor von Neumann algebra, with* $a \in A$, *then* (*Range* ad a)$' \cap A = C$ *or* $a \in Z_A$. (*Here*

$$S' = \{t \in \mathscr{L}(H) \mid st = ts\ \forall s \in S\}.)$$

Proof. Suppose $a \notin Z_A$. If $b \in (\text{Range ad } a)' \cap A$ then $(\text{ad } b)((\text{ad } a)(x)) = [b, [a, x]] = 0$. Hence ad $b = 0$ or ad $a = 0$ since factors are prime. But $a \notin Z_A$ so ad $a \neq 0$. Hence ad $b = 0$ or $b \in Z_A = C$. ∎

THEOREM 1. *Let* A *be a* C^**-algebra and* $d\colon A \to A$ *a derivation for which* $[p(d(x)), x] = 0\ \forall x \in A$. *Then* $p(d(x)) = 0\ \forall x$.

Proof. First assume A is a von Neumann factor. Then $d(x) = (\text{ad } a)(x)$ for some $a \in A$ [8, Theorem 4.1.6]. By Lemma 1, $[p(\text{ad } a)(x), y] = -[p(\text{ad } a)(y), x]$ $\forall x, y \in A$. Hence $y \in (\text{Range } p(\text{ad } a)' \cap A$ iff $p(\text{ad } a)(y) \in Z_A$ iff $p(\text{ad } a)(y) = 0$ since Z_A contains no nonzero commutators and $c_0 = 0$. Hence $[\text{Range } p(\text{ad } a)]' \cap A = \text{Ker } p(\text{ad } a)$. Now $(\text{ad } a)(p(\text{ad } a)(x)) = 0\ \forall x$ implies $0 = (\text{ad } a) \circ p(\text{ad } a) = p(\text{ad } a) \circ (\text{ad } a)$ so that $\text{Range}(\text{ad } a) \subseteq \text{Ker } p(\text{ad } a)$. Therefore $[\text{Range}(\text{ad } a)]' \cap A \supseteq [\text{Ker } p(\text{ad } a)]' \cap A = ([\text{Range } p(\text{ad } a)]' \cap A)' \supseteq \text{Range } p(\text{ad } a)$. By the corollary to Lemma 2, ad $a = 0$ or $[\text{Range}(\text{ad } a)]' \cap A = C$. Hence $a \in Z_A$ or $\text{Range } p(\text{ad } a) \subseteq C$. The latter implies $p(\text{ad } a) = 0$ since Z_A contains no nonzero commutators.

Now let A be any C^*-algebra and $\pi: A \to \pi(A)$ an irreducible representation. In particular $\pi(A)^{-uw}$ is a factor. Moreover, since d is uniformly continuous [8, Lemma 4.1.3], and if I is a uniformly closed, twosided ideal in A,

$$\left\{ \sum_{i=1}^{k} a_i b_i \;\middle|\; a_i, b_i \in I \right\}^- = I,$$

we have $d(I) \subseteq I$. Thus d induces a derivation $\bar{d}: A/\operatorname{Ker}\pi \to A/\operatorname{Ker}\pi$. Let $\pi_0: A/\operatorname{Ker}\pi \to \pi(A)$ be the canonical isomorphism induced by π. π_0 is then a faithful irreducible representation and $d_1 = \pi_0 \circ \bar{d} \circ \pi_0^{-1}$ is a derivation on $\pi(A) = B$ with the property that $[p(d_1)(x), x] = 0 \;\forall x \in B$. There exists $b \in B^{-uw}$ such that $d_1(x) = [b, x] = (\operatorname{ad} b)(x) \;\forall x \in B$. Since multiplication is uw continuous in one variable we have $[p(\operatorname{ad} b)(x), x] = 0 \;\forall x \in B^{-uw}$. By the first part of the argument, $p(\operatorname{ad} b)(x) = 0$ so that $p(d_1)(x) = 0$. Thus $p(\bar{d}) = 0$. That is, $p(d)$ maps into $\operatorname{Ker}\pi$. Since $\cap_\pi \operatorname{Ker}\pi = \{0\}$ where the intersection is taken over all irreducible π, we have $p(d) = 0$. ▮

4. Algebraic Derivations

In this section we study derivations, ad a, for which $p(\operatorname{ad} a) = 0$ and show that for certain algebras a is algebraic over the centre. In [3], Herstein showed that if A is a simple ring and $(\operatorname{ad} a)^n = 0$ there exists $\lambda \in Z_A$ such that $(a - \lambda)^n = 0$. If A is a complex, topologically simple Banach algebra with identity e, the argument simplifies and shows

Theorem 2. *If $p(\operatorname{ad} a) = 0$, there exists $\alpha \in C$ such that $p(a - \alpha e) = 0$ or $p(\alpha e - a) = 0$.*

Proof. By a standard argument A is algebraically simple.

For each $\alpha \in C$, $\operatorname{ad} a = \operatorname{ad}(a - \alpha e)$. Hence $p(\operatorname{ad}(a - \alpha e)) = 0$. Since the spectrum of a in A is non-empty, we can replace a by $a - \alpha e$, for appropriate $\alpha \in C$, so as to assume $p(\operatorname{ad} a) = 0$ and a is not regular. By Lemma 1(i), for some $a_1 \in A$, $0 = p(\operatorname{ad} a)x = p(a)x + a_1 x a$. Hence $p(a) \in Aa$, so that $Ap(a)A \subseteq Aa$. Since $Ap(a)A$ is a 2-sided ideal in A we have $p(a) = 0$ or $A = Aa$, namely a is left-regular. Similarly, $xp(-a) \in aA$, so that $Ap(-a)A \subseteq aA$, and $p(-a) = 0$ or a is right regular. Thus, unless $p(a) = 0$ or $p(-a) = 0$, a is both left and right regular, hence regular. But a is not regular and the theorem follows. ▮

We now turn our attention to operator algebras. The following lemma is similar to [3, Teorema].

Lemma 3. *Let H be a complex Banach space and A an algebraically irreducible algebra of bounded operators on H containing the identity operator e. If $p(\operatorname{ad} a) = 0$, $a \in A$, then for each $u \in H$, the subspace $W =$ Span $(u, au, \ldots, a^n u)$ reduces a.*

Proof. Algebraic irreducibility means that A is 1-fold transitive on H. By [2, Lemma 2] A is then n-fold transitive if H is infinite dimensional or of dimension at least n.

It suffices to show that the vectors $u, au, \ldots, a^n u$ are linearly dependent. Suppose to the contrary that they are linearly independent. Let $\alpha \in C$, $v \in H$. By $(n+1)$-fold transitivity, choose $x \in A$ such that $xa^k u = \alpha^k v$ for $k = 0, \ldots, n$. By Lemma 1(i) we have

$$0 = \sum_{k=0}^{n} (-1)^k p^{(k)}(a)\, xa^k u/k! = p(a - \alpha e)v,$$

by Taylor's formula. Thus $p(a - \alpha e) = 0$ $\forall \alpha \in C$. The Hahn–Banach theorem implies that if $p(a - \alpha e) = 0$ for $n + 1$ distinct values α, then all coefficients of $p(a - \alpha e)$ are zero. But $c_n \neq 0$, a contradiction. ▌

THEOREM 3. *Let A be an algebraically irreducible algebra of bounded operators on a complex Banach space H containing the identity e. If, for $a \in A$, $p(\mathrm{ad}\, a) = 0$ then the point spectrum $PS(a)$ is non-empty and for each $\alpha \in PS(a)$, $p(a - \alpha e) = 0$.*

Proof. Lemma 3 shows $PS(a) \neq \phi$ since there are non-trivial finite-dimensional subspaces of H invariant under a. Hence there exist $\alpha \in C$ and $0 \neq v \in H$ such that $(a - \alpha e)v = 0$. Hence $(a - \alpha e)^k v = 0$ for all positive integers k so that by Lemma 1, $0 = p(\mathrm{ad}(a - \alpha e))xv = p(a - \alpha e)xv$. Since A is 1-fold transitive, $p(a - \alpha e) = 0$. ▌

COROLLARY. *Let A be a simple C^*-algebra with identity e, and $a \in A$ be such that $p(\mathrm{ad}\, a) = 0$. There exists $\alpha \in C$ such that $p(a - \alpha e) = 0$.*

Proof. Since A is simple it has a faithful irreducible *-representation which is algebraically irreducible by [1, 2.8.4]. ▌

THEOREM 4. *Let A be a C^*-algebra of operators acting on a complex Hilbert space H, and assume A contains the identity operator e. Let $R = A''$ the ultra-weak closure of A. If $p(\mathrm{ad}\, a) = 0$, $a \in A$, there exists $z \in Z_R$ such that $p(a - z) = 0$.*

Proof. We recall a few facts about representations of C^*-algebras. Let M be the universal enveloping von Neumann algebra of R. If ϕ is any *-representation of R and π the natural injection of R into M, there exists a normal *-representation $\tilde{\phi}$ of M such that $\phi(x) = \tilde{\phi}(\pi(x))$ for all $x \in R$. The image $\tilde{\phi}(M)$ is the ultra-weak closure of $\phi(R)$. If ϕ is irreducible, then $\tilde{\phi}(M) = \phi(R)^{-uw} = \mathscr{L}(H_\phi)$ where H_ϕ is the representation space of ϕ. Moreover, if $\tilde{\phi}$ is a normal homomorphism of M onto a von Neumann algebra N, there exists a central projection $c \in M$ and a *-isomorphism $\tilde{\psi}$ of M_c onto N such that $\tilde{\phi}(x) = \tilde{\psi}(xc)$ $\forall x \in M$.

Now R has a complete set of irreducible *-representations ϕ_β, and in the above notation, $\phi_\beta(x) = \tilde{\phi}_\beta(\pi(x))$ where $\tilde{\phi}_\beta$ is a normal *-homomorphism of M on $\mathscr{L}(H_{\phi_\beta})$.

There exist central projections c_β in M and *-isomorphisms $\tilde{\psi}_\beta$ of M_{c_β} on $\mathscr{L}(H_{\phi_\beta})$ such that $\tilde{\phi}_\beta(x) = \tilde{\psi}_\beta(xc_\beta)\ \forall x \in M$. Since $\tilde{\phi}_\beta$ is an irreducible representation (and hence algebraically irreducible by [1: 2.8.4]) we have by Theorem 3 that there exists $\alpha_\beta \in C$ such that $p(\phi_\beta(a) - \alpha\beta) = 0$. One has $|\alpha_\beta| \leqslant \|\phi_\beta(a)\| \leqslant \|a\|$, and, for each β, $p((\pi(a) - \alpha_\beta)c_\beta) = 0$. By the completeness of the set of irreducible representations, LUB $c_\beta = e$ the identity operator on the representation space of M. Choose a collection of mutually orthogonal central projections $d_\beta \in M$ such that $d_\beta \leqslant c_\beta$ and $\sum d_\beta = e$. Let $s = \sum \alpha_\beta d_\beta$. Then $s \in Z_M$ and $p(\pi(a) - s) = 0$.

Let i: $R \to R$ be the identity map, and $\bar{\imath}$ the normal homomorphism of M on R such that $\bar{\imath}(\pi(x)) = i(x)\ (=x)\ \forall x \in R$. Let c be a central projection in M and $\bar{\jmath}$ an isomorphism of M_c on R such that $\bar{\imath}(\pi(x) = \bar{\jmath}(\pi(x)c)\ \forall x \in R$. $\bar{\jmath}$ sets up an isomorphism of the centre of M_c with Z_R, so there exists $z \in Z_R$ such that $\bar{\jmath}(sc) = z$. Hence $\bar{\imath}(s) = z$, and $0 = \bar{\imath}(p(\pi(a) - s)) = p(\bar{\imath}(\pi(a)) - \bar{\imath}(s)) = p(a - z)$. ∎

5. Centralizing Automorphisms

Let M be a von Neumann algebra and ϕ: $M \to M$ a *-automorphism of M onto M.

Lemma 4. *If $p \in M$ is a projection $\phi(\mathbf{p}) = \phi(p)$ and $\phi(\bar{p}) = \phi(p)$.*

Proof. ϕ preserves order, projections, and Z_M. ∎

Corollary. *If M is a von Neumann algebra which has no abelian central summands and ϕ a *-automorphism such that $\phi(p) = p$ for all core-free projections (i.e. projections p for which $\mathbf{p} = 0$) then ϕ is the identity automorphism.*

Proof. If $c \in M$ is a central projection then $c = \bar{p}$ for a core-free projection $p \in M$ [5, Lemma 4]. Hence $\phi(c) = \phi(\bar{p}) = \overline{\phi(p)} = \bar{p} = c$. If p is any projection, $p - \mathbf{p}$ is core-free so $p - \mathbf{p} = \phi(p - \mathbf{p}) = \phi(p) - \phi(\mathbf{p}) = \phi(p) - \mathbf{p}$. Hence $\phi(p) = p$ for all projections. Since ϕ is ultra-weakly continuous and M the ultra-weak closure of the linear span of its projections the result follows. ∎

Now suppose ϕ is a *-automorphism of the von Neumann algebra M such that $[\phi(x), x] \in Z_M\ \forall x \in M$.

Lemma 5. $[\phi(x), x] = 0\ \forall x$.

Proof. If $x = x^*$ then $[x, [\phi(x), x]] = 0$ implies $[\phi(x), x] = 0$ by [7, Lemma 6]. By the *-linearity of ϕ the result holds for all x. ∎

Lemma 6. *If $\phi(x_0) \neq x_0$ there exists a nonzero central projection c_0 such that $c_0x_0 \in Z_M$. If, in addition, $c_0x_0 = 0$ then $x_0 = (1 - c_0)\,\phi(x_0)$.*

Proof. We follow [4, Lemma 1]. $[\phi(x), x] = 0$ implies $[\phi(x+y), x+y] = 0$. Hence $[\phi(x), y] = [x, \phi(y)]\ \forall x, y \in M$. Replacing y by xy, $[\phi(x), xy] = [x, \phi(xy)] = [x, \phi(x)\,\phi(y)] = \phi(x)[x, \phi(y)] + [x, \phi(x)]\,\phi(y) = \phi(x)[x, \phi(y)]$. Moreover, $[\phi(x), xy] = x[\phi(x), y] + [\phi(x), x]y = x[\phi(x), y] = x[x, \phi(y)]$. This implies $(\phi(x) - x)[x, \phi(y)] = 0\ \forall y$ so that $(\phi(x) - x)[x, t] = 0\ \forall t$ since ϕ is onto. Finally replacing t by st we get

$$(\phi(x) - x)\, s[x, t] = 0 \qquad \forall x, s, t \in M. \tag{2}$$

If $\phi(x_0) - x_0 \neq 0$ then $(M(\phi(x_0) - x_0)M)^-$ is a nonzero ultra-weakly closed two-sided ideal in M. There exists a central projection $c_0 \in M$ such that $(M(\phi(x_0) - x_0)M)^- = M_{c_0}$. From (2) we have that $M(\phi(x_0) - x_0)\, M[x_0, t] = 0$ $\forall t \in M$ so that $M_{c_0}[x_0, t] = 0\ \forall t$. In particular, $[c_0x_0, c_0t] = 0\ \forall t \in M$ so that $c_0x_0 \in Z_{M_{c_0}} = (Z_M)_{c_0} \subseteq Z_M$.

Now M contains the identity operator which implies $\phi(x_0) - x_0 \in M_{c_0}$. Hence $\phi(x_0) - x_0 = c_0t$ for some $t \in M$. If $c_0x_0 = 0$ then $0 = (1 - c_0)(\phi(x_0) - x_0) = (1 - c_0)\,\phi(x_0) - x_0$. ∎

LEMMA 7. *If p is a core-free projection in M then $\phi(p) = p$ or $\bar{p}\phi(p) = p$.*

Proof. Suppose $\phi(p) \neq p$ and let $\bar{p} = c$. By Lemma 4 there exists a nonzero central projection c_0 such that $c_0p \in Z_M$. Since p is core-free and $c_0p \leqslant p$ we have $c_0p = 0$. Hence $0 = \overline{c_0p} = c_0\bar{p} = c_0c$. Also by Lemma 4, $p = (1 - c_0)\phi(p)$ so that $p = cp = c(1 - c_0)\,\phi(p) = c\phi(p) = \bar{p}\phi(p)$. ∎

Our intention is to show that ϕ is the identity on one summand of M and the other summand is abelian. Since ϕ takes abelian subalgebras of M onto abelian subalgebras and central projections onto central projections it will take the I_1 summand (= largest abelian central summand) of M onto the I_1 summand. Hence we assume that M has no abelian summands. By the corollary to Lemma 2, if ϕ is not the identity on M there exists a core-free projection p such that $\phi(p) \neq p$. Let $\{p_\alpha\}$ be a maximal collection of parallel core-free projections such that $\phi(p_\alpha) \neq p_\alpha$ and set $p = \sum p_\alpha,\ c = \bar{p}$.

LEMMA 8. *$\phi\,|_{M_{1-c}}$ is the identity automorphism.*

Proof. If $\phi\,|_{M_{1-c}}$ is not the identity on M_{1-c} there exists a core free $p_0 = p_0(1 - c) \in M_{1-c}$ such that $\phi(p_0(1 - c)) \neq p_0(1 - c)$. But if $p_0(1 - c)$ is core-free in M_{1-c} it is core-free in M. By the maximality of $\{p_\alpha\}$, $p_0(1 - c) = 0$. But then $\phi(p_0(1 - c)) = p_0(1 - c)$. ∎

COROLLARY. *ϕ maps M_c onto M_c.*

Proof. $1 - c = \phi(1 - c) = \phi(1) - \phi(c) = 1 - \phi(c)$. $\phi(xc) = \phi(x)\,\phi(c) = \phi(x)c$. ∎

LEMMA 9. $M_c = \{0\}$.

Proof. We have $p_\alpha = \bar{p}_\alpha \phi(p_\alpha)$ and $c = \bar{p} = \overline{\sum p_\beta} = \sum \bar{p}_\beta$. Claim: $\bar{p}_\beta \phi(p_\alpha) = 0$ if $\beta \neq \alpha$. For, $p_\beta = \bar{p}_\beta \phi(p_\beta)$ so that $\bar{p}_\beta = \bar{p}_\beta \overline{\phi(p_\beta)} = \bar{p}_\beta \phi(\bar{p}_\beta)$. Hence $\bar{p}_\beta \phi(p_\alpha) = \bar{p}_\beta \phi(\bar{p}_\beta) \phi(p_\alpha) = \bar{p}_\beta \phi(\bar{p}_\beta p_\alpha) = 0$ since p_α and p_β are parallel for $\alpha \neq \beta$. Since ϕ maps M_c on M_c, $\phi(p_\alpha) = \phi(p_\alpha)c = \phi(p_\alpha) \sum \bar{p}_\beta = \phi(p_\alpha) \bar{p}_\alpha = p_\alpha$ contradicting the choice of the p_α. ∎

THEOREM 5. *If ϕ is a centralizing *-automorphism of the von Neumann algebra M and d the largest central projection of M for which M_x is abelian then $\phi |_{M_{1-d}}$ is the identity automorphism on M_{1-d}.*

Remark 1. In the proof of Theorem 1 for prime rings essential use is made of the result that if d_1, d_2 and $d_1 d_2$ are derivations of a prime ring then $d_1 = 0$ or $d_2 = 0$. If M is a von Neumann algebra, p and q nonzero core-free projections with $\bar{p}\bar{q} = 0$ but $\bar{p} \neq 0$ and $\bar{q} \neq 0$ then $[p, [q, x]] = 0$ for all $x \in M$. That is, if $d_1(x) = [p, x]$, $d_2(x) = [q, x]$ then $d_1 d_2 = 0$ is a derivation but neither d_1 nor d_2 is zero.

Remark 2. It would be desirable to prove Theorem 2 without the assumption that ϕ preserves adjoints. We can do this in the special case that $\phi(x) = axa^{-1}$ and $a \in M$. For, suppose ϕ has this form and $[\phi(x), x] = 0$ $\forall x \in M$. As before this implies $[\phi(x), y] = [x, \phi(y)]$ $\forall x, y \in M$. In particular $[\phi(x), a] = [x, \phi(a)] = [x, a]$ $\forall x \in M$, or $[axa^{-1}, a] = [x, a]$. Hence, multiplying this relation out, we get $2ax = xa + a^2xa^{-1}$ and multiplying this on left by a^{-1} we get

$$2x = a^{-1}xa + axa^{-1} \qquad \forall x \in M. \tag{3}$$

Let $x = p$, p an idempotent. Then $a^{-1}pa = r$ and $apa^{-1} = q$ are idempotents and (3) becomes

$$2p = r + q. \tag{4}$$

This implies

$$2pr = r + qr \qquad \text{and} \qquad 2rp = r + rq. \tag{5}$$

Now $[apa^{-1}, p] = [p, apa^{-1}]$ so that $apa^{-1}p - papa^{-1} = papa^{-1} - apa^{-1}p$ or $2apa^{-1}p = 2papa^{-1}$. That is $rp = pr$. From (5) $rq = qr$.

Now $4p = (2p)^2 = (r + q)^2 = r + 2rq + q = 2p + 2rq$ so that $p = rq$. From (4) $2rq = r + q$ so that $2rq = rq + q$ or $q = rq$. Similarly $r = rq$. Hence $p = r = q$. Now $p = r = a^{-1}pa$ so that $ap = pa$ for all idempotents $p \in M$. Hence $a \in Z_M$.

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