Differential calculus on the Faber polynomials

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Abstract
The Faber polynomials are presented as a coordinate system to study the geometry of the manifold of coefficients of univalent functions.
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Résumé
Les polynômes de Faber sont présentés comme un système de coordonnées pour étudier la géométrie de la variété des coefficients des fonctions univalentes.
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1. Introduction
We show how the methods introduced in [2] and [3] allow to do differential calculus on the manifold of coefficients of univalent functions. The Faber polynomials \((F_k)_{k \geq 1}\) are given by the identity [5,12]

\[
1 + b_1 w + b_2 w^2 + \cdots + b_k w^k + \cdots = \exp \left( - \sum_{k=1}^{+\infty} \frac{F_k(b_1, b_2, \ldots, b_k)}{k} w^k \right).
\]

(1.1)
The polynomials \((G_m)_{m \geq 1}\) and \((K_n^p)_{n \geq 1}, p \in \mathbb{Z}\) are given by

\[
\frac{1}{1 + b_1 w + b_2 w^2 + \cdots + b_k w^k + \cdots} = 1 + \sum_{m=1}^{+\infty} G_m(b_1, b_2, \ldots, b_m) w^m,
\]

\[
(1 + b_1 w + b_2 w^2 + \cdots + b_k w^k + \cdots)^p = 1 + \sum_{n \geq 1} K_n^p(b_1, b_2, \ldots, b_n) w^n,
\]

then \(G_m = K_m^{-1}\) and \(K_m^1 = b_m\). Important polynomials are also the \((P_n^k)_{n \geq 2}\), see [1, (A.1.7)]. If \(f(z) = z h(z)\),

\[
\left(\frac{zf'(z)}{f(z)}\right)^2 [h(z)]^k = \sum_{n \geq 2} P_n^{n+k} z^n \quad \text{for } k \in \mathbb{Z}.
\]

The polynomials \((F_n)_{n \geq 0}, (G_n)_{n \geq 0}, (K_n^p)_{n \geq 0}, (P_n^k)_{n \geq 2}\) are homogeneous of degree \(n\) in the variables \((b_1, b_2, \ldots)\) where \(b_k\) has weight \(k\). As in [1–3,7] let the function of the infinite number of variables

\[(b_1, b_2, \ldots, b_k, \ldots) \rightarrow h(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_k z^k + \cdots.\]

On the infinite dimensional manifold of coefficients \(M = \{(b_1, b_2, \ldots, b_k, \ldots)\}\) of univalent functions, consider the operators [3],

\[
W_j = -\frac{\partial}{\partial b_j} - b_1 \frac{\partial}{\partial b_{j+1}} - \cdots - b_k \frac{\partial}{\partial b_{j+k}} - \cdots \quad \text{for } j \geq 1.
\]

For \(j \geq 1\), it holds \(\frac{\partial}{\partial b_j}[h(z)] = z^j\) and \(W_j[h(z)] = -z^j h(z)\). We have (see [3])

\[
W_j(F_m) = m \delta_{jm} \quad \text{and} \quad W_j(G_m) = G_{m-j} \times 1_{m \geq j},
\]

\[
\frac{\partial}{\partial b_p} W_j - W_j \frac{\partial}{\partial b_p} = -\frac{\partial}{\partial b_{j+p}} \quad \text{and} \quad W_p W_q = W_q W_p,
\]

\[
W_p W_q + W_{p+q} = \sum_{k \geq 0} \sum_{m \geq 0} b_k b_m \frac{\partial^2}{\partial b_{q+m} \partial b_{k+p}}.
\]

For \(k \in \mathbb{Z}\), let

\[
V_j^k = -\sum_{n \geq 0} K_n^{k+1} \frac{\partial}{\partial b_{n+j}}.
\]

We consider for \(j \geq 1\) and \(a \in \mathbb{Z}\), the operators \((V_j^a)_{j \geq 1}\) and for \(a = 1\), we put

\[
V_j = -\sum_{n \geq 0} K_n^{j+1} \frac{\partial}{\partial b_{n+j}}.
\]

Then \(W_j = V_j^0\), \(V_j^{-1} = -\frac{\partial}{\partial b_j}\), \(V_j = V_j^1\), \(j \geq 1\) and

\[
V_j^k V_p^s - V_p^s V_j^k = (k-s)V_{j+p}^{k+s} \quad \text{for } p \geq 1, \ j \geq 1.
\]

The differential operators \((V_j^a)_{j \geq 1}, k \in \mathbb{Z}\) form an algebra and for \(a \in \mathbb{Z}\), the set of \((V_j^a)_{j \geq 1}\) is a subalgebra since

\[
V_j^a V_p^m - V_p^m V_j^a = a(j-p)V_j^{a(j+p)}.
\]
Let $f(z) = zh(z)$. For $j \geq 1$, the vector field $V_j$ is the image through the map $f \to f^{-1}$ of the Kirillov operator

$$L_j = \frac{\partial}{\partial b_j} + \sum_{n \geq 1} (n+1)b_n \frac{\partial}{\partial b_{n+j}}. \quad (1.13)$$

Let $x_1, x_2, \ldots, x_n$ be the roots of $\xi^n + b_1 \xi^{n-1} + b_2 \xi^{n-2} + \cdots + b_{n-1} \xi + b_n = 0$ and consider Newton symmetric functions $\pi_k = x_1^k + x_2^k + \cdots + x_n^k$, $k \geq 1$, it was proved in [3] that

$$\pi_k(b_1, b_2, \ldots, b_n) = F_k(b_1, b_2, \ldots, b_n) \quad \text{for } k \leq n,$$

where $(F_k)_{k \geq 1}$ are the Faber polynomials. This is a consequence of

$$\log(1 + b_1 w + b_2 w^2 + \cdots + b_k w^k + \cdots) = -\sum_{k=1}^{+\infty} \frac{F_k(b_1, b_2, \ldots, b_k)}{k} w^k \quad (1.15)$$

or equivalently (1.1). With this identification, the exact coefficients of the polynomial $F_k(b_1, b_2, \ldots, b_k)$, $k \geq 1$, have been calculated in [3]. The polynomials $(F_k)_{k \geq 1}$ are completely determined as homogeneous polynomial solutions of the system of partial differential equations (1.6) involving $(W_j)_{j \geq 1}$ (See [3]). The exact coefficients of the polynomials $(G_n)$ and of all the $(K^n_p)$ have been given in [3].

The object of this note is to prove that the polynomials $(K^n_p)$ are all obtained as partial derivatives of the Faber polynomials and show how some of the recursion formulae on the polynomials are related to elementary differential calculus on $\mathcal{M}$. This is a step towards the classification of Faber type polynomials (see [8]). In the last section, we give the example of the conformal map from the exterior of the unit disk onto the exterior of $[-2, +2]$. This shows how to introduce non trivial second order differential operators on the manifold $\mathcal{M}$.

**Main Theorem.** We have for $n \geq 1$, $k \geq 1$,

$$\frac{\partial F_n}{\partial b_k} = -nG_{n-k} \times 1_{n \geq k}, \quad (1.16)$$

$$\frac{\partial}{\partial b_k} G_n = -K^{-1}_{n-k} \times 1_{k \leq n}, \quad \frac{\partial}{\partial b_k} K^n_p = pK^{p-1}_{n-k} \times 1_{k \leq n}, \quad \frac{\partial^3 F_j}{\partial b_k \partial b_p} = -2jK^{-3}_{j-(p+k+1)} \times 1_{j \geq k+p+r}.$$

For $j \geq k_1 + k_2 + \cdots + k_s$, $k_1 \geq 1$, $1 \leq k_2, \ldots, k_s \geq 1$ and $s \geq 1$,

$$\frac{\partial^s F_j}{\partial b_{k_1} \partial b_{k_2} \cdots \partial b_{k_s}} = (-1)^s (s-1)!jK^{-s}_{j-(k_1+k_2+\cdots+k_s)} \quad \text{(T1)}.$$

Moreover for $n \geq 1$, $k \geq 1$,

$$K^n_k(b_1, b_2, \ldots, b_n) = K^{-k}_n(G_1(b_1), G_2(b_1, b_2), \ldots, G_j(b_1, b_2, \ldots, b_j), \ldots, G_n(b_1, b_2, \ldots, b_n)) \quad \text{(T2)}.$$

In the notation, all functions are functions of $(b_1, b_2, \ldots, b_n, \ldots)$.

The first $(F_n)$ for $1 \leq n \leq 11$, as well as the first $(G_n)$ are shown in [3]. The $(K^n_p)$, $p \in \mathbb{Z}$, $p \neq 0$ and $n \leq 5$ are in [2]. In [3], an exact expression of the coefficients of all the polynomials $(K^n_p)$ has been given. We explicit the first $K^n_p$,
An expression of $K^\beta_n$ can be obtained as follows, let

$$
\phi_n(z) = b_1z + b_2z^2 + b_3z^3 + \cdots + b_nz^n.
$$
The line integral \( D_k^n = \frac{1}{2i\pi} \int \phi_n(\xi) k^{\xi n} + 1 d\xi \) is equal to the coefficient of \( z^n \) in \( \phi_n(z)^k \) and is of course independent of \( p \). For any \( p \in \mathbb{Z} \), we have,
\[
K_p^n = pb_n + \frac{p(p - 1)}{2} D_n^2 + \frac{p!}{(p - 3)!} D_n^3 + \frac{p!}{(p - 4)!} D_n^4 + \cdots + \frac{p!}{(p - n)!} D_n^n. \tag{T3}
\]
\( D_k^n \) is the sum of terms having \( k \) factors in \( K_k^n \) and
\[
C_p^n = \frac{p!}{n!(p - n)!} = \frac{p(p - 1) \cdots (p - n + 1)}{n!}
\]
is the binomial coefficient. If \( b_1 \neq 0 \),
\[
D_k^n(b_1, b_2, \ldots, b_n) = b_k K_{n-k}^k \left( \frac{b_2}{b_1}, \ldots, \frac{b_{n-k+1}}{b_1} \right). \tag{T4}
\]
Replacing in (T3) and iterating the procedure permits to obtain the exact expression on \( K_p^n \), see [3] and Section 4.2 below.

We have relations between the partial derivatives of the Faber polynomials as
\[
\frac{\partial G_n}{\partial b_k} = -K_{n-k}^{n+1} \times 1_{n \geq k} \quad \text{for all} \ n \geq 1, \ k \geq 1, \ \text{and} \ p \geq 0, \quad \tag{1.17}
\]
\[
\frac{\partial^2 F_n}{\partial b_1^2} = -n \frac{\partial G_n}{\partial b_2}, \quad \frac{\partial^2 F_n}{\partial b_r \partial b_s} = -n \frac{\partial G_n}{\partial b_{r+s}}. \tag{1.18}
\]

Let
\[
X_0 = -\sum_{j \geq 1} b_j \frac{\partial}{\partial b_j} = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \cdots - b_k \frac{\partial}{\partial b_k} - \cdots = -\sum_{j \geq 1} G_j W_j, \tag{1.19}
\]
then
\[
X_0 F_n = -n G_n \quad \text{and} \quad \frac{\partial^2 F_n}{\partial b_r \partial b_s} = \frac{\partial}{\partial b_{r+s}} (X_0 F_n) \ \forall r \geq 1, s \geq 1, \tag{1.20}
\]
\[
\frac{\partial}{\partial b_j} X_0 - X_0 \frac{\partial}{\partial b_j} = -\frac{\partial}{\partial b_j}. \tag{1.21}
\]
This leads to the construction of differential operators on \( \mathcal{M} \) which transform one polynomial into the other. See Section 7.

On the other hand, from (T1), we obtain, see (3.19) and Section 4,

**Main Corollary.** The coefficients of the Schwarzian derivative of \( f(z) = z + b_1 z^2 + b_2 z^3 + \cdots + b_n z^{n+1} + \cdots \) are given in terms of Faber polynomials and their second derivatives as
\[
z^2 S(f)(z) = z^2 \left( \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 \right) = \sum_{k \geq 2} \mathcal{P}_k z^k \]
where
\[
\mathcal{P}_k = -(k - 2) F_k(2b_1, 3b_2, \ldots, (j + 1)b_j, \ldots)
- \frac{1}{2} K_k^2 \left( F_1(2b_1), F_2(2b_1, 3b_2), \ldots, F_k(2b_1, 3b_2, \ldots, (k + 1)b_k) \right) \tag{C1}
\]
and \( K_k^2(F_1(2b_1), F_2(2b_1, 3b_2), \ldots) = -\frac{1}{k+2} \frac{\partial^2 F_{k+2}}{\partial b_1^2}(c_1, c_2, \ldots, c_k) \) is the second derivative \( F_{k+2} \) calculated at the point
(c_1, c_2, \ldots, c_k) = (G_1(F_1(2b_1)), G_2(F_1(2b_1), F_2(2b_1, 3b_2)), \ldots, G_k(F_1(2b_1), \ldots, F_k(2b_1, 3b_2, \ldots, (k+1)b_k))).

The tool is the composition of maps on the manifold $\mathcal{M}$. We have

$$\frac{h'(w)}{h(w)} = -\sum_{k=1}^{+\infty} F_k(b_1, b_2, \ldots, b_k)w^{k-1} = -(F_1 + F_2w + F_3w^2 + \cdots). \tag{1.22}$$

The function $f(w) = wh(w) = w + b_1w^2 + b_2w^3 + \cdots + b_nw^{n+1} + \cdots$ satisfies

$$w\frac{f'(w)}{f(w)} = 1 + w\frac{h'(w)}{h(w)} = 1 - \sum_{k \geq 1} F_k(b_1, b_2, \ldots, b_k)w^k \tag{1.23}$$

and $g(w) = \frac{w}{h(w)} = w + \sum_{n \geq 1} G_n(b_1, b_2, \ldots, b_n)w^{n+1} + \cdots$ satisfies

$$w\frac{g'(w)}{g(w)} = 1 - w\frac{h'(w)}{h(w)} = 1 + \sum_{k \geq 1} F_k(b_1, b_2, \ldots, b_k)w^k. \tag{1.24}$$

From (1.23) and (1.24), we deduce

$$F_n(G_1(b_1), G_2(b_1, b_2), \ldots, G_n(b_1, b_2, \ldots, b_n)) + F_n(b_1, b_2, \ldots, b_n) = 0. \tag{1.25}$$

We consider the following maps from $\mathcal{M} \to \mathcal{M}$,

$$F : (b_1, b_2, \ldots, b_n, \ldots) \to (F_1(b_1), F_2(b_1, b_2), \ldots, F_n(b_1, b_2, \ldots, b_n), \ldots)$$

$$F^{-1} : (b_1, b_2, \ldots, b_n, \ldots) \to (c_1, c_2, \ldots, c_n, \ldots) \text{ such that } F_1(c_1) = b_1, \quad F_2(c_1, c_2) = (b_1, b_2), \quad \ldots,$$

$$F_n(c_1, c_2, \ldots, c_n) = (b_1, b_2, \ldots, b_n). \quad \ldots$$

$$G : (b_1, b_2, \ldots, b_n, \ldots) \to (G_1(b_1), G_2(b_1, b_2), \ldots, G_n(b_1, b_2, \ldots, b_n), \ldots),$$

$$S : (b_1, b_2, \ldots, b_n, \ldots) \to (-b_1, -b_2, \ldots, -b_n, \ldots).$$

The relation (1.25) means that $G$ is obtained as the composition of maps

$$G = F^{-1} \circ S \circ F. \tag{1.26}$$

The first polynomials $(F_n^{-1})_{n \geq 1}$ defined by the map $F^{-1}$ are given by

$$F_1^{-1}(b_1) = -b_1 \quad \text{and} \quad F_2^{-1}(b_1, b_2) = \frac{1}{2}(b_1^2 - b_2),$$

$$F_3^{-1}(b_1, b_2, b_3) = \frac{1}{6}(-b_3^3 + 3b_1b_2 - 2b_3),$$

$$F_4^{-1}(b_1, b_2, b_3, b_4) = \frac{1}{4!}(b_1^4 - 6b_1^2b_2 + 3b_1^2b_3 + b_3^2 + 8b_1b_3 - 6b_4),$$

$$F_5^{-1} = \frac{1}{5!}(-b_1^5 - 15b_1b_2^2 + 10b_1^2b_2 - 3b_1b_2b_3 - 20b_2b_3 + 20b_2b_3 + 30b_1b_4 - 24b_5),$$

$$F_6^{-1} = \frac{1}{6!}(b_1^6 + 144b_1b_5 - 15b_3^2 + 45b_2^2b_1 - 15b_2b_1^2 - 12b_1b_2b_3 + 90b_2b_4$$
$$+ 40b_1^3b_3 - 120b_6 + 40b_2^2 - 90b_1^2b_4).$$
\[ F_{7}^{-1} = \frac{1}{7!} (-b_1^7 - 504b_1^5b_5 + 504b_2b_5 + 840b_1b_6 + 21b_1^5b_2 + 420b_2^2b_3 - 70b_1^6) \\
\left. - 280b_1b_3^2 - 210b_2b_3 - 105b_2^2b_1 + 105b_1b_3b_2 + 420b_3b_4 - 720b_7 \\+ 210b_1^2b_4 - 630b_1b_2b_4, \right\] \\
\[ F_{8}^{-1} = \frac{1}{8!} (b_1^8 - 4032b_1b_2b_5 + 1344b_1^3b_5 + 2688b_3b_5 - 3360b_1^4b_6 + 3360b_2b_6 \\
+ 5760b_1b_7 + 105b_2^4 - 420b_3^2b_1^2 + 1680b_1b_2b_3b_2 - 1120b_1^3b_2b_3 \\
- 420b_4b_2 + 210b_3b_2^2 - 1120b_2b_3^2 + 112b_1^4b_3 + 1120b_1^2b_2^2 - 28b_1^6b_2 \\
- 5040b_8 + 1260b_4^2 + 2520b_2b_3^2b_4 - 1260b_2^3b_4 - 3360b_1b_3b_4) \]

We put \( F_{0}^{-1} = 1 \). We have \( \exp\left(-\sum_{j \geq 1} \frac{b_j}{j} z^j\right) = 1 + \sum_{k \geq 1} F_{k}^{-1}(b_1, b_2, \ldots, b_k)z^k \) and \( \frac{\partial}{\partial b_1} F_{j}^{-1} = -F_{j-1}^{-1}, \quad \forall \, j \geq 2, \)

\[ \frac{\partial}{\partial b_k} F_{p}^{-1} = 0 \quad \text{if} \quad k \geq p + 1, \quad (1.27) \]

\[ \frac{\partial}{\partial b_k} F_{k}^{-1} = -\frac{1}{k} \quad \text{and} \quad \frac{\partial}{\partial b_k} F_{p}^{-1} = -\frac{1}{k} F_{p-k}^{-1} \quad \text{if} \quad k \leq p. \quad (1.28) \]

Differentiating (1.25), we obtain systems of partial differential equations satisfied by the \((F_n)_{n \geq 1}\) and the \((F_n^{-1})\). If \( p \geq 1 \) is an integer, we denote \( p \times S \) the map

\[ p \times S: (b_1, b_2, \ldots, b_n, \ldots) \to (-pb_1, -pb_2, \ldots, -pb_n, \ldots) \]

(1.29) and \( p \times I \) the map

\[ p \times I: (b_1, b_2, \ldots, b_n, \ldots) \to (pb_1, pb_2, \ldots, pb_n, \ldots). \]

(1.30)

Consider the maps, for \( p \in Z, \, p \neq 0, \)

\[ K^p: (b_1, b_2, \ldots, b_n, \ldots) \to (K_{1}^p(b_1), K_{2}^p(b_1, b_2), \ldots, K_{j}^p(b_1, b_2, \ldots, b_j), \ldots). \]

We obtain for \( p \geq 1, \)

\[ K^{-p} = F^{-1} \circ (p \times S) \circ F, \quad (1.31) \]

\[ K^p = F^{-1} \circ (p \times I) \circ F. \quad (1.32) \]

This last relation shows that \( K^p \circ K^q = K^{pq} \circ K^p = K^{pq} \) for \( p \neq 0, \, q \neq 0, \, p, \, q \in Z \). In particular, it is enough to know the \((K^p)\) when \( p \) are prime numbers, to obtain all the other \((K^p), \, p \in Z, \) by composition of maps. From (1.31)–(1.32), we see that

\[ F_n(K_{1}^{-p}(b_1), K_{2}^{-p}(b_1, b_2), \ldots, K_{n}^{-p}(b_1, b_2), \ldots, b_n)) \]

\[ + F_n(K_{1}^p(b_1), K_{2}^p(b_1, b_2), \ldots, K_{n}^p(b_1, b_2), \ldots, b_n)) = 0 \]

which extends (1.25).

The coefficients of \( f^{-1}(z) \) the inverse map of \( f(z) = z + b_1z^2 + b_2z^3 + \cdots \) are given by

\[ f^{-1}(z) = z + \sum_{n \geq 1} \frac{1}{n + 1} K_{n}^{-(n+1)} z^{n+1} \quad (1.33) \]
(compare with [13]) and the coefficients of \( g^{-1}(z) \) the inverse map of \( g(z) = z + b_1 + \frac{b_2}{z} + \frac{b_3}{z^2} + \ldots + \frac{b_{n+1}}{z^n} + \ldots \) are given by
\[
g^{-1}(z) = z - b_1 - \sum_{n \geq 1} \frac{1}{n} K_{n+1}^n \frac{1}{z^n}.
\]
(1.34)

Thus the two maps \( \phi_f \) and \( \phi_g \) defined by
\[
\phi_f : (b_1, b_2, b_3, b_4, \ldots) \to \left( \frac{1}{2} K_1^{-2}, \frac{1}{3} K_2^{-3}, \frac{1}{4} K_3^{-4}, \frac{1}{5} K_4^{-5}, \ldots, \frac{1}{n+1} K_n^{-(n+1)}, \ldots \right),
\]
\[
\phi_g : (b_1, b_2, b_3, \ldots) \to \left( -b_1, -K_2^1, -\frac{1}{2} K_3^2, -\frac{1}{3} K_4^3, -\frac{1}{4} K_5^4, \ldots, -\frac{1}{n} K_{n+1}^n, \ldots \right)
\]
satisfy \( \phi_f \circ \phi_f = \text{Id}_M \) and \( \phi_g \circ \phi_g = \text{Id}_M \).

2. Identities between the polynomials

First, we recall the basic facts relative to the polynomials \((F_j)_{j \geq 0}, (G_j)_{j \geq 0}, (K_n^p), (P_n^p), n \geq 1, p \in \mathbb{Z}\) and the differential operators \((W_j)_{j \geq 1}\).

2.1. Zeroes and particular values of the polynomials

We have
\[
G_1(1) = -1, \quad G_n(1, 1, 1, \ldots, 1) = 0 \quad \text{for } n \geq 2,
\]
\[
G_1(2) = -2, \quad G_2(2, 3) = 1,
\]
\[
G_1(3) = -3, \quad G_n(2, 3, 4, \ldots, k, \ldots, n+1) = 0 \quad \text{for } n \geq 2,
\]
\[
G_2(3, 6) = 3, \quad G_3(3, 6, 10) = -1,
\]
\[
G_4(3, 6, 10, 15) = 0,
\]
\[
G_n\left(3, 6, 10, 15, 21, \ldots, \frac{(n+1)(n+2)}{2}\right) = 0 \quad \text{for } n \geq 5,
\]
\[
G_n\left(1, \frac{1}{2!}, \frac{1}{3!}, \ldots, \frac{1}{n!}\right) = (-1)^n \frac{1}{n!} \quad \text{for } n \geq 1, \tag{2.1}
\]
\[
K_1^{-2}(1) = -2, \quad K_2^{-2}(1, 1) = 1,
\]
\[
K_2^{-3}(1, 1) = 3, \quad K_3^{-3}(1, 1, 1) = -1,
\]
\[
K_n^{-2}(1, 1, \ldots, 1) = 0 \quad \text{for } n \geq 3,
\]
\[
K_n^{-3}(1, 1, \ldots, 1) = 0 \quad \text{for } n \geq 4,
\]
\[
K_n^{-2}(-4, -2, -4, -2, -4, -2, \ldots) = \begin{cases} 8(n-2) & \text{if } n \text{ is odd,} \\ 2(5n-4) & \text{if } n \text{ is even.} \end{cases} \tag{2.2}
\]

For \( n \geq 1, \)
\[
F_n(1, 1, 1, \ldots, 1) = -1, \tag{i}
\]
\[
F_n(-1, 1, \ldots, (-1)^n) = (-1)^{n+1}, \tag{ii}
\]
\[
F_n(4, 9, \ldots, (n+1)^2) = -3 + (-1)^n. \tag{iii}
\]
For any \( p \in \mathbb{Z}, p \neq 0, n \geq 1 \)

\[
F_n(2, 3, 4, 5, \ldots, n + 1) = -2, \quad \text{(iv)}
\]

\[
F_n(1, \frac{1}{2^1}, \frac{1}{3^1}, \ldots, \frac{1}{n^1}) = 0 \quad \text{for } n \geq 2. \quad \text{(v)}
\]

For any \( p \in \mathbb{Z} \)

\[
F_n^{-1}(p, p, p, \ldots, p) = (-1)^p C_n^p \quad \text{with } C_n^p = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}, \quad \text{(vi)}
\]

\[
F_n^{-1}(b_1 + 1, b_2 + 1, \ldots, b_n + 1) = F_n^{-1}(b_1, b_2, \ldots, b_n) - F_n^{-1}(b_1, b_2, \ldots, b_{n-1}). \quad \text{(vii)}
\]

\[
F_n^{-1}(-p, -p, \ldots, -p) = C_n^{n+p-1}, \quad \text{(viii)}
\]

\[
K_n^p(2, 3, 4, 5, \ldots, n + 1) = C_n^{2p+n-1} = (-1)^n C_n^{-2p}, \quad \text{(ix)}
\]

\[
K_n^p(-1, 1, -1, \ldots, (-1)^n) = C_n^{-p}. \quad \text{(x)}
\]

**Proof.** We take the particular case of the function

\[
h(z) = 1 + z + z^2 + \cdots + z^n + \cdots = \frac{1}{1-z} \quad \text{for } |z| < 1.
\]

Since \( \frac{1}{h(z)} = 1 - z \), we find \( G_n(1, 1, 1, \ldots, 1) \) for all \( n \geq 1 \). In the same way, for \( |z| < 1 \), consider

\[
h'(z) = 1 + 2z + 3z^2 + \cdots + (n+1)z^n + \cdots.
\]

Since

\[
1 \quad \frac{h''(z)}{h'(z)} = (1-z)^2 = 1 - 2z + z^2
\]

we find \( G_n(2, 3, 4, \ldots, n+1) \). We continue with

\[
\frac{1}{(1-z)^3} = \frac{h''(z)}{2} = 1 + 3z + 6z^2 + 10z^3 + 15z^4 + 21z^5 + \cdots + \frac{(n+1)(n+2)}{2}z^n + \cdots.
\]

For \( K_n^{-2} \), since \( \frac{1}{h(z)^2} = (1-z)^2 \), we obtain \( K_n^{-2}(1, 1, 1, \ldots, 1) \). For \( K_n^{-3}(1, 1, 1, \ldots) \), we use

\[
\frac{1}{h(z)^3} = 1 - 3z + 3z^2 - z^3.
\]

In this way, we find particular values of \((b_1, b_2, \ldots, b_n, \ldots)\) such that the functions \( G_n \) and \( K_n^{-p}, p \geq 1 \) are zero. We obtain the zeros of \((K_n^p)_{n \geq 1}, p \geq 1\), using the identity (T2) in the Main theorem. To find zeros of \((F_n)_{n \geq 1}\), we take \( h(z) = \exp(z) \). Moreover since for a homogeneous polynomial \( P_n \) of degree \( n \),

\[
P_n(rb_1, r^2b_2, \ldots, r^nb_n) = r^n P_n(b_1, b_2, \ldots, b_n) \quad \forall r \in \mathbb{C}
\]

we obtain for the polynomials \((G_n)_{n \geq 1}, (K_n^{-p})_{n \geq 1} \) and \((F_n)_{n \geq 1}\), curves of zeros in the manifold \( \mathcal{M} \). Of course for these polynomials, there are many other manifolds of zeros. See [4]. To find the special values for \((F_n)_{n \geq 1}\), for (i), we consider \( h(z) = \frac{1}{1-z} \) which gives \( \frac{h'}{h} = \frac{1}{1-z} \). For (ii), we take \( h(z) = \frac{1}{1+z} \). For (iii), (iv), (v) and (2.1), we take the Koebe function

\[
f(z) = \frac{z}{(1-z)^2}, \quad \text{then } \frac{zf''}{f} = 1 + \frac{2z}{1-z} = \frac{1+z}{1-z},
\]

\[
h(z) = f'(z) = \frac{1+z}{(1-z)^3} = 1 + 4z + 9z^2 + \cdots + (n+1)^2z^n + \cdots,
\]

\[
\frac{h'}{h} = \frac{f''}{f'} = \frac{1}{1+z} + \frac{3}{1-z} = 4 + 2z + 4z^2 + 2z^3 + 4z^4 + \cdots, \quad \text{(iii)}'
\]
\begin{align*}
    z^2 S_f(z) &= z^2 \left[ \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 \right] = -\frac{6z^2}{(1-z)^2} = \sum_{k \geq 2} \mathcal{P}_k z^k \\
    &= z^2 \left( 2 + 8z + 6z^2 + 16z^3 + 10z^4 + \cdots \right) \\
    &\quad - \frac{z^2}{2} \left( 16 + 16z + 36z^2 + 32z^3 + 4 \times 14z^4 + \cdots \right). 
\end{align*}

We calculate (iv) with \( \frac{\mathcal{P}}{f} \) and (iii) with (iii)''. To calculate (2.2), we use

\[
\sum_{k \geq 0} K_k^2 (-4, -2, -4, -2, -4, -2, \ldots) z^k = \left( 1 - \frac{z f''}{f'} \right)^2 = \left( \frac{1}{1 + z} - \frac{3z}{1 - z} \right)^2.
\]

To prove (vi),

\[
\exp \left( -p \sum_{j \geq 1} \frac{z^j}{j} \right) = \exp \left[ p \log (1 - z) \right] = (1 - z)^p = 1 + \sum_{n \geq 1} F_n^{-1} (p, p, \ldots, p) z^n.
\]

To prove (vii),

\[
\exp \left( - \sum_{j \geq 1} (b_j + 1) \frac{z^j}{j} \right) = 1 + \sum_{n \geq 1} F_n^{-1} (b_1 + 1, b_2 + 1, \ldots, b_n + 1) z^n
\]

\[
= \exp \left( - \sum_{j \geq 1} b_j \frac{z^j}{j} \right) \times \exp \left( - \sum_{j \geq 1} \frac{z^j}{j} \right) = (1 - z) \times \left( 1 + \sum_{n \geq 1} F_n^{-1} (b_1, b_2, \ldots, b_n) z^n \right)
\]

and we identify equal powers of \( z \). The identity (vii) generalizes the classical identity \( C_n^{p+1} = C_{n-1}^p + C_n^p \) for the binomial coefficients. See for example [6, vol. 1, II-12].

To prove (viii), we have to calculate \( F^{-1} \circ S \), it comes from

\[
\exp \left( \sum_{j \geq 1} \frac{b_j}{j} z^j \right) = 1 + \sum_{k \geq 1} F_k^{-1} (-b_1, -b_2, \ldots, -b_k) z^k.
\]

Taking \( b_j = n \) for all \( j \geq 1 \), \( \exp(n \sum_{j \geq 1} 1 \cdot \frac{z^j}{j}) = \frac{1}{(1-z)^n} = \sum_{j \geq 0} C_n^{n+j-1} z^j \).

To prove (ix), we take the Koebe function \( f(z) \), then \( h(z) = \frac{f(z)}{z} = \frac{1}{1-z} \) and \( [h(z)]^p = \frac{1}{(1-z)^p} = 1 + \sum_{n \geq 1} C_n^{2p+n-1} z^n \). We can also deduce (ix) using the composition of maps: \( K^p (2b_1, 3b_2, \ldots, (n + 1)b_n, \ldots) = F^{-1} \circ pI \circ F (2b_1, 3b_2, \ldots) \). For \( b_1 = b_2 = \cdots = 1 \), we replace \( F (2, 3, \ldots) \) using (iv), then we use (viii). To prove (x), we take \( h(z) = \frac{1}{1+z} \). \( \square \)

**Remark 2.1.** We verify the main corollary (C1) when \( f(z) \) is the Koebe function. From (C1), for the Koebe function,

\[
\mathcal{P}_k = -(k - 2) F_k (4, 9, 16, \ldots, (k + 1)^2) - \frac{1}{2} K_k^2 (-4, -2, -4, -2, -4, \ldots).
\]

If \( k \) is odd, from (iii), we have \( F_k (4, 9, 16, \ldots, (k + 1)^2) = -4 \), thus from (2.2), we find that \( \mathcal{P}_k = 0 \). If \( k \) is even, from (iii), \( F_k (4, 9, 16, \ldots, (k + 1)^2) = -2 \) and using (2.2), we find \( \mathcal{P}_k = -(k - 2) \times (-2) - (5k - 4) = -3k \). Thus \( \mathcal{P}_{2p} = -6p \). Compare with (iii)''.

We obtain values of \( F_n \) and \( G_n \) when the \((b_j)_{j \geq 1}\) are binomial coefficients,
Proposition 2.1. We have

\[ Fn\left( -\binom{n}{1}, \binom{n}{2}, -\binom{n}{3}, \ldots, (-1)^n \binom{n}{n} \right) = n, \]

\[ Gn\left( -\binom{n}{1}, \binom{n}{2}, -\binom{n}{3}, \ldots, (-1)^n \binom{n}{n} \right) = \binom{2n-1}{n}, \]

where \( \binom{n}{k} \) is the binomial coefficient. More generally, let \( q \in \mathbb{C} \), and let

\[ \left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{(1-q^n)(1-q^{n-1}) \cdots (1-q^{n-k+1})}{(1-q)(1-q^2) \cdots (1-q^k)} \]

be the Gaussian polynomial, then

\[ Gn\left( -\left[ \begin{array}{c} n \\ 1 \end{array} \right], q \left[ \begin{array}{c} n \\ 2 \end{array} \right], -q^3 \left[ \begin{array}{c} n \\ 3 \end{array} \right], \ldots, (-1)^n q \frac{n(n-1)}{2} \left[ \begin{array}{c} n \\ n \end{array} \right] \right) = \binom{2n-1}{n}. \]

Proof. We obtain \( F_n \) with

\[ \sum_{j \geq 1} \frac{w^j}{j} = \log \left( 1 - \binom{n}{1} w + \binom{n}{2} w^2 - \binom{n}{3} w^3 + \cdots + (-1)^n \binom{n}{n} w^n \right) = n \log(1-w). \]

We can also deduce the identity for \( F_n \) from (2.2)–(vi) and the composition of maps \( F \circ F^{-1} = \text{Id} \). We obtain \( G_n \) with the relation

\[ \left[ 1 - \binom{n}{1} w + \binom{n}{2} w^2 - \binom{n}{3} w^3 + \cdots + (-1)^n \binom{n}{n} w^n \right]^{-1} = \frac{1}{(1-w)^n} = \sum_{j \geq 0} \binom{n+j-1}{j} w^j \]

when \( \binom{n}{k} \) is the binomial coefficient, and in the case of the Gaussian polynomial,

\[ \left[ 1 - \binom{n}{1} w + q \binom{n}{2} w^2 - q^3 \binom{n}{3} w^3 + \cdots + (-1)^n q \frac{n(n-1)}{2} \binom{n}{n} w^n \right]^{-1} = \frac{1}{\prod_{k=0}^{n-1} (1-q^k w)} = \sum_{k \geq 0} \binom{n+k-1}{k} w^k. \]

From (1.26), we know that \( G = F^{-1} \circ S \circ F \), then \( G \circ F^{-1} = F^{-1} \circ S \), we can deduce the first identity for \( G_n \) from (2.2)–(viii). \( \square \)

2.2. Basic identities

The function \( h(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_k z^k + \cdots \) satisfies

\[ h(z) - zh'(z) = 1 - \sum_{n \geq 2} (n-1) b_n z^n, \]

\[ h(z) - zh'(z) + \frac{z^2 h''(z)}{2} = 1 + \sum_{n \geq 3} \frac{(n-1)(n-2)}{2} b_n z^n, \]
\[ h(z) - zh'(z) + \frac{z^2h''(z)}{2} - \frac{z^3h'''(z)}{3!} = 1 - \sum_{n \geq 4} \frac{(n-1)(n-2)(n-3)}{3!} b_n z^n, \]
\[ \ldots \]

### 2.3. Relations between the polynomials

We differentiate (1.1) with respect to \( w \),
\[
b_1 + 2b_2 w + \cdots + kb_k w^{k-1} + \cdots = \left(1 + b_1 w + b_2 w^2 + \cdots + b_p w^p + \cdots\right) \times \left(-\sum_{j \geq 1} F_j w^{j-1}\right).
\]

We equal coefficients of same powers of \( w \), it gives the recurrence for the polynomials \((F_k)_{k \geq 0}\), \(F_0 = 1\),
\[
-k b_k = \sum_{1 \leq j \leq k} F_j b_{k-j}. \tag{2.3}
\]

With the same approach, one finds other relations between the polynomials as

**Proposition 2.2.**

\[
F_{j+1} = - \sum_{0 \leq r \leq j} (r + 1)b_{r+1}G_{j-r}, \tag{2.4}
\]
\[
n G_n = \sum_{1 \leq j \leq n} F_j G_{n-j}, \tag{2.5}
\]
\[
\frac{n}{p-1} K_n^{1-p} = \frac{1}{r-1} \sum_{1 \leq j \leq n} j K_j^{1-r} K_{n-j}^{r-p} \quad \text{for } 2 \leq r < p, \tag{2.6}
\]
\[
\frac{n}{p-1} K_n^{1-p} = \sum_{1 \leq j \leq n} F_j K_{n-j}^{1-p} \quad \forall p \neq 1, \ p \in \mathbb{Z}, \tag{2.7}
\]
\[
K_n^p = \sum_{0 \leq j \leq n} K_j^r K_{n-j}^{p-r}. \tag{2.8}
\]

**Proof of the identities (2.4)–(2.8).** To find (2.4), we consider \( h(w) = 1 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots + b_k w^k + \cdots \),
\[
h'(w) = b_1 + 2b_2 w + 3b_3 w^2 + \cdots + nb_n w^{n-1} + \cdots.
\]

Since \( \frac{1}{h'(w)} = \sum_{n \geq 0} G_n w^n \), multiplying by \( h'(w) \) gives
\[
\frac{h'(w)}{h(w)} = \sum_{r \geq 0} \left[ \sum_{0 \leq r \leq j} (r+1)b_{r+1}G_{j-r} \right] w^r.
\]

To obtain (2.4), we compare with (1.22). For (2.5), we identify the following expansions
\[
\frac{h'(w)}{h(w)^2} = \frac{h'(w)}{h(w)} \times \frac{1}{h(w)} = \left( \sum_{j \geq 0} -F_{j+1} w^j \right) \times \left( \sum_{p \geq 0} G_p w^p \right).
\]
\[ \frac{h'(w)}{h(w)^2} = -\frac{1}{dw} \frac{1}{h(w)} = -\sum_{n \geq 0} (n + 1)G_{n+1}w^n. \]

To find (2.6) and (2.7), we use that for \(0 \leq r \leq p\),
\[ \frac{h'(w)}{h(w)^p} = \frac{h'(w)}{h(w)^r} \times \frac{1}{h(w)^{p-r}}. \]  

(i)

If \(p \neq 1\) and \(r \neq 1\), (2.6) comes from
\[ \frac{1}{(p - 1)} \frac{1}{dw} \frac{1}{h(w)^{p-1}} = \frac{1}{(r - 1)} \left( \frac{1}{dw} \frac{1}{h(w)^{r-1}} \right) \times \sum_{j \geq 0} K_j^{r-p}w^j. \]

In (i), we take \(r = 1\) and we obtain (2.7) with
\[ \frac{1}{p - 1} \frac{1}{dw} \frac{1}{h(w)^{p-1}} = \left( \sum_{j \geq 0} F_{j+1}w^j \right) \times \frac{1}{h(w)^{p-1}}. \]

Remark 2.1. For \(k \geq 1\), (1.22) yields
\[ \frac{w^{1-k}h'(w)}{h(w)} + F_1w^{1-k} + F_2w^{2-k} + \cdots + F_mw^{m-k} + \cdots + F_{k-1}w^{-1} + F_k \]
\[ = -(F_{k+1}w + \cdots + F_{k+r}w^r + \cdots) \]

and
\[ (F_{k+1}w + \cdots + F_{k+r}w^r + \cdots) \times h(w) \]
\[ = \sum_{j \geq 1} (F_{k+1}b_{j-1} + F_{k+2}b_{j-2} + \cdots + F_{j+k}b_0)w^j. \]

The relations (2.4), (2.5), (2.6) and (2.7) involve the first derivative of \(h\), we can find other relations by multiplying powers of \(h\). For example (2.8) comes from \((h(w))^p = h(w)^r \times h(w)^{p-r}\).

Below, we give further identities between the polynomials. We consider the polynomials \((P_n^k)_{n \geq 1}, k \in \mathbb{Z}\) defined by (1.4). We define \(B_n^k\) by
\[ \left( \frac{zh'(z)}{h(z)} \right)^2 [h(z)]^k = \sum_{n \geq 2} B_n^{n+k}z^n. \]  

(2.9)

Since \(zf' = 1 + b'/n\), we have
\[ p_n^{k+n} = B_n^{n+k} \times 1_{n \geq 2} + \frac{2n + k}{k}K_n^k. \]  

(2.10)

Proposition 2.3. Let \(f(\xi) = \xi(1 + b_1 \xi + b_2 \xi^2 + \cdots)\). For \(k \neq 0\), we have
\[ \phi_k(\xi) = \frac{\xi f'(\xi)}{f(\xi)} \times h(\xi)^k = \sum_{n \geq 0} \frac{k + n}{k} K_n^k \xi^n. \]  

(2.11)

If \(k \geq 1\),
\[ \phi_k(\xi) = \frac{(-1)^k}{k!} \left[ \frac{\partial^k}{\partial b_1^k} \left( \sum_{n \geq 0} F_{k+n} \xi^n \right) \right] (G_1(b_1), G_2(b_1, b_2), \ldots) \]  

(2.12)
and the function $\sum_{n \geq 0} F_{k+n}(b_1, b_2, \ldots, b_{k+n}) z^n$, $k \geq 1$, is given by the line integral
\[
\sum_{n \geq 0} F_{k+n}(b_1, b_2, \ldots) z^n = -\frac{1}{2\pi i} \int \frac{h'(\zeta)}{\zeta^{k-1}(\zeta - z) h(\zeta)} \, d\zeta.
\] (2.13)

If $k \geq 1$, the function $\phi_{-k}(\zeta) = \frac{\zeta f'(\zeta)}{f(\zeta)} \times \frac{\zeta^{k}}{k} = \sum_{n \geq 0} \frac{k-n}{k} K_n^{-k} \zeta^n$ is given by
\[
\phi_{-k}(\zeta) = (\frac{1}{2\pi i} \int l(\zeta) \zeta^n + 1 d\zeta) \frac{\zeta^{k}}{2\pi i} \int l(\zeta) \zeta^{k} (\zeta - z) d\zeta.
\] (2.14)

**Proof.** For (2.11), we use the recursion formula (2.7) or give a direct proof since $\phi_k(\zeta) = h(\zeta) \frac{k}{k} + \frac{\zeta^{k}}{k} \frac{d}{d\zeta} h(\zeta)$ For (2.12), we have from (T1), for $k > 0$ and $j \geq 0$,
\[
(j + k) K_{-j}^k = (\frac{1}{k} \frac{\zeta}{k} \frac{d^{k}}{d\zeta} F_{k+j}.
\] (2.15)

The proof of (2.13) is classical for Laurent series: let $l(\zeta) = \sum_{n \in \mathbb{Z}} \alpha_n \zeta^n$, then
\[
\sum_{n \geq k} \alpha_n \zeta^n = \sum_{n \geq k} \frac{\zeta^n}{2\pi i} \int \frac{l(\zeta)}{\zeta^{n+1}} d\zeta = \frac{\zeta^{k}}{2\pi i} \int l(\zeta) \zeta^{k} (\zeta - z) d\zeta.
\]
We take $l(\zeta) = -\frac{\zeta^{h'(\zeta)}}{h(\zeta)}$. For $\phi_{-k}$, we proceed in the same way. \(\Box\)

**Remark 2.2.** By composition of maps, see (1.31)–(1.32), we can define $(K_n^k)_{n \geq 0}$ for any $k \in \mathbb{R}$ with $K_0^k = 1$ and $(K_1^k, K_2^k, \ldots, K_n^k, \ldots) = F^{-1} \circ k \times \text{Id} \circ F$. From Proposition 2.3, we see that for fixed $n \geq 1$, we have $\lim_{k \to 0} \frac{n-k}{k} K_n^{-k} = F_n$. Since $\lim_{k \to 0} K_n^{-k} = 0$, we obtain for
\[
\lim_{k \to 0} \frac{n-k}{k} K_n^{-k} = \frac{1}{n} \times F_n \quad \text{for} \quad n \geq 1.
\] (2.16)

**Proposition 2.4.** Assume that $f^{-1}$ is the inverse of $f$, $f(f^{-1}(z)) = z$. For $f(\zeta) = \zeta[1 + \sum_{n \geq 1} b_n \zeta^n]$ and $\phi_k(\zeta) = \frac{\zeta f'(\zeta)}{f(\zeta)} \times h(\zeta)^k$, then
\[
\phi_k(f^{-1}(z)) = 1 + \sum_{n \geq 1} P_n^k(b_1, b_2, \ldots, b_n) z^n \quad \text{for} \quad k \neq 0,
\] (2.17)
\[
P_n^k(b_1, \ldots, b_n)
= \sum_{0 \leq s \leq n} K_s^2(-F_1(b_1), \ldots, -F_s(b_1, \ldots, b_s)) \times K_{n-s}^{-n}(b_1, \ldots, b_{n-s}).
\] (2.18)

**Proof.** See [1, (A.1.2)]. In particular,
\[
P_n^0(b_1, \ldots, b_n) = K_n^2(-F_1(b_1), \ldots, -F_n(b_1, \ldots, b_n)),
\]
\[
P_n^0(G_1(b_1), \ldots, G_n(b_1, \ldots, b_n)) = K_n^2(F_1(b_1), \ldots, F_n(b_1, \ldots, b_n))
\]
and $P_n^0(b_1, \ldots, b_n) - P_n^0(G_1(b_1), \ldots, G_n(b_1, \ldots, b_n)) = -4 F_n(b_1, b_2, \ldots, b_n).$ \(\Box\)
(2.19) – Expressions of \( P^k_n \)

If \( n \neq k \),

\[
P^k_n(b_1, \ldots, b_n) = -\frac{1}{k - n} \times \sum_{j=0}^{n} (k - j) F_j(b_1, \ldots, b_j) K_{n-j}^{k-n}(b_1, \ldots, b_{n-j})
\]

\[
= \frac{k}{k - n} K_n^{k-n} - \frac{1}{k - n} \times \sum_{j=1}^{n} (k - j) F_j(b_1, b_2, \ldots, b_j)
\]

\[
\times K_{n-j}^{k-n}(b_1, b_2, \ldots, b_{n-j}). \quad (E)_1
\]

If \( n = k \) (with \( F_0 = -1 \))

\[
P^n_n(b_1, b_2, \ldots, b_n) = \sum_{j=0}^{n} F_j(b_1, b_2, \ldots, b_j) \times F_{n-j}(b_1, b_2, \ldots, b_{n-j}). \quad (E)_2
\]

**Remark 2.3.** If \( k = 1 \), \( f'(f^{-1}(z)) = 1 + \sum_{n \geq 1} P^1_n(b_1, b_2, \ldots) z^n \) with

\[
P^1_n = \frac{1}{n-1} \sum_{0 \leq j \leq n} (1 - j) F_j K_{n-j}^{1-n}. \quad (E)_3
\]

**Proof of (E)_1 and (E)_2.** From [1, (A.1.1)],

\[
\frac{zf'(z)}{f(z)} [h(z)]^k \times \frac{zf'(z)}{f(z)} [h(z)]^p = 1 + \sum_{n \geq 1} P^{n+k+p}_n z^n.
\]

On the other hand,

\[
\frac{zf'(z)}{f(z)} [h(z)]^k = z^{1-k} f'(z) f(z)^{k-1} = z^{1-k} \frac{1}{k} \frac{d}{dz} f(z)^k = \frac{1}{k} \sum_{n \geq 0} (n + k) K_n^k z^n.
\]

Multiplying \( \frac{zf'(z)}{f(z)} [h(z)]^k \times \frac{zf'(z)}{f(z)} [h(z)]^p \), we obtain for any \( k, p \in \mathbb{Z} \), \( k \neq 0, p \neq 0 \),

\[
P^{n+k+p}_n = \sum_{0 \leq j \leq n} (j + k)(n - j + p) \times \frac{1}{p k} K_j^k K_{n-j}^p. \quad (E)_4
\]

We make \( p \to 0 \) as in Remark 2.2, it gives \( P^n_k \). To obtain \( P^n_n \), we make \( k \to 0 \). \( \square \)

**Remark 2.4.** From \( \frac{(j+k)}{k} K_j^k = \frac{(-1)^k}{k!} \frac{\partial^k}{\partial b_1^k} F_{k+j} \) and from (E)_4, we deduce that for \( k > 0, p > 0 \),

\[
P^{n+k+p}_n(b_1, b_2, \ldots) = \sum_{0 \leq j \leq n} \frac{(-1)^{k+p}}{k! p!} \left( \frac{\partial^k}{\partial b_1^k} F_{k+j} \right) (c_1, c_2, \ldots) \times \left( \frac{\partial^p}{\partial b_1^p} F_{p+n-j} \right) (c_1, c_2, \ldots). \quad (E)_5
\]

From (E)_1, we obtain for \( k > n \),
Theorem 2.5. For the Koebe function \( f(z) = \frac{z}{(1-z)^2} \) or \( f(z) = \frac{z}{(1+z)^2} \), we have \( P_{n-k}^{-k} = P_{n+k}^k \) for \( n \geq 1, k \geq 1 \),

\[
P_{n-k}^{-k}(b_1, b_2, \ldots, b_{n-k}) = P_{n+k}^k(b_1, b_2, \ldots, b_{n+k})
\]

for \((b_1, b_2, \ldots) = (2, 3, 4, 5, 6, \ldots)\), \( b_n = n + 1 \).

Conversely, let \( f(z) = z + b_1z^2 + \cdots + b_nz^{n+1} + \cdots \), if \( P_{n-k}^{-k} = P_{n+k}^k \) for \( n \geq 1, k \geq 1 \), then \( f(z) = \frac{z}{(1-z)^2}, \epsilon = 1 \) or \( \epsilon = -1 \).

Moreover, for the Koebe function, we have \( K_{n-j}^{-n} \times 1_{n \geq j} = K_{n-j}^{-n} \times 1_{n-j \geq 0}, n, j \in \mathbb{Z} \). This last relation is the same as the classical \( C_{n-j}^{2n} = C_{n-j}^{2n} \) on the binomial coefficients.

We verify (2.20) with \( n = 3, k = 2 \). With \([1, (A.1.7)]\), we calculate \( P_{1}^{-2} = -b_1 \),

\[
P_2^{-2} = 7b_5 - 20b_1b_4 + 30b_1b_2^2 + 35b_1^2b_3 - 50b_1^2b_2 + 14b_1^3 - 16b_2b_3.
\]

When \( b_n = (n + 1) \), we find \( P_{1}^{-2} = P_2^2 = -2 \). With \([1, (A.1.7)]\),

\[
P_4^{-2}(b_1, b_2, b_3, b_4) = 6b_4 - 12b_1b_3 - 5b_2^2 + 16b_1^2b_2 - 5b_1^4
\]

and \( P_4^2(2, 3, 4, 5) = P_0^{-2} = 1 \).

Proof of Theorem 2.5. Let \( f(z) = \frac{z}{(1-z)^2} \), we have \( \frac{zf'(z)}{f(z)} = \frac{1+z}{1-z} \) and

\[
\left( \frac{zf'(z)}{f(z)} \right)^2 h(z) = \frac{(1+z)^2}{(1-z)^2k+2} = 1 + \sum_{n \geq 1} P_{n+k}^n z^n.
\]

It gives the line integral

\[
P_{n+k} = \frac{1}{2i\pi} \int \left( \frac{\xi f'\xi}{f(\xi)} \right)^2 h(\xi) \frac{d\xi}{\xi^{n+1}} - \sum_{\xi \neq 0} \text{Residue}
\]

\[
= \frac{1}{2i\pi} \int \left( \frac{1+z}{(1-z)^{2k+2}} \right) \frac{d\xi}{\xi^{n+1}} - \text{Residue at } \xi = 1.
\]

With (ii), we find \( P_{n-j}^{-j} = \frac{1}{2i\pi} \int \lambda(\xi)\xi^{-j} d\xi \) and \( P_{n+j}^j = \frac{1}{2i\pi} \int \lambda(\xi)\xi^{-j} d\xi \) with

\[
\lambda(\xi) = \left( \frac{\xi f'\xi}{f(\xi)} \right)^2 \left[ f(\xi) \right]^{-n} = \frac{(1+z)^2}{(1-z)^{2-2n}z^n}
\]

since for \( n \geq 1 \), the function \( \lambda(\xi) \) has only a pole at \( \xi = 0 \). We can calculate the two line integrals on the circle \( |\xi| = 1 \). Since the function \( \lambda(\xi) \) is such that \( \lambda \left( \frac{1}{\xi} \right) = \lambda(\xi) \), we put \( \xi = \frac{1}{\xi} \) and we see that the two line integrals \( P_{n-j}^{-j} \) and \( P_{n+j}^j \) are equal. Consider any \( f(z) \) and let the function
\( k(z) = f(\frac{1}{z}) \), then \( \left( \frac{z^{k'(z)}}{k(z)} \right)^2 = \left( \frac{uf'(u)}{f(u)} \right)^2 \) at \( u = \frac{1}{z} \). The Koebe function satisfies \( f(z) = f(\frac{1}{z}) \). This proves (2.20).

Conversely, assume that \( P^{−k}_{n−k} = P^k_{n+k} \) for \( n \geq 1, k \geq 1 \).

Taking \( n = 1 \), we find \( P^1_2 = P^0_{−1} = 1 \) and \( P^j_{1+j} = 0 \) for \( j \geq 2 \). It gives \( P^{n−1}_n = 0 \) for \( n \geq 3 \) and

\[
\frac{1}{f(z)} \times \frac{z^2 f'(z)^2}{f(z)^2} = \frac{1}{z} + P^0_1 + z
\]

(2.21)

then \( f(z) = f(\frac{1}{z}) \).

Taking \( n = 2 \), we obtain \( P^2_4 = 1 \) and \( P^j_{2+j} = 0 \) for \( j \geq 3 \). Thus \( P^{n−2}_n = 0 \) for \( n \geq 4 \) and from (1.4),

\[
\frac{1}{f(z)^2} \times \frac{z^2 f'(z)^2}{f(z)^2} = \frac{1}{z^2} + P^{-1}_1 \frac{1}{z} + P^0_2 + P^1_3 z + z^2.
\]

(2.22)

We have \( P^{−1}_1 = P^1_3 \) \( (n = 2, j = 1) \). Taking the ratio of (2.20) by (2.21), we see that

\[
f(z) = z \times \frac{1 + P^0_3 z + z^2}{1 + P^1_1 z + P^0_2 z^2 + P^{-1}_1 z^3 + z^4}.
\]

Let \( f(z) = z h(z) \) and \( l(z) = \frac{1}{h'(z)} = \frac{1 + P^1_1 z + P^0_2 z^2 + P^{-1}_1 z^3 + z^4}{1 + P^0_2 z + z^2} \). With (2.21), \( l(z) = 1 + G_1 z + G_2 z^2 + \cdots + G_n z^n + \cdots \) must satisfy

\[
l(z) (1 - \frac{z l'(z)}{l(z)})^2 = 1 + P^0_1 z + z^2
\]

(i)

then

\[
(l(z) - z l'(z))^2 = 1 + P^0_1 + P^0_2 z^2 + P^{-1}_1 z^3 + z^4.
\]

(ii)

Using the identity (2.2), we have \( P^{−1}_1 = 0 \). With (ii), we see that \( 1 + P^0_1 z + z^2 \) must have a double root. It implies that \( (P^0_1)^2 = 4 \). Identifying the coefficients in (ii) gives \( P^0_2 = −2 \) and \( f(z) = \frac{z}{(1−\epsilon z)^2}, \epsilon = 1 \) or \( \epsilon = −1 \).

For the Koebe function, we prove \( K^{−n}_{n−j} = K^0_{n+j} \) in the same way and then apply (2.2)(ix).

**Remark 2.5.** Eq. (2.21) has other solutions than the Koebe function, but (2.21) and \( K^{−1}_{1−j} = K^{−1}_{1+j}, \forall j \geq 1 \) or equivalently \( K^{−1}_2 = 1 \), \( K^{−1}_n = 0, \forall n \geq 3 \), implies that \( f(z) = \frac{z}{(1−\epsilon z)^2}, \epsilon = +1, −1 \). According to (4.8) below, the condition \( K^{−1}_n = 0 \) for \( n \geq 3 \) means that \( \frac{\partial}{\partial b_1} F_n = 0 \) for \( n \geq 4 \).

**Remark 2.6.** When we write the expressions (E)1, (E)2, \ldots\ of \( (P^k_n) \) in the case of the Koebe function, with (2.2)(iii), (iv), (x) we obtain relations between the binomial coefficients.

3. The composition of maps

We consider the polynomials

\[
(b_1, b_1, \ldots, b_n) \rightarrow F_n(b_1, b_1, \ldots, b_n)
\]

\[
(b_1, b_1, \ldots, b_n) \rightarrow G_n(b_1, b_1, \ldots, b_n) \quad n \geq 1,
\]
as functions of \((b_1, b_2, \ldots, b_n)\) and we take composition of maps. We denote \(F_n(G_1, \ldots, G_n)\) the composition of maps

\[
(b_1, b_2, \ldots, b_n) \rightarrow F_n(G_1(b_1), G_2(b_1, b_2), \ldots, G_n(b_1, b_2, \ldots, b_n)).
\]

In the same way, \(G_n(G_1, \ldots, G_n)\) is the composition of maps

\[
(b_1, b_2, \ldots, b_n) \rightarrow G_n(G_1(b_1), G_2(b_1, b_2), \ldots, G_n(b_1, b_2, \ldots, b_n))
\]

and \(G_n(F_1, \ldots, F_n)\) is the composition of maps

\[
(b_1, b_2, \ldots, b_n) \rightarrow G_n(F_1(b_1), F_2(b_1, b_2), \ldots, F_n(b_1, b_2, \ldots, b_n)).
\]

For example, we have \(F_1(G_1)(b_1) = F_1(-b_1) = b_1, \ldots\)

\[
\begin{align*}
G_1(F_1)(b_1) &= b_1, \\
G_2(F_1, F_2)(b_1, b_2) &= F_1^2 - F_2 = 2b_2, \\
G_3(F_1, F_2, F_3)(b_1, b_2, b_3) &= b_1b_2 + 3b_3, \\
G_4(F_1, F_2, F_3, F_4) &= 2b_2^2 + 2b_1b_3 + 4b_4, \\
G_5(F_1, F_2, F_3, F_4, F_5) &= b_1b_2^2 + 7b_3b_2 + 3b_1b_4 + 5b_5, \\
G_6(F_1, F_2, F_3, F_4, F_5, F_6) &= 4b_1b_2b_3 + 2b_2^3 + 6b_3^2 + 10b_2b_4 + 4b_1b_5 + 6b_6, \\
G_7(F_1, F_2, F_3, F_4, F_5, F_6, F_7) &= 17b_3b_4 + b_1b_2^3 + 13b_2b_5 + 11b_2^2b_3 + 4b_1b_3^2 + 5b_1b_6 + 6b_1b_2b_4 + 7b_7, \\
G_8(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8) &= 6b_3b_1b_2^2 + 8b_5b_1b_2 + 12b_4b_1b_3 + 2b_4^2 + 12b_4b_2 + 16b_4b_2^2 + 16b_6b_2 \\
&\quad+ 22b_5b_3 + 20b_2^2b_2 + 6b_1b_7 + 8b_8.
\end{align*}
\]

**Proposition 3.1.** For \(n \geq 1,\)

\[
\begin{align*}
G_n(G_1, G_2, \ldots, G_n) &= b_n, \\
F_n(b_1, b_2, \ldots, b_n) + F_n(G_1, G_2, \ldots, G_n) &= 0.
\end{align*}
\]

**Proof of (3.1) and (3.2).** For (3.1), let \(\tilde{h}(w) = \frac{1}{h(w)} = \sum_{n \geq 0} G_n w^n,\) then \(\frac{1}{\tilde{h}(w)} = h(w).\) In particular, writing \(G_2(G_1, G_2) = b_2\) and \(G_3(G_1, G_2, G_3) = b_3,\) we obtain \(G_1^2 - G_2 = b_2\) and \(-G_1^3 + 2G_1G_2 - G_3 = b_3, \ldots.\) The second relation is an immediate consequence of (1.23) and (1.24). \(\square\)

**Remark 3.1.** 1. The second formula (3.2) can also be proved by recurrence: it is true for \(n = 1.\) We shall use the two recursion formulae (2.5) and (2.3),

\[
\begin{align*}
F_n + b_1F_{n-1} + b_2F_{n-2} + \cdots + b_{n-1}F_1 + nb_n &= 0, \\
F_n + G_1F_{n-1} + G_2F_{n-2} + \cdots + G_{n-1}F_1 - nG_n &= 0.
\end{align*}
\]

From the first relation, we have \(F_2(G_1, G_2) + G_1F_1(G_1) + 2G_2 = 0\) and from the second relation \(F_2(b_1, b_2) + G_1F_1(b_1) - 2G_2 = 0.\) Using that (3.2) is true for \(n = 1\) and adding the two relations above, we obtain (3.2) for \(n = 2\) and the formula by recurrence on \(n.\)
2. We can also find this formula with the recursion formula for the \((G_n)_n \geq 0,\)
\[
G_1 + b_1 = 0, \quad G_2 + b_1G_1 + b_2 = 0, \quad G_3 + b_1G_2 + b_2G_1 + b_3 = 0
\]
and in general, we have \(G_n + b_1G_{n-1} + b_2G_{n-2} + \cdots + b_{n-1}G_1 + b_n = 0\) as follows. From (2.4)
\[F_2(b_1, b_2) = -(b_1G_1 + 2b_2)\] and \(F_2(G_1, G_2) = -(b_1G_1 + 2G_2)\). Adding and using (i) gives the result. We proceed in the same way for \(F_n\).

3. Another proof of (3.2) is to show that \(G = F^{-1} \circ S \circ F\) as follows. From (1.23)–(1.24), we have the map \(\phi : h \to u = 1 - \frac{h^2}{n}\)
\[(b_1, b_2, \ldots, b_n, \ldots) \rightarrow (F_1(b_1), \frac{F_2(b_1, b_2)}{2}, \ldots, \frac{F_j(b_1, b_2, \ldots, b_j)}{j}, \ldots).
\]
Its inverse map gives the Schur polynomials. We calculate \(h\) from \(u\) with the relation \(h(z) = \exp(\frac{1-u(z)}{z})\). The map
\[F : (b_1, b_2, \ldots, b_n, \ldots) \rightarrow (F_1(b_1), F_2(b_1, b_2, \ldots, F_j(b_1, \ldots, b_j), \ldots)
\]
is a bijection. The map \(S : 1 - \frac{h^2}{n} \to 1 + \frac{h^2}{n} = 1 - \frac{h^2}{n^2}\) with \(\tilde{h} = \frac{1}{n}\) is also a bijection. Then the map \(\phi^{-1} \circ S \circ \phi\) is just \(h \to \tilde{h} = \frac{1}{n}\). This gives (1.26). To calculate \(F_n^{-1}\), we have to solve the system in \((b_1, b_2, \ldots, b_n)\),
\[
F_1(b_1) = c_1, \quad F_2(b_1, b_2) = c_2, \quad F_n(b_1, b_2, \ldots, b_n) = c_n, \ldots
\]

**Proof of (1.31)–(1.32).** For \(p \geq 1\), we consider the map \(\phi_p : h \to u = 1 - p\frac{h^2}{n}\) which allows us to calculate \(h^p\).

More identities similar to (3.1) and (3.2) can be found.

**Theorem 3.2.**
\[
F_n(-F_1(b_1), -F_2(b_1, b_2), \ldots, -F_k(b_1, b_2, \ldots, b_k), -F_n(b_1, b_2, \ldots, b_n)) = F_n(2b_1, 3b_2, 4b_3, \ldots, (n + 1)b_n) - F_n(b_1, b_2, b_3, \ldots, b_n),
\]
\[F_n(F_1, F_2, \ldots, F_n) = F_n(0, -b_2, -2b_3, -3b_4, \ldots, -(n - 1)b_n) - F_n(b_1, b_2, \ldots, b_n),
\]
\[
G_n(F_1, F_2, \ldots, F_n) = \sum_{k=0}^{n} b_k G_{n-k}(2b_1, 3b_2, 4b_3, \ldots, (j + 1)b_j, \ldots),
\]
\[
G_n(-F_1, -F_2, \ldots, -F_n) = \sum_{k=0}^{n} b_k G_{n-k}(0, -b_2, -2b_3, \ldots, -(j - 1)b_j, \ldots).
\]
For \(p \in Z, p \neq 0\),
\[
K_n^p(-F_1, -F_2, \ldots, -F_n) = \sum_{k=0}^{n} K_n^{-p}(b_1, b_2, \ldots, b_n)
\]
\[
\times K_k^p(2b_1, 3b_2, \ldots, (j + 1)b_j, \ldots).
\]
\[ K_n^p(F_1, F_2, \ldots, F_n) = \sum_{k=0}^{n} K_{n-k}^{-p}(b_1, b_2, \ldots, b_n) \times K_k^p(0, -2b_3, \ldots, -(j-1)b_j, \ldots). \tag{3.10} \]

**Remark 3.2.** Consider the maps \( D^1 \) and \( D^{-1} \) from \( \mathcal{M} \) to \( \mathcal{M} \),

\[
D^1 : (b_1, b_2, \ldots, b_k, \ldots) \to (2b_1, 3b_2, 4b_3, \ldots, (n+1)b_n, \ldots),
\]

\[
D^{-1} : (b_1, b_2, \ldots, b_k, \ldots) \to (0, b_2, 2b_3, \ldots, (n-1)b_n, \ldots),
\]

then (3.5)–(3.7) can be written as \( F \circ S \circ F = F \circ D^1 - F \) and \( F \circ F = F \circ S \circ D^{-1} \). Remark that \( K^p, F, G, \) and \( S \) are bijection while \( D^{-1} \) is not.

**Proof of Theorem 3.2.** To prove (3.5), we consider \( f(z) = zh(z) \).

\[
\frac{(zf')'}{zf'} = - \sum_{k \geq 1} F_k(-F_1, -F_2, \ldots, -F_k)z^{k-1}. \tag{i} \]

On the other hand

\[
\frac{d}{dz} \log \left( \frac{zf'}{f} \right) = \frac{1}{z} + \frac{(f')'}{f'} = \frac{1}{z} + \left( \frac{f''}{f} - \frac{(f')^2}{f^2} \right) \times \frac{f}{f'} = \frac{1}{z} + \frac{f''}{f'} - \frac{f'}{f} = \frac{1}{z} - \frac{f'}{f} - \sum_{k \geq 1} F_k(2b_1, 3b_2, \ldots, (k+1)b_k)z^{k-1}.
\]

Using the expression of \( \frac{1}{z} - \frac{f'}{f} \), we deduce

\[
\frac{(zf')'}{zf'} = \sum_{k \geq 1} F_k(b_1, b_2, \ldots, b_k)z^{k-1} - \sum_{k \geq 1} F_k(2b_1, 3b_2, \ldots, (k+1)b_k)z^{k-1}. \tag{ii} \]

The comparison of (i) and (ii) gives (3.5).

For (3.6), let \( g(z) = \frac{z}{h(z)} \). Then \( u(z) = \frac{g'(z)}{g(z)} = 1 + \sum_{n \geq 1} F_n(b_1, b_2, \ldots, b_n)z^n \) satisfies

\[
\frac{u'(z)}{u(z)} = - \sum_{k \geq 1} F_k(F_1(b_1), F_2(b_1, b_2), \ldots, F_k(b_1, b_2, \ldots, b_k))z^{k-1}. \tag{i} \]

Thus we obtain \( F_k(F_1, F_2, \ldots, F_k) \) from this expansion. On the other hand, we calculate

\[
\frac{u'(z)}{u(z)} = \frac{d}{dz} \log(u(z)) = \frac{1}{z} \left( - \sum_{k \geq 1} F_kz^k + z \frac{g''(z)}{g'(z)} \right).
\]

Since \( g'(z) = \frac{h(z) - zh'(z)}{h(z)^2} \), we obtain

\[
\frac{z}{g'(z)} \frac{g''(z)}{g'(z)} = \frac{(h - zh')'}{h - zh'} - 2z \frac{h'}{h} = \frac{(h - zh')'}{h - zh'} + 2 \sum_{k \geq 1} F_kz^k.
\]
Using $h(z) - zh'(z) = 1 + \sum_{n \geq 2} (1 - n) b_n z^n$, we deduce
\[
\frac{u'(z)}{u(z)} = \frac{1}{z} \left( \sum_{k \geq 1} F_k z^k - \sum_{k \geq 1} F_k (0, -b_2, -2b_3, \ldots, -(k - 1)b_k) z^k \right).
\]

(ii) Then we compare the two identities (i) and (ii).

To prove (3.7), we consider
\[
\frac{h(w)}{h(w) - wh'(w)} = \frac{1}{1 + b_1 w + b_2 w^2 + \cdots} \times \sum_{n \geq 0} G_n(0, -b_2, \ldots, (n - 1)b_n, \ldots) w^n
\]
since
\[
h(w) - wh'(w) = (1 + b_1 w + b_2 w^2 + \cdots + b_n w^n + \cdots) - (b_1 w + 2b_2 w^2 + \cdots + nb_n w^n + \cdots) = 1 - b_2 w^2 - 2b_3 w^3 - \cdots - (n - 1)b_n w^n - \cdots.
\]
To prove (3.8), we write
\[
\frac{h(w)}{h(w) - wh'(w)} = \frac{1 + b_1 w + b_2 w^2 + \cdots}{1 + b_2 w^2 - b_2 w^3 - \cdots (n - 1)b_n w^n - \cdots}
\]

This is also equal to \(\frac{(h(z) + zh'(z))^p}{h(z)^p}\). For (3.10), we take \(1 - z \frac{h'(z)}{h(z)^p}\). □

**Remark 3.3.** With (3.7)–(3.10) we define differential operators on Faber polynomials. For (3.7)–(3.8), we have (see (1.16)) $G_{n-k} = -\frac{1}{n} \frac{\partial}{\partial b_k} F_n$, thus
\[
G_n(-F_1, -F_2, \ldots, -F_n) = -\frac{1}{n + 1} \left[ \sum_{k \geq 0} b_k k + 1 \frac{\partial F_{n+1}}{\partial b_{k+1}} \right] (2b_1, 3b_2, 4b_3, \ldots, (n + 1)b_n).
\]

For (3.9), we take for example $p = 2$. See Proposition 2.4. With (1.16), we deduce that
\[
K_{n-k}^{-2} = \frac{1}{n + 1} \frac{\partial}{\partial b_k} \left( \frac{\partial}{\partial b_1} F_{n+1} \right) = \frac{1}{n + 2} \frac{\partial}{\partial b_k} \left( \frac{\partial}{\partial b_2} F_{n+2} \right)
\]
and
\[
K_n^2(-F_1, -F_2, \ldots, -F_n) = \frac{1}{n + 1} U_2 \left( \frac{\partial}{\partial b_1} F_{n+1} \right) + K_{n-2}(b_1, \ldots, b_n)
\]
\[
= \frac{1}{n + 2} U_2 \left( \frac{\partial}{\partial b_1} F_{n+2} \right) + \frac{1}{n + 2} \frac{\partial^2 F_{n+2}}{\partial b_1^2},
\]
where $U_2$ is the differential operator
\[
U_2 = \sum_{k \geq 1} K_k^2 (2b_1, 3b_2, \ldots, (k + 1)b_k) \frac{\partial}{\partial b_k}
\]
\[
= 4b_1 \frac{\partial}{\partial b_1} + (6b_2 + 4b_1^2) \frac{\partial}{\partial b_2} + (8b_3 + 12b_1b_2) \frac{\partial}{\partial b_3} + \cdots.
\]
We have
\[ U_2 \frac{\partial}{\partial b_1} - \frac{\partial}{\partial b_1} U_2 = -4L_1 \]  
where \( L_1 \) is given by (1.13). In the same way, in (3.10), we put
\[ T_2 = \sum_{k \geq 1} K_n^2(0, -b_2, -2b_3, \ldots, -(k - 1)b_k) \frac{\partial}{\partial b_k} \]
\[ = -2b_2 \frac{\partial}{\partial b_2} - 4b_3 \frac{\partial}{\partial b_3} + (b_2^2 - 6b_4) \frac{\partial}{\partial b_4} + (4b_2b_3 - 8b_5) \frac{\partial}{\partial b_5} + \cdots. \]  

Then \( T_2[h(z)] = (h(z) - z h'(z))^2 - 1 \). We have
\[ K_n^2(F_1, F_2, \ldots, F_n) = \frac{1}{n + 1} T_2 \frac{\partial}{\partial b_1} F_{n+1} + K_n^{-2}(b_1, \ldots, b_n) \]
\[ = \frac{1}{n + 1} \frac{\partial}{\partial b_1} T_2 F_{n+1} + K_n^{-2}(b_1, b_2, \ldots, b_n) \]
\[ = \frac{1}{n + 2} T_2 \frac{\partial}{\partial b_2} F_{n+2} + \frac{1}{n + 2} \frac{\partial^2 F_{n+2}}{\partial b_1^2}. \]  

By (2.18), \( K_n^2(-F_1, -F_2, \ldots) - K_n^2(F_1, F_2, \ldots) = -4F_n = -\frac{4}{n+1}L_1(F_{n+1}) \), see (7.11). With (3.14), it gives \( \frac{\partial}{\partial b_1}(U_2 - T_2)F_n = 0 \), thus \( (U_2 - T_2)F_n \) does not depend on \( b_1 \) for \( n \geq 2 \). For \( p \geq 1 \), \( T_2 \frac{\partial}{\partial b_p} - \frac{\partial}{\partial b_p} T_2 = -2(p - 1) \sum_{k \geq p} (k - p - 1)b_{k-p} \frac{\partial}{\partial b_k} \frac{\partial}{\partial b_p} \)
\[ T_2 \frac{\partial}{\partial b_p} - \frac{\partial}{\partial b_p} T_2 = -2(p - 1) \sum_{k \geq p} (k - p - 1)b_{k-p} \frac{\partial}{\partial b_{p+k}} = -2(p - 1)(L_p + 2W_p). \]  

**Corollary 3.3.** Let \((P_k)_{k \geq 2}\) be the coefficients of the Schwarzian derivative as in (C1), and let
\[ \mathcal{H} = T_2 \frac{\partial}{\partial b_2} + \frac{\partial^2}{\partial b_1^2} \]  
then \( P_k(b_1, b_2, b_3, \ldots, b_k) + (k - 2)F_k(2b_1, 3b_2, \ldots, (k + 1)b_k) \) is equal to
\[ -\frac{1}{2(k + 2)} [\mathcal{H}F_{k+2}](2b_1, 3b_2, \ldots, (j + 1)b_j, \ldots). \]  

**Corollary 3.4.** Let \( T = \frac{\partial^2}{\partial b_1^2} + T_2 \frac{\partial}{\partial b_2} + (\frac{\partial}{\partial b_2} - b_1 \frac{\partial}{\partial b_3} + \frac{\partial}{\partial b_4}) \). The condition (2.21) is equivalent to
\[ (T F_n)(G_1, G_2, \ldots, G_{n-2}) = 0 \quad \forall n \geq 4. \]  

**Proof.** The condition (2.21) is the same as
\[ K_n^2(-F_1, -F_2, \ldots) = b_n + b_1 b_{n-1} + b_{n-2} \quad \forall n \geq 1. \]  
From (1.26), we have \( K^p \circ S \circ F = K^p \circ F \circ G \). Thus
\[ K_n^2(-F_1, -F_2, \ldots) = \frac{1}{n + 2} \left( T_2 \frac{\partial}{\partial b_2} F_{n+2} \right) \circ G + K_n^{-2} \circ G. \]
Since \( G \circ G = \text{Identity} \) and \( G_1(b_1) = -b_1 \), we can write the right side in (3.21) as \((G_n - b_1 G_{n-1} + G_{n-2})\) at the point \((G_1, G_2, \ldots, G_n)\). With (1.16), we have

\[
G_n - b_1 G_{n-1} + G_{n-2} = -\frac{1}{n+2} \left( \frac{\partial}{\partial b_2} - b_1 \frac{\partial}{\partial b_3} + \frac{\partial}{\partial b_4} \right) F_{n+2}.
\]

Then (3.22) and (3.23) imply (3.20). The condition (3.20) is always satisfied for \( n = 1, 2, 3 \). For \( n = 4 \), it gives \( 3b_2 - 2b_1^2 = 1 \), for \( n = 5 \), \( 5b_3 - 3b_1 b_2 - b_1 = 0 \).

4. The polynomials and their derivatives. Proof of the Main Theorem

4.1. The partial derivatives \( \left( \frac{\partial}{\partial b_k} \right)_{k \geq 1} \)

**Theorem 4.1.** For \( p \geq 1, n \geq 0 \),

\[
(n + p)G_n = -\frac{\partial F_{n+p}}{\partial b_p}.
\]

In particular, \( \frac{\partial F_p}{\partial b_p} = -p \), and \( \frac{\partial F_n}{\partial b_k} = -n G_{n-k} \) if \( k \leq n \). Let \((F_{n-1})_{n \geq 1}\) be the inverse Faber polynomials, then

\[
\frac{\partial}{\partial b_k} F_{p-1} = -\frac{1}{k} F_{p-k} \times 1_{k \leq p}.
\]

**Proof.** Let \( \psi(w) = w + b_1 + b_2 \frac{w}{w_2} + b_3 \frac{w^2}{w_3} + \cdots + b_{p-1} \frac{w^{p-1}}{w_{p-1}} + \cdots \) and

\[
\psi_p(w) = \psi(w) - \frac{t}{w^{p-1}} \quad p \geq 1.
\]

We have \( w^p \psi'_p(w) = w^p \psi'(w) + (p - 1)t \),

\[
\frac{w \psi'_p(w)}{\psi_p(w)} = 1 + \sum_{n \geq 1} F_n(b_1, \ldots, b_{p-1}, b_p - t, b_{p+1}, \ldots) \times \frac{1}{w^n}.
\]

We differentiate this equation with respect to \( t \) and we make \( t = 0 \),

\[
\phi(w) = \frac{d}{dt} \bigg|_{t=0} \frac{w \psi'_p(w)}{\psi_p(w)} = \sum_{n \geq 1} \frac{\partial F_n}{\partial b_p} \times \frac{1}{w^n}.
\]

On the other hand

\[
\frac{w \psi'_p(w)}{\psi_p(w)} = \frac{w^p \psi'(w) + (p - 1)t}{w^{p-1} \psi(w) - t}.
\]

We calculate \( \phi \) with this expression

\[
\frac{d}{dt} \frac{w \psi'_p(w)}{\psi_p(w)} = w \left[ \frac{(p - 1)w^{p-2} \psi(w) + w^{p-1} \psi'(w)}{(w^{p-1} \psi(w) - t)^2} \right] = -w \frac{d}{dw} \left( \frac{1}{w^{p-1} \psi(w)} \right).
\]

At \( t = 0 \),

\[
\phi(w) = -w \frac{d}{dw} \left( \frac{1}{w^{p-1} \psi(w)} \right) = -w \frac{d}{dw} \sum_{n \geq 0} G_n \times \frac{1}{w^{n+p}} = \sum_{n \geq 0} G_n(n + p) \frac{1}{w^{n+p}}.
\]
Comparing the two expressions of $\phi$ and since $F_n$ does not contain $b_p$ when $n < p$, we obtain the result. To calculate the derivatives of the map $F^{-1}$, we take

$$h(z) = 1 + \sum_{j \geq 1} F_{j}^{-1}(b_1, b_2, b_3, \ldots, b_j)z^j$$

since $F \circ F^{-1} = \text{Identity}$, we have $\frac{d}{dz} \log(h(z)) = -\sum_{k \geq 1} b_k z^{k-1}$. We differentiate with respect to $b_k$, for $k \geq 1$,

$$-z^{k-1} = \frac{\partial}{\partial b_k} \left( \frac{h'(z)}{h(z)} \right) = \frac{d}{dz} \left( \frac{\partial}{\partial b_k} \frac{h(z)}{h(z)} \right).$$

We integrate this identity with respect to $z$,

$$-\frac{1}{k} z^k = \frac{\partial}{\partial b_k} \frac{h(z)}{h(z)} + C(b_1, b_2, \ldots)$$

where $C(b_1, b_2, \ldots)$ is constant in $z$. Making $z = 0$, we see that $C = 0$. thus

$$-\frac{1}{k} z^k \times h(z) = \frac{\partial}{\partial b_k} h(z) = \sum_{j \geq 1} \left( \frac{\partial}{\partial b_k} F_{j}^{-1} \right) z^j.$$

Since $-\frac{1}{k} z^k \times h(z) = -\frac{1}{k} z^k (1 + \sum_{j \geq 1} F_{j}^{-1} z^j)$, we obtain the partial derivatives of $F^{-1}$. □

**Corollary 4.2.** For $n \geq 0$, $p \geq 1$,

$$\frac{\partial}{\partial b_p} (G_1(b_1), G_2(b_1, b_2), \ldots, G_n(b_1, b_2, \ldots, b_n)) = -(n + p)b_n. \quad (4.3)$$

**Proof.** Since $G \circ G = \text{Identity}$, it is a consequence of (4.1). □

**Corollary 4.3.**

$$\frac{\partial G_n}{\partial b_k} = -K_{n-k}^{-2} \times 1_{n \geq k}. \quad (4.4)$$

**Proof.** We differentiate $G = F^{-1} \circ S \circ F$.

$$\frac{\partial G_n}{\partial b_k} = \sum_{j \geq 1} \frac{\partial}{\partial b_j} (S \circ F) \times \left( -\frac{\partial}{\partial b_k} F_{j}^{-1} \right) = \sum_{1 \leq j \leq n} \left( -\frac{1}{j} F_{n-j}^{-1} (S \circ F) \right) \times (jG_{j-k}).$$

After simplification by $j$, and since $F^{-1} \circ S \circ F = G$, we find

$$\frac{\partial G_n}{\partial b_k} = -\sum_{1 \leq j \leq n} G_{n-j} G_{j-k} = -K_{n-k}^{-2}. \quad \Box$$

The following operators up to a minus sign, $(Z_k)_{k \geq 0}$ were introduced in [2].

**Corollary 4.4.** With the recursion $F_{j+1} = -\sum_{0 \leq r \leq j} (r + 1)b_{r+1}G_{j-r}$, see (2.4), for $k \geq 0$, we deduce

$$Z_k = \sum_{r \geq 0} (r + 1)b_{r+1} \frac{\partial}{\partial b_{r+k+1}} \quad \text{and} \quad (j + k + 1)F_{j+1} = Z_k F_{j+k+1}.$$
Proof. From Theorem 4.1, \((j + k + 1)G_{j-r} = -\frac{\partial}{\partial b_{r+k+1}}F_{j+k+1}\). Thus, if \(k \geq 0\), with the recursion formula (2.4) where we replace \(G_{j-r}\), we find
\[
(j + k + 1)F_{j+1} = \sum_{0 \leq r < j} (r + 1)b_{r+1}\frac{\partial}{\partial b_{r+k+1}}F_{j+k+1} = Z_k(F_{j+k+1}).
\]
For \(k < 0\), then \(\frac{\partial}{\partial b_{r+k+1}}\) is defined only if \(r + k \geq 0\), i.e. \(r \geq -k\). We decompose the sum \(F_{j+1} = -\sum_{0 \leq r < -k}(r + 1)b_{r+1}G_{j-r} - \sum_{-k \leq r \leq j}(r + 1)b_{r+1}G_{j-r}. \quad \Box
\]

4.2. Proof of the main results

Proof of the Main Theorem.

\[
\frac{\partial}{\partial b_k}\left(\frac{1}{h(z)}\right) = -\frac{z^k}{h(z)^2} = -z^k - \sum_{n \geq 1} K_n^{-2}z^{n+k}.
\]

On the other hand, \(\frac{\partial}{\partial b_k}\left(\frac{1}{h(z)}\right) = \sum_{n \geq 1} \frac{\partial}{\partial b_k}G_n z^n\). In these two last expressions, we identify the coefficients of equal power of \(z\). It gives (1.17). We have \((n + p)G_n = -\frac{\partial F_{n+p}}{\partial b_p}\), then differentiating this expression with respect to \(k\), we obtain
\[
(p + k + n)K_n^{-2} = \frac{\partial^2 F_{p+k+n}}{\partial b_k \partial b_p} \quad \forall p \geq 1, \forall k \geq 1.
\]

We deduce higher order partial derivatives of \(F_j\) from
\[
\frac{\partial K_n^{-p}}{\partial b_k} = -pK_n^{-1} \times 1_n \geq k \quad \text{for} \ n \geq 1, \ k \geq 1, \ p \neq 0, \ p \in \mathbb{Z}. \tag{4.5}
\]

\(K_0^p = 1\) for any \(p\). The proof of (4.5) or equivalently \(\frac{\partial K_n^p}{\partial b_k} = pK_{n-k}^{p-1} \times 1_n \geq k\) is as follows,
\[
\frac{\partial}{\partial b_k}\left(\frac{1}{h(z)^p}\right) = \sum_{n \geq 1} \frac{\partial}{\partial b_k}(K_n^{-p})z^n. \quad \text{ (i)}
\]

On the other hand
\[
\frac{\partial}{\partial b_k}\left(\frac{1}{h(z)^p}\right) = -p\frac{\partial}{\partial b_k}(h(z)) h(z)^{p+1} = -pz^k h(z)^{p+1} = -pz^k \sum_{n \geq 1} K_n^{-1}z^{n+1} = -p \sum_{q \geq 1} K_q^{-1}z^{q+k}. \quad \text{(ii)}
\]

The identification of the coefficients of \(z^q\) in the two expressions (i) and (ii) gives (4.5). We see that one can calculate as derivatives of Faber polynomials all the \((K_n^{-p})_{n \geq 1}\) for \(p \geq 1. \quad \Box
\]

Proof of (T2). We wish to calculate \(K_n^p\) for \(p \geq 2\). Let \(\tilde{h}(z) = \frac{1}{h(z)} = 1 + G_1z + G_2z^2 + \cdots + G_nz^n + \cdots\). Then (T2) is obtained with the identification of coefficients of equal powers of \(z\) in \(\tilde{h}(z)^{-p} = 1 + \sum_{n \geq 1} K_n^{-p}(G_1, G_2, \ldots, G_n)z^n = h(z)^p\) with \(h(z)^p = 1 + \sum_{n \geq 1} K_n^p(b_1, b_2, \ldots, b_n)z^n\). We can also give a proof with the composition of maps \(K_n^p = F^{-1} \circ pI \circ F\) and
\[
K_n^{-p} \circ G = F^{-1} \circ pS \circ F \circ F^{-1} \circ S \circ F = F^{-1} \circ pI \circ F = K_n^p. \quad \Box
\]
Corollary 4.5. All the $K^p_n, n \geq 1, p \in \mathbb{Z}$ can be obtained as derivatives of Faber polynomials. For $p \geq 1$,

$$(-1)^p(p-1)! (n + k_1 + k_2 + \cdots + k_p) K^{-p}_n(b_1, b_2, \ldots, b_n) \frac{\partial^p F_{n+k_1+\cdots+k_p}}{\partial b_{k_1} \partial b_{k_2} \cdots \partial b_{k_p}} (b_1, b_2, \ldots, b_n, \ldots, b_q, \ldots).$$

(4.6)

Let

$$\phi(b_1, b_2, \ldots, b_n, \ldots, b_q, \ldots) = \frac{\partial^p F_{n+k_1+\cdots+k_p}}{\partial b_{k_1} \partial b_{k_2} \cdots \partial b_{k_p}} (b_1, b_2, \ldots, b_n, \ldots, b_q, \ldots),$$

for $p \geq 1$, we have

$$(-1)^p(p-1)! (n + k_1 + k_2 + \cdots + k_p) K^p_n(b_1, b_2, \ldots, b_n) = \phi(G_1(b_1, b_2, \ldots), G_2(b_1, b_2, \ldots), \ldots, G_q(b_1, b_2, \ldots, \ldots)).$$

(4.7)

Corollary 4.6. For $p \geq 1$,

$$(-1)^p(p-1)! (n + p) K^{-p}_n(b_1, b_2, \ldots, b_n) = \frac{\partial^p F_{n+p}}{\partial b_1^p} (b_1, b_2, \ldots, b_n, \ldots, b_{n+p}).$$

(4.8)

$$(-1)^p(p-1)! (n + p) K^p_n(b_1, b_2, \ldots, b_n)$$

$$= \frac{\partial^p F_{n+p}}{\partial b_1^p} (G_1(b_1, G_2(b_1, b_2), \ldots, G_{n+p}(b_1, b_2, \ldots, b_{n+p})).$$

(4.9)

In particular

$$K^{(n+1)}_n = \frac{(-1)^{n+1}}{n!(2n+1)} \frac{\partial^{n+1} F_{2n+1}}{\partial b_1^{n+1}} (b_1, b_2, \ldots, b_q, \ldots),$$

(4.10)

$$K^{n+1}_n = \frac{(-1)^n}{(n-1)!(2n+1)} \left( \frac{\partial^n F_{2n+1}}{\partial b_1^n} \right) (G_1(b_1), G_2(b_1, b_2), \ldots, G_q(b_1, b_2, \ldots, b_q)).$$

(4.11)

Proof of the Main Corollary. Let $f(z) = z + b_1 z^2 + b_2 z^3 + \cdots + b_n z^{n+1} + \cdots$.

$$\frac{f''(z)}{f'(z)} = -\sum_{k \geq 1} F_k(2b_1, 3b_2, \ldots, (k+1)b_k) z^{k-1},$$

$$\left( \frac{f''(z)}{f'(z)} \right)' = -\sum_{k \geq 2} (k-1) F_k(2b_1, 3b_2, \ldots, (k+1)b_k) z^{k-2}. \quad \text{(i)}$$

On the other hand

$$\frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 = -\frac{1}{2z^2} \left( \frac{z f''(z)}{f'(z)} \right)^2$$

$$= -\frac{1}{2z^2} \left( \sum_{k \geq 1} F_k(2b_1, 3b_2, \ldots, (k+1)b_k) z^k + 1 - 1 \right)^2$$
\[
\begin{align*}
&= -\frac{1}{2z^2} \left( 1 + \sum_{k \geq 1} F_k(2b_1, 3b_2, \ldots) z^k \right)^2 + \frac{1}{z^2} \left( 1 + \sum_{k \geq 1} F_k(2b_1, 3b_2, \ldots) z^k \right) - \frac{1}{2z^2} \\
&= -\frac{1}{2z^2} \sum_{k \geq 2} K_k^2(F_1(2b_1), F_2(2b_1, 3b_2), \ldots) z^k + \frac{1}{z^2} \sum_{k \geq 2} F_k(2b_1, 3b_2, \ldots) z^k. 
\end{align*}
\]

We add the two expressions (i) and (ii) to obtain the Main Corollary. \(\square\)

**Remark 4.1.** Let \(H_k(b_1, b_2, \ldots, b_k) = F_k(2b_1, 3b_2, 4b_3, \ldots, (k+1)b_k).\) With the expressions of the \((F_n)_{n \geq 1}\) in [3], we find \(H_1(b_1) = F_1(2b_1) = -2b_1\) and \(H_2(b_1, b_2) = F_2(2b_1, 3b_2) = 2(2b_1^2 - 3b_2)\)

\[
H_3(b_1, b_2, b_3) = F_3(2b_1, 3b_2, 4b_3) = 2(-4b_1^3 + 9b_1b_2 - 6b_3),
\]

\[
H_4(b_1, b_2, b_3, b_4) = F_4(2b_1, 3b_2, 4b_3, 5b_4) = 2(8b_1^4 - 24b_1^2b_2 + 9b_2^2 + 16b_1b_3 - 10b_4).
\]

We can calculate \(P_k\) with \(P_k = -(k-1)H_k - \frac{1}{2} \sum_{j=1}^{k-1} H_{k-j}H_j\) or we can use (C1).

**Proof of (T3).** \(h(z)^p = \sum_{n \geq 0} K_n^p z^n = (1 + b_1z + b_2z^2 + \cdots + b_nz^n + \cdots)^p,\)

\[
K_n^p = \frac{1}{2\pi i} \int \frac{(1 + \phi_n(\xi) + b_{n+1}\xi^{n+1} + \cdots)^p}{\xi^{n+1}} d\xi = \frac{1}{2\pi} \int \frac{(1 + \phi_n(\xi))^p}{\xi^{n+1}} d\xi
\]

and we write Newton binomial formula \((1 + \phi_n(\xi))^p = 1 + p\phi_n(\xi) + \frac{p(p-1)}{2} \phi_n(\xi)^2 + \cdots.\) \(\square\)

**Proof of (T4).** If \(b_1 \neq 0,\) then \(\phi_n(z) = b_1z + b_2z^2 + b_3z^3 + \cdots + b_nz^n = b_1z(1 + \frac{b_2}{b_1}z + \cdots + \frac{b_n}{b_1}z^{n-1}).\) Thus

\[
(\phi_n(z))^k = b_1^k z^k \left( 1 + \frac{b_2}{b_1}z + \cdots + \frac{b_n}{b_1}z^{n-1} \right)^k = b_1^k z^k \sum_{j=1}^{k} K_j^k \left( \frac{b_2}{b_1}, \ldots, \frac{b_{j+1}}{b_1} \right) z^j.
\]

The coefficient of \(z^n\) in this expression is obtained for \(j + k = n.\) This gives (T4). We can obtain the exact expression of \(K_n^p\) as follows,

\[
K_n^p = \sum_{1 \leq k_1 \leq n} C_{k_1}^p D_{n-k_1}^p = \sum_{1 \leq k_1 \leq n} C_{k_1}^p b_1^{k_1} \sum_{1 \leq k_2 \leq n-k_1} C_{k_2}^p \left( \frac{b_2}{b_1} \right)^{k_2} D_{n-k_1}^p \left( \frac{b_3}{b_2} \right).\]

If \(b_1 \neq 0,\)

\[
K_1^p = C_1^p b_1, \quad K_2^p = C_1^p b_1 + C_2^p b_2, \quad K_3^p = C_1^p b_1 + C_2^p b_2^2 K_1^p \left( \frac{b_2}{b_1} \right) + C_3^p b_3,
\]

\[
K_4^p = C_1^p b_4 + C_2^p b_1^2 K_2^p \left( \frac{b_2}{b_1} \right) + C_3^p b_3 K_1^p \left( \frac{b_2}{b_1} \right) + C_4^p b_4^2.
\]

\[
K_5^p = C_1^p b_5 + C_2^p b_1^2 K_3^p \left( \frac{b_2}{b_1} \right) + C_3^p b_3 K_2^p \left( \frac{b_2}{b_1} \right) + C_4^p b_4 K_1^p \left( \frac{b_2}{b_1} \right) + C_5^p b_5,
\]

\[
K_6^p = C_1^p b_6 + C_2^p b_1^2 K_4^p \left( \frac{b_2}{b_1} \right) + C_3^p b_3 K_3^p \left( \frac{b_2}{b_1} \right) + C_4^p b_4 K_2^p \left( \frac{b_2}{b_1} \right) + C_5^p b_5 K_1^p \left( \frac{b_2}{b_1} \right) + C_6^p b_6.
\]
5. Identities related to the \((W_j)_{j \geq 1}\), the \((V_j^k)_{j \geq 1}\), \(k \in \mathbb{Z}\) and the \((V_j)_{j \geq 1}\)

It has been proved in [3] that
\[
W_j W_q = W_q W_j \quad \text{for } j \geq 1, \quad q \geq 1. \tag{5.1}
\]
For \(j \geq 1, \ m \geq 0, \)
\[
W_j F_m = m \delta_{j,m} \quad \text{and} \quad W_j G_m = G_{m-j} \tag{5.2}
\]
with the convention \(G_p = 0\) if \(p < 0\). Moreover \(K^n_p = 1\) and
\[
W_j \left( K^n_p \right) = 0 \quad \text{for } n < j, \quad W_j \left( K^n_p \right) = -p K^n_{n-j} \quad \text{for } n \geq j, \quad p \in \mathbb{Z}. \tag{5.3}
\]

**Proof of (5.1)–(5.3).** For (5.1), we remark that \(W_j W_p [h(z)] = z^{p+j} h(z)\). For the other identities, let \((b_1, \ldots, b_k, \ldots) \rightarrow h(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_k z^k + \cdots\), then \(\frac{\partial}{\partial b_j} [h(z)] = z^j\). It is enough to calculate \(W_j [h(z)]\), use (1.1), then calculate \(W_j \left[ \frac{1}{h(z)} \right]\), use (1.3) by equating coefficients of similar powers of \(z\). This is done as follows. Let
\[
(b_1, b_2, \ldots, b_k, \ldots) \rightarrow \phi (b_1, b_2, \ldots, b_k, \ldots).
\]
Since \(W_j\) is a differential operator, \(W_j [\exp(\phi)] = \exp(\phi) \times W_j [\phi]\). With (1.1), we have \(W_j [h(z)] = h(z) \times (- \sum_{k=1}^{+\infty} \frac{W_k F_k}{k} z^k)\). Comparing this last expression with \(W_j [h(z)] = -z^j h(z)\), we deduce that \(z^j = \sum_{k=1}^{+\infty} \frac{W_k F_k}{k} z^k\). Equating the coefficients of \(z^k\) gives \(W_j (F_m)\). To obtain \(W_j [G_m]\), we calculate \(W_j \left[ \frac{1}{h(z)} \right] = - \frac{W_j [h(z)]}{h(z)^2}\). With \(W_j [h(z)] = -z^j h(z)\), it gives \(W_j \left[ \frac{1}{h(z)} \right] = \frac{z^j}{h(z)}\). With (1.2), it gives
\[
\frac{z^j}{h(z)} = \sum_{m \geq 1} W_j G_m z^m. \tag{i}
\]
In (i), we replace \(\frac{1}{h(z)}\) by (1.2), thus \(z^j (\sum_{m \geq 0} G_m z^m) = \sum_{m \geq 1} W_j [G_m] z^m\). In this identity, equating the coefficients of \(z^m\) gives \(W_j (G_m)\). In the same way,
\[
W_j [h(z)^p] = ph(z)^{p-1} W_j [h(z)] = -pz^j h(z)^p = -pz^j \left( 1 + \sum_{n \geq 1} K^n_p z^n \right).
\]
On the other hand, \(W_j [h(z)^p] = \sum_{s \geq 1} W_j [K^s_p] z^s\). Identifying the two expressions of \(W_j [h(z)^p]\), we obtain \(-pz^j (1 + \sum_{n \geq 1} K^n_p z^n) = \sum_{s \geq 1} W_j [K^s_p] z^s\). Equating the coefficients of equal powers of \(z^j\) gives \(W_j (K^n_p)\). \(\square\)
Theorem 5.1. The operators \((V_k^j)_{j \geq 1}, k \in \mathbb{Z}\) satisfy \((1.11),
\begin{equation}
V_k^q V_s^p + (p + 1)V_s^{p+q} = \sum_{n \geq 0, j \geq 0} K_n^{p+1} K_j^{q+1} \frac{\partial^2}{\partial b_{n+s} \partial b_{k+j}}
\end{equation}
and \(V_j = \sum_{n \geq 0} K_n^j W_{j+n} \)
\begin{equation}
V_j^k (F_{p}) = p K_j^{k-p}, \quad V_j^k (K_s^q) = -q K_{s-j}^{q+k},
\end{equation}
\begin{equation}
V_j^k [h(z)] = -z^j [h(z)]^{k+1}.
\end{equation}
The polynomials \((P_k^p)_{n \geq 0}, P_0^p = 1\) (see \((1.4)) satisfy
\begin{equation}
V_j^k (P_n^{p+j}) = -(2k + j) P_n^{p+j} + 2(k - p)(n - p + k + j) K_k^{p+j} \quad \text{if } k + j \neq 0,
\end{equation}
\begin{equation}
V_j^{-j} (P_n^{p+j}) = j P_n^{p+j} + 2(j + p) F_n^{p+j},
\end{equation}
\begin{equation}
V_j (P_n^{p+j}) = -(2k + j) P_n^{p+j} - p (p + 1) f(z)^{p-1} f'(z).
\end{equation}
and the Neretin polynomials \((C_1), z^2 S(f)(z) = \sum_{k \geq 2} P_k z^k,
\begin{equation}
V_k (P_j) = -(k^3 - k) P_j^{k-j}.
\end{equation}
Proof of \((5.7)-(5.8).\) Since \(V_p [h(z)] = -z^p h(z)^{p+1},\) we deduce for \(p \geq 1,
\begin{equation}
V_p [f(z)] = -f(z)^{p+1}, \quad V_p \left( \frac{f'(z)}{f(z)} \right) = -f(z)^{p-1} f'(z)
\end{equation}
and \(V_p \left( \frac{f''(z)}{f(z)} \right) = -p (p + 1) f(z)^{p-1} f'(z).\) We obtain \((5.7)-(5.8)) by identification of coefficients. \(\square\)

In \([3],\) the homogeneity operator \(L_0 = b_1 \frac{\partial}{\partial b_1} + 2b_2 \frac{\partial}{\partial b_2} + \cdots + k b_k \frac{\partial}{\partial b_k} + \cdots\) is expressed as \(L_0 = \sum_{j \geq 1} F_j W_j.\)

Lemma 5.2. We have \(G_n = K_n^{-1}, b_n = K_n^1,\)
\begin{equation}
k G_k = \sum_{1 \leq j \leq k} F_j G_{k-j} \quad \text{and} \quad n K_n^p = -p \sum_{1 \leq j \leq n} F_j K_n^{p-j}.
\end{equation}
Proof of \((5.10).\) From the recurrence relation for the polynomials \((F_k)_{k \geq 0},\)
\begin{equation}
L_0 = \sum_{k \geq 1} k b_k \frac{\partial}{\partial b_k} = - \sum_{j \geq 1} F_j \left( \sum_{k \geq j} b_{k-j} \frac{\partial}{\partial b_k} \right) = \sum_{j \geq 1} F_j W_j.
\end{equation}
\(K_n^p\) is homogeneous of degree \(n,\) thus \(L_0 K_n^p = n K_n^p.\) Since \(L_0 K_n^p = \sum_{j \geq 1} F_j W_j K_n^{p-j} = - \sum_{1 \leq j \leq n} F_j p K_n^{p-j},\) we obtain the recursion formula for \(K_n^p.\) See \((2.5), (2.7).\) \(\square\)

Theorem 5.3. The \((\frac{\partial}{\partial b_j})_{j \geq 1}\) are given in terms of the \((W_j)_{j \geq 1}\) with
\[
\frac{\partial}{\partial b_j} = -W_j - \sum_{k \geq 1} G_k W_{j+k}, \quad (5.11)
\]

\[
X_0 = -\sum_{j \geq 1} b_j \frac{\partial}{\partial b_j} = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \cdots - b_k \frac{\partial}{\partial b_k} - \cdots \quad (5.12)
\]
satisfies

\[
X_0 = -\sum_{j \geq 1} G_j W_j. \quad (5.13)
\]

In particular,

\[
X_0(F_n) = -nG_n. \quad (5.14)
\]

**Proof.** For \((\frac{\partial}{\partial b_j})_{j \geq 1}\), it is enough to verify that

\[
\frac{\partial}{\partial b_j} [h(z)] = -W_j [h(z)] - \sum_{k \geq 1} G_k W_{j+k} [h(z)]. \quad (i)
\]

Since \(W_j[h(z)] = -z^j h(z)\), we have

\[
-W_j [h(z)] - \sum_{k \geq 1} G_k W_{j+k} [h(z)] = z^j (1 + G_1 z + G_2 z^2 + \cdots + G_k z^k + \cdots) \times h(z).
\]

Thus (i) is the same as \(z^j = z^j (1 + G_1 z + G_2 z^2 + \cdots + G_k z^k + \cdots) \times h(z)\). It is the immediate consequence of (1.3). To prove (5.13), we see that \(X_0[h(z)] = 1 - h(z)\). We write \(X_0 = \sum_{j \geq 1} H_j W_j\). Applied to \(h(z)\), it gives

\[
X_0[h(z)] = 1 - h(z) = -\left[ \sum_{j \geq 1} H_j z^j \right] h(z).
\]

Thus

\[
\sum_{j \geq 1} H_j z^j = \frac{h(z) - 1}{h(z)} = 1 - \frac{1}{h(z)} = 1 - \sum_{n \geq 0} G_n z^n.
\]

By identification of the coefficient of \(z^j\), we find \(H_j = -G_j\). \(\Box\)

**Remark 5.1.** If we calculate \(W_{p+k} F_{n+p}\) with (5.2), we obtain with (5.11) another proof of (1.16),

\[
\frac{\partial}{\partial b_p} F_{n+p} = -W_p F_{n+p} - \sum_{k \geq 1} G_k W_{p+k} F_{n+p} = -(n + p)G_n.
\]

**Theorem 5.4.** For \(k, p \in \mathbb{Z}, j \geq 1\),

\[
V^k_j = \sum_{n \geq 1} K_{n-j}^{k-p} V_n^p \quad \text{and} \quad X_0 = \sum_{n \geq 1} \left[ K_n^p - K_n^{-1-p} \right] V_n^p.
\]

We can also express the \((V^k_j)\) in terms of \((V_j)_{j \geq 1}\) with the inverse function \(f^{-1}(z)\).
6. The composition of differentials on coefficients

6.1. The inverse function

The inverse function is important in the study of coefficients regions, see [11, p. 104]. Also asymptotics of the derivatives of the Faber polynomials are calculated with inverse functions, see [10]. Let

\[ f(w) = wh(w) = w + b_1 w^2 + b_2 w^3 + \cdots + b_n w^{n+1} + \cdots. \]

\[ w \frac{f'(w)}{f(w)} = 1 + w \frac{h'(w)}{h(w)} = 1 - \sum_{k \geq 1} F_k(b_1, b_2, \ldots, b_k) w^k. \]

We denote \( k(z) = f^{-1}(z) \) the inverse of \( f \), we have \((f \circ k)(z) = z\), letting \( w = k(z) \),

\[ \frac{k'(z)}{k(z)} = \frac{1}{1 - \sum_{p \geq 1} F_p(b_1, b_2, \ldots, b_p) k(z)^p} \]

\[ = 1 + \sum_{m=1}^{+\infty} G_m(-F_1, -F_2, \ldots, -F_m) k(z)^m \]

we have \( k(z) = f^{-1}(z) = z - b_1 z^2 - (b_2 - 2b_1^2) z^3 - (b_3 + 5b_1^3 - 5b_1 b_2) z^4 - (b_4 - 14b_1^4 + 21b_1^2 b_2 - 6b_1 b_3 - 3b_2^2) z^5 + \cdots \).

By a residue calculus, we know that

\[ f^{-1}(z) = \frac{1}{2i \pi} \int_{|\zeta| = \rho} \frac{\xi f'((\xi))}{f((\xi)) - z} d\xi = \sum_{n \geq 1} \left( \frac{1}{2i \pi} \int_{\mathbb{C}} \frac{\xi f'((\xi))}{f((\xi))} \frac{d\xi}{f((\xi))^n} \right) z^n. \quad (6.1) \]

**Theorem 6.1.** Let \( f(z) = z + b_1 z^2 + \cdots + b_n z^n + \cdots \). The inverse function of \( f \), \( f^{-1}(f(z)) = z \) is given in terms of the derivatives of the Faber polynomials of \( f(z) \) with

\[ f^{-1}(z) = z + \sum_{n \geq 1} \frac{1}{n+1} K_n z^{n+1} \]

\[ = z + \sum_{n \geq 1} \frac{(-1)^{n+1}}{(2n+1)} \times \frac{1}{(n+1)!} \left[ \frac{\partial^{n+1}}{\partial b_1^{n+1}} F_{2n+1}(b_1, b_2, \ldots, b_q, \ldots) \right] z^{n+1}. \quad (6.2) \]

Let \( g(z) = zh(\frac{1}{z}) = z + b_1 + \frac{b_2}{z} + \frac{b_3}{z^2} + \cdots + \frac{b_n}{z^{n-1}} + \cdots \), then the inverse function of \( g \) is

\[ g^{-1}(z) = z - b_1 - \sum_{n \geq 1} \frac{1}{n} K_{n+1} \frac{1}{z^n} \]

\[ = z - b_1 + \sum_{n \geq 1} \frac{(-1)^{n+1}}{(2n+1)} \times \frac{1}{n!} \left[ \frac{\partial^n}{\partial b_1^n} F_{2n+1} \right] (G_1(b_1), G_2(b_1, b_2), \ldots, G_q(b_1, b_2, \ldots), \ldots) \frac{1}{z^n}. \quad (6.3) \]
Proof. $\frac{zf'(z)}{f(z)} = -\sum_{k\geq 0} F_k z^k$ with $F_0 = -1$ and $\frac{z^n}{f(z)^n} = \sum_{p \geq 0} K_{p-n}^{-n} z^p$, we deduce that

$$\frac{zf'(z)}{f(z)} \frac{1}{f(z)^n} = -\sum_{p \geq 0, k \geq 0} F_k K_{p-n}^{-n} z^{p+k-n}.$$ 

The residue is obtained for $p + k - n = -1$ and is equal to $-\sum_{0 \leq k \leq n-1} F_k K_{n-k-1}^{-n}$. This is the coefficient of $z^n$ in the expression of $f^{-1}(z)$. Thus the coefficient of $z^{n+1}$ is

$$K_{n}^{-(n+1)} - \sum_{1 \leq k \leq n} F_k K_{n-k}^{-(n+1)} = K_{n}^{-(n+1)} - \frac{n}{n+1} K_{n+1}^{-(n+1)}$$

where we have used the recursion formula (2.7). For example the coefficients of $z^4$ is given when $n = 3$ by $\frac{1}{4}K_3^{-4} = 5b_1b_2 - b_3 - 5b_3^3$. The expressions of the coefficients of $f^{-1}(z)$ in terms of the $(K_k^n)$ were found in an other way in [2, (1.2.8)–(1.2.9)]. For $g^{-1}(z)$, we use [2, (1.2.8)] and (T2). \(\square\)

**Proposition 6.2.** We have

$$h(f^{-1}(z)) = 1 + b_1 z - \sum_{n \geq 2} \frac{1}{n-1} K_{n-1-n}^{1-n} z^n$$

(6.4)

with $f(z) = z h(z)$. Assume that $p \geq 2$, then

$$[h(f^{-1}(z))]^p = 1 + \sum_{1 \leq n \leq p-1} \frac{p}{p - n} K_{n}^{p-n} z^n - F_p z^p - \sum_{n \geq p+1} \frac{p}{n - p} K_{n}^{p-n} z^n.$$ 

(6.5)

Assume that $p \geq 1$, then

$$[h(f^{-1}(z))]^{-p} = 1 + \sum_{n \geq 1} \frac{p}{n + p} K_{n}^{-(p+n)} z^n.$$ 

(6.6)

The function $\psi(z) = h(f^{-1}(z))$ has been considered in [13]. The coefficients of $[h(f^{-1}(z))]^p$, $p \in \mathbb{Z}$ have been given in [2, (1.2.4) and (0.7)].

**Proof.** By a residue calculus,

$$h(f^{-1}(z)) = \frac{1}{2i\pi} \int \frac{h(\xi) f'(\xi)}{f(\xi) - z} \frac{d\xi}{\xi} = \sum_{n \geq 0} \left( \frac{1}{2i\pi} \int \frac{\xi f'(\xi)}{f(\xi)} \frac{1}{f(\xi)^{n-1}} \frac{d\xi}{\xi^2} \right) z^n$$

$$= 1 + b_1 z + \sum_{n \geq 2} \left( \frac{1}{2i\pi} \int \frac{\xi f'(\xi)}{f(\xi)} \frac{1}{f(\xi)^{n-1}} \frac{d\xi}{\xi^2} \right) z^n.$$ 

Since with (2.7), $\sum_{1 \leq k \leq n} F_k K_{n-k}^{1-n} = \frac{n}{n-1} K_{n-1}^{1-n}$, the coefficient of $z^n$ in the expansion of $h(f^{-1}(z))$ is given by $K_{n}^{1-n} - \frac{n}{n-1} K_{n-1}^{1-n} = -\frac{1}{n-1} K_{n}^{1-n}$. Then we finish the proof as in Theorem 6.1. In the same way, we have

$$[h(f^{-1}(z))]^p = \sum_{n \geq 0} \left( \frac{1}{2i\pi} \int \frac{\xi f'(\xi)}{f(\xi)} \frac{1}{f(\xi)^{n-p}} \frac{d\xi}{\xi^{p+1}} \right) z^n.$$
We have
\[ \frac{\xi f'(\xi)}{f(\xi)} - \frac{1}{f(\xi)^{-p}} = - \sum_{k \geq 0, j \geq 0} F_k K_j^{p-n} \xi^{k+j+p-n}, \]
the coefficient of $\xi^p$ in this expression is obtained when $k + j = n$ and is equal to
\[ - \sum_{0 \leq k \leq n} F_k K_n^{p-n} - \sum_{1 \leq k \leq n} F_k K_{n-k}^{p-n} = - \frac{p}{n-p} K_n^{p-n} \xi^n \]
when $p \neq n$. If $p = n$, we take the coefficient $-F_p$ of $\xi^p$ in $\xi f'(\xi) f(\xi)$.

**Remark 6.1.** Following [13], for any $p \in \mathbb{Z}$, $p \neq 0$,
\[ \frac{[h(f^{-1}(z))^p}{f'(f^{-1}(z))} = \sum_{n \geq 0} K_n^{p-(n+1)} z^n. \]

**Proof.** We have
\[ \frac{[h(f^{-1}(z))^p}{f'(f^{-1}(z))} = \frac{1}{2i\pi} \int \frac{h(\xi)^p}{f(\xi) - z} d\xi = \frac{1}{2i\pi} \int \frac{h(\xi)^{p-1}}{1 - \frac{z}{f(\xi)}} \frac{d\xi}{\xi} \]
\[ = \sum_{n \geq 0} \left( \frac{1}{2i\pi} \int \frac{h(\xi)^{p-1}}{\xi^n h(\xi)^n} \frac{d\xi}{\xi} \right) z^n = \sum_{n \geq 0} \frac{1}{2i\pi} \int h(\xi)^{p-1} \frac{d\xi}{\xi} \xi^n \frac{1}{\xi^{n+1}} z^n. \]
Since $h(\xi)^{p-1} = \sum_{j \geq 0} K_j^{p-(n+1)} \frac{\xi^j}{\xi^{n+1}}$, the residue is $K_n^{p-(n+1)}$.

**Remark 6.2.** With the inverse function, we obtain also expressions of $P_k^n$ (see Proposition 2.4),
\[ P_k^n(b_1, b_2, \ldots, b_n) = \sum_{0 \leq s \leq n} \frac{k+s}{k} K_s^k(b_1, b_2, \ldots, b_s) \times K_{n-s}^s \left( \frac{1}{2} K_1^{-2}(b_1), \frac{1}{3} K_2^{-3}(b_1, b_2), K_3^{-4}, \ldots, \frac{1}{p+1} K_p^{-(p+1)}, \ldots \right). \]

**Proof.** From Proposition 2.3, $\phi_k(\zeta) = \sum_{n \geq 0} \frac{k+n}{k} K_n^k \zeta^n$. We put $\xi = f^{-1}(z)$. From Theorem 6.1,
\[ f^{-1}(z)^n = z^n \sum_{j \geq 0} K_j^j \left( \frac{1}{2} K_1^{-2}, \frac{1}{3} K_2^{-3}, K_3^{-4}, \ldots, \frac{1}{p+1} K_p^{-(p+1)}, \ldots \right) z^j. \]
This gives the first expression of $P_k^n$.

**Remark 6.3.**
\[ (f^{-1})'(z) = 1 + \sum_{k \geq 1} \frac{(-1)^k}{k!} z^k \psi_k(z) \quad \text{with} \quad \psi_k(z) = \sum_{p \geq 0} \frac{(2k+p)!}{(k+p)!} D_{k+p}^{f(z)}. \]

**Proof.** We use (T3).
\[
\frac{d}{dz} f^{-1}(z) = 1 + \sum_{n \geq 1} K_n^{-(n+1)} z^n = 1 + \sum_{1 \leq k \leq n, 1 \leq \ell \leq n} (-1)^k \frac{(n + k)!}{n!} D_n^{k} z^{n-k} \\
= 1 + \sum_{k \geq 1} \frac{(-1)^k}{k!} z^k \left( \sum_{n \geq k} \frac{(n + k)!}{n!} D_n^{k} z^{n-k} \right).
\]

\[\tag{6.11}\]

**Remark 6.4.** For \( s \geq 1, p \in \mathbb{Z} \), then
\[
V_p^s \left[ f^{-1}(z) \right] = z^{1+s} \sum_{n \geq 0} K_n^{p-s-(n+1)} z^n = \frac{z^{1+s} [h(f^{-1}(z))]^{p-s}}{f'(f^{-1}(z))}.
\]

Let \( \Phi_k(\xi) = \frac{\xi f'(\xi)}{f(\xi)} \times h(\xi)^k = \sum_{n \geq 0} \frac{k+n+s}{n} K_n^{k} \xi^n \), we have
\[
V_p^s \left[ \Phi_k(\xi) \right] = -\xi^s \sum_{n \geq 0} \frac{k(k+n+s)}{n+s} K_n^{k+p} \xi^n.
\]

**6.2. Composition of derivations and recurrence formulae**

We know (see 1.25) that \( F_n(b_1, b_2, \ldots, b_n) + F_n(G_1, G_2, \ldots, G_n) = 0 \). In the following, we show how the differentiation of this identity yields the recursion formula (2.8) with \( p = -1 \) and \( r = -2 \), i.e. \( K_n^{-1} = \sum_{0 \leq j \leq n} K_j^{-2} K_n^{1-j} \). Then we prove that it gives a partial differential equation satisfied by the \( (F_n)_{n \geq 1} \).

The differentiation of (1.25) with respect to \( b_k \) gives
\[
\frac{\partial}{\partial b_k} F_n(b_1, b_2, \ldots, b_n) + \sum_{j=1}^{n} \left( \frac{\partial F_n}{\partial b_j} \right) (G_1, G_2, \ldots, G_n) \times \frac{\partial G_j}{\partial b_k} (b_1, b_2, \ldots) = 0.
\]

We know from (1.16) that \( \frac{\partial F_n}{\partial b_k} (b_1, b_2, \ldots, b_n) = -n G_{n-j} (b_1, b_2, \ldots, b_n) \). This expression calculated at the point \( (G_1(b_1), G_2(b_2), \ldots, G_n(b_1, b_2, \ldots, b_n)) \) gives
\[
\left( \frac{\partial F_n}{\partial b_j} \right) (G_1, G_2, \ldots, G_n) = -n G_{n-j} (G_1, G_2, \ldots, G_n) = -n b_{n-j}.
\]

We replace in (6.12), we obtain
\[
\frac{\partial}{\partial b_k} F_n(b_1, b_2, \ldots, b_n) - \sum_{j=1}^{n-1} nb_{n-j} \frac{\partial G_j}{\partial b_k} (b_1, b_2, \ldots, b_n) - n \frac{\partial G_n}{\partial b_k} = 0.
\]

or equivalently
\[
\frac{\partial}{\partial b_k} F_n(b_1, b_2, \ldots, b_n) - \sum_{j=1}^{n-1} nb_j \frac{\partial G_{n-j}}{\partial b_k} (b_1, b_2, \ldots, b_n) - n \frac{\partial G_n}{\partial b_k} = 0.
\]

On the other hand, \( -n G_j = \frac{\partial F_n}{\partial b_{n-j}} \). We replace in (6.14), we obtain
\[
\frac{\partial}{\partial b_k} F_n(b_1, b_2, \ldots, b_n) + \sum_{j=k}^{n-1} b_{n-j} \frac{\partial^2 F_n}{\partial b_{n-j} \partial b_k} (b_1, b_2, \ldots, b_n) - n \frac{\partial G_n}{\partial b_k} = 0.
\]

\[\tag{6.15}\]
We go back to the expressions of the partial derivatives of \( F_n \) in terms of the \( K^p \) to see that (6.15) is the same as \( K^{-1}_n = \sum_{0 \leq j \leq n} K^{-2}_j K^1_{n-j} \).

**Lemma 6.3.**

\[
-K^{-2}_{n-k} = \frac{\partial G_n}{\partial b_k} \quad \forall n, k, n \leq k, \text{ and }
\]

\[
\frac{\partial^2 F_n}{\partial b_r \partial b_s} = -n \frac{\partial G_n}{\partial b_{r+s}} \quad r, s \geq 1, \quad n \geq 1.
\]

**Proof.** From (T1). □

**Theorem 6.4.** Let \( X_0 = -\sum_{j \geq 1} b_j \frac{\partial}{\partial b_j} = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \cdots - b_k \frac{\partial}{\partial b_k} - \cdots \). Then the identity

\[
K^{-1}_n = \sum_{0 \leq j \leq n} K^{-2}_j K^1_{n-j}
\]

is the same as

\[
\frac{\partial^2 F_n}{\partial b_r \partial b_s} = \frac{\partial}{\partial b_{r+s}} (X_0 F_n) \quad \forall r, s \geq 1, \quad n \geq 1.
\]

**Proof.** From (6.15), \( \frac{\partial F_n}{\partial b_k} + \sum_{j \geq 1} b_j \frac{\partial^2 F_n}{\partial b_j \partial b_k} = n \frac{\partial G_n}{\partial b_k} \). The left side of this equation is

\[
\frac{\partial}{\partial b_k} \left( \sum_{j \geq 1} b_j \frac{\partial}{\partial b_j} F_n \right) = -\frac{\partial}{\partial b_k} (X_0 F_n).
\]

On the other hand, \( n \frac{\partial G_n}{\partial b_k} \) is given by Lemma 6.3. This proves the theorem. □

7. **First order differential operators on \( \mathcal{M} \)**

We have seen that the operators \((W_j)_{j \geq 1}, X_0, \frac{\partial}{\partial b_j}\) allow to pass from polynomials \((F_k)_{k \geq 1}\) to polynomials \((G_m)_{m \geq 1}\). In particular, we found \((n+p)G_n = -\frac{\partial F_{n+p}}{\partial b_p}\) and \(W_j G_m = G_{m-j}\). Operators \((Z_k)\) in [2] are of this type. In [9], family of vector fields related to the Virasoro algebra have been considered. We found that the operators \((V_k)_{k \geq 1}\) transforms the Neretin polynomials \(P_j\) into \(-k^3 - k) P_{j+k}.\) In the following, we construct first order differential operators on the manifold \( \mathcal{M} \) which permit to pass from one polynomial to the other.

7.1. **The operators \((X_k)_{k \in \mathbb{Z}}\)**

The operator \(X_0 = -\sum_{j \geq 1} G_j W_j = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \cdots - b_n \frac{\partial}{\partial b_n} - \cdots\) has appeared to be a natural operator on \( \mathcal{M} \). We have

\[
X_0 \frac{\partial}{\partial b_p} - \frac{\partial}{\partial b_p} X_0 = b_p \frac{\partial}{\partial b_p}.
\]

(7.1)

On the other hand, \( \frac{\partial}{\partial b_k} = -\sum_{j \geq 1} G_j W_{j+k}.\)
**Definition 7.1.** For \( k \geq 1 \), we put

\[
X_k = \frac{\partial}{\partial b_k} = - \sum_{j \geq 1} G_j W_{j+k} \quad \text{and} \quad X_{-k} = - \sum_{j \geq 1} G_{j+k} W_j. \quad (7.2)
\]

Since \( W_j = \sum_{p \geq 0} G_p \frac{\partial}{\partial G_{j+p}} \), we deduce

\[
X_{-k} = - \sum_{j \geq 1, p \geq 0} G_{j+k} G_p \frac{\partial}{\partial G_{j+p}} = - \left[ \sum_{0 \leq i \leq r-1} G_{r-i+k} G_i \right] \frac{\partial}{\partial G_r}. \quad (7.3)
\]

**Proposition 7.2.** For \( n, k \geq 1 \), we have \( X_0(F_n) = -nG_n \) and

\[
X_k(F_n) = -nG_{n-k} \times 1_{k \leq n} \quad \text{and} \quad X_{-k}(F_n) = -nG_{n+k}. \quad (7.4)
\]

**Proof.** From \( W_j F_p = p\delta_{j,p} \). \( \square \)

**Remark 7.1.** In terms of the coordinates \((b_k)_{k \geq 1}\),

\[
X_{-1} = -(b_2 - b_1^2) \frac{\partial}{\partial b_1} - (b_3 - b_1 b_2) \frac{\partial}{\partial b_2} - \cdots - (b_{n+1} - b_1 b_n) \frac{\partial}{\partial b_n} - \cdots,
\]

\[
X_{-2} = -(b_3^2 - 2b_1 b_2 + b_3) \frac{\partial}{\partial b_1} - \cdots - (b_2^2 b_n - b_2 b_n - b_1 b_{n+1} + b_{n+2}) \frac{\partial}{\partial b_n} - \cdots.
\]

In terms of the coordinates \((G_k)_{k \geq 1}\),

\[
X_0 = -G_1^{\partial}{\partial G_1} - (G_2 + G_1^2) \frac{\partial}{\partial G_2} + (G_n - K_{n+1}(G_1, G_2, \ldots)) \frac{\partial}{\partial G_n} + \cdots,
\]

\[
X_{-1} = -G_2^{\partial}{\partial G_1} - (G_2 G_1 + G_3) \frac{\partial}{\partial G_2} - (G_2^2 + G_3 G_1 + G_4) \frac{\partial}{\partial G_3}
\]

\[
- (G_5 + 2G_2 G_3 + G_1 G_4) \frac{\partial}{\partial G_5} - (G_6 + 2G_4 G_2 + G_1 G_5 + G_3^2) \frac{\partial}{\partial G_6} - \cdots
\]

\[
= \sum_{n \geq 1} (G_{n+1} + G_n G_1 - K_{n+1}^2(G_1, G_2, \ldots, G_n, G_{n+1})) \frac{\partial}{\partial G_n}.
\]

For \( k \geq 2 \), \( X_{-k} = \sum_{n \geq 1} H_n \frac{\partial}{\partial G_n} \) with

\[
H_n = G_{n+k} + G_{n+k-1} G_1 + G_{n+k-2} G_2 + \cdots + G_n G_k - K_{n+k}^2(G_1, G_2, \ldots, G_n, G_{n+k}).
\]

From our main theorem, we see that the coefficient \( H_n \) is a sum of partial derivatives of Faber polynomials.

**Lemma 7.3.** The condition \( X_{-k}(F_n) = -nG_{n+k} \) for \( n \geq 1 \) and \( k \geq 0 \) determines the operators \( X_{-k} \) in a unique way. Consider differential operators \( (\tilde{X}_{-k}) \), \( k \geq 0 \), of the form

\[
\tilde{X}_{-k} = B_1^k \frac{\partial}{\partial b_1} + B_2^k \frac{\partial}{\partial b_2} + \cdots + B_n^k \frac{\partial}{\partial b_n} + \cdots \quad \text{for} \quad k \geq 0,
\]

where the \( B_n^k \) are homogeneous polynomials in the variables \((b_1, b_2, \ldots, b_n, \ldots)\) of degree \( n + k \) and such that \( \tilde{X}_{-k}(F_n) = -nG_{n+k} \) for \( n \geq 1 \), \( k \geq 0 \), then \( \tilde{X}_{-k} = X_{-k} \).
Moreover, \( X_0[h(z)] = -h(z) + 1 \), \( X_{-1}[h(z)] = -\frac{h(z)}{z} + \frac{1}{z} + b_1 h(z) \),
\[
X_{-2}[h(z)] = -\frac{h(z)}{z^2} + \frac{1}{z^2} - \frac{G_1 h(z)}{z} - G_2 h(z),
\]
\[
X_{-3}[h(z)] = -\frac{h(z)}{z^3} + \frac{1}{z^3} - \frac{G_1 h(z)}{z^2} - \frac{G_2 h(z)}{z} - G_3 h(z),
\]
\[
\ldots
don'to
\[
X_{-j}[h(z)] = \frac{1}{z^j} - \sum_{0 \leq k \leq j} \frac{G_k}{z^{j-k}} \times h(z).
\]

**Proof.** Let \( h(z) = 1 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots \). For \( X_0 \), the condition \( X_0[F_n] = -nG_n \) for \( n \geq 1 \) implies that
\[
X_0 \frac{h'}{h} = -\sum_{k \geq 1} X_0(F_k)z^{k-1} = \sum_{k \geq 1} kG_kz^{k-1} = \frac{d}{dz} \left( \frac{1}{h(z)} \right).
\]
Exchanging the order of derivation \( X_0 \) and \( \frac{d}{dz} \), we have \( \frac{d}{dz} \frac{X_0(h)}{h} = \frac{d}{dz} \left( \frac{1}{h} \right) \). Integrating with respect to \( z \) gives \( \frac{X_0(h)}{h} = \frac{1}{h} + \alpha \) where \( \alpha \) is a constant. If we take \( \alpha = -1 \), then \( X_0(h) = 1 - h \). To express \( X_0 \) in the \((b_k)\) coordinates, we have
\[
X_0[h(z)] = -b_1 z - b_2 z^2 - \ldots - b_n z^n - \ldots = -b_1 \frac{\partial}{\partial b_1} h(z) - b_2 \frac{\partial}{\partial b_2} h(z) - \ldots.
\]
To get \( X_0 \) in terms of the \((G_k)_{k \geq 1}\) coordinates, we consider \( \tilde{h}(z) = \frac{1}{h(z)} \). We have
\[
X_0[\tilde{h}(z)] = X_0 \left[ \frac{1}{h(z)} \right] = \tilde{h}(z) - \tilde{h}(z)^2 = \sum_{n \geq 1} [G_n - K_n^2(G_1, \ldots, G_k, \ldots)] z^n.
\]
Since \( z^n = \frac{\partial}{\partial G_n}[\tilde{h}(z)] \), we obtain the result. For \( X_{-1} \), the method is the same. From \( X_{-1}(F_n) = -nG_{n+1} \) for \( n \geq 1 \), we deduce
\[
X_{-1} \frac{h'}{h} = -\sum_{k \geq 1} X_{-1}(F_k)z^{k-1} = \sum_{k \geq 1} kG_{k+1}z^{k-1}
\]
\[
= \frac{1}{z} \sum_{k \geq 1} kG_kz^{k-1} - \frac{1}{z^2} \sum_{k \geq 1} G_kz^k = \frac{1}{z} \frac{d}{dz} \left( \frac{1}{h(z)} \right) - \frac{1}{z^2} \left( \frac{1}{h(z)} \right) + \frac{1}{z^2}.
\]
Exchanging the order of derivation \( X_{-1} \) and \( \frac{d}{dz} \), we have
\[
\frac{d}{dz} \frac{X_{-1}(h)}{h} = \frac{d}{dz} \left( \frac{1}{zh(z)} - \frac{1}{z} \right).
\]
Integrating with respect to \( z \) gives \( \frac{X_{-1}(h)}{h} = \frac{1}{zh(z)} - \frac{1}{z} + \text{constant} \). Taking the constant equal to \( b_1 \) gives \( X_{-1} \). In the same way, \( X_{-j} F_n = -nG_{n+j} \), for \( n \geq 1 \) implies that
\[
X_{-j} \left( \frac{h'}{h} \right) = -\sum_{k \geq 1} W_{-j} F_k z^{k-1} = \sum_{k \geq 1} kG_{k+j}z^{k-1}
\]
\[
= \frac{1}{z^j} \sum_{k \geq 1} (k + j)G_{k+j}z^{k+j-1} - \frac{j}{z^{j+1}} \sum_{k \geq 1} G_{k+j}z^{k+j}.
\]
\[
\frac{d}{dz} \left( \frac{1}{z^j h(z)} - \frac{1}{z^j} \sum_{0 \leq k \leq j} G_k z^k \right). \quad \Box
\]

### 7.2. The operators \((M_k)_{k \in \mathbb{Z}}\)

We have \(L_0 = \sum_{k \geq 1} k b_k \frac{\partial}{\partial b_k} = \sum_{j \geq 1} F_j W_j\).

**Definition 7.2.** For \(k \geq 1\), let

\[
M_k = \sum_{j \geq 1} F_j W_{j+k} \quad \text{and} \quad M_{-k} = \sum_{j \geq 1} F_{j+k} W_j. \tag{7.5}
\]

**Proposition 7.4.** For \(k \geq 1\), \([M_k F_n] = n \delta_{n,k}.\) For the last identity, we verify that \(M_k F_n = (k - p) M_{k+p} F_n\) for \(p, k \in \mathbb{Z}\).

**Proof.** From \(W_j(F_n) = n \delta_{n,j}\). For the last identity, we verify that \([M_k F_n] = n \delta_{n,k}\).

**Lemma 7.5.** In terms of the coordinates \((b_k)_{k \geq 1}\), for \(k \geq 1\),

\[
M_k = \sum_{j \geq 1} F_j W_{j+k} = b_1 \frac{\partial}{\partial b_{k+1}} + 2b_2 \frac{\partial}{\partial b_{k+2}} + \cdots + pb_p \frac{\partial}{\partial b_{k+p}} + \cdots. \tag{7.6}
\]

In particular if \(h(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_p z^p + \cdots\), we have

\[
M_k[h(z)] = z^{k+1} h'(z) \quad \text{for } k \geq 1. \tag{7.7}
\]

**Proof.** We verify the identity on \(h(z)\). From \(W_j[h(z)] = -z^j h(z)\). Thus

\[
M_k[h(z)] = -z^k \left( \sum_{j \geq 1} F_j z^j \right) \times h(z) = z^{k+1} h'(z).
\]

Since \((b_1 \frac{\partial}{\partial b_{k+1}} + 2b_2 \frac{\partial}{\partial b_{k+2}} + \cdots + pb_p \frac{\partial}{\partial b_{k+p}} + \cdots) h(z) = z^{k+1} h'(z)\), we obtain (7.6). For \(k \geq 0\), the operators \(M_{-k} = \sum_{j \geq 1} F_{j+k} W_j\) are given by \(M_0 = L_0\),

\[
M_{-1} = (2b_2 - b_1^2) \frac{\partial}{\partial b_1} + (3b_3 - b_1 b_2) \frac{\partial}{\partial b_2} + \cdots + ((n+1)b_{n+1} - b_1 b_n) \frac{\partial}{\partial b_n} + \cdots,
\]

\[
M_{-2} = \sum_{j \geq 1} [(j+2)b_{j+2} - b_{j+1} b_1 + b_j (b_1^2 - 2b_2)] \frac{\partial}{\partial b_j},
\]

\[
\ldots
\]

\[
M_{-k} = \sum_{j \geq 1} (b_j F_k + b_{j+1} F_{k-1} + \cdots + b_{j+k-1} F_1 + (j+k) b_{j+k}) \frac{\partial}{\partial b_j}. \quad \Box
\]

**Remark 7.2.** On \(M\), define the differential operators

\[
L_k = M_k - W_k \quad \text{for } k \geq 1. \tag{7.8}
\]
With the convention $F_0 = -1$,

$$L_k = \sum_{j \geq 0} F_j W_{j+k} = F_0 W_k + F_1 W_{k+1} + F_2 W_{k+2} + \cdots \quad (7.9)$$

then $L_k = M_k - W_k$, $k \geq 1$, is the Kirillov operator

$$L_k = \frac{\partial}{\partial b_k} + \sum_{n \geq 1} (n+1)b_n \frac{\partial}{\partial b_{n+k}}. \quad (7.10)$$

For $f(z) = zh(z) = z + b_1 z^2 + b_2 z^3 + \cdots + b_p z^{p+1} + \cdots$, we have $L_k[f(z)] = z^{1+k} f'(z)$ and

$$L_k(F_n) = nF_{n-k} \times 1_{n \geq k}. \quad (7.11)$$

### 7.3. The operators $(V_j)_{j \geq 1}$ and $(V^k_j)_{j \geq 1}$

We do not stay anymore in the class of polynomials $(F_n)$, $(G_n)$. For $j \geq 1$ and $k \in \mathbb{Z}$, see (1.9)–(1.10),

$$V_j = - \sum_{n \geq 1} K_{n-j}^{j+1} \frac{\partial}{\partial b_{n+j}} \quad \text{and} \quad V^k_j = - \sum_{n \geq 0} K_{n-j}^{k+1} \frac{\partial}{\partial b_{n+j}}. \quad (7.12)$$

The polynomials $(P^k_n)_{n \geq 0}$, see Proposition 2.4 and [1, (A.1.2)], are given by

$$\frac{zf'(z)}{f(z)} h(z)^k = 1 + \sum_{n \geq 1} P^k_n f(z)^n \quad (7.13)$$

where $f(z) = zh(z)$. We have the recursion formulas, for $q \in \mathbb{Z}$,

$$(n+1)b_n = \sum_{0 \leq j \leq n} P^q_j K_{n-j}^{j+1-q}, \quad \frac{n+1}{k+q} K_{n-j}^{k+q} = \sum_{j=0}^{n} P^q_j K_{n-j}^{j+k}, \quad (7.14)$$

$$-F_n = \sum_{0 \leq j \leq n} P^q_j K_{n-j}^{j-q}. \quad (7.15)$$

With (7.14), we replace $(n+1)b_n$ in (7.10). It gives for $L_k$, $k \geq 1$ (with $b_0 = 1$),

$$L_k = \sum_{0 \leq n \leq 0 \leq j \leq n} P^q_j K_{n-j}^{j+1-q} \frac{\partial}{\partial b_{n+k}} = \sum_{0 \leq j \leq n} P^q_j \left[ \sum_{n \geq j} K_{n-j}^{j+1-q} \frac{\partial}{\partial b_{n+k}} \right] = - \sum_{j \geq 0} P^q_j V^q_{j+k}. \quad (7.16)$$

For $q = -k$ and $k \geq 1$, we obtain

$$L_k = - \sum_{j \geq 0} P^{-k}_j V_{j+k} = - \sum_{j \geq 1} P^{-k}_j \times 1_{j \geq k} V_j. \quad (7.16)$$

**Definition 7.3.** For any $k \in \mathbb{Z}$, with the convention $P^k_n = 0$ if $n < 0$, we put

$$L_k = - \sum_{j \geq 1} P^{-k}_j V_j. \quad (7.17)$$
7.4. The Kirillov operators \((L_{-p})_{p \geq 1}\)

It has been proved in [1, (A.4.5)] that the vector fields \((L_{-p})_{p \geq 0}\) obtained by Kirillov in [7] are such that for \(f(z) = z h(z)\), it holds
\[
L_{-p}[f(z)] = \sum_{j \geq 0} P^{p}_{1+j+p} f(z)^{j+2}.
\]
(7.18)

**Proposition 7.6.** Let \(L_{-p}, p \geq 1\) the operator defined by (7.18), then \(L_{-p}\) is given by (7.17), we have
\[
L_{-p} = - \sum_{j \geq 1} P^{p}_{j+p} V_{j}.
\]
(7.19)

**Proof.** From (5.9), \(V_{j}[f(z)] = -f(z)^{j+1}\). \(\square\)

**Remark 7.3.** We have
\[
L_{-p} = \sum_{r \geq 1} A^{p}_{r+1} \frac{\partial}{\partial b_{r+1}} \quad \text{with} \quad A^{p}_{r+1} = \sum_{0 \leq j \leq r} P^{p}_{j+j+p} K_{r-j}^{j+1}.
\]
(7.20)

**Proof.** From (7.18), \(L_{-p} = \sum_{j \geq 0, n \geq 0} P^{p}_{1+j+p} K_{n}^{j+2} \frac{\partial}{\partial b_{n+j+1}}\). Thus
\[
L_{-p} = \sum_{r \geq 0} A^{p}_{r+1} \frac{\partial}{\partial b_{r+1}} \quad \text{with} \quad A^{p}_{r+1} = \sum_{0 \leq j \leq r} P^{p}_{j+j+p} K_{r-j}^{j+2}.
\]
(7.21)

This proves (7.20). We obtain (7.21) with (5.11),
\[
L_{-p} = - \sum_{j \geq 0, n \geq 0, k \geq 0} P^{p}_{1+j+p} K_{n}^{j+2} G_{k} W_{n+j+k+1}
\]
\[
= - \sum_{j \geq 0, s \geq 0} P^{p}_{1+j+p} K_{s}^{j+1} W_{j+s+1}.
\]
\(\square\)

**Remark 7.4.** We have \(L_{-k} = M_{-k} - Y_{-k}\) with
\[
Y_{-k} = - \sum_{r \geq 1} J^{k}_{r} W_{r} \quad \text{and} \quad J^{k}_{r} = \sum_{s=0}^{k} P^{k}_{s} K_{r+k-s}^{s-k}.
\]
(7.22)

In particular \(L_{-1} = M_{-1} - X_{-1}\).

**Proof.** From (7.15), \(M_{-k} = \sum_{r \geq 1} F_{j+k} W_{j} = - \sum_{r \geq 1} [\sum_{0 \leq s \leq j+k} P^{q}_{s} K_{k+j-s}^{s-q}] W_{j}\)
\[
= \sum_{0 \leq s \leq j+k} P^{q}_{s} K_{k+j-s}^{s-q} - \sum_{0 \leq s \leq k} P^{q}_{s} K_{j-s}^{k+s-q} + \sum_{1 \leq s \leq j} P^{q}_{s} K_{j-s}^{k+s-q}.
\]

With \(k = q\), the second sum is \(J^{k}_{j}\) as in (7.21). The first sum gives \(Y_{-k}\). \(\square\)
Remark 7.5. With (1.11) and (5.7), we find for any \( p \in \mathbb{Z}, \ j \geq 1, \)

\[
L_{-p} V_j - V_j L_{-p} = \sum_{1 \leq s \leq j} (V_j (P^p_{s+p})) V_s.
\]

(7.23)

8. Second order differential operators

Let \( \triangle_0 = \sum_{p \geq 1, q \geq 1} F_{p+q} (W_{p+q} + W_p W_q) \).

Proposition 8.1. Let \( L_0 = \sum_{j \geq 1} F_j W_j \) be the homogeneity operator, then

\[
\triangle_0 F_n = n(n - 1)F_n \quad \text{and} \quad (\triangle_0 + L_0)F_n = n^2F_n.
\]

Proof. Because of (1.6), \( W_p W_q F_n = 0 \). On the other hand

\[
\sum_{p \geq 1, q \geq 1} F_{p+q} W_{p+k} F_n = n \times \left( \sum_{p \geq 1, q \geq 1} \delta_{p+q,n} \right) F_n.
\]

Then we remark that \( (\sum_{p \geq 1, q \geq 1} \delta_{p+q,n}) = n - 1. \)

Definition 8.1. We consider \( Q_j = \sum_{p \geq 1, q \geq 1, p+q=j} W_p W_q \) for \( j \geq 2 \).

Because of (1.8), \( \triangle_0 \) and \( Q_j \) are second order differential operators on the manifold \( M \). With the expression (1.8) of \( W_{p+q} + W_p W_q \), we have

\[
\frac{\partial}{\partial b_j} (W_p W_q + W_{p+q}) - (W_p W_q + W_{p+q}) \frac{\partial}{\partial b_j} = W_p \frac{\partial}{\partial b_{q+p}} + W_q \frac{\partial}{\partial b_{j+p}}.
\]

Since \( W_p \) and \( W_q \) commute, we have \( Q_2 = W_1^2 = K_2^2(W_1, 0), Q_3 = 2W_1 W_2 = K_3^2(W_1, W_2, 0), \)
\( Q_4 = 2W_1 W_3 + W_2^2 = K_4^2(W_1, W_2, 2, 0), \ldots, Q_n = K_n^2(W_1, W_2, \ldots, W_{n-1}, 0) \) and \( Q_j W_p = W_p Q_j \) for \( j \geq 2 \) and \( p \geq 1. \)

Since \( W_{j-k} W_k G_n = W_{j-p} W_p G_n, \) for \( k \leq j, \ p \leq j, \) we have \( Q_2 G_j = G_{j-2}, \ Q_3 G_j = 2G_{j-3}, \ldots, Q_n G_j = (n - 1)G_{j-n}. \)

The operator \( \triangle_0 \) decomposes into \( \triangle_0 = \triangle_1 + \triangle_2 \) with

\[
\triangle_1 = \sum_{j \geq 2} F_j Q_j,
\]

\[
\triangle_2 = \sum_{j \geq 2} \left( \sum_{p \geq 1, q \geq 1, p+q=j} 1 \right) F_j W_j = \sum_{j \geq 2} (j - 1)F_j W_j.
\]

Since \( W_j F_n = n \delta_{j,n}, \) we have \( Q_j F_n = 0, \ j \geq 2 \) and since \( Q_j (G_n) = (j - 1)G_{n-j}, \) we find

\[
\triangle_1 F_n = 0,
\]

\[
\triangle_2 F_n = n(n - 1)F_n,
\]

and

\[
\triangle_1 G_n = \triangle_2 G_n = \sum_{j \geq 2} (j - 1)F_j G_{n-j}.
\]

We deduce that
\[ \Delta_1 Q_j = Q_j \Delta_1, \quad W_j \Delta_1 - \Delta_1 W_j = j Q_j, \]
\[ \Delta_2 Q_j = Q_j \Delta_2, \quad W_j \Delta_2 - \Delta_2 W_j = j(j - 1) W_j, \]

and \[ \Delta_2 \Delta_1 - \Delta_1 \Delta_2 = \sum_{k \geq 2} k(k - 1) F_k Q_k. \]

**Lemma 8.2.** Let \( X_0 = - \sum_{j \geq 1} G_j W_j \) and \( L_0 = \sum_{j \geq 1} F_j W_j \), then
\[ X_0 G_n = G_n - K_n^2 (G_1, G_2, \ldots, G_n), \quad X_0 F_n = -n G_n, \] (8.1)
\[ X_0 L_0 = L_0 X_0, \] (8.2)
\[ L_0 \Delta_2 = \Delta_2 L_0 \quad \text{and} \quad L_0 \Delta_1 - \Delta_1 L_0 = \sum_{k \geq 2} k F_k Q_k. \] (8.3)

**Proof.** (8.1) results from the expression of \( X_0 \) in the \( (G_n)_{n \geq 1} \) coordinates. Because of (1.6), we have
\[ L_0 X_0 = \sum_{q \geq 1} [\sum_{j \geq 1} F_j G_{q-j}] W_q + \sum_{j \geq 1, q \geq 1} F_j G_q W_j W_q \]
and
\[ X_0 L_0 = \sum_{q \geq 1} q G_q W_q + \sum_{j \geq 1, q \geq 1} F_j G_q W_j W_q. \]

Since \[ \sum_{j \geq 1} F_j G_{q-j} = q G_q, \] see (2.5), it proves that \( X_0 L_0 = L_0 X_0 \). The identities (8.3) are consequence of (1.6). \( \square \)

9. The conformal map from the exterior of the unit disk onto the exterior of \([-2, +2]\)

Let \( \psi(w) = w + \frac{1}{w} \) be the conformal map from the exterior of the unit disk onto the exterior of \([-2, 2]\). The Faber polynomials \( F_n(z) \) of \([-2, 2]\) are given by
\[ \frac{w^2 - 1}{w^2 - wz + 1} = \sum_{n=0}^{\infty} F_n(z) w^{-n}. \]

They satisfy the differential equation
\[ (z^2 - 4) F''_n(z) + z F'_n(z) = n^2 F_n(z). \] (9.1)

In the following, we consider Faber polynomials \( F_n(b_1, b_2, 0, 0, \ldots, 0) \). All the \( b_j \) are zero except \( b_1 \) and \( b_2 \). We have \( F_1(b_1) = -b_1, F_2(b_1, b_2) = b_1^2 - 2b_2, F_3(b_1, b_2, 0) = -b_1^3 + 3b_1 b_2, F_4(b_1, b_2, 0, 0) = b_1^4 - 4b_1^2 b_2 + 2b_2^2, \ldots. \)

**Theorem 9.1.** Faber polynomials associated to \( \psi(w) = w + b_1 + \frac{b_2}{w} \) verify
\[ ((z - b_1)^2 - 4b_2) F''_n(z) + (z - b_1) F'_n(z) = n^2 F_n(z). \] (9.2)

In particular, if \( b_1 = 0 \) and \( b_2 = 1 \), we obtain (9.1).

To prove the theorem, we need the following lemmas.

**Lemma 9.2.** Let \( L = \frac{\partial^2}{\partial b_1} + \frac{\partial}{\partial b_2} + \sum_{k \geq 1} b_k \frac{\partial^2}{\partial b_2 \partial b_k} \), then \( LF_n = 0 \).

**Proof.** From (6.15). \( \square \)
Lemma 9.3. Consider $\Delta_0 = \sum_{p \geq 1, q \geq 1} F_{p+q}(W_{p+q} + W_pW_q)$ and let $\phi(b_1, b_2)$ be a function defined on $M$, which depends only of $b_1, b_2$, then

$$\Delta_0 \phi =\left[(b_1^2 - 2b_2)\frac{\partial^2}{\partial b_1^2} + 2b_1b_2\frac{\partial^2}{\partial b_1 \partial b_2} + 2b_2^2\frac{\partial^2}{\partial b_2^2}\right] \phi.$$

Proof. Let $\phi$ depend only on the variables $b_1$ and $b_2$. If $p > 2$ or $q > 2$, we have $[W_{p+q} + W_pW_q] \phi = 0$. If $p = 2$, $q = 2$, then $\phi = W_2^2 \phi = \frac{\partial^2}{\partial b_2^2} \phi$. If $p = 2$, $q = 1$ or $p = 1$, $q = 2$, $(W_3 + W_2W_1) \phi = \frac{\partial}{\partial b_1} (\frac{\partial}{\partial b_1} + b_1 \frac{\partial}{\partial b_2}) \phi$. If $p = 1$, $q = 1$, then $(W_2 + W_1) \phi = [-\frac{\partial}{\partial b_2} + (\frac{\partial}{\partial b_1} + b_1 \frac{\partial}{\partial b_2})^2] \phi$.

We calculate $\Delta_0 \phi = [F_2(W_2 + W_1^2) + 2F_3(W_3 + W_2W_1) + F_4(W_4 + W_2^2)] \phi$. This gives

$$\Delta_0 \phi = \left[F_2\left(-\frac{\partial}{\partial b_2} + \left(\frac{\partial}{\partial b_1} + b_1 \frac{\partial}{\partial b_2}\right)^2\right) + 2F_3\left(\frac{\partial}{\partial b_2} \left(\frac{\partial}{\partial b_1} + b_1 \frac{\partial}{\partial b_2}\right) + F_4 \frac{\partial^2}{\partial b_2^2}\right)\right] \phi$$

or equivalently

$$\Delta_0 \phi = \left[F_2\left(\frac{\partial^2}{\partial b_1^2} + 2b_1 \frac{\partial^2}{\partial b_1 \partial b_2} + b_2^2 \frac{\partial^2}{\partial b_2^2}\right) + 2F_3\left(\frac{\partial^2}{\partial b_1 \partial b_2} + b_1 \frac{\partial^2}{\partial b_2^2}\right) + F_4 \frac{\partial^2}{\partial b_2^2}\right] \phi.$$

Replacing $F_2, F_3, F_4$, we find Lemma 9.3.

Proof of the theorem. From Lemma 9.2, we know that

$$\left(2b_1b_2 \frac{\partial^2}{\partial b_1 \partial b_2} + 2b_2^2 \frac{\partial^2}{\partial b_2^2}\right) F_n = \left(-2b_2 \frac{\partial}{\partial b_2} - 2b_2 \frac{\partial^2}{\partial b_2^2}\right) F_n.$$

We replace the right hand side in the expression of $\Delta_0$ and we find

$$\left[(b_1^2 - 4b_2) \frac{\partial^2}{\partial b_1^2} - 2b_2 \frac{\partial}{\partial b_2}\right] F_n = n(n - 1) F_n. \tag{9.3}$$

Since $F_n$ is homogeneous, $b_1 \frac{\partial}{\partial b_1} F_n + 2b_2 \frac{\partial}{\partial b_2} F_n = n F_n$. Replacing $-2b_2 \frac{\partial}{\partial b_2} F_n$ in (9.2), we find

$$(b_1^2 - 4b_2) \frac{\partial F_n}{\partial b_1} + b_1 \frac{\partial F_n}{\partial b_1} = n^2 F_n. \tag{9.3}$$

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References


