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Differential calculus on the Faber polynomials

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Abstract

The Faber polynomials are presented as a coordinate system to study the geometry of the manifold of coefficients of univalent functions.

Résumé

Les polynômes de Faber sont présentés comme un système de coordonnées pour étudier la géométrie de la variété des coefficients des fonctions univalentes.

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1. Introduction

We show how the methods introduced in [2] and [3] allow to do differential calculus on the manifold of coefficients of univalent functions. The Faber polynomials $(F_k)_{k\geqslant 1}$ are given by the identity [5,12]

$$1 + b_1 w + b_2 w^2 + \cdots + b_k w^k + \cdots = \exp\left(-\sum_{k=1}^{+\infty} \frac{F_k(b_1, b_2, \ldots, b_k)}{k} w^k\right). \tag{1.1}$$

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The polynomials $(G_m)_{m\geqslant 1}$ and $(K_n^p)_{n\geqslant 1}$, $p\in Z$ are given by

$$\frac{1}{1+b_1w+b_2w^2+\cdots+b_kw^k+\cdots}=1+\sum_{m=1}^{+\infty}G_m(b_1,b_2,\ldots,b_m)w^m, \tag{1.2}$$

$$\left(1+b_1w+b_2w^2+\cdots+b_kw^k+\cdots\right)^p=1+\sum_{n\geqslant 1}K_n^p(b_1,b_2,\ldots,b_n)w^n, \tag{1.3}$$

then $G_m=K_m^{-1}$ and $K_m^1=b_m$. Important polynomials are also the $(P_n^k)_{n\geqslant 2}$, see [1, (A.1.7)]. If $f(z)=zh(z)$,

$$\left(\frac{zf'(z)}{f(z)}\right)^2\left[h(z)\right]^k=\sum_{n\geqslant 2}P_n^{n+k}z^n \quad \text{for } k\in Z. \tag{1.4}$$

The polynomials $(F_n)_{n\geqslant 0}$, $(G_n)_{n\geqslant 0}$, $(K_n^p)_{n\geqslant 0}$, $(P_n^k)_{n\geqslant 2}$ are homogeneous of degree n in the variables $(b_1,b_2,\ldots)$ where b_k has weight k. As in [1–3,7] let the function of the infinite number of variables

$$(b_1,b_2,\ldots,b_k,\ldots)\to h(z)=1+b_1z+b_2z^2+\cdots+b_kz^k+\cdots.$$

On the infinite dimensional manifold of coefficients $\mathcal{M}=\{(b_1,b_2,\ldots,b_k,\ldots)\}$ of univalent functions, consider the operators [3],

$$W_j=-\frac{\partial}{\partial b_j}-b_1\frac{\partial}{\partial b_{j+1}}-\cdots-b_k\frac{\partial}{\partial b_{j+k}}-\cdots \quad \text{for } j\geqslant 1. \tag{1.5}$$

For $j\geqslant 1$, it holds $\frac{\partial}{\partial b_j}[h(z)]=z^j$ and $W_j[h(z)]=-z^jh(z)$. We have (see [3])

$$W_j(F_m)=m\delta_{jm} \quad \text{and} \quad W_j(G_m)=G_{m-j}\times 1_{m\geqslant j}, \tag{1.6}$$

$$\frac{\partial}{\partial b_p}W_j-W_j\frac{\partial}{\partial b_p}=-\frac{\partial}{\partial b_{j+p}} \quad \text{and} \quad W_pW_q=W_qW_p, \tag{1.7}$$

$$W_pW_q+W_{p+q}=\sum_{k\geqslant 0}\sum_{m\geqslant 0}b_kb_m\frac{\partial^2}{\partial b_{q+m}\partial b_{k+p}}. \tag{1.8}$$

For $k\in Z$, let

$$V_j^k=-\sum_{n\geqslant 0}K_n^{k+1}\frac{\partial}{\partial b_{n+j}}. \tag{1.9}$$

We consider for $j\geqslant 1$ and $a\in Z$, the operators $(V_j^{aj})_{j\geqslant 1}$ and for $a=1$, we put

$$V_j=-\sum_{n\geqslant 0}K_n^{j+1}\frac{\partial}{\partial b_{n+j}}. \tag{1.10}$$

Then $W_j=V_j^0$, $V_j^{-1}=-\frac{\partial}{\partial b_j}$, $V_j=V_j^j$, $j\geqslant 1$ and

$$V_j^kV_p^s-V_p^sV_j^k=(k-s)V_{j+p}^{k+s} \quad \text{for } p\geqslant 1,\ j\geqslant 1. \tag{1.11}$$

The differential operators $(V_j^k)_{j\geqslant 1}$, $k\in Z$ form an algebra and for $a\in Z$, the set of $(V_j^{aj})_{j\geqslant 1}$ is a subalgebra since

$$V_j^{aj}V_p^{ap}-V_p^{ap}V_j^{aj}=a(j-p)V_{j+p}^{a(j+p)}. \tag{1.12}$$

Let $f(z) = zh(z)$. For $j \geqslant 1$, the vector field V_j is the image through the map $f \to f^{-1}$ of the Kirillov operator

$$L_j = \frac{\partial}{\partial b_j} + \sum_{n\geqslant 1}(n+1)b_n \frac{\partial}{\partial b_{n+j}}. \tag{1.13}$$

Let $x_1, x_2, \ldots, x_n$, be the roots of $\xi^n + b_1\xi^{n-1} + b_2\xi^{n-2} + \cdots + b_{n-1}\xi + b_n = 0$ and consider Newton symmetric functions $\pi_k = x_1^k + x_2^k + \cdots + x_n^k$, $k \geqslant 1$, it was proved in [3] that

$$\pi_k(b_1, b_2, \ldots, b_n) = F_k(b_1, b_2, \ldots, b_n) \quad \text{for } k \leqslant n, \tag{1.14}$$

where $(F_k)_{k\geqslant 1}$ are the Faber polynomials. This is a consequence of

$$\log\bigl(1 + b_1 w + b_2 w^2 + \cdots + b_k w^k + \cdots\bigr) = -\sum_{k=1}^{+\infty} \frac{F_k(b_1, b_2, \ldots, b_k)}{k} w^k \tag{1.15}$$

or equivalently (1.1). With this identification, the exact coefficients of the polynomial $F_k(b_1, b_2, \ldots, b_k)$, $k \geqslant 1$, have been calculated in [3]. The polynomials $(F_k)_{k\geqslant 1}$ are completely determined as homogeneous polynomial solutions of the system of partial differential equations (1.6) involving $(W_j)_{j\geqslant 1}$ (See [3]). The exact coefficients of the polynomials (G_n) and of all the (K_n^p) have been given in [3].

The object of this note is to prove that the polynomials (K_n^p) are all obtained as partial derivatives of the Faber polynomials and show how some of the recursion formulae on the polynomials are related to elementary differential calculus on $\mathcal{M}$. This is a step towards the classification of Faber type polynomials (see [8]). In the last section, we give the example of the conformal map from the exterior of the unit disk onto the exterior of $[-2, +2]$. This shows how to introduce non trivial second order differential operators on the manifold $\mathcal{M}$.

Main Theorem. *We have for $n \geqslant 1$, $k \geqslant 1$,*

$$\frac{\partial F_n}{\partial b_k} = -nG_{n-k} \times 1_{n\geqslant k}, \tag{1.16}$$

$$\frac{\partial}{\partial b_k} G_n = \frac{\partial}{\partial b_k} K_n^{-1} = -K_{n-k}^{-2} \times 1_{k\leqslant n}, \qquad \frac{\partial}{\partial b_k} K_n^p = pK_{n-k}^{p-1} \times 1_{k\leqslant n},$$

$$\frac{\partial^2 F_j}{\partial b_k \partial b_p} = jK_{j-(p+k)}^{-2} \times 1_{j\geqslant k+p}, \qquad \frac{\partial^3 F_j}{\partial b_r \partial b_k \partial b_p} = -2jK_{j-(p+k+r)}^{-3} \times 1_{j\geqslant k+p+r}.$$

For $j \geqslant k_1 + k_2 + \cdots + k_s$, $k_1 \geqslant 1, \ldots, k_s \geqslant 1$ and $s \geqslant 1$,

$$\frac{\partial^s F_j}{\partial b_{k_1} \partial b_{k_2} \cdots \partial b_{k_s}} = (-1)^s (s-1)!\, jK_{j-(k_1+k_2+\cdots+k_s)}^{-s}. \tag{T1}$$

Moreover for $n \geqslant 1$, $k \geqslant 1$,

$$K_n^k(b_1, b_2, \ldots, b_n) = K_n^{-k}\bigl(G_1(b_1), G_2(b_1, b_2), \ldots, G_j(b_1, b_2, \ldots, b_j), \ldots, G_n(b_1, b_2, \ldots, b_n)\bigr). \tag{T2}$$

In the notation, all functions are functions of $(b_1, b_2, \ldots, b_n, \ldots)$.

The first (F_n) for $1 \leqslant n \leqslant 11$, as well as the first (G_n) are shown in [3]. The (K_n^p), $p \in Z$, $p \neq 0$ and $n \leqslant 5$ are in [2]. In [3], an exact expression of the coefficients of all the polynomials (K_n^p) has been given. We explicit the first K_n^p,

$$K_1^2 = 2b_1, \qquad K_5^2 = 2b_5 + 2b_1b_4 + 2b_2b_3,$$
$$K_2^2 = 2b_2 + b_1^2, \qquad K_6^2 = 2b_5b_1 + 2b_4b_2 + b_3^2 + 2b_6,$$
$$K_3^2 = 2b_3 + 2b_1b_2, \qquad K_7^2 = 2b_5b_2 + 2b_7 + 2b_4b_3 + 2b_1b_6,$$
$$K_4^2 = 2b_4 + 2b_1b_3 + b_2^2, \qquad K_8^2 = 2b_7b_1 + 2b_5b_3 + 2b_2b_6 + 2b_8 + b_4^2,$$
$$K_1^3 = 3b_1, \qquad K_3^3 = 6b_1b_2 + b_1^3 + 3b_3,$$
$$K_2^3 = 3b_1^2 + 3b_2, \qquad K_4^3 = 6b_1b_3 + 3b_1^2b_2 + 3b_4 + 3b_2^2,$$
$$K_5^3 = 6b_1b_4 + 3b_1^2b_3 + 3b_1b_2^2 + 3b_5 + 6b_2b_3,$$
$$K_6^3 = 6b_5b_1 + 6b_4b_2 + 3b_3^2 + 3b_6 + 3b_1^2b_4 + 6b_1b_2b_3 + b_2^3,$$
$$K_7^3 = 6b_5b_2 + 3b_5b_1^2 + 3b_7 + 6b_4b_3 + 6b_1b_6 + 6b_1b_4b_2 + 3b_1b_3^2 + 3b_2^2b_3,$$
$$K_8^3 = 6b_7b_1 + 6b_1b_5b_2 + 6b_1b_4b_3 + 3b_1^2b_6 + 6b_5b_3 + 3b_4b_2^2$$
$$+ 3b_2b_3^2 + 6b_2b_6 + 3b_8 + 3b_4^2,$$
$$K_1^4 = 4b_1, \qquad K_3^4 = 12b_1b_2 + 4b_1^3 + 4b_3,$$
$$K_2^4 = 6b_1^2 + 4b_2, \qquad K_4^4 = 12b_1b_3 + 12b_1^2b_2 + 6b_2^2 + b_1^4 + 4b_4,$$
$$K_5^4 = 12b_1b_4 + 12b_3b_1^2 + 12b_1b_2^2 + 12b_2b_3 + 4b_2b_1^3 + 4b_5,$$
$$K_6^4 = 12b_1b_5 + 12b_1^2b_4 + 24b_1b_2b_3 + 12b_2b_4 + 6b_3^2 + 4b_1^3b_3 + 4b_2^3 + 6b_1^2b_2^2 + 4b_6,$$
$$K_1^{-2} = -2b_1, \qquad K_3^{-2} = -2b_3 + 6b_1b_2 - 4b_1^3,$$
$$K_2^{-2} = 3b_1^2 - 2b_2, \qquad K_4^{-2} = 5b_1^4 + 6b_1b_3 + 3b_2^2 - 12b_1^2b_2 - 2b_4,$$
$$K_5^{-2} = -12b_1b_2^2 - 4b_5 + 20b_2b_1^3 + 6b_1b_4 - 6b_1^5 - 12b_3b_1^2 + 6b_2b_3,$$
$$K_6^{-2} = 3b_3^2 + 30b_2^2b_1^2 + 12b_5b_1 - 12b_1^2b_4 - 4b_2^3 - 24b_1b_2b_3 + 6b_2b_4$$
$$+ 7b_1^6 - 2b_6 + 20b_1^3b_3 - 30b_2b_1^4,$$
$$K_7^{-2} = 12b_5b_2 + 6b_1b_6 - 12b_1b_3^2 - 12b_2^2b_3 - 24b_1b_4b_2 + 42b_2b_1^5 + 6b_4b_3 - 2b_7$$
$$- 8b_1^7 + 60b_2b_3b_1^2 - 30b_1^4b_3 - 60b_2^2b_1^3 + 20b_2^3b_1 + 20b_1^3b_4 - 24b_5b_1^2,$$
$$K_8^{-2} = -12b_4b_2^2 - 60b_2^3b_1^2 - 12b_2b_3^2 + 105b_2^2b_1^4 - 30b_1^4b_4 - 120b_2b_3b_1^3 - 12b_1^2b_6$$
$$+ 5b_2^4 + 60b_1^2b_4b_2 - 2b_8 + 6b_7b_1 + 12b_5b_3 + 6b_2b_6 + 60b_2^2b_3b_1 + 42b_3b_1^5$$
$$- 48b_1b_5b_2 + 30b_1^2b_3^2 + 3b_4^2 + 9b_1^8 - 56b_2b_1^6 - 24b_1b_4b_3 + 40b_1^3b_5,$$
$$K_1^{-3} = -3b_1, \qquad K_3^{-3} = -3b_3 - 10b_1^3 + 12b_1b_2,$$
$$K_2^{-3} = 6b_1^2 - 3b_2, \qquad K_4^{-3} = 15b_1^4 + 6b_2^2 - 30b_2b_1^2 - 3b_4 + 12b_1b_3,,$$
$$K_5^{-3} = 12b_1b_4 + 12b_2b_3 + 60b_2b_1^3 - 30b_1b_2^2 - 30b_3b_1^2 - 21b_1^5 - 3b_5,$$
$$K_6^{-3} = 12b_4b_2 + 60b_3^2 + 90b_2^2b_1^2 - 3b_6 - 30b_4b_1^2 + 28b_1^6 - 60b_3b_1b_2 + 60b_3b_1^3$$
$$- 105b_1^4b_2 - 10b_2^3 + 12b_1b_5,$$
$$K_3^{-4} = 4 \times \left(-5b_1^3 + 5b_1b_2 - b_3\right),$$
$$K_4^{-5} = 5 \times \left(14b_1^4 - 21b_2b_1^2 + 6b_1b_3 - b_4 + 3b_2^2\right).$$

An expression of K_n^p can be obtained as follows, let

$$\phi_n(z) = b_1z + b_2z^2 + b_3z^3 + \cdots + b_nz^n.$$

The line integral $D_n^k = \frac{1}{2i\pi}\int \frac{\phi_n(\xi)^k}{\xi^{n+1}}\,d\xi$ is equal to the coefficient of z^n in $\phi_n(z)^k$ and is of course independent of p. For any $p \in Z$, we have,

$$K_n^p = pb_n + \frac{p(p-1)}{2}D_n^2 + \frac{p!}{(p-3)!3!}D_n^3 + \frac{p!}{(p-4)!4!}D_n^4 + \cdots + \frac{p!}{(p-n)!n!}D_n^n. \quad \text{(T3)}$$

D_n^k is the sum of terms having k factors in K_n^k and

$$C_n^p = \frac{p!}{n!(p-n)!} = \frac{p(p-1)\cdots(p-n+1)}{n!}$$

is the binomial coefficient. If $b_1 \neq 0$,

$$D_n^k(b_1, b_2, \ldots, b_n) = b_1^k K_{n-k}^k\left(\frac{b_2}{b_1}, \ldots, \frac{b_{n-k+1}}{b_1}\right). \quad \text{(T4)}$$

Replacing in (T3) and iterating the procedure permits to obtain the exact expression on K_n^p, see [3] and Section 4.2 below.

We have relations between the partial derivatives of the Faber polynomials as

$$\frac{\partial G_n}{\partial b_k} = \frac{\partial G_{n+p}}{\partial b_{k+p}} = -K_{n-k}^{-2} \times 1_{n\geqslant k} \quad \text{for all } n \geqslant 1,\ k \geqslant 1, \text{ and } p \geqslant 0, \quad (1.17)$$

$$\frac{\partial^2 F_n}{\partial b_1^2} = -n\frac{\partial G_n}{\partial b_2}, \qquad \frac{\partial^2 F_n}{\partial b_r \partial b_s} = -n\frac{\partial G_n}{\partial b_{r+s}}. \quad (1.18)$$

Let

$$X_0 = -\sum_{j\geqslant 1} b_j \frac{\partial}{\partial b_j} = -b_1\frac{\partial}{\partial b_1} - b_2\frac{\partial}{\partial b_2} - \cdots - b_k\frac{\partial}{\partial b_k} - \cdots = -\sum_{j\geqslant 1} G_j W_j, \quad (1.19)$$

then

$$X_0 F_n = -nG_n \quad \text{and} \quad \frac{\partial^2 F_n}{\partial b_r \partial b_s} = \frac{\partial}{\partial b_{r+s}}(X_0 F_n) \quad \forall r \geqslant 1, s \geqslant 1, \quad (1.20)$$

$$\frac{\partial}{\partial b_j}X_0 - X_0\frac{\partial}{\partial b_j} = -\frac{\partial}{\partial b_j}. \quad (1.21)$$

This leads to the construction of differential operators on $\mathcal{M}$ which transform one polynomial into the other. See Section 7.

On the other hand, from (T1), we obtain, see (3.19) and Section 4,

Main Corollary. *The coefficients of the Schwarzian derivative of* $f(z) = z + b_1z^2 + b_2z^3 + \cdots + b_nz^{n+1} + \cdots$ *are given in terms of Faber polynomials and their second derivatives as* $z^2S(f)(z) = z^2[(\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2] = \sum_{k\geqslant 2}\mathcal{P}_k z^k$ *where*

$$\mathcal{P}_k = -(k-2)F_k\big(2b_1, 3b_2, \ldots, (j+1)b_j, \ldots\big) - \frac{1}{2}K_k^2\big(F_1(2b_1), F_2(2b_1, 3b_2), \ldots, F_k\big(2b_1, 3b_2, \ldots, (k+1)b_k\big)\big) \quad \text{(C1)}$$

and $K_k^2(F_1(2b_1), F_2(2b_1, 3b_2), \ldots) = -\frac{1}{k+2}\frac{\partial^2 F_{k+2}}{\partial b_1^2}(c_1, c_2, \ldots, c_k)$ *is the second derivative* F_{k+2} *calculated at the point*

$$(c_1, c_2, \dots, c_k) = \big(G_1\big(F_1(2b_1)\big), G_2\big(F_1(2b_1), F_2(2b_1, 3b_2)\big), \dots, G_k\big(F_1(2b_1), \dots, F_k\big(2b_1, 3b_2, \dots, (k+1)b_k\big)\big)\big).$$

The tool is the composition of maps on the manifold $\mathcal{M}$. We have

$$\frac{h'(w)}{h(w)} = -\sum_{k=1}^{+\infty} F_k(b_1, b_2, \dots, b_k) w^{k-1} = -\big(F_1 + F_2 w + F_3 w^2 + \cdots\big). \tag{1.22}$$

The function $f(w) = wh(w) = w + b_1 w^2 + b_2 w^3 + \cdots + b_n w^{n+1} + \cdots$ satisfies

$$w\frac{f'(w)}{f(w)} = 1 + w\frac{h'(w)}{h(w)} = 1 - \sum_{k\geqslant 1} F_k(b_1, b_2, \dots, b_k) w^k \tag{1.23}$$

and $g(w) = \frac{w}{h(w)} = w + \sum_{n\geqslant 1} G_n(b_1, b_2, \dots, b_n) w^{n+1} + \cdots$ satisfies

$$w\frac{g'(w)}{g(w)} = 1 - w\frac{h'(w)}{h(w)} = 1 + \sum_{k\geqslant 1} F_k(b_1, b_2, \dots, b_k) w^k. \tag{1.24}$$

From (1.23) and (1.24), we deduce

$$F_n\big(G_1(b_1), G_2(b_1, b_2), \dots, G_n(b_1, b_2, \dots, b_n)\big) + F_n(b_1, b_2, \dots, b_n) = 0. \tag{1.25}$$

We consider the following maps from $\mathcal{M} \to \mathcal{M}$,

$$\begin{aligned}
&F : (b_1, b_2, \dots, b_n, \dots) \to \big(F_1(b_1), F_2(b_1, b_2), \dots, F_n(b_1, b_2, \dots, b_n), \dots\big)\\
&F^{-1} : (b_1, b_2, \dots, b_n, \dots) \to (c_1, c_2, \dots, c_n, \dots) \quad \text{such that}\\
&F_1(c_1) = b_1, \quad F_2(c_1, c_2) = (b_1, b_2), \quad \dots,\\
&F_n(c_1, c_2, \dots, c_n) = (b_1, b_2, \dots, b_n), \quad \dots,\\
&G : (b_1, b_2, \dots, b_n, \dots) \to \big(G_1(b_1), G_2(b_1, b_2), \dots, G_n(b_1, b_2, \dots, b_n), \dots\big),\\
&S : (b_1, b_2, \dots, b_n, \dots) \to (-b_1, -b_2, \dots, -b_n, \dots).
\end{aligned}$$

The relation (1.25) means that G is obtained as the composition of maps

$$G = F^{-1} \circ S \circ F. \tag{1.26}$$

The first polynomials $(F_n^{-1})_{n\geqslant 1}$ defined by the map F^{-1} are given by

$$\begin{aligned}
&F_1^{-1}(b_1) = -b_1 \quad \text{and} \quad F_2^{-1}(b_1, b_2) = \frac{1}{2}\big(b_1^2 - b_2\big),\\
&F_3^{-1}(b_1, b_2, b_3) = \frac{1}{6}\big(-b_1^3 + 3b_1 b_2 - 2b_3\big),\\
&F_4^{-1}(b_1, b_2, b_3, b_4) = \frac{1}{4!}\big(b_1^4 - 6b_1^2 b_2 + 3b_2^2 + 8b_1 b_3 - 6b_4\big),\\
&F_5^{-1} = \frac{1}{5!}\big(-b_1^5 - 15b_1 b_2^2 + 10b_1^3 b_2 - 20b_1^2 b_3 + 20b_2 b_3 + 30b_1 b_4 - 24b_5\big),\\
&F_6^{-1} = \frac{1}{6!}\big(b_1^6 + 144b_1 b_5 - 15b_2^3 + 45b_2^2 b_1^2 - 15b_2 b_1^4 - 120b_1 b_2 b_3 + 90b_2 b_4\\
&\qquad + 40b_1^3 b_3 - 120b_6 + 40b_3^2 - 90b_1^2 b_4\big),
\end{aligned}$$

$$F_7^{-1} = \frac{1}{7!}\big(-b_1^7 - 504b_1^2b_5 + 504b_2b_5 + 840b_1b_6 + 21b_1^5b_2 + 420b_2b_1^2b_3 - 70b_1^4b_3 - 280b_1b_3^2 - 210b_2^2b_3 - 105b_2^2b_1^3 + 105b_1b_2^3 + 420b_3b_4 - 720b_7 + 210b_1^3b_4 - 630b_1b_2b_4\big),$$

$$F_8^{-1} = \frac{1}{8!}\big(b_1^8 - 4032b_1b_2b_5 + 1344b_1^3b_5 + 2688b_3b_5 - 3360b_1^2b_6 + 3360b_2b_6 + 5760b_1b_7 + 105b_2^4 - 420b_2^3b_1^2 + 1680b_1b_2^2b_3 - 1120b_1^3b_2b_3 - 420b_1^4b_4 + 210b_1^4b_2^2 - 1120b_2b_3^2 + 112b_1^5b_3 + 1120b_1^2b_3^2 - 28b_1^6b_2 - 5040b_8 + 1260b_4^2 + 2520b_2b_1^2b_4 - 1260b_2^2b_4 - 3360b_1b_3b_4\big).$$

We put $F_0^{-1} = 1$. We have $\exp(-\sum_{j\geqslant 1} \frac{b_j}{j} z^j) = 1 + \sum_{k\geqslant 1} F_k^{-1}(b_1, b_2, \ldots, b_k) z^k$ and $\frac{\partial}{\partial b_1} F_j^{-1} = -F_{j-1}^{-1}$, $\forall j \geqslant 2$,

$$\frac{\partial}{\partial b_k} F_p^{-1} = 0 \quad \text{if } k \geqslant p+1, \tag{1.27}$$

$$\frac{\partial}{\partial b_k} F_k^{-1} = -\frac{1}{k} \quad \text{and} \quad \frac{\partial}{\partial b_k} F_p^{-1} = -\frac{1}{k} F_{p-k}^{-1} \quad \text{if } k \leqslant p. \tag{1.28}$$

Differentiating (1.25), we obtain systems of partial differential equations satisfied by the $(F_n)_{n\geqslant 1}$ and the (F_n^{-1}). If $p \geqslant 1$ is an integer, we denote $p \times S$ the map

$$p \times S : (b_1, b_2, \ldots, b_n, \ldots) \to (-pb_1, -pb_2, \ldots, -pb_n, \ldots) \tag{1.29}$$

and $p \times I$ the map

$$p \times I : (b_1, b_2, \ldots, b_n, \ldots) \to (pb_1, pb_2, \ldots, pb_n, \ldots). \tag{1.30}$$

Consider the maps, for $p \in Z$, $p \neq 0$,

$$K^p : (b_1, b_2, \ldots, b_n, \ldots) \to \big(K_1^p(b_1), K_2^p(b_1, b_2), \ldots, K_j^p(b_1, b_2, \ldots, b_j), \ldots\big).$$

We obtain for $p \geqslant 1$,

$$K^{-p} = F^{-1} \circ (p \times S) \circ F, \tag{1.31}$$

$$K^p = F^{-1} \circ (p \times I) \circ F. \tag{1.32}$$

This last relation shows that $K^p \circ K^q = K^q \circ K^p = K^{pq}$ for $p \neq 0, q \neq 0, p, q \in Z$. In particular, it is enough to know the (K^p) when p are prime numbers, to obtain all the other K^p, $p \in Z$, by composition of maps. From (1.31)–(1.32), we see that

$$F_n\big(K_1^{-p}(b_1), K_2^{-p}(b_1, b_2), \ldots, K_n^{-p}(b_1, b_2, \ldots, b_n)\big) + F_n\big(K_1^p(b_1), K_2^p(b_1, b_2), \ldots, K_n^p(b_1, b_2, \ldots, b_n)\big) = 0$$

which extends (1.25).

The coefficients of $f^{-1}(z)$ the inverse map of $f(z) = z + b_1 z^2 + b_2 z^3 + \cdots$ are given by

$$f^{-1}(z) = z + \sum_{n\geqslant 1} \frac{1}{n+1} K_n^{-(n+1)} z^{n+1} \tag{1.33}$$

(compare with [13]) and the coefficients of $g^{-1}(z)$ the inverse map of $g(z) = z + b_1 + \frac{b_2}{z} + \frac{b_3}{z^2} + \cdots + \frac{b_{n+1}}{z^n} + \cdots$ are given by

$$g^{-1}(z) = z - b_1 - \sum_{n\geqslant 1} \frac{1}{n} K_{n+1}^n \frac{1}{z^n}. \tag{1.34}$$

Thus the two maps ϕ_f and ϕ_g defined by

$$\phi_f : (b_1, b_2, b_3, b_4, \ldots) \to \left(\frac{1}{2}K_1^{-2}, \frac{1}{3}K_2^{-3}, \frac{1}{4}K_3^{-4}, \frac{1}{5}K_4^{-5}, \ldots, \frac{1}{n+1}K_n^{-(n+1)}, \ldots\right),$$
$$\phi_g : (b_1, b_2, b_3, \ldots) \to \left(-b_1, -K_2^1, -\frac{1}{2}K_3^2, -\frac{1}{3}K_4^3, -\frac{1}{4}K_5^4, \ldots, -\frac{1}{n}K_{n+1}^n, \ldots\right)$$

satisfy $\phi_f \circ \phi_f = \mathrm{Id}_{\mathcal{M}}$ and $\phi_g \circ \phi_g = \mathrm{Id}_{\mathcal{M}}$.

2. Identities between the polynomials

First, we recall the basic facts relative to the polynomials $(F_j)_{j\geqslant 0}$, $(G_j)_{j\geqslant 0}$, (K_n^p), (P_n^p), $n \geqslant 1$, $p \in Z$ and the differential operators $(W_j)_{j\geqslant 1}$.

2.1. Zeroes and particular values of the polynomials

We have

$$\begin{aligned}
&G_1(1) = -1, \qquad G_n(1,1,1,\ldots,1) = 0 \quad \text{for } n \geqslant 2,\\
&G_1(2) = -2, \qquad G_2(2,3) = 1,\\
&G_1(3) = -3, \qquad G_n(2,3,4,\ldots,k,\ldots,n+1) = 0 \quad \text{for } n \geqslant 2,\\
&G_2(3,6) = 3, \qquad G_3(3,6,10) = -1,\\
&G_4(3,6,10,15) = 0,\\
&G_n\left(3,6,10,15,21,\ldots,\frac{(n+1)(n+2)}{2}\right) = 0 \quad \text{for } n \geqslant 5,\\
&G_n\left(1,\frac{1}{2!},\frac{1}{3!},\ldots,\frac{1}{n!}\right) = (-1)^n\frac{1}{n!} \quad \text{for } n \geqslant 1, \qquad (2.1)\\
&K_1^{-2}(1) = -2, \qquad K_2^{-2}(1,1) = 1,\\
&K_2^{-3}(1,1) = 3, \qquad K_3^{-3}(1,1,1) = -1,\\
&K_n^{-2}(1,1,1,\ldots,1) = 0 \quad \text{for } n \geqslant 3,\\
&K_n^{-3}(1,1,\ldots,1) = 0 \quad \text{for } n \geqslant 4,\\
&K_n^2(-4,-2,-4,-2,-4,-2,\ldots) = \begin{cases} 8(n-2) & \text{if } n \text{ is odd},\\ 2(5n-4) & \text{if } n \text{ is even}.\end{cases} \qquad (2.2)
\end{aligned}$$

For $n \geqslant 1$,

$$F_n(1,1,1,\ldots,1) = -1, \tag{i}$$
$$F_n(-1,1,\ldots,(-1)^n) = (-1)^{n+1}, \tag{ii}$$
$$F_n(4,9,\ldots,(n+1)^2) = -3 + (-1)^n, \tag{iii}$$

$$F_n(2,3,4,5,\dots,n+1) = -2, \tag{iv}$$

$$F_n\left(1, \tfrac{1}{2!}, \tfrac{1}{3!}, \dots, \tfrac{1}{n!}\right) = 0 \quad \text{for } n \geqslant 2. \tag{v}$$

For any $p \in Z$, $p \neq 0$, $n \geqslant 1$

$$F_n^{-1}(p,p,p,\dots,p) = (-1)^n C_n^p \quad \text{with } C_n^p = \tfrac{p(p-1)(p-2)\cdots(p-n+1)}{n!}, \tag{vi}$$

$$F_n^{-1}(b_1+1, b_2+1, \dots, b_n+1) = F_n^{-1}(b_1,b_2,\dots,b_n) - F_{n-1}^{-1}(b_1,b_2,\dots,b_{n-1}), \tag{vii}$$

$$F_n^{-1}(-p,-p,\dots,-p) = C_n^{n+p-1}, \tag{viii}$$

$$K_n^p(2,3,4,5,\dots,n+1) = C_n^{2p+n-1} = (-1)^n C_n^{-2p}, \tag{ix}$$

$$K_n^p\left(-1,1,-1,\dots,(-1)^n\right) = C_n^{-p}. \tag{x}$$

Proof. We take the particular case of the function

$$h(z) = 1 + z + z^2 + \cdots + z^n + \cdots = \frac{1}{1-z} \quad \text{for } |z| < 1.$$

Since $\frac{1}{h(z)} = 1 - z$, we find $G_n(1,1,1,\dots,1)$ for all $n \geqslant 1$. In the same way, for $|z| < 1$, consider $h'(z) = 1 + 2z + 3z^2 + \cdots + (n+1)z^n + \cdots$. Since

$$\frac{1}{h'(z)} = (1-z)^2 = 1 - 2z + z^2$$

we find $G_n(2,3,4,\dots,n+1)$. We continue with

$$\frac{1}{(1-z)^3} = \frac{h''(z)}{2} = 1 + 3z + 6z^2 + 10z^3 + 15z^4 + 21z^5 + \cdots + \frac{(n+1)(n+2)}{2} z^n + \cdots.$$

For K_n^{-2}, since $\frac{1}{h(z)^2} = (1-z)^2$, we obtain $K_n^{-2}(1,1,1,\dots,1)$. For $K_n^{-3}(1,1,1,\dots)$, we use $\frac{1}{h(z)^3} = 1 - 3z + 3z^2 - z^3$. In this way, we find particular values of $(b_1, b_2, \dots, b_n, \dots)$ such that the functions G_n and K_n^{-p}, $p \geqslant 1$ are zero. We obtain the zeros of $(K_n^p)_{n \geqslant 1}$, $p \geqslant 1$, using the identity (T2) in the Main theorem. To find zeros of $(F_n)_{n \geqslant 1}$, we take $h(z) = \exp(z)$. Moreover since for a homogeneous polynomial P_n of degree n,

$$P_n\left(rb_1, r^2 b_2, \dots, r^n b_n\right) = r^n P_n(b_1, b_2, \dots, b_n) \quad \forall r \in C$$

we obtain for the polynomials $(G_n)_{n \geqslant 1}$, $(K_n^{-p})_{n \geqslant 1}$ and $(F_n)_{n \geqslant 1}$, curves of zeros in the manifold $\mathcal{M}$. Of course for these polynomials, there are many other manifolds of zeros. See [4]. To find the special values for $(F_n)_{n \geqslant 1}$, for (i), we consider $h(z) = \frac{1}{1-z}$ which gives $\frac{h'}{h} = \frac{1}{1-z}$. For (ii), we take $h(z) = \frac{1}{1+z}$. For (iii), (iv), (v) and (2.1), we take the Koebe function

$$f(z) = \frac{z}{(1-z)^2}, \quad \text{then } \frac{zf'}{f} = 1 + \frac{2z}{1-z} = \frac{1+z}{1-z},$$

$$h(z) = f'(z) = \frac{1+z}{(1-z)^3} = 1 + 4z + 9z^2 + \cdots + (n+1)^2 z^n + \cdots,$$

$$\frac{h'}{h} = \frac{f''}{f'} = \frac{1}{1+z} + \frac{3}{1-z} = 4 + 2z + 4z^2 + 2z^3 + 4z^4 + \cdots, \tag{iii$'$}$$

$$\begin{aligned} z^2 S_f(z) &= z^2\left[\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right] = -\frac{6z^2}{(1-z^2)^2} = \sum_{k\geqslant 2}\mathcal{P}_k z^k \\ &= z^2\big(2+8z+6z^2+16z^3+10z^4+\cdots\big) \\ &\quad - \frac{z^2}{2}\big(16+16z+36z^2+32z^3+4\times 14z^4+\cdots\big). \end{aligned} \qquad \text{(iii)}''$$

We calculate (iv) with $\frac{zf'}{f}$ and (iii) with (iii)′. To calculate (2.2), we use

$$\sum_{k\geqslant 0} K_k^2(-4,-2,-4,-2,-4,-2,\ldots)z^k = \left(1 - z\frac{f''}{f'}\right)^2 = \left(\frac{1}{1+z} - \frac{3z}{1-z}\right)^2.$$

To prove (vi),

$$\exp\left(-p\sum_{j\geqslant 1}\frac{z^j}{j}\right) = \exp\big[p\log(1-z)\big] = (1-z)^p = 1 + \sum_{n\geqslant 1} F_n^{-1}(p,p,\ldots,p)z^n.$$

To prove (vii),

$$\begin{aligned} \exp\left(-\sum_{j\geqslant 1}(b_j+1)\frac{z^j}{j}\right) &= 1 + \sum_{n\geqslant 1} F_n^{-1}(b_1+1,b_2+1,\ldots,b_n+1)z^n \\ &= \exp\left(-\sum_{j\geqslant 1} b_j\frac{z^j}{j}\right)\times\exp\left(-\sum_{j\geqslant 1}\frac{z^j}{j}\right) = (1-z)\times\left(1+\sum_{n\geqslant 1}F_n^{-1}(b_1,b_2,\ldots,b_n)z^n\right) \end{aligned}$$

and we identify equal powers of z. The identity (vii) generalizes the classical identity $C_n^{p+1} = C_{n-1}^p + C_n^p$ for the binomial coefficients. See for example [6, vol. 1, II-12].

To prove (viii), we have to calculate $F^{-1}\circ S$, it comes from

$$\exp\left(\sum_{j\geqslant 1}\frac{b_j}{j}z^j\right) = 1 + \sum_{k\geqslant 1} F_k^{-1}(-b_1,-b_2,\ldots,-b_k)z^k.$$

Taking $b_j = n$ for all $j\geqslant 1$, $\exp(n\sum_{j\geqslant 1}\frac{1}{j}z^j) = \frac{1}{(1-z)^n} = \sum_{j\geqslant 0} C_j^{n+j-1}z^j$.

To prove (ix), we take the Koebe function $f(z)$, then $h(z) = \frac{f(z)}{z} = \frac{1}{(1-z)^2}$ and $[h(z)]^p = \frac{1}{(1-z)^{2p}} = 1 + \sum_{n\geqslant 1} C_n^{2p+n-1}z^n$. We can also deduce (ix) using the composition of maps: $K^p(2b_1, 3b_2,\ldots,(n+1)b_n,\ldots) = F^{-1}\circ pI\circ F(2b_1,3b_2,\ldots)$. For $b_1 = b_2 = \cdots = 1$, we replace $F(2,3,\ldots)$ using (iv), then we use (viii). To prove (x), we take $h(z) = \frac{1}{1+z}$. □

Remark 2.1. We verify the main corollary (C1) when $f(z)$ is the Koebe function. From (C1), for the Koebe function,

$$\mathcal{P}_k = -(k-2)F_k\big(4,9,16,\ldots,(k+1)^2\big) - \frac{1}{2}K_k^2(-4,-2,-4,-2,-4,\ldots).$$

If k is odd, from (iii), we have $F_k(4,9,16,\ldots,(k+1)^2) = -4$, thus from (2.2), we find that $\mathcal{P}_k = 0$. If k is even, from (iii), $F_k(4,9,16,\ldots,(k+1)^2) = -2$ and using (2.2), we find $\mathcal{P}_k = -(k-2)\times(-2) - (5k-4) = -3k$. Thus $\mathcal{P}_{2p} = -6p$. Compare with (iii)″.

We obtain values of F_n and G_n when the $(b_j)_{j\geqslant 1}$ are binomial coefficients,

Proposition 2.1. *We have*

$$F_n\left(-\binom{n}{1},\binom{n}{2},-\binom{n}{3},\ldots,(-1)^n\binom{n}{n}\right)=n,$$
$$G_n\left(-\binom{n}{1},\binom{n}{2},-\binom{n}{3},\ldots,(-1)^n\binom{n}{n}\right)=\binom{2n-1}{n},$$

where $\binom{n}{k}=C_k^n$ *is the binomial coefficient. More generally, let* $q\in C$, *and let*

$$\begin{bmatrix} n\\ k\end{bmatrix}=\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}$$

be the Gaussian polynomial, then

$$G_n\left(-\begin{bmatrix} n\\ 1\end{bmatrix},q\begin{bmatrix} n\\ 2\end{bmatrix},-q^3\begin{bmatrix} n\\ 3\end{bmatrix},\ldots,(-1)^n q^{\frac{n(n-1)}{2}}\begin{bmatrix} n\\ n\end{bmatrix}\right)=\begin{bmatrix} 2n-1\\ n\end{bmatrix}.$$

Proof. We obtain F_n with

$$-n\sum_{j\geqslant 1}\frac{w^j}{j}=\log\left(1-\binom{n}{1}w+\binom{n}{2}w^2-\binom{n}{3}w^3+\cdots+(-1)^n\binom{n}{n}w^n\right)$$
$$=n\log(1-w).$$

We can also deduce the identity for F_n from (2.2)–(vi) and the composition of maps $F\circ F^{-1}=\mathrm{Id}$. We obtain G_n with the relation

$$\left[1-\binom{n}{1}w+\binom{n}{2}w^2-\binom{n}{3}w^3+\cdots+(-1)^n\binom{n}{n}w^n\right]^{-1}$$
$$=\frac{1}{(1-w)^n}=\sum_{j\geqslant 0}\binom{n+j-1}{j}w^j$$

when $\binom{n}{k}$ is the binomial coefficient, and in the case of the Gaussian polynomial,

$$\left[1-\begin{bmatrix} n\\ 1\end{bmatrix}w+q\begin{bmatrix} n\\ 2\end{bmatrix}w^2-q^3\begin{bmatrix} n\\ 3\end{bmatrix}w^3+\cdots+(-1)^n q^{\frac{n(n-1)}{2}}\begin{bmatrix} n\\ n\end{bmatrix}w^n\right]^{-1}$$
$$=\frac{1}{\prod_{k=0}^{n-1}(1-q^k w)}=\sum_{k\geqslant 0}\begin{bmatrix} n+k-1\\ k\end{bmatrix}w^k.$$

From (1.26), we know that $G=F^{-1}\circ S\circ F$, then $G\circ F^{-1}=F^{-1}\circ S$, we can deduce the first identity for G_n from (2.2)–(viii). □

2.2. Basic identities

The function $h(z)=1+b_1z+b_2z^2+\cdots+b_kz^k+\cdots$ satisfies

$$h(z)-zh'(z)=1-\sum_{n\geqslant 2}(n-1)b_nz^n,$$
$$h(z)-zh'(z)+\frac{z^2h''(z)}{2}=1+\sum_{n\geqslant 3}\frac{(n-1)(n-2)}{2}b_nz^n,$$

$$h(z) - zh'(z) + \frac{z^2h''(z)}{2} - \frac{z^3h'''(z)}{3!} = 1 - \sum_{n\geqslant 4} \frac{(n-1)(n-2)(n-3)}{3!} b_n z^n,$$

$$\ldots$$

2.3. Relations between the polynomials

We differentiate (1.1) with respect to w,

$$b_1 + 2b_2w + \cdots + kb_kw^{k-1} + \cdots$$
$$= \left(1 + b_1w + b_2w^2 + \cdots + b_pw^p + \cdots\right) \times \left(-\sum_{j\geqslant 1} F_j w^{j-1}\right).$$

We equal coefficients of same powers of w, it gives the recurrence for the polynomials $(F_k)_{k\geqslant 0}$, $F_0 = 1$,

$$-kb_k = \sum_{1\leqslant j\leqslant k} F_j b_{k-j}. \tag{2.3}$$

With the same approach, one find other relations between the polynomials as

Proposition 2.2.

$$F_{j+1} = -\sum_{0\leqslant r\leqslant j} (r+1)b_{r+1}G_{j-r}, \tag{2.4}$$

$$nG_n = \sum_{1\leqslant j\leqslant n} F_j G_{n-j}, \tag{2.5}$$

$$\frac{n}{p-1}K_n^{1-p} = \frac{1}{r-1}\sum_{1\leqslant j\leqslant n} jK_j^{1-r}K_{n-j}^{r-p} \quad \textit{for } 2\leqslant r < p, \tag{2.6}$$

$$\frac{n}{p-1}K_n^{1-p} = \sum_{1\leqslant j\leqslant n} F_j K_{n-j}^{1-p} \quad \forall p \neq 1,\ p\in Z, \tag{2.7}$$

$$K_n^p = \sum_{0\leqslant j\leqslant n} K_j^r K_{n-j}^{p-r}. \tag{2.8}$$

Proof of the identities (2.4)–(2.8). To find (2.4), we consider $h(w) = 1 + b_1w + b_2w^2 + b_3w^3 + \cdots + b_kw^k + \cdots$,

$$h'(w) = b_1 + 2b_2w + 3b_3w^2 + \cdots + nb_nw^{n-1} + \cdots.$$

Since $\frac{1}{h(w)} = \sum_{n\geqslant 0} G_n w^n$, multiplying by $h'(w)$ gives

$$\frac{h'(w)}{h(w)} = \sum_{r\geqslant 0}\left[\sum_{0\leqslant r\leqslant j} (r+1)b_{r+1}G_{j-r}\right]w^r.$$

To obtain (2.4), we compare with (1.22). For (2.5), we identify the two following expansions

$$\frac{h'(w)}{h(w)^2} = \frac{h'(w)}{h(w)} \times \frac{1}{h(w)} = \left(\sum_{j\geqslant 0} -F_{j+1}w^j\right) \times \left(\sum_{p\geqslant 0} G_p w^p\right),$$

$$\frac{h'(w)}{h(w)^2} = -\frac{\mathrm{d}}{\mathrm{d}w}\frac{1}{h(w)} = -\sum_{n\geqslant 0}(n+1)G_{n+1}w^n.$$

To find (2.6) and (2.7), we use that for $0 \leqslant r \leqslant p$,

$$\frac{h'(w)}{h(w)^p} = \frac{h'(w)}{h(w)^r} \times \frac{1}{h(w)^{p-r}}. \tag{i}$$

If $p \neq 1$ and $r \neq 1$, (2.6) comes from

$$\frac{1}{(p-1)}\frac{\mathrm{d}}{\mathrm{d}w}\frac{1}{h(w)^{p-1}} = \frac{1}{(r-1)}\left(\frac{\mathrm{d}}{\mathrm{d}w}\frac{1}{h(w)^{r-1}}\right) \times \sum_{j\geqslant 0} K_j^{r-p} w^j.$$

In (i), we take $r = 1$ and we obtain (2.7) with

$$\frac{1}{p-1}\frac{\mathrm{d}}{\mathrm{d}w}\frac{1}{h(w)^{p-1}} = \left(\sum_{j\geqslant 0} F_{j+1}w^j\right) \times \frac{1}{h(w)^{p-1}}. \qquad \square$$

Remark 2.1. For $k \geqslant 1$, (1.22) yields

$$\begin{aligned}&\frac{w^{1-k}h'(w)}{h(w)} + F_1 w^{1-k} + F_2 w^{2-k} + \cdots + F_m w^{m-k} + \cdots + F_{k-1}w^{-1} + F_k \\ &\quad = -\big(F_{k+1}w + \cdots + F_{k+r}w^r + \cdots\big)\end{aligned}$$

and

$$\begin{aligned}&\big(F_{k+1}w + \cdots + F_{k+r}w^r + \cdots\big) \times h(w) \\ &\quad = \sum_{j\geqslant 1}(F_{k+1}b_{j-1} + F_{k+2}b_{j-2} + \cdots + F_{j+k}b_0)w^j.\end{aligned}$$

The relations (2.4), (2.5), (2.6) and (2.7) involve the first derivative of h, we can find other relations by multiplying powers of h. For example (2.8) comes from $h(w)^p = h(w)^r \times h(w)^{p-r}$.

Below, we give further identities between the polynomials. We consider the polynomials $(P_n^k)_{n\geqslant 1}$, $k \in Z$ defined by (1.4). We define B_n^k by

$$\left(\frac{zh'(z)}{h(z)}\right)^2 \big[h(z)\big]^k = \sum_{n\geqslant 2} B_n^{n+k} z^n. \tag{2.9}$$

Since $\frac{zf'}{f} = 1 + \frac{zh'}{h}$, we have

$$P_n^{k+n} = B_n^{n+k} \times 1_{n\geqslant 2} + \frac{2n+k}{k}K_n^k. \tag{2.10}$$

Proposition 2.3. *Let $f(\zeta) = \zeta(1 + b_1\zeta + b_2\zeta^2 + \cdots)$. For $k \neq 0$, we have*

$$\phi_k(\zeta) = \frac{\zeta f'(\zeta)}{f(\zeta)} \times h(\zeta)^k = \sum_{n\geqslant 0}\frac{k+n}{k}K_n^k\zeta^n. \tag{2.11}$$

If $k \geqslant 1$,

$$\phi_k(\zeta) = \frac{(-1)^k}{k!}\left[\frac{\partial^k}{\partial b_1^k}\left(\sum_{n\geqslant 0}F_{k+n}\zeta^n\right)\right]\big(G_1(b_1), G_2(b_1, b_2), \ldots\big) \tag{2.12}$$

and the function $\sum_{n\geqslant 0} F_{k+n}(b_1, b_2, \ldots, b_{k+n})z^n$, $k \geqslant 1$, *is given by the line integral*

$$\sum_{n\geqslant 0} F_{k+n}(b_1, b_2, \ldots)z^n = -\frac{1}{2\mathrm{i}\pi}\int \frac{h'(\zeta)}{\zeta^{k-1}(\zeta - z)h(\zeta)}\,\mathrm{d}\zeta. \tag{2.13}$$

If $k \geqslant 1$, *the function* $\phi_{-k}(\zeta) = \frac{\zeta f'(\zeta)}{f(\zeta)} \times \frac{\zeta^k}{f(\zeta)^k} = \sum_{n\geqslant 0} \frac{k-n}{k} K_n^{-k}\zeta^n$ *is given by*

$$\phi_{-k}(\zeta) = \frac{(-1)^k}{k!}\left[\frac{\partial^k}{\partial b_1^k}\left(\sum_{n\geqslant 0}\frac{k-n}{k+n}F_{k+n}\zeta^n\right)\right](b_1, b_2, \ldots). \tag{2.14}$$

Proof. For (2.11), we use the recursion formula (2.7) or give a direct proof since $\phi_k(\zeta) = h(\zeta)^k + \frac{\zeta}{k}\frac{\mathrm{d}}{\mathrm{d}\zeta}h(\zeta)^k$. For (2.12), we have from (T1), for $k > 0$ and $j \geqslant 0$,

$$\frac{(j+k)}{k}K_j^{-k} = \frac{(-1)^k}{k!}\frac{\partial^k}{\partial b_1^k}F_{k+j}. \tag{2.15}$$

The proof of (2.13) is classical for Laurent series: let $l(\zeta) = \sum_{n\in Z}\alpha_n z^n$, then

$$\sum_{n\geqslant k}\alpha_n z^n = \sum_{n\geqslant k}\frac{z^n}{2\mathrm{i}\pi}\int\frac{l(\zeta)}{\zeta^{n+1}}\,\mathrm{d}\zeta = \frac{z^k}{2i\pi}\int\frac{l(\zeta)}{\zeta^k(\zeta - z)}\,\mathrm{d}\zeta.$$

We take $l(\zeta) = -\frac{\zeta h'(\zeta)}{h(\zeta)}$. For ϕ_{-k}, we proceed in the same way. □

Remark 2.2. By composition of maps, see (1.31)–(1.32), we can define $(K_n^k)_{n\geqslant 0}$ for any $k \in R$ with $K_0^k = 1$ and $(K_1^k, K_2^k, \ldots, K_n^k, \ldots) = F^{-1} \circ k \times \mathrm{Id} \circ F$. From Proposition 2.3, we see that for fixed $n \geqslant 1$, we have $\lim_{k\to 0}\frac{n-k}{k}K_n^{-k} = F_n$. Since $\lim_{k\to 0}K_n^{-k} = 0$, we obtain for

$$\lim_{k\to 0}\frac{1}{k}K_n^{-k} = \frac{1}{n} \times F_n \quad \text{for } n \geqslant 1. \tag{2.16}$$

Proposition 2.4. *Assume that* f^{-1} *is the inverse of* f, $f(f^{-1}(z)) = z$. *For* $f(\zeta) = \zeta[1 + \sum_{n\geqslant 1} b_n\zeta^n]$ *and* $\phi_k(\zeta) = \frac{\zeta f'(\zeta)}{f(\zeta)} \times h(\zeta)^k$, *then*

$$\phi_k\bigl(f^{-1}(z)\bigr) = 1 + \sum_{n\geqslant 1} P_n^k(b_1, b_2, \ldots, b_n)z^n \quad \textit{for } k \neq 0, \tag{2.17}$$

$$P_n^k(b_1, \ldots, b_n) = \sum_{0\leqslant s\leqslant n} K_s^2\bigl(-F_1(b_1), \ldots, -F_s(b_1, \ldots, b_s)\bigr) \times K_{n-s}^{k-n}(b_1, \ldots, b_{n-s}). \tag{2.18}$$

Proof. See [1, (A.1.2)]. In particular,

$$P_n^n(b_1, \ldots, b_n) = K_n^2\bigl(-F_1(b_1), \ldots, -F_n(b_1, \ldots, b_n)\bigr),$$
$$P_n^n\bigl(G_1(b_1), \ldots, G_n(b_1, \ldots, b_n)\bigr) = K_n^2\bigl(F_1(b_1), \ldots, F_n(b_1, \ldots, b_n)\bigr)$$

and $P_n^n(b_1, \ldots, b_n) - P_n^n(G_1(b_1), \ldots, G_n(b_1, \ldots, b_n)) = -4F_n(b_1, b_2, \ldots, b_n)$. □

(2.19) – Expressions of (P_n^k)

If $n \neq k$,

$$P_n^k(b_1,\dots,b_n) = -\frac{1}{k-n} \times \sum_{j=0}^{n} (k-j) F_j(b_1,\dots,b_j) K_{n-j}^{k-n}(b_1,\dots,b_{n-j})$$

$$= \frac{k}{k-n} K_n^{k-n} - \frac{1}{k-n} \times \sum_{j=1}^{n} (k-j) F_j(b_1,b_2,\dots,b_j) \times K_{n-j}^{k-n}(b_1,b_2,\dots,b_{n-j}). \qquad (\mathrm{E})_1$$

If $n = k$ (with $F_0 = -1$)

$$P_n^n(b_1,b_2,\dots,b_n) = \sum_{j=0}^{n} F_j(b_1,b_2,\dots,b_j) \times F_{n-j}(b_1,b_2,\dots,b_{n-j}). \qquad (\mathrm{E})_2$$

Remark 2.3. If $k = 1$, $f'(f^{-1}(z)) = 1 + \sum_{n\geqslant 1} P_n^1(b_1,b_2,\dots)z^n$ with

$$P_n^1 = \frac{1}{n-1} \sum_{0\leqslant j\leqslant n} (1-j) F_j K_{n-j}^{1-n}. \qquad (\mathrm{E})_3$$

Proof of $(\mathrm{E})_1$ **and** $(\mathrm{E})_2$**.** From [1, (A.1.1)],

$$\frac{zf'(z)}{f(z)}\big[h(z)\big]^k \times \frac{zf'(z)}{f(z)}\big[h(z)\big]^p = 1 + \sum_{n\geqslant 1} P_n^{n+k+p} z^n.$$

On the other hand,

$$\frac{zf'(z)}{f(z)}\big[h(z)\big]^k = z^{1-k} f'(z) f(z)^{k-1} = z^{1-k} \frac{1}{k} \frac{\mathrm{d}}{\mathrm{d}z} f(z)^k = \frac{1}{k} \sum_{n\geqslant 0} (n+k) K_n^k z^n.$$

Multiplying $\frac{zf'(z)}{f(z)}[h(z)]^k \times \frac{zf'(z)}{f(z)}[h(z)]^p$, we obtain for any $k, p \in Z$, $k \neq 0$, $p \neq 0$,

$$P_n^{n+k+p} = \sum_{0\leqslant j\leqslant n} (j+k)(n-j+p) \times \frac{1}{pk} K_j^k K_{n-j}^p. \qquad (\mathrm{E})_4$$

We make $p \to 0$ as in Remark 2.2, it gives P_n^k. To obtain P_n^n, we make $k \to 0$. □

Remark 2.4. From $\frac{(j+k)}{k} K_j^{-k} = \frac{(-1)^k}{k!} \frac{\partial^k}{\partial b_1^k} F_{k+j}$ and from $(\mathrm{E})_4$, we deduce that for $k > 0$, $p > 0$,

$$P_n^{n+k+p}(b_1,b_2,\dots) = \sum_{0\leqslant j\leqslant n} \frac{(-1)^{k+p}}{k!p!} \left(\frac{\partial^k}{\partial b_1^k} F_{k+j}\right)(c_1,c_2,\dots) \times \left(\frac{\partial^p}{\partial b_1^p} F_{p+n-j}\right)(c_1,c_2,\dots). \qquad (\mathrm{E})_5$$

From $(\mathrm{E})_1$, we obtain for $k > n$,

$$P_n^k(b_1, b_2, \dots, b_n) = \frac{(-1)^{k-n+1}}{(k-n)!} \sum_{j=0}^{n} F_j(b_1, b_2, \dots, b_j) \frac{\partial^{k-n}}{\partial b_1^{k-n}} F_{k-j}(c_1, c_2, \dots, c_{k-j}) \tag{E$_6$}$$

with $(c_1, c_2, \dots, c_k, \dots) = (G_1(b_1), G_2(b_1, b_2), \dots, G_k(b_1, b_2, \dots), \dots)$. When $k < n$, expressions of P_n^k in terms of partial derivatives of the Faber polynomials are more complicated.

Theorem 2.5. *For the Koebe function $f(z) = \frac{z}{(1-z)^2}$ or $f(z) = \frac{z}{(1+z)^2}$, we have $P_{n-k}^{-k} = P_{n+k}^k$ for $n \geqslant 1$, $k \geqslant 1$,*

$$P_{n-k}^{-k}(b_1, b_2, \dots, b_{n-k}) = P_{n+k}^k(b_1, b_2, \dots, b_{n+k})$$
$$\text{for } (b_1, b_2, \dots) = (2, 3, 4, 5, 6, \dots), \quad b_n = n + 1. \tag{2.20}$$

Conversely, let $f(z) = z + b_1 z^2 + \cdots + b_n z^{n+1} + \cdots$, if $P_{n-k}^{-k} = P_{n+k}^k$ for $n \geqslant 1$, $k \geqslant 1$, then $f(z) = \frac{z}{(1-\epsilon z)^2}$, $\epsilon = 1$ or $\epsilon = -1$.

Moreover, for the Koebe function, we have $K_{n-j}^{-n} \times 1_{n \geqslant j} = K_{n+j}^{-n} \times 1_{n+j \geqslant 0}$, $n, j \in Z$. This last relation is the same as the classical $C_{n-j}^{2n} = C_{n+j}^{2n}$ on the binomial coefficients.

We verify (2.20) with $n = 3$, $k = 2$. With [1, (A.1.7)], we calculate $P_1^{-2} = -b_1$,

$$P_5^2 = 7b_5 - 20b_1 b_4 + 30b_1 b_2^2 + 35b_1^2 b_3 - 50b_1^3 b_2 + 14b_1^5 - 16b_2 b_3.$$

When $b_n = (n + 1)$, we find $P_1^{-2} = P_5^2 = -2$. With [1, (A.1.7)],

$$P_4^2(b_1, b_2, b_3, b_4) = 6b_4 - 12b_1 b_3 - 5b_2^2 + 16b_1^2 b_2 - 5b_1^4$$

and $P_4^2(2, 3, 4, 5) = P_0^{-2} = 1$.

Proof of Theorem 2.5. Let $f(z) = \frac{z}{(1-z)^2}$, we have $\frac{zf'(z)}{f(z)} = \frac{1+z}{1-z}$ and

$$\left(\frac{zf'(z)}{f(z)}\right)^2 \big[h(z)\big]^k = \frac{(1+z)^2}{(1-z)^{2k+2}} = 1 + \sum_{n \geqslant 1} P_n^{n+k} z^n. \tag{i}$$

It gives the line integral

$$P_n^{n+k} = \frac{1}{2i\pi} \int \left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 \big[h(\zeta)\big]^k \frac{d\zeta}{\zeta^{n+1}} - \sum_{\zeta \neq 0} \text{Residue}$$
$$= \frac{1}{2i\pi} \int \frac{(1+\zeta)^2}{(1-\zeta)^{2k+2}} \frac{d\zeta}{\zeta^{n+1}} - \text{Residue at } \zeta = 1. \tag{ii}$$

With (ii), we find $P_{n-j}^{-j} = \frac{1}{2i\pi} \int \lambda(\zeta) \zeta^j \frac{d\zeta}{\zeta}$ and $P_{n+j}^j = \frac{1}{2i\pi} \int \lambda(\zeta) \zeta^{-j} \frac{d\zeta}{\zeta}$ with

$$\lambda(\zeta) = \left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 \big[f(\zeta)\big]^{-n} = \frac{(1+\zeta)^2}{(1-\zeta)^{2-2n} \zeta^n} \tag{iii}$$

since for $n \geqslant 1$, the function $\lambda(\zeta)$ has only a pole at $\zeta = 0$. We can calculate the two line integrals on the circle $|\zeta| = 1$. Since the function $\lambda(\zeta)$ is such that $\lambda(\frac{1}{\zeta}) = \lambda(\zeta)$, we put $\zeta = \frac{1}{\xi}$ and we see that the two line integrals P_{n-j}^{-j} and P_{n+j}^j are equal. Consider any $f(z)$ and let the function

$k(z)=f(\frac{1}{z})$, then $(\frac{zk'(z)}{k(z)})^2=(\frac{uf'(u)}{f(u)})^2$ at $u=\frac{1}{z}$. The Koebe function satisfies $f(z)=f(\frac{1}{z})$. This proves (2.20).

Conversely, assume that $P_{n-k}^{-k}=P_{n+k}^{k}$ for $n\geqslant 1$, $k\geqslant 1$.

Taking $n=1$, we find $P_2^1=P_0^{-1}=1$ and $P_{1+j}^{j}=0$ for $j\geqslant 2$. It gives $P_n^{n-1}=0$ for $n\geqslant 3$ and

$$\frac{1}{f(z)}\times\frac{z^2f'(z)^2}{f(z)^2}=\frac{1}{z}+P_1^0+z \tag{2.21}$$

then $f(z)=f(\frac{1}{z})$.

Taking $n=2$, we obtain $P_4^2=1$ and $P_{2+j}^{j}=0$ for $j\geqslant 3$. Thus $P_n^{n-2}=0$ for $n\geqslant 4$ and from (1.4),

$$\frac{1}{f(z)^2}\times\frac{z^2f'(z)^2}{f(z)^2}=\frac{1}{z^2}+P_1^{-1}\frac{1}{z}+P_2^0+P_3^1z+z^2. \tag{2.22}$$

We have $P_1^{-1}=P_3^1$ $(n=2,\ j=1)$. Taking the ratio of (2.20) by (2.21), we see that

$$f(z)=z\times\frac{1+P_1^0z+z^2}{1+P_1^{-1}z+P_2^0z^2+P_1^{-1}z^3+z^4}.$$

Let $f(z)=zh(z)$ and $l(z)=\frac{1}{h(z)}=\frac{1+P_1^{-1}z+P_2^0z^2+P_1^{-1}z^3+z^4}{1+P_1^0z+z^2}$. With (2.21), $l(z)=1+G_1z+G_2z^2+\cdots+G_nz^n+\cdots$ must satisfy

$$l(z)(1-\frac{zl'(z)}{l(z)})^2=1+P_1^0z+z^2 \tag{i}$$

then

$$\big(l(z)-zl'(z)\big)^2=1+P_1^{-1}+P_2^0z^2+P_1^{-1}z^3+z^4. \tag{ii}$$

Using the identity (2.2), we have $P_1^{-1}=0$. With (ii), we see that $1+P_1^0z+z^2$ must have a double root. It implies that $(P_1^0)^2=4$. Identifying the coefficients in (ii) gives $P_2^0=-2$ and $f(z)=\frac{z}{(1-\epsilon z)^2}$, $\epsilon=1$ or $\epsilon=-1$. □

For the Koebe function, we prove $K_{n-j}^{-n}=K_{n+j}^{-n}$ in the same way and then apply (2.2)(ix).

Remark 2.5. Eq. (2.21) has other solutions than the Koebe function, but (2.21) and $K_{1-j}^{-1}=K_{1+j}^{-1}$, $\forall j\geqslant 1$ or equivalently $K_2^{-1}=1$, $K_n^{-1}=0$, $\forall n\geqslant 3$, implies that $f(z)=\frac{z}{(1-\epsilon z)^2}$, $\epsilon=+1$, -1. According to (4.8) below, the condition $K_n^{-1}=0$ for $n\geqslant 3$ means that $\frac{\partial}{\partial b_1}F_n=0$ for $n\geqslant 4$.

Remark 2.6. When we write the expressions $(E)_1$, $(E)_2$, … of (P_n^k) in the case of the Koebe function, with (2.2)(iii), (iv), (x) we obtain relations between the binomial coefficients.

3. The composition of maps

We consider the polynomials

$$\left.\begin{array}{l}(b_1,b_1,\ldots,b_n)\to F_n(b_1,b_1,\ldots,b_n)\\(b_1,b_1,\ldots,b_n)\to G_n(b_1,b_1,\ldots,b_n)\end{array}\right.\quad n\geqslant 1,$$

as functions of $(b_1, b_2, \dots, b_n)$ and we take composition of maps. We denote $F_n(G_1, \dots, G_n)$ the composition of maps

$$(b_1, b_2, \dots, b_n) \to F_n\big(G_1(b_1), G_2(b_1, b_2), \dots, G_n(b_1, b_2, \dots, b_n)\big).$$

In the same way, $G_n(G_1, \dots, G_n)$ is the composition of maps

$$(b_1, b_2, \dots, b_n) \to G_n\big(G_1(b_1), G_2(b_1, b_2), \dots, G_n(b_1, b_2, \dots, b_n)\big)$$

and $G_n(F_1, \dots, F_n)$ is the composition of maps

$$(b_1, b_2, \dots, b_n) \to G_n\big(F_1(b_1), F_2(b_1, b_2), \dots, F_n(b_1, b_2, \dots, b_n)\big).$$

For example, we have $F_1(G_1)(b_1) = F_1(-b_1) = b_1, \dots$.

$$\begin{aligned}
&G_1(F_1)(b_1) = b_1,\\
&G_2(F_1, F_2)(b_1, b_2) = F_1^2 - F_2 = 2b_2,\\
&G_3(F_1, F_2, F_3)(b_1, b_2, b_3) = b_1 b_2 + 3b_3,\\
&G_4(F_1, F_2, F_3, F_4) = 2b_2^2 + 2b_1 b_3 + 4b_4,\\
&G_5(F_1, F_2, F_3, F_4, F_5) = b_1 b_2^2 + 7b_3 b_2 + 3b_1 b_4 + 5b_5,\\
&G_6(F_1, F_2, F_3, F_4, F_5, F_6) = 4b_1 b_2 b_3 + 2b_2^3 + 6b_3^2 + 10b_2 b_4 + 4b_1 b_5 + 6b_6,\\
&G_7(F_1, F_2, F_3, F_4, F_5, F_6, F_7)\\
&\quad = 17b_3 b_4 + b_1 b_2^3 + 13b_2 b_5 + 11b_2^2 b_3 + 4b_1 b_3^2 + 5b_1 b_6 + 6b_1 b_2 b_4 + 7b_7,\\
&G_8(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8)\\
&\quad = 6b_3 b_1 b_2^2 + 8b_5 b_1 b_2 + 12b_4 b_1 b_3 + 2b_2^4 + 12b_4^2 + 16b_4 b_2^2 + 16b_6 b_2\\
&\qquad + 22b_5 b_3 + 20b_3^2 b_2 + 6b_1 b_7 + 8b_8.
\end{aligned}$$

Proposition 3.1. *For $n \geqslant 1$,*

$$G_n(G_1, G_2, \dots, G_n) = b_n, \tag{3.1}$$

$$F_n(b_1, b_2, \dots, b_n) + F_n(G_1, G_2, \dots, G_n) = 0. \tag{3.2}$$

Proof of (3.1) and (3.2). For (3.1), let $\tilde{h}(w) = \frac{1}{h(w)} = \sum_{n\geqslant 0} G_n w^n$, then $\frac{1}{\tilde{h}(w)} = h(w)$. In particular, writing $G_2(G_1, G_2) = b_2$ and $G_3(G_1, G_2, G_3) = b_3$, we obtain $G_1^2 - G_2 = b_2$ and $-G_1^3 + 2G_1 G_2 - G_3 = b_3, \dots$. The second relation is an immediate consequence of (1.23) and (1.24). □

Remark 3.1. 1. The second formula (3.2) can also be proved by recurrence: it is true for $n = 1$. We shall use the two recursion formulae (2.5) and (2.3),

$$\begin{aligned}
&F_n + b_1 F_{n-1} + b_2 F_{n-2} + \cdots + b_{n-1} F_1 + n b_n = 0,\\
&F_n + G_1 F_{n-1} + G_2 F_{n-2} + \cdots + G_{n-1} F_1 - n G_n = 0.
\end{aligned} \tag{3.3}$$

From the first relation, we have $F_2(G_1, G_2) + G_1 F_1(G_1) + 2G_2 = 0$ and from the second relation $F_2(b_1, b_2) + G_1 F_1(b_1) - 2G_2 = 0$. Using that (3.2) is true for $n = 1$ and adding the two relations above, we obtain (3.2) for $n = 2$ and the formula by recurrence on n.

2. We can also find this formula with the recursion formula for the $(G_n)_{n\geqslant 0}$,

$$G_1 + b_1 = 0, \quad G_2 + b_1 G_1 + b_2 = 0, \quad G_3 + b_1 G_2 + b_2 G_1 + b_3 = 0 \tag{i}$$

and in general, we have $G_n + b_1 G_{n-1} + b_2 G_{n-2} + \cdots + b_{n-1} G_1 + b_n = 0$ as follows. From (2.4) $F_2(b_1, b_2) = -(b_1 G_1 + 2b_2)$ and $F_2(G_1, G_2) = -(b_1 G_1 + 2G_2)$. Adding and using (i) gives the result. We proceed in the same way for F_n.

3. Another proof of (3.2) is to show that $G = F^{-1} \circ S \circ F$ as follows. From (1.23)–(1.24), we have the map $\phi : h \to u = 1 - z\frac{h'}{h}$

$$(b_1, b_2, \ldots, b_n, \ldots) \to \left(F_1(b_1), \frac{F_2(b_1, b_2)}{2}, \ldots, \frac{F_j(b_1, b_2, \ldots, b_j)}{j}, \ldots\right).$$

Its inverse map gives the Schur polynomials. We calculate h from u with the relation $h(z) = \exp(\frac{1-u(z)}{z})$. The map

$$F : (b_1, b_2, \ldots, b_n, \ldots) \to \big(F_1(b_1), F_2(b_1, b_2), \ldots, F_j(b_1, \ldots, b_j), \ldots\big)$$

is a bijection. The map $S : 1 - z\frac{h'}{h} \to 1 + z\frac{h'}{h} = 1 - z\frac{\tilde{h}'}{\tilde{h}}$ with $\tilde{h} = \frac{1}{h}$ is also a bijection. Then the map $\phi^{-1} \circ S \circ \phi$ is just $h \to \tilde{h} = \frac{1}{h}$. This gives (1.26). To calculate F_n^{-1}, we have to solve the system in $(b_1, b_2, \ldots, b_n)$,

$$F_1(b_1) = c_1, \qquad F_2(b_1, b_2) = c_2, \qquad F_n(b_1, b_2, \ldots, b_n) = c_n, \quad \ldots \tag{3.4}$$

Proof of (1.31)–(1.32). For $p \geqslant 1$, we consider the map $\phi_p : h \to u = 1 - pz\frac{h'}{h}$ which allows us to calculate h^p. □

More identities similar to (3.1) and (3.2) can be found.

Theorem 3.2.

$$\begin{aligned} &F_n\big(-F_1(b_1), -F_2(b_1, b_2), \ldots, -F_k(b_1, b_2, \ldots, b_k), -F_n(b_1, b_2, \ldots, b_n)\big) \\ &\quad = F_n\big(2b_1, 3b_2, 4b_3, \ldots, (n+1)b_n\big) - F_n(b_1, b_2, b_3, \ldots, b_n), \end{aligned} \tag{3.5}$$

$$\begin{aligned} F_n(F_1, F_2, \ldots, F_n) = {} & F_n\big(0, -b_2, -2b_3, -3b_4, \ldots, -(n-1)b_n\big) \\ & - F_n(b_1, b_2, \ldots, b_n), \end{aligned} \tag{3.6}$$

$$G_n(-F_1, -F_2, \ldots, -F_n) = \sum_{k=0}^{n} b_k G_{n-k}\big(2b_1, 3b_2, 4b_3, \ldots, (j+1)b_j, \ldots\big), \tag{3.7}$$

$$G_n(F_1, F_2, \ldots, F_n) = \sum_{k=0}^{n} b_k G_{n-k}\big(0, -b_2, -2b_3, \ldots, -(j-1)b_j, \ldots\big). \tag{3.8}$$

For $p \in Z$, $p \neq 0$,

$$\begin{aligned} K_n^p(-F_1, -F_2, \ldots, -F_n) = \sum_{k=0}^{n} & K_{n-k}^{-p}(b_1, b_2, \ldots, b_n) \\ & \times K_k^p\big(2b_1, 3b_2, \ldots, (j+1)b_j, \ldots\big), \end{aligned} \tag{3.9}$$

$$K_n^p(F_1, F_2, \dots, F_n) = \sum_{k=0}^{n} K_{n-k}^{-p}(b_1, b_2, \dots, b_n) \times K_k^p\big(0, -b_2, -2b_3, \dots, -(j-1)b_j, \dots\big). \tag{3.10}$$

Remark 3.2. Consider the maps D^1 and D^{-1} from $\mathcal{M}$ to $\mathcal{M}$,

$$D^1 : (b_1, b_2, \dots, b_k, \dots) \to \big(2b_1, 3b_2, 4b_3, \dots, (n+1)b_n, \dots\big),$$
$$D^{-1} : (b_1, b_2, \dots, b_k, \dots) \to \big(0, b_2, 2b_3, \dots, (n-1)b_n, \dots\big),$$

then (3.5)–(3.7) can be written as $F \circ S \circ F = F \circ D^1 - F$ and $F \circ F = F \circ S \circ D^{-1}$. Remark that K^p, F, G, and S are bijection while D^{-1} is not.

Proof of Theorem 3.2. To prove (3.5), we consider $f(z) = zh(z)$.

$$\frac{\left(\frac{zf'}{f}\right)'}{\frac{zf'}{f}} = -\sum_{k\geqslant 1} F_k(-F_1, -F_2, \dots, -F_k)z^{k-1}. \tag{i}$$

On the other hand

$$\frac{\mathrm{d}}{\mathrm{d}z}\log\left(\frac{zf'}{f}\right) = \frac{1}{z} + \frac{\left(\frac{f'}{f}\right)'}{\frac{f'}{f}} = \frac{1}{z} + \left(\frac{f''}{f} - \frac{(f')^2}{f^2}\right) \times \frac{f}{f'}$$
$$= \frac{1}{z} + \frac{f''}{f'} - \frac{f'}{f} = \frac{1}{z} - \frac{f'}{f} - \sum_{k\geqslant 1} F_k\big(2b_1, 3b_2, \dots, (k+1)b_k\big)z^{k-1}.$$

Using the expression of $\frac{1}{z} - \frac{f'}{f}$, we deduce

$$\frac{\left(\frac{zf'}{f}\right)'}{\frac{zf'}{f}} = \sum_{k\geqslant 1} F_k(b_1, b_2, \dots, b_k)z^{k-1} - \sum_{k\geqslant 1} F_k\big(2b_1, 3b_2, \dots, (k+1)b_k\big)z^{k-1}. \tag{ii}$$

The comparison of (i) and (ii) gives (3.5).

For (3.6), let $g(z) = \frac{z}{h(z)}$. Then $u(z) = \frac{zg'(z)}{g(z)} = 1 + \sum_{n\geqslant 1} F_n(b_1, b_2, \dots, b_n)z^n$ satisfies

$$\frac{u'(z)}{u(z)} = -\sum_{k\geqslant 1} F_k\big(F_1(b_1), F_2(b_1, b_2), \dots, F_k(b_1, b_2, \dots, b_k)\big)z^{k-1}. \tag{i}$$

Thus we obtain $F_k(F_1, F_2, \dots, F_k))$ from this expansion. On the other hand, we calculate

$$\frac{u'(z)}{u(z)} = \frac{\mathrm{d}}{\mathrm{d}z}\log\big(u(z)\big) = \frac{1}{z}\left(-\sum_{k\geqslant 1} F_k z^k + z\frac{g''(z)}{g'(z)}\right).$$

Since $g'(z) = \frac{h(z) - zh'(z)}{h(z)^2}$, we obtain

$$z\frac{g''(z)}{g'(z)} = z\frac{(h - zh')'}{h - zh'} - 2z\frac{h'}{h} = z\frac{(h - zh')'}{h - zh'} + 2\sum_{k\geqslant 1} F_k z^k.$$

Using $h(z) - zh'(z) = 1 + \sum_{n\geqslant 2}(1-n)b_n z^n$, we deduce

$$\frac{u'(z)}{u(z)} = \frac{1}{z}\left(\sum_{k\geqslant 1} F_k z^k - \sum_{k\geqslant 1} F_k\big(0, -b_2, -2b_3, \ldots, -(k-1)b_k\big) z^k\right). \tag{ii}$$

Then we compare the two identities (i) and (ii).

To prove (3.8), we write $\frac{h(w)}{h(w)-wh'(w)}$ in two different ways,

$$\frac{1}{1-\frac{wh'(w)}{h(w)}} = \sum_{n\geqslant 0} G_n\big(F_1(b_1), F_2(b_1,b_2), \ldots, F_n(b_1, b_2, \ldots, b_n)\big) w^n,$$

$$\frac{h(w)}{h(w)-wh'(w)} = \big(1 + b_1 w + b_2 w^2 + \cdots\big) \times \sum_{n\geqslant 0} G_n\big(0, -b_2, \ldots, -(n-1)b_n, \ldots\big) w^n$$

since

$$\begin{aligned} h(w) - wh'(w) &= \big(1 + b_1 w + b_2 w^2 + \cdots + b_n w^n + \cdots\big) \\ &\quad - \big(b_1 w + 2b_2 w^2 + \cdots + n b_n w^n + \cdots\big) \\ &= 1 - b_2 w^2 - 2b_3 w^3 - \cdots - (n-1) b_n w^n - \cdots. \end{aligned}$$

To prove (3.7), we consider $\frac{h(w)}{h(w)+wh'(w)}$. To prove (3.9),

$$\left(1 + z\frac{h'(z)}{h(z)}\right)^p = \left(1 - \sum_{k\geqslant 1} F_k z^k\right)^p = \sum_{n\geqslant 0} K_n^p(-F_1, -F_2, \ldots, -F_n) z^n.$$

This is also equal to $\frac{(h(z)+zh'(z))^p}{h(z)^p}$. For (3.10), we take $(1 - z\frac{h'(z)}{h(z)})^p$. □

Remark 3.3. With (3.7)–(3.10) we define differential operators on Faber polynomials. For (3.7)–(3.8), we have (see (1.16)) $G_{n-k} = -\frac{1}{n}\frac{\partial}{\partial b_k} F_n$, thus

$$\begin{aligned} &G_n(-F_1, -F_2, \ldots, -F_n) \\ &\quad = -\frac{1}{n+1}\left[\sum_{k\geqslant 0} \frac{b_k}{k+1}\frac{\partial F_{n+1}}{\partial b_{k+1}}\right]\big(2b_1, 3b_2, 4b_3, \ldots, (n+1)b_n\big). \end{aligned} \tag{3.11}$$

For (3.9), we take for example $p = 2$. See Proposition 2.4. With (1.16), we deduce that

$$K_{n-k}^{-2} = \frac{1}{n+1}\frac{\partial}{\partial b_k}\left(\frac{\partial}{\partial b_1} F_{n+1}\right) = \frac{1}{n+2}\frac{\partial}{\partial b_k}\left(\frac{\partial}{\partial b_2} F_{n+2}\right)$$

and

$$\begin{aligned} K_n^2(-F_1, -F_2, \ldots, -F_n) &= \frac{1}{n+1} U_2\left(\frac{\partial}{\partial b_1} F_{n+1}\right) + K_n^{-2}(b_1, \ldots, b_n) \\ &= \frac{1}{n+2} U_2\left(\frac{\partial}{\partial b_1} F_{n+2}\right) + \frac{1}{n+2}\frac{\partial^2 F_{n+2}}{\partial b_1^2}, \end{aligned} \tag{3.12}$$

where U_2 is the differential operator

$$\begin{aligned} U_2 &= \sum_{k\geqslant 1} K_k^2\big(2b_1, 3b_2, \ldots, (k+1)b_k\big)\frac{\partial}{\partial b_k} \\ &= 4b_1\frac{\partial}{\partial b_1} + \big(6b_2 + 4b_1^2\big)\frac{\partial}{\partial b_2} + (8b_3 + 12b_1 b_2)\frac{\partial}{\partial b_3} + \cdots. \end{aligned} \tag{3.13}$$

We have

$$U_2\frac{\partial}{\partial b_1}-\frac{\partial}{\partial b_1}U_2=-4L_1 \tag{3.14}$$

where L_1 is given by (1.13). In the same way, in (3.10), we put

$$\begin{aligned}T_2&=\sum_{k\geqslant 1}K_k^2\big(0,-b_2,-2b_3,\ldots,-(k-1)b_k\big)\frac{\partial}{\partial b_k}\\&=-2b_2\frac{\partial}{\partial b_2}-4b_3\frac{\partial}{\partial b_3}+\big(b_2^2-6b_4\big)\frac{\partial}{\partial b_4}+(4b_2b_3-8b_5)\frac{\partial}{\partial b_5}+\cdots.\end{aligned} \tag{3.15}$$

Then $\mathcal{T}_2[h(z)]=(h(z)-zh'(z))^2-1$. We have

$$\begin{aligned}K_n^2(F_1,F_2,\ldots,F_n)&=\frac{1}{n+1}T_2\frac{\partial}{\partial b_1}F_{n+1}+K_n^{-2}(b_1,\ldots,b_n)\\&=\frac{1}{n+1}\frac{\partial}{\partial b_1}T_2F_{n+1}+K_n^{-2}(b_1,b_2,\ldots,b_n)\\&=\frac{1}{n+2}T_2\frac{\partial}{\partial b_2}F_{n+2}+\frac{1}{n+2}\frac{\partial^2F_{n+2}}{\partial b_1^2}.\end{aligned} \tag{3.16}$$

By (2.18), $K_n^2(-F_1,-F_2,\ldots)-K_n^2(F_1,F_2,\ldots)=-4F_n=-\frac{4}{n+1}L_1(F_{n+1})$, see (7.11). With (3.14), it gives $\frac{\partial}{\partial b_1}(U_2-T_2)F_n=0$, thus $(U_2-T_2)F_n$ does not depend on b_1 for $n\geqslant 2$. For $p\geqslant 1$, $T_2\frac{\partial}{\partial b_p}-\frac{\partial}{\partial b_p}T_2=-2(p-1)\sum_{k\geqslant p}(k-p-1)b_{k-p}\frac{\partial}{\partial b_k}$

$$T_2\frac{\partial}{\partial b_p}-\frac{\partial}{\partial b_p}T_2=-2(p-1)\sum_{n\geqslant 0}(n-1)b_n\frac{\partial}{\partial b_{p+n}}=-2(p-1)(L_p+2W_p). \tag{3.17}$$

Corollary 3.3. *Let $(\mathcal{P}_k)_{k\geqslant 2}$ be the coefficients of the Schwarzian derivative as in* (C1)*, and let*

$$\mathcal{H}=T_2\frac{\partial}{\partial b_2}+\frac{\partial^2}{\partial b_1^2} \tag{3.18}$$

then $\mathcal{P}_k(b_1,b_2,b_3,\ldots,b_k)+(k-2)F_k(2b_1,3b_2,\ldots,(k+1)b_k)$ is equal to

$$-\frac{1}{2(k+2)}[\mathcal{H}F_{k+2}]\big(2b_1,3b_2,\ldots,(j+1)b_j,\ldots\big). \tag{3.19}$$

Corollary 3.4. *Let $\mathcal{T}=\frac{\partial^2}{\partial b_1^2}+T_2\frac{\partial}{\partial b_2}+(\frac{\partial}{\partial b_2}-b_1\frac{\partial}{\partial b_3}+\frac{\partial}{\partial b_4})$. The condition* (2.21) *is equivalent to*

$$(\mathcal{T}F_n)(G_1,G_2,\ldots,G_{n-2})=0\quad\forall n\geqslant 4. \tag{3.20}$$

Proof. The condition (2.21) is the same as

$$K_n^2(-F_1,-F_2,\ldots)=b_n+b_1b_{n-1}+b_{n-2}\quad\forall n\geqslant 1. \tag{3.21}$$

From (1.26), we have $K^p\circ S\circ F=K^p\circ F\circ G$. Thus

$$K_n^2(-F_1,-F_2,\ldots)=\frac{1}{n+2}\left(T_2\frac{\partial}{\partial b_2}F_{n+2}\right)\circ G+K_n^{-2}\circ G. \tag{3.22}$$

Since $G \circ G = \text{Identity}$ and $G_1(b_1) = -b_1$, we can write the right side in (3.21) as $(G_n - b_1 G_{n-1} + G_{n-2})$ at the point $(G_1, G_2, \ldots, G_n)$. With (1.16), we have

$$G_n - b_1 G_{n-1} + G_{n-2} = -\frac{1}{n+2}\left(\frac{\partial}{\partial b_2} - b_1 \frac{\partial}{\partial b_3} + \frac{\partial}{\partial b_4}\right) F_{n+2}. \tag{3.23}$$

Then (3.22) and (3.23) imply (3.20). The condition (3.20) is always satisfied for $n = 1, 2, 3$. For $n = 4$, it gives $3b_2 - 2b_1^2 = 1$, for $n = 5$, $5b_3 - 3b_1 b_2 - b_1 = 0$. □

4. The polynomials and their derivatives. Proof of the Main Theorem

4.1. The partial derivatives $(\frac{\partial}{\partial b_k})_{k \geqslant 1}$

Theorem 4.1. *For* $p \geqslant 1$, $n \geqslant 0$,

$$(n+p) G_n = -\frac{\partial F_{n+p}}{\partial b_p}. \tag{4.1}$$

In particular, $\frac{\partial F_p}{\partial b_p} = -p$, *and* $\frac{\partial F_n}{\partial b_k} = -n G_{n-k}$ *if* $k \leqslant n$. *Let* $(F_n^{-1})_{n \geqslant 1}$ *be the inverse Faber polynomials, then*

$$\frac{\partial}{\partial b_k} F_p^{-1} = -\frac{1}{k} F_{p-k}^{-1} \times 1_{k \leqslant p}. \tag{4.2}$$

Proof. Let $\psi(w) = w + b_1 + \frac{b_2}{w} + \frac{b_3}{w^2} + \cdots + \frac{b_p}{w^{p-1}} + \cdots$ and

$$\psi_p(w) = \psi(w) - \frac{t}{w^{p-1}} \quad p \geqslant 1.$$

We have $w^p \psi_p'(w) = w^p \psi'(w) + (p-1)t$,

$$\frac{w \psi_p'(w)}{\psi_p(w)} = 1 + \sum_{n \geqslant 1} F_n(b_1, \ldots, b_{p-1}, b_p - t, b_{p+1}, \ldots) \times \frac{1}{w^n}.$$

We differentiate this equation with respect to t and we make $t = 0$,

$$\phi(w) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \frac{w \psi_p'(w)}{\psi_p(w)} = \sum_{n \geqslant 1} \frac{\partial F_n}{\partial b_p} \times \frac{1}{w^n}.$$

On the other hand

$$\frac{w \psi_p'(w)}{\psi_p(w)} = \frac{w^p \psi'(w) + (p-1)t}{w^{p-1}\psi(w) - t}.$$

We calculate ϕ with this expression

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{w \psi_p'(w)}{\psi_p(w)} = w \left[\frac{(p-1) w^{p-2} \psi(w) + w^{p-1} \psi'(w)}{(w^{p-1}\psi(w) - t)^2}\right] = -w \frac{\mathrm{d}}{\mathrm{d}w} \frac{1}{(w^{p-1}\psi(w) - t)}.$$

At $t = 0$,

$$\phi(w) = -w \frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{1}{w^{p-1}\psi(w)}\right) = -w \frac{\mathrm{d}}{\mathrm{d}w} \sum_{n \geqslant 0} G_n \times \frac{1}{w^{n+p}} = \sum_{n \geqslant 0} G_n (n+p) \frac{1}{w^{n+p}}.$$

Comparing the two expressions of ϕ and since F_n does not contain b_p when $n < p$, we obtain the result. To calculate the derivatives of the map F^{-1}, we take

$$h(z) = 1 + \sum_{j\geqslant 1} F_j^{-1}(b_1, b_2, b_3, \dots, b_j) z^j$$

since $F \circ F^{-1} = \text{Identity}$, we have $\frac{\mathrm{d}}{\mathrm{d}z}\log(h(z)) = -\sum_{k\geqslant 1} b_k z^{k-1}$. We differentiate with respect to b_k, for $k \geqslant 1$,

$$-z^{k-1} = \frac{\partial}{\partial b_k}\left(\frac{h'(z)}{h(z)}\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\frac{\partial}{\partial b_k}h(z)}{h(z)}\right).$$

We integrate this identity with respect to z,

$$-\frac{1}{k}z^k = \frac{\frac{\partial}{\partial b_k}h(z)}{h(z)} + C(b_1, b_2, \dots)$$

where $C(b_1, b_2, \dots)$ is constant in z. Making $z = 0$, we see that $C = 0$. thus

$$-\frac{1}{k}z^k \times h(z) = \frac{\partial}{\partial b_k}h(z) = \sum_{j\geqslant 1}\left(\frac{\partial}{\partial b_k}F_j^{-1}\right)z^j.$$

Since $-\frac{1}{k}z^k \times h(z) = -\frac{1}{k}z^k(1 + \sum_{j\geqslant 1} F_j^{-1} z^j)$, we obtain the partial derivatives of F^{-1}. □

Corollary 4.2. *For $n \geqslant 0$, $p \geqslant 1$,*

$$\frac{\partial F_{n+p}}{\partial b_p}\bigl(G_1(b_1), G_2(b_1, b_2), \dots, G_n(b_1, b_2, \dots, b_n)\bigr) = -(n+p)b_n. \tag{4.3}$$

Proof. Since $G \circ G = \text{Identity}$, it is a consequence of (4.1). □

Corollary 4.3.

$$\frac{\partial G_n}{\partial b_k} = -K_{n-k}^{-2} \times 1_{n\geqslant k}. \tag{4.4}$$

Proof. We differentiate $G = F^{-1} \circ S \circ F$.

$$\frac{\partial G_n}{\partial b_k} = \sum_{j\geqslant 1}\frac{\partial F_n^{-1}}{\partial b_j}(S\circ F) \times \left(-\frac{\partial F_j}{\partial b_k}\right) = \sum_{1\leqslant j\leqslant n}\left(-\frac{1}{j}F_{n-j}^{-1}(S\circ F)\right) \times (jG_{j-k}).$$

After simplification by j, and since $F^{-1} \circ S \circ F = G$, we find

$$\frac{\partial G_n}{\partial b_k} = -\sum_{1\leqslant j\leqslant n} G_{n-j}G_{j-k} = -K_{n-k}^{-2}. \qquad \square$$

The following operators up to a minus sign, $(Z_k)_{k\geqslant 0}$ were introduced in [2].

Corollary 4.4. *With the recursion $F_{j+1} = -\sum_{0\leqslant r\leqslant j}(r+1)b_{r+1}G_{j-r}$, see* (2.4), *for $k \geqslant 0$, we deduce*

$$Z_k = \sum_{r\geqslant 0}(r+1)b_{r+1}\frac{\partial}{\partial b_{r+k+1}} \quad \textit{and} \quad (j+k+1)F_{j+1} = Z_k F_{j+k+1}.$$

Proof. From Theorem 4.1, $(j+k+1)G_{j-r} = -\frac{\partial}{\partial b_{r+k+1}} F_{j+k+1}$. Thus, if $k \geqslant 0$, with the recursion formula (2.4) where we replace G_{j-r}, we find

$$(j+k+1)F_{j+1} = \sum_{0\leqslant r\leqslant j} (r+1)b_{r+1} \frac{\partial}{\partial b_{r+k+1}} F_{j+k+1} = Z_k(F_{j+k+1}).$$

For $k<0$, then $\frac{\partial}{\partial b_{r+k+1}}$ is defined only if $r+k \geqslant 0$, i.e. $r \geqslant -k$. We decompose the sum $F_{j+1} = -\sum_{0\leqslant r<-k}(r+1)b_{r+1}G_{j-r} - \sum_{-k\leqslant r\leqslant j}(r+1)b_{r+1}G_{j-r}$. □

4.2. Proof of the main results

Proof of the Main Theorem.

$$\frac{\partial}{\partial b_k}\left(\frac{1}{h(z)}\right) = -\frac{z^k}{h(z)^2} = -z^k - \sum_{n\geqslant 1} K_n^{-2} z^{n+k}.$$

On the other hand, $\frac{\partial}{\partial b_k}(\frac{1}{h(z)}) = \sum_{n\geqslant 1} \frac{\partial}{\partial b_k} G_n z^n$. In these two last expressions, we identify the coefficients of equal power of z. It gives (1.17). We have $(n+p)G_n = -\frac{\partial F_{n+p}}{\partial b_p}$, then differentiating this expression with respect to k, we obtain

$$(p+k+n)K_n^{-2} = \frac{\partial^2 F_{p+k+n}}{\partial b_k \partial b_p} \quad \forall p \geqslant 1,\ \forall k \geqslant 1.$$

We deduce higher order partial derivatives of F_j from

$$\frac{\partial K_n^{-p}}{\partial b_k} = -pK_{n-k}^{-(p+1)} \times 1_{n\geqslant k} \quad \text{for } n \geqslant 1,\ k \geqslant 1,\ p \neq 0,\ p \in Z. \tag{4.5}$$

$K_0^p = 1$ for any p. The proof of (4.5) or equivalently $\frac{\partial K_n^p}{\partial b_k} = pK_{n-k}^{p-1} \times 1_{n\geqslant k}$ is as follows,

$$\frac{\partial}{\partial b_k}\left(\frac{1}{h(z)^p}\right) = \sum_{n\geqslant 1} \frac{\partial}{\partial b_k}\left(K_n^{-p}\right) z^n. \tag{i}$$

On the other hand

$$\frac{\partial}{\partial b_k}\left(\frac{1}{h(z)^p}\right) = \frac{-p\frac{\partial}{\partial b_k}(h(z))}{h(z)^{p+1}} = \frac{-pz^k}{h(z)^{p+1}} = -pz^k \sum_{n\geqslant 1} K_n^{-(p+1)} z^n$$
$$= -p\sum_{q\geqslant 1} K_q^{-(p+1)} z^{q+k}. \tag{ii}$$

The identification of the coefficients of z^q in the two expressions (i) and (ii) gives (4.5). We see that one can calculate as derivatives of Faber polynomials all the $(K_n^{-p})_{n\geqslant 1}$ for $p \geqslant 1$. □

Proof of (T2). We wish to calculate K_n^p for $p \geqslant 2$. Let $\tilde{h}(z) = \frac{1}{h(z)} = 1 + G_1 z + G_2 z^2 + \cdots + G_n z^n + \cdots$. Then (T2) is obtained with the identification of coefficients of equal powers of z in $\tilde{h}(z)^{-p} = 1 + \sum_{n\geqslant 1} K_n^{-p}(G_1, G_2, \ldots, G_n) z^n = h(z)^p$ with $h(z)^p = 1 + \sum_{n\geqslant 1} K_n^p(b_1, b_2, \ldots, b_n) z^n$. We can also give a proof with the composition of maps $K_n^p = F^{-1} \circ pI \circ F$ and

$$K_n^{-p} \circ G = F^{-1} \circ pS \circ F \circ F^{-1} \circ S \circ F = F^{-1} \circ pI \circ F = K_n^p. \qquad \square$$

Corollary 4.5. *All the K_n^p, $n \geqslant 1$, $p \in Z$ can be obtained as derivatives of Faber polynomials. For $p \geqslant 1$,*

$$\begin{aligned}&(-1)^p(p-1)!(n+k_1+k_2+\cdots+k_p)K_n^{-p}(b_1,b_2,\ldots,b_n)\\&\quad=\frac{\partial^p F_{n+k_1+\cdots+k_p}}{\partial b_{k_1}\partial b_{k_2}\cdots\partial b_{k_p}}(b_1,b_2,\ldots,b_n,\ldots,b_q,\ldots).\end{aligned}\tag{4.6}$$

Let

$$\phi(b_1,b_2,\ldots,b_n,\ldots,b_q,\ldots)=\frac{\partial^p F_{n+k_1+\cdots+k_p}}{\partial b_{k_1}\partial b_{k_2}\cdots\partial b_{k_p}}(b_1,b_2,\ldots,b_n,\ldots,b_q,\ldots),$$

for $p \geqslant 1$, we have

$$\begin{aligned}&(-1)^p(p-1)!(n+k_1+k_2+\cdots+k_p)K_n^{p}(b_1,b_2,\ldots,b_n)\\&\quad=\phi\big(G_1(b_1,b_2,\ldots),G_2(b_1,b_2,\ldots),\ldots,G_q(b_1,b_2,\ldots),\ldots\big).\end{aligned}\tag{4.7}$$

Corollary 4.6. *For $p \geqslant 1$,*

$$(-1)^p(p-1)!(n+p)K_n^{-p}(b_1,b_2,\ldots,b_n)=\frac{\partial^p F_{n+p}}{\partial b_1^p}(b_1,b_2,\ldots,b_n,\ldots,b_{n+p}),\tag{4.8}$$

$$\begin{aligned}&(-1)^p(p-1)!(n+p)K_n^{p}(b_1,b_2,\ldots,b_n)\\&\quad=\frac{\partial^p F_{n+p}}{\partial b_1^p}\big(G_1(b),G_2(b_1,b_2),\ldots,G_{n+p}(b_1,b_2,\ldots,b_{n+p})\big).\end{aligned}\tag{4.9}$$

In particular

$$K_n^{-(n+1)}=\frac{(-1)^{n+1}}{n!(2n+1)}\frac{\partial^{n+1}F_{2n+1}}{\partial b_1^{n+1}}(b_1,b_2,\ldots,b_q,\ldots),\tag{4.10}$$

$$K_{n+1}^{n}=\frac{(-1)^n}{(n-1)!(2n+1)}\left(\frac{\partial^n F_{2n+1}}{\partial b_1^n}\right)\big(G_1(b_1),G_2(b_1,b_2),\ldots,G_q(b_1,b_2,\ldots,b_q)\big).\tag{4.11}$$

Proof of the Main Corollary. Let $f(z)=z+b_1z^2+b_2z^3+\cdots+b_nz^{n+1}+\cdots$.

$$\frac{f''(z)}{f'(z)}=-\sum_{k\geqslant 1}F_k\big(2b_1,3b_2,\ldots,(k+1)b_k\big)z^{k-1},$$

$$\left(\frac{f''(z)}{f'(z)}\right)'=-\sum_{k\geqslant 2}(k-1)F_k\big(2b_1,3b_2,\ldots,(k+1)b_k\big)z^{k-2}.\tag{i}$$

On the other hand

$$\begin{aligned}-\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2&=-\frac{1}{2z^2}\left(\frac{zf''(z)}{f'(z)}\right)^2\\&=-\frac{1}{2z^2}\left(\sum_{k\geqslant 1}F_k\big(2b_1,3b_2,\ldots,(k+1)b_k\big)z^k+1-1\right)^2\end{aligned}$$

$$= -\frac{1}{2z^2}\Bigl(1+\sum_{k\geqslant 1} F_k(2b_1, 3b_2, \ldots)z^k\Bigr)^2 + \frac{1}{z^2}\Bigl(1+\sum_{k\geqslant 1} F_k(2b_1, 3b_2, \ldots)z^k\Bigr) - \frac{1}{2z^2}$$
$$= -\frac{1}{2z^2}\sum_{k\geqslant 2} K_k^2\bigl(F_1(2b_1), F_2(2b_1, 3b_2), \ldots\bigr)z^k + \frac{1}{z^2}\sum_{k\geqslant 2} F_k(2b_1, 3b_2, \ldots)z^k. \qquad \text{(ii)}$$

We add the two expressions (i) and (ii) to obtain the Main Corollary. □

Remark 4.1. Let $H_k(b_1, b_2, \ldots, b_k) = F_k(2b_1, 3b_2, 4b_3, \ldots, (k+1)b_k)$. With the expressions of the $(F_n)_{n\geqslant 1}$ in [3], we find $H_1(b_1) = F_1(2b_1) = -2b_1$ and $H_2(b_1, b_2) = F_2(2b_1, 3b_2) = 2(2b_1^2 - 3b_2)$

$$H_3(b_1, b_2, b_3) = F_3(2b_1, 3b_2, 4b_3) = 2\bigl(-4b_1^3 + 9b_1b_2 - 6b_3\bigr),$$
$$H_4(b_1, b_2, b_3, b_4) = F_4(2b_1, 3b_2, 4b_3, 5b_4) = 2\bigl(8b_1^4 - 24b_1^2b_2 + 9b_2^2 + 16b_1b_3 - 10b_4\bigr).$$

We can calculate $\mathcal{P}_k$ with $\mathcal{P}_k = -(k-1)H_k - \frac{1}{2}\sum_{j=1}^{k-1} H_{k-j}H_j$ or we can use (C1).

Proof of (T3). $h(z)^p = \sum_{n\geqslant 0} K_n^p z^n = (1 + b_1z + b_2z^2 + \cdots + b_nz^n + \cdots)^p$,

$$K_n^p = \frac{1}{2\mathrm{i}\pi}\int \frac{(1+\phi_n(\xi) + b_{n+1}\xi^{n+1} + \cdots)^p}{\xi^{n+1}}\,\mathrm{d}\xi = \frac{1}{2\mathrm{i}\pi}\int \frac{(1+\phi_n(\xi))^p}{\xi^{n+1}}\,\mathrm{d}\xi$$

and we write Newton binomial formula $(1+\phi_n(\xi))^p = 1 + p\phi_n(\xi) + \frac{p(p-1)}{2}\phi_n(\xi)^2 + \cdots$. □

Proof of (T4). If $b_1 \neq 0$, then $\phi_n(z) = b_1z + b_2z^2 + b_3z^3 + \cdots + b_nz^n = b_1z(1 + \frac{b_2}{b_1}z + \cdots + \frac{b_n}{b_1}z^{n-1})$. Thus

$$\bigl(\phi_n(z)\bigr)^k = b_1^kz^k\Bigl(1 + \frac{b_2}{b_1}z + \cdots + \frac{b_n}{b_1}z^{n-1}\Bigr)^k = b_1^kz^k\sum_{j\geqslant 1} K_j^k\Bigl(\frac{b_2}{b_1}, \ldots, \frac{b_{j+1}}{b_1}\Bigr)z^j.$$

The coefficient of z^n in this expression is obtained for $j + k = n$. This gives (T4). We can obtain the exact expression of K_n^p as follows,

$$K_n^p = \sum_{1\leqslant k_1\leqslant n} C_{k_1}^p D_n^{k_1} = \sum_{1\leqslant k_1\leqslant n} C_{k_1}^p b_1^{k_1} \sum_{1\leqslant k_2\leqslant n-k_1} C_{k_2}^{k_1}\Bigl(\frac{b_2}{b_1}\Bigr)^{k_2} D_{n-k_1}^{k_2}\Bigl(\frac{b_3}{b_2}, \ldots\Bigr).$$

If $b_1 \neq 0$,

$$K_1^p = C_1^pb_1, \qquad K_2^p = C_1^pb_2 + C_2^pb_1^2, \qquad K_3^p = C_1^pb_3 + C_2^pb_1^2K_1^2\Bigl(\frac{b_2}{b_1}\Bigr) + C_3^pb_1^3,$$
$$K_4^p = C_1^pb_4 + C_2^pb_1^2K_2^2\Bigl(\frac{b_2}{b_1}, \frac{b_3}{b_1}\Bigr) + C_3^pb_1^3K_1^3\Bigl(\frac{b_2}{b_1}\Bigr) + C_4^pb_1^4,$$
$$K_5^p = C_1^pb_5 + C_2^pb_1^2K_3^2\Bigl(\frac{b_2}{b_1}, \frac{b_3}{b_1}, \frac{b_4}{b_1}\Bigr) + C_3^pb_1^3K_2^3\Bigl(\frac{b_2}{b_1}, \frac{b_3}{b_1}\Bigr) + C_4^pb_1^4K_1^4\Bigl(\frac{b_2}{b_1}\Bigr) + C_5^pb_1^5,$$
$$K_6^p = C_1^pb_6 + C_2^pb_1^2K_4^2\Bigl(\frac{b_2}{b_1}, \frac{b_3}{b_1}, \frac{b_4}{b_1}, \frac{b_5}{b_1}\Bigr) + C_3^pb_1^3K_3^3\Bigl(\frac{b_2}{b_1}, \frac{b_3}{b_1}, \frac{b_4}{b_1}\Bigr)$$
$$+ C_4^pb_1^4K_2^4\Bigl(\frac{b_2}{b_1}, \frac{b_3}{b_1}\Bigr) + C_5^pb_1^5K_1^5\Bigl(\frac{b_2}{b_1}\Bigr) + C_6^pb_1^6.$$

If $b_1=0$, and $b_2\neq 0$,

$$K_2^p=C_1^pb_2,\qquad K_3^p=C_1^pb_3,\qquad K_4^p=C_1^pb_4+C_2^pb_2^2,$$
$$K_5^p=C_1^pb_5+C_2^pb_2K_1^2\left(\frac{b_3}{b_2}\right),\qquad K_6^p=C_1^pb_6+C_2^pb_2^2K_2^2\left(\frac{b_3}{b_2},\frac{b_4}{b_2}\right)+C_3^pb_2^3,$$
$$K_7^p=C_1^pb_7+C_2^pb_2^2K_3^2\left(\frac{b_3}{b_2},\frac{b_4}{b_2},\frac{b_5}{b_2}\right)+C_3^pb_2^3K_1^3\left(\frac{b_3}{b_2}\right),\quad \ldots. \qquad \square$$

5. Identities related to the $(W_j)_{j\geqslant 1}$, the $(V_j^k)_{j\geqslant 1}$, $k\in Z$ and the $(V_j)_{j\geqslant 1}$

It has been proved in [3] that

$$W_jW_q=W_qW_j\quad \text{for } j\geqslant 1,\ q\geqslant 1. \tag{5.1}$$

For $j\geqslant 1$, $m\geqslant 0$,

$$W_jF_m=m\delta_{j,m}\quad \text{and}\quad W_jG_m=G_{m-j} \tag{5.2}$$

with the convention $G_p=0$ if $p<0$. Moreover $K_0^p=1$ and

$$W_j\big(K_n^p\big)=0\quad \text{for } n<j,\qquad W_j\big(K_n^p\big)=-pK_{n-j}^p\quad \text{for } n\geqslant j,\ p\in Z. \tag{5.3}$$

Proof of (5.1)–(5.3). For (5.1), we remark that $W_jW_p[h(z)]=z^{p+j}h(z)$. For the other identities, let $(b_1,\ldots,b_k,\ldots)\to h(z)=1+b_1z+b_2z^2+\cdots+b_kz^k+\cdots$, then $\frac{\partial}{\partial b_j}[h(z)]=z^j$. It is enough to calculate $W_j[h(z)]$, use (1.1), then calculate $W_j[\frac{1}{h(z)}]$, use (1.3) by equating coefficients of similar powers of z. This is done as follows. Let

$$(b_1,b_2,\ldots,b_k,\ldots)\to\phi(b_1,b_2,\ldots,b_k,\ldots).$$

Since W_j is a differential operator, $W_j[\exp(\phi)]=\exp(\phi)\times W_j[\phi]$. With (1.1), we have $W_j[h(z)]=h(z)\times(-\sum_{k=1}^{+\infty}\frac{W_jF_k}{k}z^k)$. Comparing this last expression with $W_j[h(z)]=-z^jh(z)$, we deduce that $z^j=\sum_{k=1}^{+\infty}\frac{W_jF_k}{k}z^k$. Equating the coefficients of z^k gives $W_j(F_m)$. To obtain $W_j[G_m]$, we calculate $W_j[\frac{1}{h(z)}]=-\frac{W_j[h(z)]}{h(z)^2}$. With $W_j[h(z)]=-z^jh(z)$, it gives $W_j[\frac{1}{h(z)}]=\frac{z^j}{h(z)}$. With(1.2), it gives

$$\frac{z^j}{h(z)}=\sum_{m\geqslant 1}W_jG_mz^m. \tag{i}$$

In (i), we replace $\frac{1}{h(z)}$ by (1.2), thus $z^j(\sum_{m\geqslant 0}G_mz^m)=\sum_{m\geqslant 1}W_j[G_m]z^m$. In this identity, equating the coefficients of z^m gives $W_j(G_m)$. In the same way,

$$W_j\big[h(z)^p\big]=ph(z)^{p-1}W_j\big[h(z)\big]=-pz^jh(z)^p=-pz^j\left(1+\sum_{n\geqslant 1}K_n^pz^n\right).$$

On the other hand, $W_j[h(z)^p]=\sum_{s\geqslant 1}W_j[K_s^p]z^s$. Identifying the two expressions of $W_j[h(z)^p]$, we obtain $-pz^j(1+\sum_{n\geqslant 1}K_n^pz^n)=\sum_{s\geqslant 1}W_j[K_s^p]z^s$. Equating the coefficients of equal powers of z^j gives $W_j(K_n^p)$. $\square$

Theorem 5.1. *The operators* $(V_j^k)_{j\geqslant 1}$, $k \in Z$ *satisfy* (1.11),

$$V_k^q V_s^p + (p+1)V_{s+k}^{p+q} = \sum_{n\geqslant 0, j\geqslant 0} K_n^{p+1} K_j^{q+1} \frac{\partial^2}{\partial b_{n+s}\partial b_{k+j}} \tag{5.4}$$

and $V_j^k = \sum_{n\geqslant 0} K_n^k W_{j+n}$

$$V_j^k(F_p) = pK_{p-j}^k, \qquad V_j^k\big(K_s^q\big) = -qK_{s-j}^{q+k}, \tag{5.5}$$

$$V_j^k\big[h(z)\big] = -z^j\big[h(z)\big]^{k+1}. \tag{5.6}$$

The polynomials $(P_n^k)_{n\geqslant 0}$, $P_0^k = 1$ (*see* (1.4)) *satisfy*

$$V_p^k\big(P_n^{n+j}\big) = -(2k+j)P_{n-p}^{n-p+j+k} + \frac{2(k-p)(n-p+k+j)}{k+j}K_{n-p}^{k+j} \quad \text{if } k+j \neq 0,$$

$$V_p^{-j}\big(P_n^{n+j}\big) = jP_{n-p}^{n-p} + 2(j+p)F_{n-p},$$

$$V_k\big(P_n^{n+j}\big) = -(2k+j)P_{n-k}^{n+j} \tag{5.7}$$

and the Neretin polynomials (C1), $z^2S(f)(z) = \sum_{k\geqslant 2}\mathcal{P}_k z^k$,

$$V_k(\mathcal{P}_j) = -(k^3-k)P_{j-k}^j. \tag{5.8}$$

Proof of (5.7)–(5.8). Since $V_p[h(z)] = -z^p h(z)^{p+1}$, we deduce for $p \geqslant 1$,

$$V_p\big[f(z)\big] = -f(z)^{p+1}, \qquad V_p\left(\frac{f'(z)}{f(z)}\right) = -f(z)^{p-1}f'(z) \tag{5.9}$$

and $V_p(\frac{f''(z)}{f'(z)}) = -p(p+1)f(z)^{p-1}f'(z)$. We obtain (5.7)–(5.8) by identification of coefficients. □

In [3], the homogeneity operator $L_0 = b_1\frac{\partial}{\partial b_1} + 2b_2\frac{\partial}{\partial b_2} + \cdots + kb_k\frac{\partial}{\partial b_k} + \cdots$ is expressed as $L_0 = \sum_{j\geqslant 1} F_j W_j$.

Lemma 5.2. *We have* $G_n = K_n^{-1}$, $b_n = K_n^1$,

$$kG_k = \sum_{1\leqslant j\leqslant k} F_j G_{k-j} \quad \text{and} \quad nK_n^p = -p\sum_{1\leqslant j\leqslant n} F_j K_{n-j}^p. \tag{5.10}$$

Proof of (5.10). From the recurrence relation for the polynomials $(F_k)_{k\geqslant 0}$,

$$L_0 = \sum_{k\geqslant 1} kb_k\frac{\partial}{\partial b_k} = -\sum_{j\geqslant 1} F_j\left(\sum_{k\geqslant j} b_{k-j}\frac{\partial}{\partial b_k}\right) = \sum_{j\geqslant 1} F_j W_j.$$

K_n^p is homogeneous of degree n, thus $L_0 K_n^p = nK_n^p$. Since $L_0K_n^p = \sum_{j\geqslant 1} F_j W_j K_n^p = -\sum_{1\leqslant j\leqslant n} F_j pK_{n-j}^p$, we obtain the recursion formula for K_n^p. See (2.5), (2.7). □

Theorem 5.3. *The* $(\frac{\partial}{\partial b_j})_{j\geqslant 1}$ *are given in terms of the* $(W_j)_{j\geqslant 1}$ *with*

$$\frac{\partial}{\partial b_j} = -W_j - \sum_{k\geqslant 1} G_k W_{j+k}, \tag{5.11}$$

$$X_0 = -\sum_{j\geqslant 1} b_j \frac{\partial}{\partial b_j} = -b_1\frac{\partial}{\partial b_1} - b_2\frac{\partial}{\partial b_2} - \cdots - b_k\frac{\partial}{\partial b_k} - \cdots \tag{5.12}$$

satisfies

$$X_0 = -\sum_{j\geqslant 1} G_j W_j. \tag{5.13}$$

In particular,

$$X_0(F_n) = -nG_n. \tag{5.14}$$

Proof. For $(\frac{\partial}{\partial b_j})_{j\geqslant 1}$, it is enough to verify that

$$\frac{\partial}{\partial b_j}\big[h(z)\big] = -W_j\big[h(z)\big] - \sum_{k\geqslant 1} G_k W_{j+k}\big[h(z)\big]. \tag{i}$$

Since $W_j[h(z)] = -z^j h(z)$, we have

$$-W_j\big[h(z)\big] - \sum_{k\geqslant 1} G_k W_{j+k}\big[h(z)\big] = z^j\big(1 + G_1 z + G_2 z^2 + \cdots + G_k z^k + \cdots\big) \times h(z).$$

Thus (i) is the same as $z^j = z^j(1 + G_1 z + G_2 z^2 + \cdots + G_k z^k + \cdots) \times h(z)$. It is the immediate consequence of (1.3). To prove (5.13), we see that $X_0[h(z)] = 1 - h(z)$. We write X_0 as $X_0 = \sum_{j\geqslant 1} H_j W_j$. Applied to $h(z)$, it gives

$$X_0\big[h(z)\big] = 1 - h(z) = -\Big[\sum_{j\geqslant 1} H_j z^j\Big] h(z).$$

Thus

$$\sum_{j\geqslant 1} H_j z^j = \frac{h(z) - 1}{h(z)} = 1 - \frac{1}{h(z)} = 1 - \sum_{n\geqslant 0} G_n z^n.$$

By identification of the coefficient of z^j, we find $H_j = -G_j$. □

Remark 5.1. If we calculate $W_{p+k} F_{n+p}$ with (5.2), we obtain with (5.11) another proof of (1.16), $\frac{\partial}{\partial b_p} F_{n+p} = -W_p F_{n+p} - \sum_{k\geqslant 1} G_k W_{p+k} F_{n+p} = -(n+p)G_n$.

Theorem 5.4. *For $k,\ p \in Z,\ j \geqslant 1$,*

$$V_j^k = \sum_{n\geqslant 1} K_{n-j}^{k-p} V_n^p \quad \textit{and} \quad X_0 = \sum_{n\geqslant 1} \big[K_n^{-p} - K_n^{-1-p}\big] V_n^p.$$

We can also express the (V_j^k) in terms of $(V_j)_{j\geqslant 1}$ with the inverse function $f^{-1}(z)$.

6. The composition of differentials on coefficients

6.1. The inverse function

The inverse function is important in the study of coefficients regions, see [11, p. 104]. Also asymptotics of the derivatives of the Faber polynomials are calculated with inverse functions, see [10]. Let $f(w) = wh(w) = w + b_1w^2 + b_2w^3 + \cdots + b_nw^{n+1} + \cdots$.

$$w\frac{f'(w)}{f(w)} = 1 + w\frac{h'(w)}{h(w)} = 1 - \sum_{k\geqslant 1} F_k(b_1, b_2, \ldots, b_k)w^k.$$

We denote $k(z) = f^{-1}(z)$ the inverse of f, we have $(f \circ k)(z) = z$, letting $w = k(z)$, $z\frac{k'(z)}{k(z)} = \frac{f(w)}{wf'(w)}$ and

$$\begin{aligned} z\frac{k'(z)}{k(z)} &= \frac{1}{1 - \sum_{p\geqslant 1} F_p(b_1, b_2, \ldots, b_p)k(z)^p} \\ &= 1 + \sum_{m=1}^{+\infty} G_m(-F_1, -F_2, \ldots, -F_m)k(z)^m \end{aligned}$$

we have $k(z) = f^{-1}(z) = z - b_1z^2 - (b_2 - 2b_1^2)z^3 - (b_3 + 5b_1^3 - 5b_1b_2)z^4 - (b_4 - 14b_1^4 + 21b_1^2b_2 - 6b_1b_3 - 3b_2^2)z^5 + \cdots$. By a residue calculus, we know that

$$f^{-1}(z) = \frac{1}{2\mathrm{i}\pi}\int_{|\zeta|=\rho} \frac{\zeta f'(\zeta)}{f(\zeta) - z}\mathrm{d}\zeta = \sum_{n\geqslant 1}\left(\frac{1}{2\mathrm{i}\pi}\int \frac{\zeta f'(\zeta)}{f(\zeta)}\frac{\mathrm{d}\zeta}{f(\zeta)^n}\right)z^n. \tag{6.1}$$

Theorem 6.1. *Let $f(z) = z + b_1z^2 + \cdots + b_nz^n + \cdots$. The inverse function of f, $f^{-1}(f(z)) = z$ is given in terms of the derivatives of the Faber polynomials of $f(z)$ with*

$$\begin{aligned} f^{-1}(z) &= z + \sum_{n\geqslant 1} \frac{1}{n+1}K_n^{-(n+1)}z^{n+1} \\ &= z + \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{(2n+1)} \times \frac{1}{(n+1)!}\left[\frac{\partial^{n+1}}{\partial b_1^{n+1}}F_{2n+1}(b_1, b_2, \ldots, b_q, \ldots)\right]z^{n+1}. \end{aligned} \tag{6.2}$$

Let $g(z) = zh(\frac{1}{z}) = z + b_1 + \frac{b_2}{z} + \frac{b_3}{z^2} + \cdots + \frac{b_{n+1}}{z^n} + \cdots$, then the inverse function of g is

$$\begin{aligned} g^{-1}(z) &= z - b_1 - \sum_{n\geqslant 1} \frac{1}{n}K_{n+1}^n\frac{1}{z^n} \\ &= z - b_1 + \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{(2n+1)} \\ &\quad \times \frac{1}{n!}\left[\frac{\partial^n}{\partial b_1^n}F_{2n+1}\right]\big(G_1(b_1), G_2(b_1, b_2), \ldots, G_q(b_1, b_2, \ldots), \ldots\big)\frac{1}{z^n}. \end{aligned} \tag{6.3}$$

Proof. $\frac{zf'(z)}{f(z)} = -\sum_{k\geqslant 0} F_k z^k$ with $F_0 = -1$ and $\frac{z^n}{f(z)^n} = \sum_{p\geqslant 0} K_p^{-n} z^p$, we deduce that

$$\frac{zf'(z)}{f(z)}\frac{1}{f(z)^n} = -\sum_{p\geqslant 0, k\geqslant 0} F_k K_p^{-n} z^{p+k-n}.$$

The residue is obtained for $p+k-n=-1$ and is equal to $-\sum_{0\leqslant k\leqslant n-1} F_k K_{n-k-1}^{-n}$. This is the coefficient of z^n in the expression of $f^{-1}(z)$. Thus the coefficient of z^{n+1} is

$$K_n^{-(n+1)} - \sum_{1\leqslant k\leqslant n} F_k K_{n-k}^{-(n+1)} = K_n^{-(n+1)} - \frac{n}{n+1} K_n^{-(n+1)}$$

where we have used the recursion formula (2.7). For example the coefficients of z^4 is given when $n=3$ by $\frac{1}{4}K_3^{-4} = 5b_1b_2 - b_3 - 5b_1^3$. The expressions of the coefficients of $f^{-1}(z)$ in terms of the (K_n^k) were found in an other way in [2, (1.2.8)–(1.2.9)]. For $g^{-1}(z)$, we use [2, (1.2.8)] and (T2). □

Proposition 6.2. *We have*

$$h\big(f^{-1}(z)\big) = 1 + b_1 z - \sum_{n\geqslant 2} \frac{1}{n-1} K_n^{1-n} z^n \tag{6.4}$$

with $f(z) = zh(z)$. *Assume that* $p \geqslant 2$, *then*

$$\big[h\big(f^{-1}(z)\big)\big]^p = 1 + \sum_{1\leqslant n\leqslant p-1} \frac{p}{p-n} K_n^{p-n} z^n - F_p z^p - \sum_{n\geqslant p+1} \frac{p}{n-p} K_n^{p-n} z^n. \tag{6.5}$$

Assume that $p \geqslant 1$, *then*

$$\big[h\big(f^{-1}(z)\big)\big]^{-p} = 1 + \sum_{n\geqslant 1} \frac{p}{n+p} K_n^{-(p+n)} z^n. \tag{6.6}$$

The function $\psi(z) = h(f^{-1}(z))$ has been considered in [13]. The coefficients of $[h(f^{-1}(z))]^p$, $p \in Z$ have been given in [2, (I.2.4) and (0.7)].

Proof. By a residue calculus,

$$\begin{aligned} h\big(f^{-1}(z)\big) &= \frac{1}{2\mathrm{i}\pi}\int \frac{h(\xi)f'(\xi)}{f(\xi)-z}\,\mathrm{d}\xi = \sum_{n\geqslant 0}\bigg(\frac{1}{2\mathrm{i}\pi}\int_{|\xi|=\rho} \frac{\xi f'(\xi)}{f(\xi)}\frac{1}{f(\xi)^{n-1}}\frac{\mathrm{d}\xi}{\xi^2}\bigg) z^n \\ &= 1 + b_1 z + \sum_{n\geqslant 2}\bigg(\frac{1}{2\mathrm{i}\pi}\int_{|\xi|=\rho} \frac{\xi f'(\xi)}{f(\xi)}\frac{1}{f(\xi)^{n-1}}\frac{\mathrm{d}\xi}{\xi^2}\bigg) z^n. \end{aligned}$$

Since with (2.7), $\sum_{1\leqslant k\leqslant n} F_k K_{n-k}^{1-n} = \frac{n}{n-1} K_n^{1-n}$, the coefficient of z^n in the expansion of $h(f^{-1}(z))$ is given by $K_n^{1-n} - \frac{n}{n-1} K_n^{1-n} = -\frac{1}{n-1} K_n^{1-n}$. Then we finish the proof as in Theorem 6.1. In the same way, we have

$$\big[h\big(f^{-1}(z)\big)\big]^p = \sum_{n\geqslant 0} \frac{1}{2\mathrm{i}\pi}\bigg(\int \frac{\xi f'(\xi)}{f(\xi)}\frac{1}{f(\xi)^{n-p}}\frac{\mathrm{d}\xi}{\xi^{p+1}}\bigg) z^n.$$

We have $\frac{\xi f'(\xi)}{f(\xi)}\frac{1}{f(\xi)^{n-p}} = -\sum_{k\geqslant 0, j\geqslant 0} F_k K_j^{p-n}\xi^{k+j+p-n}$, the coefficient of ξ^p in this expression is obtained when $k+j=n$ and is equal to

$$-\sum_{0\leqslant k\leqslant n} F_k K_{n-k}^{p-n} = K_n^{p-n} - \sum_{1\leqslant k\leqslant n} F_k K_{n-k}^{p-n} = -\frac{p}{n-p}K_n^{p-n}z^n$$

when $p\neq n$. If $p=n$, we take the coefficient $-F_p$ of ξ^p in $\xi\frac{f'(\xi)}{f(\xi)}$. □

Remark 6.1. Following [13], for any $p\in Z$, $p\neq 0$,

$$\frac{[h(f^{-1}(z))]^p}{f'(f^{-1}(z))} = \sum_{n\geqslant 0} K_n^{p-(n+1)} z^n. \tag{6.7}$$

Proof. We have

$$\frac{[h(f^{-1}(z))]^p}{f'(f^{-1}(z))} = \frac{1}{2\mathrm{i}\pi}\int \frac{h(\xi)^p}{f(\xi)-z}\,\mathrm{d}\xi = \frac{1}{2\mathrm{i}\pi}\int \frac{h(\xi)^{p-1}}{1-\frac{z}{f(\xi)}}\frac{\mathrm{d}\xi}{\xi}$$
$$= \sum_{n\geqslant 0}\left(\frac{1}{2\mathrm{i}\pi}\int \frac{h(\xi)^{p-1}}{\xi^n h(\xi)^n}\frac{\mathrm{d}\xi}{\xi}\right)z^n = \sum_{n\geqslant 0}\frac{1}{2\mathrm{i}\pi}\int h(\xi)^{p-n-1}\frac{\mathrm{d}\xi}{\xi^{n+1}}z^n.$$

Since $h(\xi)^{p-n-1}\frac{1}{\xi^{n+1}} = \sum_{j\geqslant 0} K_j^{p-n-1}\frac{\xi^j}{\xi^{n+1}}$, the residue is $K_n^{p-(n+1)}$. □

Remark 6.2. With the inverse function, we obtain also expressions of P_n^k (see Proposition 2.4),

$$P_n^k(b_1,b_2,\ldots,b_n) = \sum_{0\leqslant s\leqslant n}\frac{k+s}{k}K_s^k(b_1,b_2,\ldots,b_s)$$
$$\times K_{n-s}^s\left(\frac{1}{2}K_1^{-2}(b_1), \frac{1}{3}K_2^{-3}(b_1,b_2), K_3^{-4},\ldots,\frac{1}{p+1}K_p^{-(p+1)},\ldots\right). \tag{6.8}$$

Proof. From Proposition 2.3, $\phi_k(\zeta) = \sum_{n\geqslant 0}\frac{k+n}{k}K_n^k\zeta^n$. We put $\zeta = f^{-1}(z)$. From Theorem 6.1,

$$f^{-1}(z)^n = z^n\sum_{j\geqslant 0}K_j^n\left(\frac{1}{2}K_1^{-2},\frac{1}{3}K_2^{-3},K_3^{-4},\ldots,\frac{1}{p+1}K_p^{-(p+1)},\ldots\right)z^j. \tag{6.9}$$

This gives the first expression of P_n^k. □

Remark 6.3.

$$(f^{-1})'(z) = 1+\sum_{k\geqslant 1}\frac{(-1)^k}{k!}z^k\psi_k(z) \quad \text{with } \psi_k(z) = \sum_{p\geqslant 0}\frac{(2k+p)!}{(k+p)!}D_{k+p}^k z^p. \tag{6.10}$$

Proof. We use (T3).

$$\frac{\mathrm{d}}{\mathrm{d}z}f^{-1}(z)=1+\sum_{n\geqslant 1}K_n^{-(n+1)}z^n=1+\sum_{1\leqslant k\leqslant n,1\leqslant n}(-1)^k\frac{(n+k)!}{n!k!}D_n^k z^n$$
$$=1+\sum_{k\geqslant 1}\frac{(-1)^k}{k!}z^k\left(\sum_{n\geqslant k}\frac{(n+k)!}{n!}D_n^k z^{n-k}\right). \qquad \square$$

Remark 6.4. For $s\geqslant 1$, $p\in Z$, then

$$V_s^p\left[f^{-1}(z)\right]=z^{1+s}\sum_{n\geqslant 0}K_n^{p-s-(n+1)}z^n=\frac{z^{1+s}[h(f^{-1}(z))]^{p-s}}{f'(f^{-1}(z))}$$
$$=\left(\frac{f(\zeta)^{1+s}h(\zeta)^{p-s}}{f'(\zeta)}\right)\bigg|_{\zeta=f^{-1}(z)}.$$

Let $\phi_k(\zeta)=\frac{\zeta f'(\zeta)}{f(\zeta)}\times h(\zeta)^k=\sum_{n\geqslant 0}\frac{k+n}{k}K_n^k\zeta^n$, we have

$$V_s^p\left[\phi_k(\zeta)\right]=-\zeta^s\sum_{n\geqslant 0}\frac{k(k+n+s)}{n+s}K_n^{k+p}\zeta^n. \tag{6.11}$$

6.2. *Composition of derivations and recurrence formulae*

We know (see 1.25) that $F_n(b_1,b_2,\dots,b_n)+F_n(G_1,G_2,\dots,G_n)=0$. In the following, we show how the differentiation of this identity yields the recursion formula (2.8) with $p=-1$ and $r=-2$, i.e. $K_n^{-1}=\sum_{0\leqslant j\leqslant n}K_j^{-2}K_{n-j}^1$. Then we prove that it gives a partial differential equation satisfied by the $(F_n)_{n\geqslant 1}$.

The differentiation of (1.25) with respect to b_k gives

$$\frac{\partial}{\partial b_k}F_n(b_1,b_2,\dots,b_n)+\sum_{j=1}^{n}\left(\frac{\partial F_n}{\partial b_j}\right)(G_1,G_2,\dots,G_n)\times\frac{\partial G_j}{\partial b_k}(b_1,b_2,\dots)=0. \tag{6.12}$$

We know from (1.16) that $\frac{\partial F_n}{\partial b_i}(b_1,b_2,\dots,b_n)=-nG_{n-i}(b_1,b_2,\dots,b_n)$. This expression calculated at the point $(G_1(b_1),G_2(b_1,b_2),\dots,G_n(b_1,b_2,\dots,b_n))$ gives

$$\left(\frac{\partial F_n}{\partial b_j}\right)(G_1,G_2,\dots,G_n)=-nG_{n-j}(G_1,G_2,\dots,G_n)=-nb_{n-j}. \tag{6.13}$$

We replace in (6.12), we obtain

$$\frac{\partial F_n}{\partial b_k}(b_1,b_2,\dots,b_n)-\sum_{j=1}^{n-1}nb_{n-j}\frac{\partial G_j}{\partial b_k}(b_1,b_2,\dots,b_n)-n\frac{\partial G_n}{\partial b_k}=0, \tag{6.14}$$

or equivalently

$$\frac{\partial F_n}{\partial b_k}(b_1,b_2,\dots,b_n)-\sum_{j=1}^{n-1}nb_j\frac{\partial G_{n-j}}{\partial b_k}(b_1,b_2,\dots,b_n)-n\frac{\partial G_n}{\partial b_k}=0.$$

On the other hand, $-nG_j=\frac{\partial F_n}{\partial b_{n-j}}$. We replace in (6.14), we obtain

$$\frac{\partial F_n}{\partial b_k}(b_1,b_2,\dots,b_n)+\sum_{j=k}^{n-1}b_{n-j}\frac{\partial^2 F_n}{\partial b_{n-j}\partial b_k}(b_1,b_2,\dots,b_n)-n\frac{\partial G_n}{\partial b_k}=0. \tag{6.15}$$

We go back to the expressions of the partial derivatives of F_n in terms of the K_n^p to see that (6.15) is the same as $K_n^{-1} = \sum_{0\leqslant j\leqslant n} K_j^{-2} K_{n-j}^1$.

Lemma 6.3.

$$-K_{n-k}^{-2} = \frac{\partial G_n}{\partial b_k} \quad \forall n, k, n \leqslant k, \quad \textit{and}$$

$$\frac{\partial^2 F_n}{\partial b_r \partial b_s} = -n \frac{\partial G_n}{\partial b_{r+s}} \quad r, s \geqslant 1,\ n \geqslant 1.$$

Proof. From (T1). □

Theorem 6.4. *Let* $X_0 = -\sum_{j\geqslant 1} b_j \frac{\partial}{\partial b_j} = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \cdots - b_k \frac{\partial}{\partial b_k} - \cdots$. *Then the identity*

$$K_n^{-1} = \sum_{0\leqslant j\leqslant n} K_j^{-2} K_{n-j}^1 \tag{6.16}$$

is the same as

$$\frac{\partial^2 F_n}{\partial b_r \partial b_s} = \frac{\partial}{\partial b_{r+s}} (X_0 F_n) \quad \forall r, s \geqslant 1,\ n \geqslant 1. \tag{6.17}$$

Proof. From (6.15), $\frac{\partial F_n}{\partial b_k} + \sum_{j\geqslant 1} b_j \frac{\partial^2 F_n}{\partial b_j \partial b_k} = n \frac{\partial G_n}{\partial b_k}$. The left side of this equation is

$$\frac{\partial}{\partial b_k} \left(\sum_{j\geqslant 1} b_j \frac{\partial}{\partial b_j} F_n \right) = -\frac{\partial}{\partial b_k} (X_0 F_n).$$

On the other hand, $n \frac{\partial G_n}{\partial b_k}$ is given by Lemma 6.3. This proves the theorem. □

7. First order differential operators on $\mathcal{M}$

We have seen that the operators $(W_j)_{j\geqslant 1}$, X_0, $\frac{\partial}{\partial b_j}$ allow to pass from polynomials $(F_k)_{k\geqslant 1}$ to polynomials $(G_m)_{m\geqslant 1}$. In particular, we found $(n + p)G_n = -\frac{\partial F_{n+p}}{\partial b_p}$ and $W_j G_m = G_{m-j}$. Operators (Z_k) in [2] are of this type. In [9], family of vector fields related to the Virasoro algebra have been considered. We found that the operators $(V_k)_{k\geqslant 1}$ transforms the Neretin polynomials $\mathcal{P}_j$ into $-(k^3 - k) P_{j-k}^k$. In the following, we construct first order differential operators on the manifold $\mathcal{M}$ which permit to pass from one polynomial to the other.

7.1. The operators $(X_k)_{k\in Z}$

The operator $X_0 = -\sum_{j\geqslant 1} G_j W_j = -b_1 \frac{\partial}{\partial b_1} - b_2 \frac{\partial}{\partial b_2} - \cdots - b_n \frac{\partial}{\partial b_n} - \cdots$ has appeared to be a natural operator on $\mathcal{M}$. We have

$$X_0 \frac{\partial}{\partial b_p} - \frac{\partial}{\partial b_p} X_0 = b_p \frac{\partial}{\partial b_p}. \tag{7.1}$$

On the other hand, $\frac{\partial}{\partial b_k} = -\sum_{j\geqslant 1} G_j W_{j+k}$.

Definition 7.1. For $k \geqslant 1$, we put

$$X_k = \frac{\partial}{\partial b_k} = -\sum_{j\geqslant 1} G_j W_{j+k} \quad \text{and} \quad X_{-k} = -\sum_{j\geqslant 1} G_{j+k} W_j. \tag{7.2}$$

Since $W_j = \sum_{p\geqslant 0} G_p \frac{\partial}{\partial G_{j+p}}$, we deduce

$$X_{-k} = -\sum_{j\geqslant 1, p\geqslant 0} G_{j+k} G_p \frac{\partial}{\partial G_{j+p}} = -\left[\sum_{0\leqslant i\leqslant r-1} G_{r-i+k} G_i\right] \frac{\partial}{\partial G_r}. \tag{7.3}$$

Proposition 7.2. *For $n, k \geqslant 1$, we have $X_0(F_n) = -nG_n$ and*

$$X_k(F_n) = -nG_{n-k} \times 1_{k\leqslant n} \quad \textit{and} \quad X_{-k}(F_n) = -nG_{n+k}. \tag{7.4}$$

Proof. From $W_j F_p = p\delta_{j,p}$. $\square$

Remark 7.1. In terms of the coordinates $(b_k)_{k\geqslant 1}$,

$$X_{-1} = -\left(b_2 - b_1^2\right)\frac{\partial}{\partial b_1} - (b_3 - b_1 b_2)\frac{\partial}{\partial b_2} - \cdots - (b_{n+1} - b_1 b_n)\frac{\partial}{\partial b_n} - \cdots,$$
$$X_{-2} = -\left(b_1^3 - 2b_1 b_2 + b_3\right)\frac{\partial}{\partial b_1} - \cdots - \left(b_1^2 b_n - b_2 b_n - b_1 b_{n+1} + b_{n+2}\right)\frac{\partial}{\partial b_n} - \cdots.$$

In terms of the coordinates $(G_k)_{k\geqslant 1}$,

$$X_0 = -G_1\frac{\partial}{\partial G_1} - \left(G_2 + G_1^2\right)\frac{\partial}{\partial G_2} + \left(G_n - K_n^2(G_1, G_2, \ldots)\right)\frac{\partial}{\partial G_n} + \cdots,$$
$$\begin{aligned} X_{-1} &= -G_2\frac{\partial}{\partial G_1} - (G_2G_1 + G_3)\frac{\partial}{\partial G_2} - \left(G_2^2 + G_3G_1 + G_4\right)\frac{\partial}{\partial G_3} \\ &\quad - (G_5 + 2G_2G_3 + G_1G_4)\frac{\partial}{\partial G_4} - \left(G_6 + 2G_4G_2 + G_1G_5 + G_3^2\right)\frac{\partial}{\partial G_5} - \cdots \\ &= \sum_{n\geqslant 1}\left(G_{n+1} + G_nG_1 - K_{n+1}^2(G_1, G_2, \ldots, G_n, G_{n+1})\right)\frac{\partial}{\partial G_n}. \end{aligned}$$

For $k \geqslant 2$, $X_{-k} = \sum_{n\geqslant 1} H_n \frac{\partial}{\partial G_n}$ with

$$H_n = G_{n+k} + G_{n+k-1}G_1 + G_{n+k-2}G_2 + \cdots + G_nG_k - K_{n+k}^2(G_1, G_2, \ldots, G_{n+k}).$$

From our main theorem, we see that the coefficient H_n is a sum of partial derivatives of Faber polynomials.

Lemma 7.3. *The condition $X_{-k}(F_n) = -nG_{n+k}$ for $n \geqslant 1$ and $k \geqslant 0$ determines the operators X_{-k} in a unique way. Consider differential operators $(\widetilde{X}_{-k})$, $k \geqslant 0$, of the form*

$$\widetilde{X}_{-k} = B_1^k\frac{\partial}{\partial b_1} + B_2^k\frac{\partial}{\partial b_2} + \cdots + B_n^k\frac{\partial}{\partial b_n} + \cdots \quad \textit{for } k \geqslant 0,$$

where the B_n^k are homogeneous polynomials in the variables $(b_1, b_2, \ldots, b_n, \ldots)$ of degree $n+k$ and such that $\widetilde{X}_{-k}[F_n] = -nG_{n+k}$ for $n \geqslant 1$, $k \geqslant 0$, then $\widetilde{X}_{-k} = X_{-k}$.

Moreover, $X_0[h(z)] = -h(z) + 1$, $X_{-1}[h(z)] = -\frac{h(z)}{z} + \frac{1}{z} + b_1 h(z)$,

$$X_{-2}\big[h(z)\big] = -\frac{h(z)}{z^2} + \frac{1}{z^2} - \frac{G_1 h(z)}{z} - G_2 h(z),$$
$$X_{-3}\big[h(z)\big] = -\frac{h(z)}{z^3} + \frac{1}{z^3} - \frac{G_1 h(z)}{z^2} - \frac{G_2 h(z)}{z} - G_3 h(z),$$
$$\ldots$$
$$X_{-j}\big[h(z)\big] = \frac{1}{z^j} - \sum_{0\leqslant k\leqslant j} \frac{G_k}{z^{j-k}} \times h(z).$$

Proof. Let $h(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots$. For X_0, the condition $X_0[F_n] = -nG_n$ for $n \geqslant 1$ implies that

$$X_0 \frac{h'}{h} = -\sum_{k\geqslant 1} X_0(F_k) z^{k-1} = \sum_{k\geqslant 1} kG_k z^{k-1} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{h(z)}\right).$$

Exchanging the order of derivation X_0 and $\frac{\mathrm{d}}{\mathrm{d}z}$, we have $\frac{\mathrm{d}}{\mathrm{d}z}\frac{X_0(h)}{h} = \frac{\mathrm{d}}{\mathrm{d}z}(\frac{1}{h})$. Integrating with respect to z gives $\frac{X_0(h)}{h} = \frac{1}{h} + \alpha$ where α is a constant. If we take $\alpha = -1$, then $X_0(h) = 1 - h$. To express X_0 in the (b_k) coordinates, we have

$$X_0\big[h(z)\big] = -b_1 z - b_2 z^2 - \cdots - b_n z^n - \cdots = -b_1 \frac{\partial}{\partial b_1} h(z) - b_2 \frac{\partial}{\partial b_2} h(z) - \cdots.$$

To get X_0 in terms of the $(G_k)_{k\geqslant 1}$ coordinates, we consider $\tilde{h}(z) = \frac{1}{h(z)}$. We have

$$X_0\big[\tilde{h}(z)\big] = X_0\left[\frac{1}{h(z)}\right] = \tilde{h}(z) - \tilde{h}(z)^2 = \sum_{n\geqslant 1}\big[G_n - K_n^2(G_1, \ldots, G_k, \ldots)\big] z^n.$$

Since $z^n = \frac{\partial}{\partial G_n}[\tilde{h}(z)]$, we obtain the result. For X_{-1}, the method is the same. From $X_{-1}(F_n) = -nG_{n+1}$ for $n \geqslant 1$, we deduce

$$X_{-1}\frac{h'}{h} = -\sum_{k\geqslant 1} X_{-1}(F_k) z^{k-1} = \sum_{k\geqslant 1} kG_{k+1} z^{k-1}$$
$$= \frac{1}{z}\sum_{k\geqslant 1} kG_k z^{k-1} - \frac{1}{z^2}\sum_{k\geqslant 1} G_k z^k = \frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{h(z)}\right) - \frac{1}{z^2}\left(\frac{1}{h(z)}\right) + \frac{1}{z^2}.$$

Exchanging the order of derivation X_{-1} and $\frac{\mathrm{d}}{\mathrm{d}z}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{X_{-1}(h)}{h} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{zh(z)} - \frac{1}{z}\right).$$

Integrating with respect to z gives $\frac{X_{-1}(h)}{h} = \frac{1}{zh(z)} - \frac{1}{z} + \text{constant}$. Taking the constant equal to b_1 gives X_{-1}. In the same way, $X_{-j}F_n = -nG_{n+j}$, for $n \geqslant 1$ implies that

$$X_{-j}\left(\frac{h'}{h}\right) = -\sum_{k\geqslant 1} W_{-j} F_k z^{k-1} = \sum_{k\geqslant 1} kG_{k+j} z^{k-1}$$
$$= \frac{1}{z^j}\sum_{k\geqslant 1}(k+j) G_{k+j} z^{k+j-1} - \frac{j}{z^{j+1}}\sum_{k\geqslant 1} G_{k+j} z^{k+j}$$

$$= \frac{d}{dz}\left(\frac{1}{z^j h(z)} - \frac{1}{z^j}\sum_{0\leqslant k\leqslant j} G_k z^k\right). \qquad \square$$

7.2. The operators $(M_k)_{k\in Z}$

We have $L_0 = \sum_{k\geqslant 1} k b_k \frac{\partial}{\partial b_k} = \sum_{j\geqslant 1} F_j W_j$.

Definition 7.2. For $k \geqslant 1$, let

$$M_k = \sum_{j\geqslant 1} F_j W_{j+k} \quad \text{and} \quad M_{-k} = \sum_{j\geqslant 1} F_{j+k} W_j. \tag{7.5}$$

Proposition 7.4. *For $k \geqslant 1$, $M_{-k}F_n = nF_{n+k}$, $M_k(F_n) = nF_{n-k} \times 1_{n\geqslant k}$. Moreover $M_kM_p - M_pM_k = (k-p)M_{k+p}$ for $p, k \in Z$.*

Proof. From $W_j(F_n) = n\delta_{n,j}$. For the last identity, we verify that $M_kM_pF_n - M_pM_kF_n = (k-p)M_{k+p}F_n$. $\square$

Lemma 7.5. *In terms of the coordinates $(b_k)_{k\geqslant 1}$, for $k \geqslant 1$,*

$$M_k = \sum_{j\geqslant 1} F_j W_{j+k} = b_1 \frac{\partial}{\partial b_{k+1}} + 2b_2 \frac{\partial}{b_{k+2}} + \cdots + pb_p \frac{\partial}{\partial b_{k+p}} + \cdots. \tag{7.6}$$

In particular if $h(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_p z^p + \cdots$, we have

$$M_k\big[h(z)\big] = z^{k+1}h'(z) \quad \textit{for } k \geqslant 1. \tag{7.7}$$

Proof. We verify the identity on $h(z)$. From $W_j[h(z)] = -z^j h(z)$. Thus

$$M_k\big[h(z)\big] = -z^k\left(\sum_{j\geqslant 1} F_j z^j\right) \times h(z) = z^{k+1}h'(z).$$

Since $(b_1 \frac{\partial}{\partial b_{k+1}} + 2b_2 \frac{\partial}{b_{k+2}} + \cdots + pb_p \frac{\partial}{\partial b_{k+p}} + \cdots)h(z) = z^{k+1}h'(z)$, we obtain (7.6). For $k \geqslant 0$, the operators $M_{-k} = \sum_{j\geqslant 1} F_{j+k} W_j$ are given by $M_0 = L_0$,

$$M_{-1} = \big(2b_2 - b_1^2\big)\frac{\partial}{\partial b_1} + (3b_3 - b_1b_2)\frac{\partial}{\partial b_2} + \cdots + \big((n+1)b_{n+1} - b_1b_n\big)\frac{\partial}{\partial b_n} + \cdots,$$

$$M_{-2} = \sum_{j\geqslant 1}\big[(j+2)b_{j+2} - b_{j+1}b_1 + b_j\big(b_1^2 - 2b_2\big)\big]\frac{\partial}{\partial b_j},$$

$$\ldots$$

$$M_{-k} = \sum_{j\geqslant 1}\big(b_jF_k + b_{j+1}F_{k-1} + \cdots + b_{j+k-1}F_1 + (j+k)b_{j+k}\big)\frac{\partial}{\partial b_j}. \qquad \square$$

Remark 7.2. On $\mathcal{M}$, define the differential operators

$$L_k = M_k - W_k \quad \text{for } k \geqslant 1. \tag{7.8}$$

With the convention $F_0 = -1$,

$$L_k = \sum_{j\geqslant 0} F_j W_{j+k} = F_0 W_k + F_1 W_{k+1} + F_2 W_{k+2} + \cdots \tag{7.9}$$

then $L_k = M_k - W_k$, $k \geqslant 1$, is the Kirillov operator

$$L_k = \frac{\partial}{\partial b_k} + \sum_{n\geqslant 1}(n+1) b_n \frac{\partial}{\partial b_{n+k}}. \tag{7.10}$$

For $f(z) = zh(z) = z + b_1 z^2 + b_2 z^3 + \cdots + b_p z^{p+1} + \cdots$, we have $L_k[f(z)] = z^{1+k} f'(z)$ and

$$L_k(F_n) = nF_{n-k} \times 1_{n\geqslant k}. \tag{7.11}$$

7.3. The operators $(V_j)_{j\geqslant 1}$ and $(V_j^k)_{j\geqslant 1}$

We do not stay anymore in the class of polynomials (F_n), (G_n). For $j \geqslant 1$ and $k \in Z$, see (1.9)–(1.10),

$$V_j = -\sum_{n\geqslant 0} K_n^{j+1} \frac{\partial}{\partial b_{n+j}} \quad \text{and} \quad V_j^k = -\sum_{n\geqslant 0} K_n^{k+1} \frac{\partial}{\partial b_{n+j}}. \tag{7.12}$$

The polynomials $(P_n^k)_{n\geqslant 0}$, see Proposition 2.4 and [1, (A.1.2)], are given by

$$\frac{zf'(z)}{f(z)} h(z)^k = 1 + \sum_{n\geqslant 1} P_n^k f(z)^n \tag{7.13}$$

where $f(z) = zh(z)$. We have the recursion formulas, for $q \in Z$,

$$(n+1)b_n = \sum_{0\leqslant j\leqslant n} P_j^q K_{n-j}^{j+1-q}, \qquad \frac{n+1}{k+q} K_n^{k+q} = \sum_{j=0}^{n} P_j^q K_{n-j}^{j+k}, \tag{7.14}$$

$$-F_n = \sum_{0\leqslant j\leqslant n} P_j^q K_{n-j}^{j-q}, \tag{7.15}$$

With (7.14), we replace $(n+1)b_n$ in (7.10). It gives for L_k, $k \geqslant 1$ (with $b_0 = 1$),

$$L_k = \sum_{0\leqslant n} \sum_{0\leqslant j\leqslant n} P_j^q K_{n-j}^{j+1-q} \frac{\partial}{\partial b_{n+k}} = \sum_{0\leqslant j} P_j^q \left[\sum_{n\geqslant j} K_{n-j}^{j+1-q} \frac{\partial}{\partial b_{n+k}}\right] = -\sum_{j\geqslant 0} P_j^q V_{j+k}^{j-q}.$$

For $q = -k$ and $k \geqslant 1$, we obtain

$$L_k = -\sum_{j\geqslant 0} P_j^{-k} V_{j+k} = -\sum_{j\geqslant 1} P_{j-k}^{-k} \times 1_{j\geqslant k} V_j. \tag{7.16}$$

Definition 7.3. For any $k \in Z$, with the convention $P_n^k = 0$ if $n < 0$, we put

$$L_k = -\sum_{j\geqslant 1} P_{j-k}^{-k} V_j. \tag{7.17}$$

7.4. The Kirillov operators $(L_{-p})_{p\geqslant 1}$

It has been proved in [1, (A.4.5)] that the vector fields $(L_{-p})_{p\geqslant 0}$ obtained by Kirillov in [7] are such that for $f(z)=zh(z)$, it holds

$$L_{-p}\big[f(z)\big]=\sum_{j\geqslant 0}P^p_{1+j+p}f(z)^{j+2}. \tag{7.18}$$

Proposition 7.6. *Let L_{-p}, $p\geqslant 1$ the operator defined by* (7.18), *then L_{-p} is given by* (7.17), *we have*

$$L_{-p}=-\sum_{j\geqslant 1}P^p_{j+p}V_j. \tag{7.19}$$

Proof. From (5.9), $V_j[f(z)]=-f(z)^{j+1}$. □

Remark 7.3. We have

$$L_{-p}=\sum_{r\geqslant 1}A^p_r\frac{\partial}{\partial b_r}\quad\text{with } A^p_r=\sum_{1\leqslant j\leqslant r}P^p_{j+p}K^{j+1}_{r-j}, \tag{7.20}$$

$$L_{-p}=-\sum_{r\geqslant 1}B^p_rW_r\quad\text{with } B^p_r=\sum_{1\leqslant j\leqslant r}P^p_{j+p}K^j_{r-j}. \tag{7.21}$$

Proof. From (7.18), $L_{-p}=\sum_{j\geqslant 0,n\geqslant 0}P^p_{1+j+p}K^{j+2}_n\frac{\partial}{\partial b_{n+j+1}}$. Thus

$$L_{-p}=\sum_{r\geqslant 0}A^p_{r+1}\frac{\partial}{\partial b_{r+1}}\quad\text{with } A^p_{r+1}=\sum_{0\leqslant j\leqslant r}P^p_{1+j+p}K^{j+2}_{r-j}.$$

This proves (7.20). We obtain (7.21) with (5.11),

$$\begin{aligned}L_{-p}&=-\sum_{j\geqslant 0,n\geqslant 0,k\geqslant 0}P^p_{1+j+p}K^{j+2}_nG_kW_{n+j+k+1}\\&=-\sum_{j\geqslant 0,s\geqslant 0}P^p_{1+j+p}K^{j+1}_sW_{j+s+1}.\qquad\square\end{aligned}$$

Remark 7.4. We have $L_{-k}=M_{-k}-Y_{-k}$ with

$$Y_{-k}=-\sum_{r\geqslant 1}J^k_rW_r\quad\text{and}\quad J^k_r=\sum_{s=0}^{k}P^k_sK^{s-k}_{r+k-s}. \tag{7.22}$$

In particular $L_{-1}=M_{-1}-X_{-1}$.

Proof. From (7.15), $M_{-k}=\sum_{r\geqslant 1}F_{j+k}W_j=-\sum_{r\geqslant 1}[\sum_{0\leqslant s\leqslant j+k}P^q_sK^{s-q}_{k+j-s}]W_j$

$$\sum_{0\leqslant s\leqslant j+k}P^q_sK^{s-q}_{k+j-s}=\sum_{0\leqslant s\leqslant k}P^q_sK^{s-q}_{k+j-s}+\sum_{1\leqslant s\leqslant j}P^q_{k+s}K^{k+s-q}_{j-s}.$$

With $k=q$, the second sum is J^k_j as in (7.21). The first sum gives Y_{-k}. □

Remark 7.5. With (1.11) and (5.7), we find for any $p \in Z$, $j \geqslant 1$,

$$L_{-p}V_j - V_j L_{-p} = \sum_{1\leqslant s\leqslant j} \big(V_j\big(P^p_{s+p}\big)\big)V_s. \tag{7.23}$$

8. Second order differential operators

Let $\triangle_0 = \sum_{p\geqslant 1, q\geqslant 1} F_{p+q}(W_{p+q} + W_p W_q)$.

Proposition 8.1. *Let $L_0 = \sum_{j\geqslant 1} F_j W_j$ be the homogeneity operator, then*

$$\triangle_0 F_n = n(n-1)F_n \quad \textit{and} \quad (\triangle_0 + L_0)F_n = n^2 F_n.$$

Proof. Because of (1.6), $W_p W_q F_n = 0$. On the other hand

$$\sum_{p\geqslant 1, q\geqslant 1} F_{p+q} W_{p+k} F_n = n \times \Bigg(\sum_{p\geqslant 1, q\geqslant 1} \delta_{p+q,n}\Bigg) F_n.$$

Then we remark that $(\sum_{p\geqslant 1, q\geqslant 1} \delta_{p+q,n}) = n-1$. □

Definition 8.1. We consider $\mathcal{Q}_j = \sum_{p\geqslant 1, q\geqslant 1, p+q=j} W_p W_q$ for $j \geqslant 2$.

Because of (1.8), $\triangle_0$ and $\mathcal{Q}_j$ are second order differential operators on the manifold $\mathcal{M}$. With the expression (1.8) of $W_{p+q} + W_p W_q$, we have

$$\frac{\partial}{\partial b_j}(W_p W_q + W_{p+q}) - (W_p W_q + W_{p+q})\frac{\partial}{\partial b_j} = W_p \frac{\partial}{\partial b_{q+j}} + W_q \frac{\partial}{\partial b_{j+p}}.$$

Since W_p and W_q commute, we have $\mathcal{Q}_2 = W_1^2 = K_2^2(W_1, 0)$, $\mathcal{Q}_3 = 2W_1 W_2 = K_3^2(W_1, W_2, 0)$, $\mathcal{Q}_4 = 2W_1 W_3 + W_2^2 = K_4^2(W_1, W_2, W_3, 0), \ldots, \mathcal{Q}_n = K_n^2(W_1, W_2, \ldots, W_{n-1}, 0)$ and $\mathcal{Q}_j W_p = W_p \mathcal{Q}_j$ for $j \geqslant 2$ and $p \geqslant 1$.

Since $W_{j-k} W_k G_n = W_{j-p} W_p G_n$, for $k \leqslant j$, $p \leqslant j$, we have $\mathcal{Q}_2 G_j = G_{j-2}$, $\mathcal{Q}_3 G_j = 2G_{j-3}, \ldots, \mathcal{Q}_n G_j = (n-1)G_{j-n}$.

The operator $\triangle_0$ decomposes into $\triangle_0 = \triangle_1 + \triangle_2$ with

$$\triangle_1 = \sum_{j\geqslant 2} F_j \mathcal{Q}_j,$$

$$\triangle_2 = \sum_{j\geqslant 2}\Bigg(\sum_{p\geqslant 1, q\geqslant 1, p+q=j} 1\Bigg) F_j W_j = \sum_{j\geqslant 2} (j-1) F_j W_j.$$

Since $W_j F_n = n\delta_{j,n}$, we have $\mathcal{Q}_j F_n = 0$, $j \geqslant 2$ and since $\mathcal{Q}_j(G_n) = (j-1)G_{n-j}$, we find

$$\triangle_1 F_n = 0,$$

$$\triangle_2 F_n = n(n-1)F_n,$$

and

$$\triangle_1 G_n = \triangle_2 G_n = \sum_{j\geqslant 2} (j-1) F_j G_{n-j}.$$

We deduce that

$$\triangle_1 \mathcal{Q}_j = \mathcal{Q}_j \triangle_1, \qquad W_j \triangle_1 - \triangle_1 W_j = j\mathcal{Q}_j,$$
$$\triangle_2 \mathcal{Q}_j = \mathcal{Q}_j \triangle_2, \qquad W_j \triangle_2 - \triangle_2 W_j = j(j-1)W_j,$$

and $\triangle_2 \triangle_1 - \triangle_1 \triangle_2 = \sum_{k\geqslant 2} k(k-1)F_k \mathcal{Q}_k$.

Lemma 8.2. *Let $X_0 = -\sum_{j\geqslant 1} G_j W_j$ and $L_0 = \sum_{j\geqslant 1} F_j W_j$, then*

$$X_0 G_n = G_n - K_n^2(G_1, G_2, \ldots, G_n), \qquad X_0 F_n = -nG_n, \tag{8.1}$$
$$X_0 L_0 = L_0 X_0, \tag{8.2}$$
$$L_0 \triangle_2 = \triangle_2 L_0 \quad \text{and} \quad L_0 \triangle_1 - \triangle_1 L_0 = \sum_{k\geqslant 2} kF_k \mathcal{Q}_k. \tag{8.3}$$

Proof. (8.1) results from the expression of X_0 in the $(G_n)_{n\geqslant 1}$ coordinates. Because of (1.6), we have $L_0 X_0 = \sum_{q\geqslant 1}[\sum_{j\geqslant 1} F_j G_{q-j}]W_q + \sum_{j\geqslant 1, q\geqslant 1} F_j G_q W_j W_q$ and

$$X_0 L_0 = \sum_{q\geqslant 1} qG_q W_q + \sum_{j\geqslant 1, q\geqslant 1} F_j G_q W_j W_q.$$

Since $\sum_{j\geqslant 1} F_j G_{q-j} = qG_q$, see (2.5), it proves that $X_0 L_0 = L_0 X_0$. The identities (8.3) are consequence of (1.6). □

9. The conformal map from the exterior of the unit disk onto the exterior of [−2, +2]

Let $\psi(w) = w + \frac{1}{w}$ be the conformal map from the exterior of the unit disk onto the exterior of $[-2, 2]$. The Faber polynomials $F_n(z)$ of $[-2, 2]$ are given by

$$\frac{w^2 - 1}{w^2 - wz + 1} = \sum_{n=0}^{\infty} F_n(z) w^{-n}.$$

They satisfy the differential equation

$$(z^2 - 4)F_n''(z) + zF_n'(z) = n^2 F_n(z). \tag{9.1}$$

In the following, we consider Faber polynomials $F_n(b_1, b_2, 0, 0, \ldots, 0)$. All the b_j are zero except b_1 and b_2. We have $F_1(b_1) = -b_1$, $F_2(b_1, b_2) = b_1^2 - 2b_2$, $F_3(b_1, b_2, 0) = -b_1^3 + 3b_1 b_2$, $F_4(b_1, b_2, 0, 0) = b_1^4 - 4b_1^2 b_2 + 2b_2^2, \ldots$.

Theorem 9.1. *Faber polynomials associated to $\psi(w) = w + b_1 + \frac{b_2}{w}$ verify*

$$\big((z - b_1)^2 - 4b_2\big)F_n''(z) + (z - b_1)F_n'(z) = n^2 F_n(z). \tag{9.2}$$

In particular, if $b_1 = 0$ and $b_2 = 1$, we obtain (9.1).

To prove the theorem, we need the following lemmas.

Lemma 9.2. *Let $L = \frac{\partial^2}{\partial^2 b_1} + \frac{\partial}{\partial b_2} + \sum_{k\geqslant 1} b_k \frac{\partial^2}{\partial b_2 \partial b_k}$, then $LF_n = 0$.*

Proof. From (6.15). □

Lemma 9.3. *Consider* $\triangle_0 = \sum_{p\geqslant 1, q\geqslant 1} F_{p+q}(W_{p+q} + W_p W_q)$ *and let* $\phi(b_1, b_2)$ *be a function defined on* $\mathcal{M}$*, which depends only of* b_1, b_2*, then*

$$\triangle_0\phi = \left[\left(b_1^2 - 2b_2\right)\frac{\partial^2}{\partial b_1^2} + 2b_1 b_2 \frac{\partial^2}{\partial b_1 \partial b_2} + 2b_2^2 \frac{\partial^2}{\partial b_2^2}\right]\phi.$$

Proof. Let ϕ depend only on the variables b_1 and b_2. If $p > 2$ or $q > 2$, we have $[W_{p+q} + W_p W_q]\phi = 0$. If $p = 2$, $q = 2$, then $(W_4 + W_2^2)\phi = W_2^2\phi = \frac{\partial^2}{\partial b_2^2}\phi$. If $p = 2$, $q = 1$ or $p = 1$, $q = 2$, $(W_3 + W_2 W_1)\phi = \frac{\partial}{\partial b_2}(\frac{\partial}{\partial b_1} + b_1\frac{\partial}{\partial b_2})\phi$. If $p = 1$, $q = 1$, then $(W_2 + W_1^2)\phi = [-\frac{\partial}{\partial b_2} + (\frac{\partial}{\partial b_1} + b_1\frac{\partial}{\partial b_2})^2]\phi$.

We calculate $\triangle_0\phi = [F_2(W_2 + W_1^2) + 2F_3(W_3 + W_2 W_1) + F_4(W_4 + W_2^2)]\phi$. This gives

$$\triangle_0\phi = \left[F_2\left(-\frac{\partial}{\partial b_2} + \left(\frac{\partial}{\partial b_1} + b_1\frac{\partial}{\partial b_2}\right)^2\right) + 2F_3\left(\frac{\partial}{\partial b_2}\left(\frac{\partial}{\partial b_1} + b_1\frac{\partial}{\partial b_2}\right) + F_4\frac{\partial^2}{\partial b_2^2}\right]\phi$$

or equivalently

$$\triangle_0\phi = \left[F_2\left(\frac{\partial^2}{\partial b_1^2} + 2b_1\frac{\partial^2}{\partial b_1\partial b_2} + b_1^2\frac{\partial^2}{\partial b_2^2}\right) + 2F_3\left(\frac{\partial^2}{\partial b_1\partial b_2} + b_1\frac{\partial^2}{\partial b_2^2}\right) + F_4\frac{\partial^2}{\partial b_2^2}\right]\phi.$$

Replacing F_2, F_3, F_4, we find Lemma 9.3. □

Proof of the theorem. From Lemma 9.2, we know that

$$\left(2b_1 b_2\frac{\partial^2}{\partial b_1\partial b_2} + 2b_2^2\frac{\partial^2}{\partial b_2^2}\right)F_n = \left(-2b_2\frac{\partial}{\partial b_2} - 2b_2\frac{\partial^2}{\partial b_1^2}\right)F_n.$$

We replace the right hand side in the expression of $\triangle_0$ and we find

$$\left[\left(b_1^2 - 4b_2\right)\frac{\partial^2}{\partial b_1^2} - 2b_2\frac{\partial}{\partial b_2}\right]F_n = n(n-1)F_n. \tag{9.3}$$

Since F_n is homogeneous, $b_1\frac{\partial}{\partial b_1}F_n + 2b_2\frac{\partial}{\partial b_2}F_n = nF_n$. Replacing $-2b_2\frac{\partial}{\partial b_2}F_n$ in (9.2), we find $(b_1^2 - 4b_2)\frac{\partial^2 F_n}{\partial b_1^2} + b_1\frac{\partial F_n}{\partial b_1} = n^2 F_n$. □

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