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Physics Letters B 632 (2006) 512–516
PHYSICS LETTERS B
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Skyrme model and isospin chemical potential

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Received 15 August 2005; received in revised form 14 October 2005; accepted 31 October 2005

Available online 10 November 2005

Editor: N. Glover

Abstract

We discuss the stability of the skyrmion solution in the presence of a finite isospin chemical potential μ. Solving numerically the mass of the skyrmion as function of μ, we find a critical value $\mu_c = 222.8$ MeV where the skyrmion mass vanishes. We compare the exact numerical treatment with an analytical discussion based on a special shape for the profile of the skyrmion due to Atiyah and Manton. The extension of this ansatz for finite μ works quite well for $\mu < 121$ MeV. Then, for small values of μ, where the analytical approach is valid, we consider the possibility of having an angular deformation for the skyrmionic profile, which is possible for finite values of μ. This is, however, a small effect. Finally we introduce finite temperature corrections, which strength the instability induced by the chemical potential, finding the dependence of the critical temperature on μ.

1. Introduction

The skyrmion picture [1] has attracted the attention of many people as a possible way for understanding the hadronic dynamics, as well as the hadronic phase structure. The behavior of hadrons in the presence of a media, i.e. taking into account temperature and/or density effects, can be analyzed according to this perspective. For example, in a baryon-rich environment, it is found that coupling constants like g_A and F_π become quenched [2]. In Refs. [3,4], which extends the construction of Atiyah and Manton in [5], the authors discussed the stability of skyrmions under finite temperature conditions, showing the existence of a critical temperature T_c. The skyrmion is not longer stable for $T > T_c$.

A similar kind of discussion has been carried out in the frame of the so-called hybrid models [6]. The construction of Hadrons as a core, described trough a chiral bag, surrounded by the tail of a skyrmion, has the same instability properties for a certain critical temperature. This can be interpreted as the occurrence of a deconfining phase transition.

Recently [7], we have analyzed different topological structures in field theory when a finite isospin chemical potential (μ) is taken into account. In particular, we were able to find an upper bound μ_c for the isospin chemical potential, such that the mass of the skyrmion vanishes.

In this Letter, we improve our previous discussion [7], showing that, in fact, the skyrmion develops an instability as function of μ, in the sense that the mass of the skyrmion, $M = M(\mu)$ diminishes and vanishes at a certain critical value μ_c. We found μ_c through an exact numerical analysis of the skyrmion mass evolution, without referring to any particular radial profile. In a second stage, we compare the numerical results with an analytical treatment based on the Atiyah–Manton ansatz [5]. We would like to emphasize that this profile is obtained from a construction based on $SU(2)$ instantons in four dimensions, by computing Holonomies along lines parallel to the time axis. In this sense, this profile is definitely more fundamental than just an educated guess. The comparison between the numerical analysis, and the analytical procedure works quite well, in a wide range of μ up to a certain value of the order of $\mu \approx 110$ MeV, where in the analytical approach it is not longer possible to get a stable mass for the skyrmion.

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doi:10.1016/j.physletb.2005.10.082

In addition, we propose an extension of the radial ansatz of the skyrmion when $\mu \neq 0$, showing that a small angular deformation occurs for the minimum energy configuration, for small values of μ. This is done in the frame of a natural extension of Atiyah–Manton's ansatz. Finally we include temperature corrections, finding how the critical temperature depends on μ.

2. Isospin chemical potential and skyrmion stability

The Skyrme Lagrangian is

$$\mathcal{L} = \frac{F_\pi^2}{16} \operatorname{Tr}\left[\partial_\mu U \partial^\mu U^\dagger\right] + \frac{1}{32e^2} \operatorname{Tr}\left[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger\right]^2, \quad (1)$$

where F_π is the pion decay constant and e is a numerical parameter, following the notation used in [8]. The introduction of an isospin chemical potential can be easily done by introducing a covariant derivative of the form [9,10]

$$\partial_\nu U \to D_\nu U = \partial_\nu U - i\frac{\mu}{2}\left[\sigma^3, U\right] g_{\nu 0}. \quad (2)$$

Defining $L_\nu \equiv (\partial_\nu U)U^\dagger$, the Lagrangian density for finite μ becomes

$$\mathcal{L}_\mu = \frac{F_\pi^2}{16} \operatorname{Tr}\left\{\partial_\nu U \partial^\nu U^\dagger + \frac{\mu^2}{2}\left[\sigma_0 - U\sigma_3 U^\dagger \sigma_3\right]\right\} + \frac{1}{32e^2} \operatorname{Tr}\left\{[L_\rho, L_\nu]^2 - \frac{\mu^2}{2}\left([\varrho, L_\nu][\varrho, L^\nu]\right)\right\}, \quad (3)$$

where $\varrho = \sigma_3 - U\sigma^3 U^\dagger$ and σ_0 is the 2×2 unit matrix.

With a little algebra, we can split the Lagrangian into a zero and a finite chemical potential contributions

$$\mathcal{L}_\mu = \mathcal{L}_{\mu=0} + \frac{F_\pi^2 \mu^2}{32} \operatorname{Tr}\left[\sigma_0 - U\sigma_3 U^\dagger \sigma_3\right] + \frac{\mu^2}{64e^2} \operatorname{Tr}[\varrho, L_\nu]^2. \quad (4)$$

$\mathcal{L}_{\mu=0}$ denotes the zero chemical potential term.

The U field matrix can be parameterized in the standard way

$$U = \exp(-i\xi \vec{\sigma} \cdot \hat{n}) = \cos\xi - i(\vec{\sigma} \cdot \hat{n}) \sin\xi, \quad (5)$$

where $\vec{\sigma}$ is the sigma matrix vector and $\hat{n}^2 = 1$. This ansatz has a "Hedgehog" shape and the following boundary conditions have to be satisfied [8]

$$\xi(\vec{r}) = \xi(r), \qquad \hat{n} = \hat{r}, \qquad \xi(0) = \pi, \qquad \xi(\infty) = 0. \quad (6)$$

The mass of the skyrmion, for static solutions, develops a dependence on the isospin chemical potential as well as on the temperature. For the remaining of this section, we will concentrate on the $T = 0$ scenario. If we define the non-dimensional variable $\hat{r} = eF_\pi r$, the mass of the skyrmion will be given by

$$M_\mu = M_{\mu=0} - \frac{\mu^2}{4e^3 F_\pi} I_2 - \frac{\mu^2}{32e^3 F_\pi} I_4, \quad (7)$$

where $M_{\mu=0}$ is the zero chemical potential contribution

$$M_{\mu=0} = \frac{F_\pi}{4e}\left\{4\pi \int_0^\infty d\hat{r}\left[\frac{\hat{r}^2}{2}\left(\frac{d\xi_1}{d\hat{r}}\right)^2 + \sin^2(\xi_1)\right] + 4\pi \int_0^\infty d\hat{r} \frac{\sin^2(\xi_1)}{\hat{r}^2}\left[4\hat{r}^2\left(\frac{d\xi_1}{d\hat{r}}\right)^2 + 2\sin^2(\xi_1)\right]\right\}, \quad (8)$$

and I_2 and I_4 are the following non-dimensional integrals

$$I_2 = \int d^3\hat{r} \operatorname{Tr}\left[\sigma_0 - U\sigma_3 U^\dagger \sigma_3\right], \qquad I_4 = \int d^3\hat{r} \operatorname{Tr}[\varrho, L_\nu]^2. \quad (9)$$

We would like to emphasize that the chemical potential terms contribute with opposite sign. This implies that the solution becomes unstable above certain value of μ.

Assuming a radial symmetric profile ($\xi = \xi(r)$), we can evaluate directly the traces in (9), getting

$$\operatorname{Tr}\left[\sigma_0 - U\sigma_3 U^\dagger \sigma_3\right] = 4\sin^2\theta \sin^2\xi, \quad (10)$$

$$\operatorname{Tr}[\varrho, L_\nu]^2 = 32\sin^2\theta \sin^2\xi\left(\frac{d\xi}{d\hat{r}}\right)^2 + 4\left(\frac{32}{\hat{r}^2}\right)\sin^2\theta \sin^4\xi, \quad (11)$$

leading us to the following integrals

$$I_2 = \frac{4\pi}{3} \int d\hat{r}\, \hat{r}^2 \sin^2\xi, \quad (12)$$

$$I_4 = \frac{32\pi}{3} \int d\hat{r}\, \hat{r}^2\left[\sin^2\xi\left(\frac{d\xi}{d\hat{r}}\right)^2 + \frac{4}{\hat{r}^2}\sin^4\xi\right]. \quad (13)$$

The variational equation for the profile according to (7) is

$$\left(\frac{1}{4}\hat{r}^2 + 2\sin^2\xi\right)\frac{d^2\xi}{d\hat{r}^2} + \frac{1}{2}\hat{r}\frac{d\xi}{d\hat{r}} + \sin 2\xi\left(\frac{d\xi}{d\hat{r}}\right)^2 - \frac{1}{4}\sin 2\xi - \frac{\sin^2\xi \sin 2\xi}{\hat{r}^2} - \frac{\hat{\mu}^2\hat{r}^2\sin^2\xi}{3}\left(\frac{1}{2}\frac{d^2\xi}{d\hat{r}^2} + \frac{1}{2\sin\xi}\left(\frac{d\xi}{d\hat{r}}\right)^2 + \frac{1}{\hat{r}}\frac{d\xi}{d\hat{r}} - \frac{1}{4}\frac{\sin 2\xi}{\sin^2\xi} - \frac{2\sin 2\xi}{\hat{r}}\right) = 0, \quad (14)$$

where $\hat{\mu} = \mu/(eF_\pi)$. Eq. (14) can be solved numerically for the profile $\xi(r)$, for different values of μ (Fig. 1). Notice that the radial extension of the skyrmion grows as a function of μ.

In order to obtain the mass of the skyrmion, the profile has to be inserted, numerically, in Eq. (7). Fig. 2 shows the chemical potential dependence of the mass. The point where the mass vanishes corresponds to the critical value $\mu_c = 222.8$ MeV. This value does not depend on a model for the shape of $\xi(r)$. It is a fundamental result associated to the skyrmion picture.

The same analysis can be done using a specific profile for $\xi(r)$. Here we will use the ansatz introduced by Atiyah and

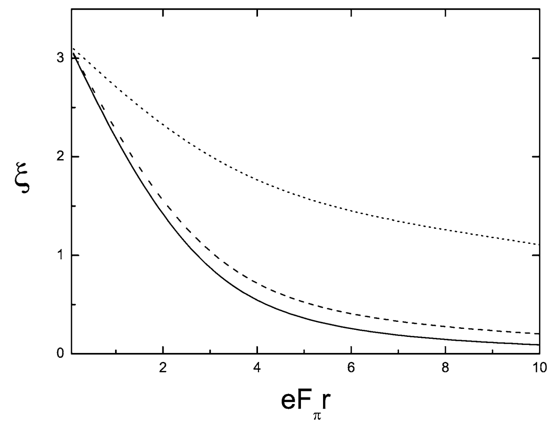

Fig. 1. Numerical solution for the profile $\xi(r)$ for different values of μ ($\mu = 12.9$ (MeV): solid line; $\mu = 25.8$ (MeV): dashed line; $\mu = 38.7$ (MeV): dotted line).

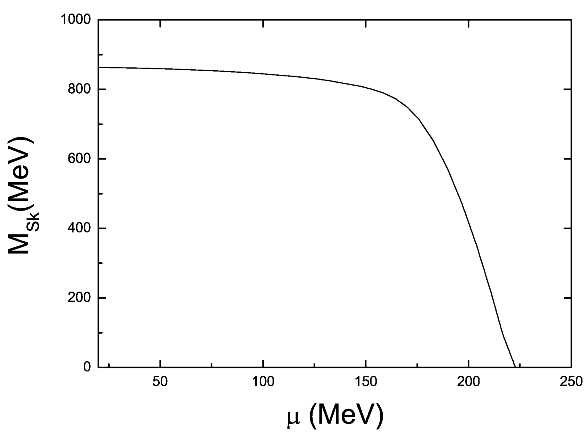

Fig. 2. The numerical solution for the skyrmion mass as function of μ.

Manton in [5], which depends on a free parameter λ:

$$\xi_\lambda(r) = \pi\left[1 - \frac{r}{\sqrt{r^2+\lambda^2}}\right]. \tag{15}$$

The variation with respect to λ enables us to find the lower mass of the skyrmion. This approach has proved to fit very accurately the numerical value of the mass of the skyrmion in the $\mu = 0$ scenario.

The function ξ can be scaled

$$\xi_\lambda(r) = \xi_1(r/\lambda), \tag{16}$$

$$\frac{d\xi_\lambda(r)}{dr} = \frac{1}{\lambda}\frac{d\xi_1(r/\lambda)}{d(r/\lambda)}, \tag{17}$$

in order to factorize λ from the integrals in (13). Using $y = r/\lambda = \hat{r}/\tilde{\lambda}$, with $\tilde{\lambda} = eF_\pi\lambda$, we obtain

$$I_2 = \tilde{\lambda}^3\left(\frac{4\pi}{3}\right)\int_0^\infty dy\, y^2 \sin^2\bigl(\xi_1(y)/2\bigr) = 1.927\left(\frac{32\pi}{3}\right)\tilde{\lambda}^3, \tag{18}$$

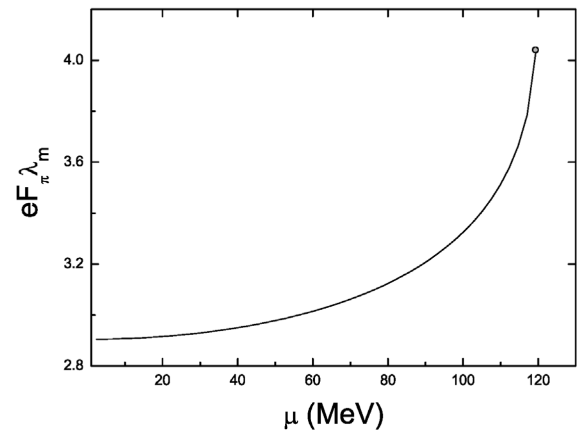

Fig. 3. The evolution of the $\tilde{\lambda}_m/F_\pi$ parameter as function of μ/F_π (the dot denotes the critical point).

$$I_4 = \tilde{\lambda}\left(\frac{4\pi}{3}\right)\int_0^\infty dy\, y^2 \times\left[\sin^2\bigl(\xi_1(y)/2\bigr)\left(\frac{d\xi_1}{dy}\right)^2 + \frac{4}{y^2}\sin^4\bigl(\xi_1(y)/2\bigr)\right] = 3.728\left(\frac{64\pi}{3}\right)\tilde{\lambda}. \tag{19}$$

Defining the dimensionless quantities $\tilde{M}_\mu \equiv M_\mu e/F_\pi$ and $\tilde{\mu} \equiv \mu/(eF_\pi)$, we find the skyrmion mass as a function of the free parameter λ

$$\tilde{M}_\mu = (4\pi)\left[0.504\tilde{\lambda} + 4.254\frac{1}{\tilde{\lambda}} - 0.101\tilde{\mu}^2\tilde{\lambda}^3 - 0.311\tilde{\mu}^2\tilde{\lambda}\right]. \tag{20}$$

The $\tilde{\lambda}_m$ which minimizes the expression takes the value

$$\tilde{\lambda}_m^2 = 0.523\frac{1}{\tilde{\mu}^2} - 0.322 - \frac{1.038}{\tilde{\mu}^2}\sqrt{0.254 - 8.511\tilde{\mu}^2 + 0.097\tilde{\mu}^4}, \tag{21}$$

where we have used the dimensionless parameter $\tilde{\lambda}$ and $e = 5.45$ according to [8]. In Fig. 3 we show the dependence of $\tilde{\lambda}_m$ on μ. In the limit $\mu \to 0$, we recover the result $\tilde{\lambda}_m = 2.904$ [5]. Notice that we have an upper bound for $\tilde{\mu} = \tilde{\mu}_c \to 0.173$, where $\tilde{\lambda}_m$ takes the value $\tilde{\lambda}_m = 4.138$. For bigger values of $\tilde{\mu}$, $\tilde{\lambda}_m$ becomes imaginary, implying, that the skyrmion is no longer stable in the regime $\tilde{\mu} > \tilde{\mu}_c$. The mass of the skyrmion, as function of μ, shows a small decrease, about 8%.

In Fig. 4 we show how the mass dependence on λ evolves for three different values of the chemical potential. Note that for $\tilde{\mu} = 0.926$, the minimum disappears and we have an inflection point signalizing the end of the stable skyrmion phase.

In Fig. 5 we compare the numerical solution for the mass with our analytical result. We see that in a wide region, up to $\mu \approx 100$ MeV, the discrepancy between both approaches is less than 2%, showing that the ansatz is a very good approximation in this region.

Notice that the critical value μ_c where the mass of the skyrmion vanishes, is much lower than the one we obtained in

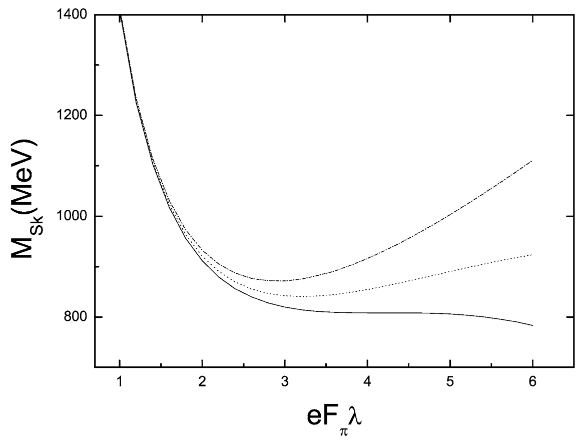

Fig. 4. The skyrmion mass as function of λ for three values of μ ($\hat{\mu} = 1.29$ (MeV): dot-dashed line; $\hat{\mu} = 64.5$ (MeV): dotted line; $\hat{\mu} = 118.68$ (MeV): continuous line).

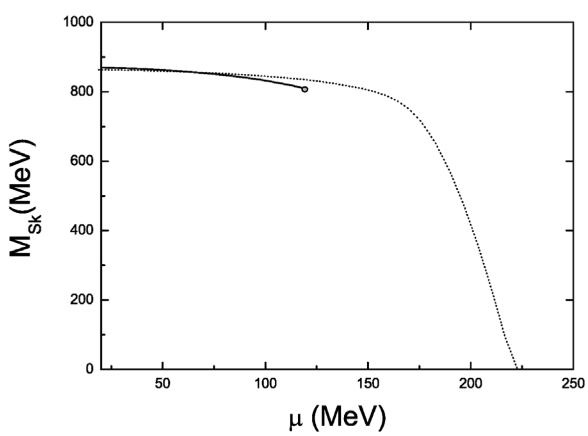

Fig. 5. Comparison between the exact numerical solution (dashed line) and the Atiyah–Manton's ansatz (continuous line) for the mass of the skyrmion.

our previous article [7], using the Atiyah–Manton's ansatz with a fixed value for λ.

3. Anisotropic skyrmion ansatz

If $\mu \neq 0$, we may expect the occurrence of anisotropic stable skyrmions, i.e., solutions where the profile ξ becomes also dependent on the azimuthal angle θ ($\xi = \xi(r, \theta)$). The discussion that follows is based on an appropriate extension of Atiyah–Manton's construction in a region of μ, where we had almost a coincidence between the numerical and analytical solution. A possible simple ansatz which decouples the angular dependence in the integrals (7), is the following

$$\xi_\lambda(r, \theta) = 2\pi \left[1 - \frac{r f(\theta)}{\sqrt{r^2 f^2(\theta) + \lambda^2}}\right], \tag{22}$$

where the function $f(\theta)$ represents a small deformation with respect to the radial configuration. A natural candidate for $f(\theta)$ is to take the first term in a Fourier expansion:

$$f(\theta) = 1 + a \sin\theta + b \cos\theta. \tag{23}$$

We found that the cosine term does not contribute to diminish the skyrmion mass, so we can take $b = 0$. In this way, the energy

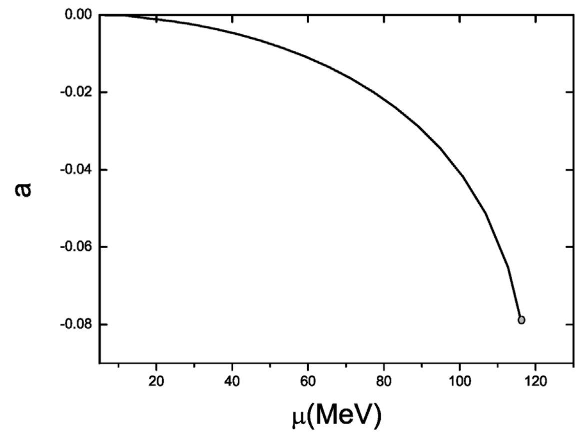

Fig. 6. The dependence of the deformation parameter a as function of μ.

equation is modified, due to the angular terms, according to

$$M_{\mu,a} = \frac{F_\pi \hat{\lambda}}{8e}(\tilde{I}_1 + \tilde{I}_2) + \frac{F_\pi}{32e\hat{\lambda}}(\tilde{I}_3 + \tilde{I}_4) - \frac{\mu^2 \hat{\lambda}^3}{4e^3 F_\pi}\tilde{I}_5 - \frac{\mu^2 \hat{\lambda}}{32 e^3 F_\pi}\tilde{I}_6. \tag{24}$$

In the previous expression, we have introduced the following set of integrals.

$$\tilde{I}_1 = 2\pi \int dr\, r^2\, d\theta \sin\theta \left[(\partial_r \xi_1)^2 + \frac{1}{r^2}(\partial_\theta \xi_1)^2\right],$$

$$\tilde{I}_2 = 2\pi \int dr\, r^2\, d\theta \sin\theta [1 - \cos\xi_1](\partial_i \hat{n})^2,$$

$$\tilde{I}_3 = \pi \int dr\, r^2\, d\theta \sin\theta [1 - \cos\xi_1](\partial_i \xi_1 \partial_j \hat{n} - \partial_j \xi_1 \partial_i \hat{n})^2,$$

$$\tilde{I}_4 = 2\pi \int dr\, r^2\, d\theta \sin\theta (1 - \cos\xi_1)^2 (\partial_i \hat{n} \times \partial_j \hat{n})^2,$$

$$\tilde{I}_5 = 8\pi \int dr\, r^2\, d\theta \sin^3\theta\theta \sin^2(\xi_1/2),$$

$$\tilde{I}_6 = 2\pi \int dr\, r^2\, d\theta \sin\theta \left\{8 \sin^2\theta \sin^2(\xi_1/2) \times \left[(\partial_r \xi_1)^2 + \frac{1}{r^2}(\partial_\theta \xi_1)^2\right] + \frac{32}{r^2} \sin^2\theta \sin^4(\xi_1/2)\right\}. \tag{25}$$

In Fig. 6 we have plotted the dependence of the deformation parameter $a(\mu)$, showing that there is a small shift in the shape of the skyrmions. The criteria for determining the dependence of $a(\mu)$ is to minimize equation (24). Such deformation is possible due to the special character of the skyrmionic solution which is a topological object that entangles the isospin with the spatial degrees of freedom and therefore, a finite isospin chemical potential has a non-trivial consequences.

4. Finite temperature effects

Finally, we will include temperature effects in our discussion, following closely [3], using their thermal profile for the

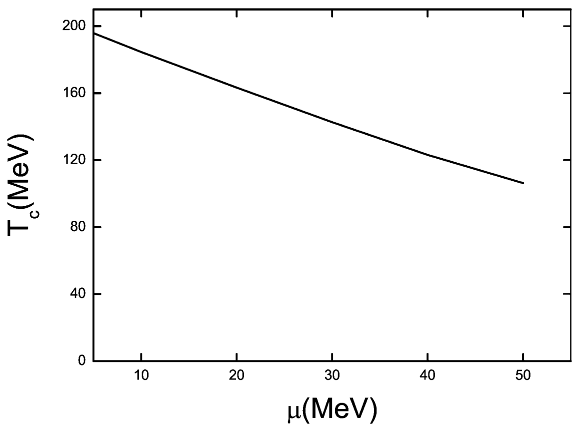

Fig. 7. The dependence of the critical temperature on μ.

skyrmion

$$\xi_\lambda^\kappa(r) = 2\pi\left[1 - \frac{r + \frac{1}{2}\lambda^2(\kappa\coth(\kappa r) - 1/r)}{\sqrt{r^2 + \frac{1}{4}\kappa^2\lambda^4 + \kappa r\lambda^2\coth(\kappa r)}}\right], \tag{26}$$

where $\kappa = 2\pi T$. In the limit $\kappa \to 0$, we recover Eq. (15). Such profile was derived through an extension to finite temperature of the instanton construction by Atiyah and Manton [5].

In this part of the analysis we will not take into account possible angular deformations of the skyrmion, since temperature does not affect the shape of the skyrmion. Also the onset of instability due to chemical potential effects will occur for lower values of μ and therefore we can disregard it.

The procedure we follow is to insert the thermal skyrmion ansatz (26) in Eq. (7). Then we minimize with respect to the parameter λ, getting for a certain μ, the critical temperature T_c where the solution cannot attain a minimum, becoming unstable.

In Fig. 7 we show how T_c depends on μ. It is clear that T_c diminishes with μ. Both parameters, temperature and chemical potential, cooperate in order to increase the occurrence of instabilities in the skyrmion solution.

In this Letter we have discussed the stability of the skyrmion configuration in the presence of finite isospin chemical potential (μ), showing the existence of a critical value μ_c where the mass vanishes. Since the idea behind the skyrmion approach is to have an effective representation of baryons, the existence of such phase transition is quite interesting. There are in the literature other discussions of phase transitions induced by density and temperature. For example, in the frame of chiral Lagrangians, a detailed analysis of the phases of pion systems [11] has been carried out, showing the existence of phase transitions with the same kind of behavior for the masses that we have presented here, in a complete different approach.

Acknowledgements

The authors would like to thank financial support from FONDECYT under grant 1051067.

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