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A time-varying generalization of the canonical factorization theorem for Toeplitz operators

Dedicated to J. Korevaar on the occasion of his 70th birthday, with respect and admiration

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ABSTRACT

The canonical factorization theorem for the symbol of a Toeplitz operator is generalized to a class of non-Toeplitz operators. The operators in this class may be described as input-output operators of time-varying linear systems. Dichotomy of difference equations plays an important role.

0. INTRODUCTION

Let $T_\Phi = [\Phi_{k-j}]_{k,j=0}^{\infty}$ be a block Toeplitz operator with $m \times m$ matrix symbol

$$(0.1) \qquad \Phi(\zeta) = \sum_{\nu=-\infty}^{\infty} \zeta^\nu \Phi_\nu, \quad |\zeta| = 1.$$

The Fourier series expansion in the right-hand side of (0.1) is assumed to be absolutely convergent. We consider T_Φ as an operator on l_m^2, the Hilbert space of norm square summable sequences with entries in $\mathbb{C}^m$. It is well known (see [GKr1]; also [GF]) that T_Φ is invertible if and only if Φ admits a right canonical factorization relative to the unit circle, that is, Φ factorizes as

$$(0.2) \qquad \Phi(\zeta) = \Phi_-(\zeta)\,\Phi_+(\zeta), \quad \zeta \in \mathbb{T},$$

where Φ_+ and $\tilde{\Phi}_-$, $\tilde{\Phi}_-(\zeta) = \Phi_-(\zeta^{-1})$, are $m \times m$ matrix functions which are analytic on the open unit disc $\mathbb{D}$, continuous $\mathbb{D} \cup \mathbb{T}$, and their determinants do not vanish on $\mathbb{D} \cup \mathbb{T}$. In this case,

$$T_\Phi = T_{\Phi_-} T_{\Phi_+},$$

which is a block upper-lower factorization of T. In this paper we study a generalization of such a factorization for a class of non-Toeplitz operators.

The generalization which we have in mind originates from the factorization theory for Toeplitz operators with a rational matrix symbol. Let Φ be such a matrix symbol, i.e., the entries of Φ are quotients of scalar polynomials. Then one may use realization theorems from mathematical systems theory (see [K]) to show that Φ admits a representation of the form

$$\Phi(\zeta) = I + C(\zeta G - A)^{-1} B, \quad \zeta \in \mathbb{T}, \tag{0.3}$$

where A and G are square matrices of which the order r may be larger than the order m of Φ, the pencil $\zeta G - A$ is regular on the unit circle $|\zeta| = 1$, i.e., $\det(\zeta G - A) \neq 0$ for $|\zeta| = 1$, and the matrices B and C have sizes $r \times m$ and $m \times r$, respectively (see [GK], Theorem 3.1). The factorization theorem for block Toeplitz operators mentioned above can be reformulated in terms of the representation (0.3). In fact, the following theorem holds (see Section 5 in [GK]).

Theorem 0.1. *Let Φ be a rational $m \times m$ matrix function given by* (0.3). *Put $A^\times = A - BC$. Then Φ admits a right canonical factorization relative to $\mathbb{T}$ if and only if the following two conditions hold*:

(i) $\det(\zeta G - A^\times) \neq 0$ *for* $|\zeta| = 1$,

(ii) $\mathbb{C}^r = \operatorname{Im} Q \oplus \operatorname{Ker} Q^\times$ *and* $\mathbb{C}^r = \operatorname{Im} P \oplus \operatorname{Ker} P^\times$.

Here r is the order of the matrices G and A, and

$$\begin{cases} Q = \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{T}} (\zeta G - A)^{-1} G \, d\zeta, & P = \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{T}} G(\zeta G - A)^{-1} \, d\zeta, \\ Q^\times = \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{T}} (\zeta G - A^\times)^{-1} G \, d\zeta, & P^\times = \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{T}} G(\zeta G - A^\times)^{-1} \, d\zeta. \end{cases} \tag{0.4}$$

In that case a right canonical factorization $\Phi(\zeta) = \Phi_-(\zeta)\,\Phi_+(\zeta)$ of Φ relative to $\mathbb{T}$ is obtained by taking

$$\Phi_-(\zeta) = I + C(\zeta G - A)^{-1} (I - \rho) B, \quad \zeta \in \mathbb{T},$$

$$\Phi_+(\zeta) = I + C\tau(\zeta G - A)^{-1} B, \quad \zeta \in \mathbb{T},$$

$$\Phi_-(\zeta)^{-1} = I - C(I - \tau)(\zeta G - A^\times)^{-1} B, \quad \zeta \in \mathbb{T},$$

$$\Phi_+(\zeta)^{-1} = I - C(\zeta G - A^\times)^{-1} \rho B, \quad \zeta \in \mathbb{T}.$$

Here τ is the projection of $\mathbb{C}^r$ along $\operatorname{Im} Q$ *onto* $\operatorname{Ker} Q^\times$ *and ρ is the projection along* $\operatorname{Im} P$ *onto* $\operatorname{Ker} P^\times$. *Furthermore, the two equalities in* (ii) *are not independent*; *in fact, the first equality in* (ii) *implies the second and conversely.*

Now let us remark that the representation (0.3) allows us to view the corresponding block Toeplitz operator T_Φ as the input-output operator of the following discrete time system:

$$(0.5)\qquad \begin{cases} Ax_{k+1} = Gx_k + Bu_k & (k = 0, 1, \ldots) \\ y_k = -Cx_{k+1} + u_k & (k = 0, 1, \ldots) \\ x_0 \in \operatorname{Im} Q \end{cases}$$

where Q is the generalized Riesz projection appearing in (0.4). Such a representation appears in [GK]. The above fact gives a hint for the class of non-Toeplitz operators that will be considered. To be more specific, we shall deal here with non-Toeplitz operators $T = [T_{ij}]_{i,j=0}^{\infty}$ that appear as input-output operators of time-varying discrete time systems. The role of the projections Q, P, $P^{\times}$ and $Q^{\times}$ in Theorem 0.3 is taken over by dichotomies for certain difference equations.

This paper consists of three sections (not counting the present introduction). In the first section we recall the notion of a dichotomy and some of its properties. The second section gives an intrinsic characterization of the class of operators that we are dealing with. The time-varying analogue of Theorem 0.3 appears in Section 3. Generalizations to input-output operators of time-varying systems of other aspects of the theory of Toeplitz operators (like invertibility and Fredholm properties) will be the topic of a different publication (see [BGK2]).

1. PRELIMINARIES ABOUT DICHOTOMY

We begin by defining the notion of a dichotomy. Let a system

$$(1.1)\qquad A_{k+1} x_{k+1} = G_k x_k \quad (k = 0, 1, \ldots),$$

be given, where $(A_{k+1})_{k=0}^{\infty}$ and $(G_k)_{k=0}^{\infty}$ are bounded sequences of $r \times r$ matrices. We consider bounded sequences of projections $(I - Q_k)_{k=0}^{\infty}$ in $\mathbb{C}^r$ satisfying the conditions:

$$(1.2)\qquad \operatorname{rank} Q_k \text{ is constant} \quad (k = 0, 1, \ldots),$$

$$(1.3)\qquad G_k(\operatorname{Ker} Q_k) \subset A_{k+1}(\operatorname{Ker} Q_{k+1}), \qquad A_{k+1}(\operatorname{Im} Q_{k+1}) \subset G_k(\operatorname{Im} Q_k)$$

and

$$(1.4)\qquad A_{k+1}(\operatorname{Ker} Q_{k+1}) \oplus G_k(\operatorname{Im} Q_k) = \mathbb{C}^r \quad (k = 0, 1, \ldots).$$

Since the rank of the sequence of projections $(I - Q_k)_{k=0}^{\infty}$ is constant, it follows from the latter direct sum condition that the mappings

$$(1.5a)\qquad G_k\big|_{\operatorname{Im} Q_k} : \operatorname{Im} Q_k \to \operatorname{Im}\,(G_k\big|_{\operatorname{Im} Q_k}) \quad (k = 0, 1, \ldots).$$

$$(1.5b)\qquad A_{k+1}\big|_{\operatorname{Ker} Q_{k+1}} : \operatorname{Ker} Q_{k+1} \to \operatorname{Im}(A_{k+1}\big|_{\operatorname{Ker} Q_{k+1}}) \quad (k = 0, 1, \ldots)$$

are invertible. Using the inclusions in (1.3) we can therefore define the following *forward and backward evolution operators* $\Lambda_{n,k}^{+}$ *and* $\Lambda_{k,n}^{-}$ $(k = 0, 1, \ldots;$ $n = k, k+1, \ldots)$ *associated with the dichotomy* $(I - Q_k)_{k=0}^{\infty}$ via

$$(1.6a)\qquad \Lambda_{n,k}^{+} = (A_n\,|_{\operatorname{Ker} Q_n})^{-1} G_{n-1} \cdots (A_{k+1}\,|_{\operatorname{Ker} Q_{k+1}})^{-1} G_k\,|_{\operatorname{Ker} Q_k} \quad (n > k),$$

(1.6b) $\quad \Lambda^-_{k,n} = (G_k\,|_{\operatorname{Im} Q_k})^{-1} A_{k+1} \cdots (G_{n-1}\,|_{\operatorname{Im} Q_{n-1}})^{-1} A_n\,|_{\operatorname{Im} Q_n} \quad (n > k),$

and

(1.6c) $\quad \Lambda^+_{k,k} = I\,|_{\operatorname{Ker} Q_k} \quad \text{and} \quad \Lambda^-_{k,k} = I\,|_{\operatorname{Im} Q_k} \quad (k = 0, 1, \ldots).$

Note that

(1.7a) $\quad \Lambda^+_{n,k} : \operatorname{Ker} Q_k \to \operatorname{Ker} Q_n, \qquad \Lambda^-_{k,n} : \operatorname{Im} Q_n \to \operatorname{Im} Q_k,$

and

(1.7b) $\quad \Lambda^+_{t,n}\Lambda^+_{n,k} = \Lambda^+_{t,k}, \quad \Lambda^-_{k,n}\Lambda^-_{n,t} = \Lambda^-_{k,t} \quad (t \geq n \geq k \geq 0).$

Let us remark that by the direct sum condition (1.4) there exists a sequence of projections $(P_k)_{k=0}^{\infty}$ in $\mathbb{C}^r$ such that

(1.8) $\quad \operatorname{Im} P_k = \operatorname{Im}(G_k\,|_{\operatorname{Im} Q_k}), \qquad \operatorname{Ker} P_k = \operatorname{Im}(A_{k+1}\,|_{\operatorname{Ker} Q_{k+1}}).$

We call $(P_k)_{k=0}^{\infty}$ the *dual sequence* of the sequence $(I - Q_k)_{k=0}^{\infty}$.

A bounded sequence of projections $(I - Q_k)_{k=0}^{\infty}$ with the properties (1.2)–(1.4) above is called a *dichotomy* for the system (1.1) if there exist positive constants a and M, with $a < 1$, such that

(1.9) $\quad \|\Lambda^+_{n,k}\| \leq Ma^{n-k}, \quad \|\Lambda^-_{k,n}\| \leq Ma^{n-k} \quad (k = 0, 1, \ldots;\ n = k, k+1, \ldots),$

and if the following three inequalities hold

(1.10) $\quad \sup_{k=0,1,\ldots} \|(A_{k+1}\,|_{\operatorname{Ker} Q_{k+1}})^{-1}\| < \infty, \qquad \sup_{k=0,1,\ldots} \|(G_k\,|_{\operatorname{Im} Q_k})^{-1}\| < \infty,$

(1.11) $\quad \sup_{k=0,1,\ldots} \|P_k\| < \infty,$

where $(P_k)_{k=0}^{\infty}$ is the dual sequence of projections defined by (1.8).

Let $(I - Q_k)_{k=0}^{\infty}$ be a dichotomy for the system (1.8). The number $\operatorname{rank}(I - Q_k)$ $(k = 0, 1, \ldots)$ which is independent of k is called the *rank of the dichotomy*. Note that the invertibility of the mappings in (1.5a)–(1.5b) and the definition (1.8) of the dual sequence $(P_k)_{k=0}^{\infty}$ show that rank P_k is equal to the rank of the dichotomy for each $k = 0, 1, \ldots$. We also call the first projection Q_0 the *initial projection* of the dichotomy $(I - Q_k)_{k=0}^{\infty}$. This definition of dichotomy appears in [BG] and [BGK1], where it is called normal dichotomy.

Let us mention two special cases that are of particular interest. We say that the system (1.1) is *dichotomically regular* if

(1.12a) $\quad A_{k+1} = I, \quad Q_k = 0 \quad (k = 0, 1, \ldots)$

(1.12b) $\quad \overline{\lim_{j\to\infty}} \left(\sup_{k=0,1,\ldots} \|G_{k+j-1} \cdots G_k\|^{1/j} \right) < 1.$

In this case, conditions (1.2)–(1.4) and the first inequality of (1.10) are fulfilled trivially, the second inequalities in (1.9) and (1.10) are vacuous, and the first inequality in (1.9) is equivalent with (1.12b). Finally, $P_k = 0$ $(k = 0, 1, \ldots)$ in this case, and hence (1.11) is also satisfied. Thus, a dichotomically regular system has the dichotomy $(I - Q_k)_{k=0}^{\infty}$ with $Q_k = 0$ for each $k = 0, 1, \ldots$.

We say that the system (2.1) is *dichotomically coregular* if

(1.13a) $G_k = I, \quad Q_k = I \quad (k = 0, 1, \ldots),$

(1.13b) $\overline{\lim_{j \to \infty}} \left(\sup_{k=0,1,\ldots} \|A_{k+1} \cdots A_{k+j}\|^{1/j} \right) < 1.$

Also in this case, conditions (1.2)–(1.4) and the second inequality of (1.10) are fulfilled trivially, the first inequalities in (1.9) and (1.10) are vacuous, and the second inequality in (1.9) is equivalent with (1.13b). Finally, $P_k = I$ $(k = 0, 1, \ldots)$, and hence (1.11) is also satisfied. In particular, a dichotomically coregular system has the dichotomy $(I - Q_k)_{k=0}^{\infty}$ with $Q_k = I$ $(k = 0, 1, \ldots)$.

A system may have different dichotomies. Theorems 1.1 and 1.2 below (which appear, respectively, as Corollary 6.5 in [BG] and as part of Theorem 1.2 in [BGK1]) describe the freedom one has in the choice of the dichotomies.

Theorem 1.1. *If the system* (1.1) *admits a dichotomy* $(I - Q_k)_{k=0}^{\infty}$, *then for* $k = 0, 1, \ldots$

$$\operatorname{Ker} Q_k = \{x_k \in \mathbb{C}^r : \exists x_{k+1}, x_{k+2}, \ldots \text{ in } \mathbb{C}^r \text{ such that } A_{n+1}x_{n+1} = G_n x_n \ (n \geq k) \text{ and } \lim_{n \to \infty} x_n = 0\}.$$

In particular, $\operatorname{Ker} Q_k$ *and* $\operatorname{Ker} P_k = \operatorname{Im}(A_{k+1}|_{\operatorname{Ker} Q_{k+1}})$ *are uniquely defined, and all the dichotomies of* (1.1) *have the same rank.*

Theorem 1.2. *If the system* (1.1) *admits a dichotomy* $(I - Q_k)_{k=0}^{\infty}$, *then for each subspace* L *of* $\mathbb{C}^r$ *with* $L \oplus \operatorname{Ker} Q_0 = \mathbb{C}^r$, *there exists a unique dichotomy* $(I - \tilde{Q}_k)_{k=0}^{\infty}$, *of* (1.1) *with* $\operatorname{Im} \tilde{Q}_0 = L$. *Furthermore, all the dichotomies of* (1.1) *are obtained in this way.*

It will be convenient to consider two types of operations on systems of the form (1.1). Consider a second system

(1.14) $\tilde{A}_{k+1}x_{k+1} = \tilde{G}_k x_k, \quad k = 0, 1, \ldots,$

where $(\tilde{A}_{k+1})_{k=0}^{\infty}$ and $(\tilde{G}_k)_{k=0}^{\infty}$ are bounded sequences of $\tilde{r} \times \tilde{r}$ matrices. The systems (1.1) and (1.14) are said to be *equivalent* if $r = \tilde{r}$ and there exist invertible $r \times r$ matrices E_k and F_k, $k = 0, 1, \ldots$, such that

(1.15a) $\sup_{k \geq 0} \{\|E_k^{\pm 1}\|, \|F_k^{\pm 1}\|\} < \infty,$

(1.15b) $\tilde{A}_{k+1} = F_k^{-1} A_{k+1} E_{k+1}, \quad \tilde{G}_k = F_k^{-1} G_k E_k \quad (k = 0, 1, \ldots).$

In this case a sequence of projections $(I - Q_k)_{k=0}^{\infty}$ is a dichotomy of (1.1) if and only if $(I - E_k^{-1} Q_k E_k)_{k=0}^{\infty}$ is a dichotomy of (1.14).

The second operation is that of forming direct sums. By definition, the *direct sum* of the systems (1.1) and (1.14) is the system

$$(1.16)\quad \begin{bmatrix} A_{k+1} & 0 \\ 0 & \tilde{A}_{k+1} \end{bmatrix} x_{k+1} = \begin{bmatrix} G_k & 0 \\ 0 & \tilde{G}_k \end{bmatrix} x_k, \quad k = 0, 1, \ldots.$$

If $(I - Q_k)_{k=0}^{\infty}$ is a dichotomy of (1.1) and $(I - \tilde{Q}_k)_{k=0}^{\infty}$ is a dichotomy of (1.14), then it is straightforward to check that the sequence of projections $(I - \Pi_k)_{k=0}^{\infty}$ where

$$\Pi_k = \begin{pmatrix} Q_k & 0 \\ 0 & \tilde{Q}_k \end{pmatrix} \quad k = 0, 1, \ldots,$$

is a dichotomy of the direct sum (1.16). The next theorem is proved in [BGK2].

Theorem 1.3. *In order that the system* (1.1) *has a dichotomy it is necessary and sufficient that* (1.1) *is equivalent to a direct sum of a dichotomically regular and a dichotomically coregular system.*

We conclude this section with some relations with operator theory. Consider the system (1.1), and let L be a subspace of $\mathbb{C}^r$. By l_r^2 we denote the Hilbert space of all norm square summable sequences with entries in $\mathbb{C}^r$, and

$$(1.17)\quad l_{r,L}^2 = \{(x_0, x_1, \ldots) \in l_r^2 \mid x_0 \in L\}.$$

We define two operators as follows:

$$(1.18)\quad G : l_{r,L}^2 \to l_r^2, \quad G(x_0, x_1, \ldots) = (G_0 x_0, G_1 x_1, \ldots),$$

$$(1.19)\quad A : l_{r,L}^2 \to l_r^2, \quad A(x_0, x_1, \ldots) = (A_1 x_1, A_2 x_2, \ldots).$$

The following result is contained in Theorem 1.1 and Proposition 2.3 of [BGK1].

Theorem 1.4. *Let A and G be as* (1.18)–(1.19), *respectively. Then the operator $G - A$ is invertible if and only if the system* (1.1) *admits a unique dichotomy $(I - Q_k)_{k=0}^{\infty}$ with* Im $Q_0 = L$. *Moreover, $\zeta G - A$ is invertible for each ζ on the unit circle $\mathbb{T}$ if and only if it is invertible for one ζ on the unit circle, and in this case*

$$\frac{1}{2\pi i} \int_{\mathbb{T}} (\zeta G - A)^{-1} G \, d\zeta = \operatorname{diag}(I\,|_L, Q_1, Q_2, \ldots),$$

and

$$\frac{1}{2\pi i} \int_{\mathbb{T}} G(\zeta G - A)^{-1} \, d\zeta = \operatorname{diag}(P_0, P_1, \ldots),$$

where $(I - Q_k)_{k=0}^{\infty}$ is the unique dichotomy of (1.1) *with* Im $Q_0 = L$, *and $(P_k)_{k=0}^{\infty}$ is its dual sequence of projections.*

The next result gives an interpretation of the dichotomy in the time invariant case, and appears as Theorem 3.3 in [BGK2].

Lemma 1.5. *Let A and G be $r \times r$ matrices. Then the system $Ax_{k+1} = Gx_k$ $(k = 0, 1, \ldots)$ admits a dichotomy if and only if $\zeta G - A$ is invertible for $|\zeta| = 1$. In this case, there exists a unique time invariant dichotomy $I - Q_k = I - Q$, where*

$$Q = \frac{1}{2\pi i} \int_{\mathbb{T}} (\zeta G - A)^{-1} G \, \mathrm{d}\zeta,$$

and the dual sequence is given by $P_k = P$ $(k = 0, 1, \ldots)$, where

$$P = \frac{1}{2\pi i} \int_{\mathbb{T}} G(\zeta G - A)^{-1} \, \mathrm{d}\zeta.$$

2. REALIZATION THEOREM

In this paper we are interested in operators that appear as input-output operators of an input-output system. The input-output systems that we have in mind are singular time-varying systems of the form

$$\Sigma \begin{cases} A_{k+1}x_{k+1} = G_k x_k + B_k u_k & (k = 0, 1, \ldots) \\ y_k = -C_{k+1}x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_o \in L. \end{cases}$$

Here, $(G_k)_{k=0}^{\infty}$ and $(A_{k+1})_{k=0}^{\infty}$, $(B_k)_{k=0}^{\infty}$ and $(C_{k+1})_{k=0}^{\infty}$ are bounded sequences of matrices of sizes $r \times r$, $r \times r$, $r \times m$, $m \times r$, respectively, and we assume that

$$(2.1) \qquad A_{k+1}x_{k+1} = G_k x_k \quad (k = 0, 1, \ldots),$$

has a dichotomy $(I - Q_k)_{k=0}^{\infty}$ with $\operatorname{Im} Q_0 = L$.

Choose an input sequence $(u_0, u_1, \ldots)$ in l_m^2. Then, by Theorem 1.4, the first equation in Σ has a unique solution $(x_0, x_1, \ldots) \in l_{r,L}^2$. Inserting the latter sequence into the second equation in Σ yields an output sequence $(y_0, y_1, \ldots) \in l_m^2$, which is uniquely determined by $(u_0, u_1, \ldots)$. It follows that Σ has a well defined input-output operator, denoted by T_Σ, which acts as a bounded linear operator on l_m^2. The latter statement also follows from Theorem 1.4 and from the fact that the sequences $(B_k)_{k=0}^{\infty}$ and $(C_{k+1})_{k=0}^{\infty}$ are bounded.

As usual for operators on l_m^2, we represent T_Σ by an infinite block matrix $T_\Sigma = [\tau_{ij}]_{ij=0}^{\infty}$, where each τ_{ij} is an $m \times m$ matrix. A straightforward application of Theorem 1.1 in [BGK1] (see also formula (4.6) of [BGK2]) shows that in this case

$$(2.2) \qquad \tau_{ij} = \begin{cases} \delta_{i,j} I_m - C_{i+1} \Lambda_{i+1,j+1}^{+} (A_{j+1} |_{\operatorname{Ker} Q_{j+1}})^{-1} (I_r - P_j) B_j & (i \geq j), \\ C_{i+1} \Lambda_{i+1,j}^{-} (G_j |_{\operatorname{Im} Q_j})^{-1} P_j B_j & (i < j). \end{cases}$$

Here $(P_k)_{k=0}^{\infty}$ is the dual sequence of projections of the dichotomy $(I - Q_k)_{k=0}^{\infty}$ and $\Lambda_{i+1,j+1}^{+}$ and $\Lambda_{i+1,j}^{-}$ denote the forward and backward evolution operators. Since the sequence $(I - Q_k)_{k=0}^{\infty}$ is a dichotomy, we can use the boundedness of the sequences $(B_k)_{k=0}^{\infty}$ and $(C_{k+1})_{k=0}^{\infty}$ and the estimates (1.9)–(1.11) to show that there exist constants $M > 0$, $0 < a < 1$, such that

$$(2.3) \qquad \|\tau_{ij}\| \leq M a^{|i-j|} \quad (i, j = 0, 1, \ldots).$$

In the next three sections, we study the invertibility of the operator T_Σ, its Fredholm properties and its UL-factorizations. In the present section we characterize the class of operators T on l_m^2 that appears as input-output operators T_Σ of the type considered here.

Consider a bounded linear operator

$$(2.4) \qquad T = [t_{ij}]_{ij=0}^{\infty} : l_m^2 \to l_m^2.$$

We say that T admits a *realization* if $T = T_\Sigma$ for some input-output system Σ of the form decribed in the first paragraph of this section.

The following result appears as Theorem 4.4 in [BGK2].

Theorem 2.1. *Let $T = [t_{ij}]_{ij=0}^{\infty}$ be a bounded linear operator in l_m^2. Then T admits a realization if and only if*

$$(2.5) \qquad \|t_{ij}\| \le M a^{|i-j|} \quad (i, j = 0, 1, \ldots)$$

for some positive constants M, a with $a < 1$, and

$$(2.6) \qquad \sup_{\nu = 0,1,\ldots} \{\operatorname{rank} H_\nu^-, \operatorname{rank} H_\nu^+\} < \infty,$$

where

$$H_\nu^- = \begin{bmatrix} t_{\nu 0} & t_{\nu 1} & \cdots & t_{\nu\nu} \\ t_{\nu+1,0} & t_{\nu+1,1} & \cdots & t_{\nu+1,\nu} \\ \vdots & \vdots & & \vdots \end{bmatrix}, \quad (\nu = 0, 1, \ldots),$$

$$H_\nu^+ = \begin{bmatrix} t_{0,\nu} & t_{0,\nu+1} & \cdots \\ \vdots & \vdots & \\ t_{\nu,\nu} & t_{\nu,\nu+1} & \cdots \end{bmatrix}, \quad (\nu = 0, 1, \ldots).$$

This theorem may be viewed as an operator version of the realization theorems in Section 5 of [GKLe], which are algebraic in nature and concern lower triangular block matrices which do not have to be related to bounded operators on an l_2-space.

Finally, we make one remark about band operators. An operator $T = [t_{ij}]_{ij=0}^{\infty}$ in l_m^2 is called a *band operator* if there exists a positive integer N such that $t_{ij} = 0$ whenever $|i - j| > N$. Theorem 2.1 shows that each band operator admits a realization.

3. CANONICAL FACTORIZATION

Let $T = [t_{ij}]_{ij=0}^{\infty}$ be a bounded operator in l_m^2 in its standard matrix representation where t_{ij} are $m \times m$ matrices. The operator T is upper triangular (respectively lower triangular) if $t_{ij} = 0$ whenever $i > j$ (respectively $i < j$). We say that T is diagonal if $t_{ij} = 0$ whenever $i \neq j$. Let us remark that if T is upper (respectively lower) triangular and invertible, then T^{-1} is also upper (respectively lower) triangular. We say that T admits a *canonical upper lower factorization* if there exist an invertible upper triangular operator T_- and an invertible lower triangular operator T_+, such that $T = T_-T_+$. We refer to [GF] and [GKr2] for canonical factorizations of operators and functions.

A necessary condition for T to admit canonical factorization is that T is invertible. This condition is not sufficient in general.

It is easily seen that if T admits two canonical upper lower factorizations $T = T_-T_+ = T'_-T'_+$, then there exists an invertible diagonal operator D such that $T'_- = T_-D$ and $T'_+ = D^{-1}T_+$. Conversely, if $T = T_-T_+$ is a canonical upper lower factorization for T, then defining T'_- and T'_+ by the above formulas we obtain another canonical factorization.

Assume that as in the introduction, $T = T_\Phi = [\Phi_{i-j}]_{ij=0}^\infty$ is a Toeplitz operator, where Φ_k are $m \times m$ matrices with $\sum_{k=-\infty}^\infty \|\Phi_k\| < \infty$. Let $\Phi(\zeta) = \sum_{k=-\infty}^\infty \Phi_k\zeta^k$ ($\zeta \in \mathbb{T}$) be the symbol of T. Assume also that Φ admits a right canonical factorization $\Phi(\zeta) = \Phi_-(\zeta)\Phi_+(\zeta)$, where Φ_+ and $\tilde{\Phi}_-$, $\tilde{\Phi}_-(\zeta) = \Phi_-(\zeta^{-1})$, are $m \times m$ matrix functions which are analytic in the open unit disc $\mathbb{D}$, continuous on $\mathbb{D} \cup \mathbb{T}$ and their determinants do not vanish on $\mathbb{D} \cup \mathbb{T}$. Put $\Phi_+(\zeta) = \sum_{k=0}^\infty \gamma_k^+\zeta^k$, $\Phi_-(\zeta) = \sum_{k=-\infty}^0 \gamma_k^-\zeta^k$, and set $\gamma_{-1}^+ = \gamma_{-2}^+ = \cdots = 0$ and $\gamma_1^- = \gamma_2^- = \cdots = 0$. Then the operators $T_{\Phi_+} = [\gamma_{i-j}^+]_{ij=0}^\infty$ and $T_{\Phi_-} = [\gamma_{i-j}^-]_{ij=0}^\infty$ are lower and upper triangular invertible operators, and the following canonical upper lower factorization holds

$$(3.1) \qquad T_\Phi = T_{\Phi_-}T_{\Phi_+}.$$

Conversely if $T_\Phi = T_-T_+$ is a canonical upper lower factorization, then in particular, T_Φ is invertible. Whence, Φ admits a right canonical factorization $\Phi = \Phi_-\Phi_+$. This factorization also induces the canonical upper lower factorization (3.1). By the uniqueness of the canonical upper lower factorization, it follows that there exists a diagonal operator D such that $T_- = T_{\Phi_-}D$ and $T_+ = D^{-1}T_{\Phi_+}$.

The above remarks show the equivalence between canonical upper lower factorization of Toeplitz operators, and right canonical factorization of matrix-valued functions in the Wiener class. By this equivalence, Theorem 0.1 is equivalent to a statement about canonical upper lower factorization of Toeplitz operators with rational symbol.

We now present our extension of this result for input-output operators of time varying systems.

Let T be the input-output operator of the system

$$\Sigma \begin{cases} A_{k+1}x_{k+1} = G_kx_k + B_ku_k & (k = 0, 1, \ldots), \\ y_k = -C_{k+1}x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_0 \in L, \end{cases}$$

where the system

$$(3.2) \qquad A_{k+1}x_{k+1} = G_kx_k \quad (k = 0, 1, \ldots)$$

admits a dichotomy $(I - Q_k)_{k=0}^\infty$ with

$$(3.3) \qquad \operatorname{Im} Q_0 = L.$$

The next result gives necessary and sufficient conditions for the existence of a canonical upper lower factorization for T. The conditions are in terms of the following *associated system*

(3.4) $\qquad A_{k+1}^{\times}x_{k+1} = G_k x_k \quad (k = 0, 1, \ldots),$

where $A_{k+1}^{\times} = A_{k+1} - B_k C_{k+1}$ $(k = 0, 1, \ldots)$.

In the statement below we use the following terminology about direct sums. Let be given a sequence of direct sum decompositions

(3.5) $\qquad V_k \oplus W_k = \mathbb{C}^r \quad (k = 0, 1, \ldots).$

We say that the direct sum decomposition (3.5) *holds uniformly* if the sequence of projections $(\Pi_k)_{k=0}^{\infty}$ defined via $\operatorname{Ker} \Pi_k = V_k$, $\operatorname{Im} \Pi_k = W_k$ $(k = 0, 1, \ldots)$, satisfies $\sup_{k \geq 0} \|\Pi_k\| < \infty$.

Theorem 3.1. *Let T be the input-output operator of the system Σ, where the system* (3.2) *admits the dichotomy $(I - Q_k)_{k=0}^{\infty}$ with* (3.3), *and let $(P_k)_{k=0}^{\infty}$ be the corresponding dual sequence of projections. Then the following conditions are equivalent*:

(I) *The operator T admits a canonical upper lower factorization.*

(II) *The associated system* (3.4) *admits a dichotomy $(I - Q_k^{\times})_{k=0}^{\infty}$ such that the following direct sum holds uniformly*

(3.6) $\qquad \operatorname{Im} Q_k \oplus \operatorname{Ker} Q_k^{\times} = \mathbb{C}^r \quad (k = 0, 1, \ldots).$

(III) *The associated system* (3.4) *admits a dichotomy whose dual sequence of projections $(P_k^{\times})_{k=0}^{\infty}$ satisfies the following direct sum condition uniformly*

(3.7) $\qquad \operatorname{Im} P_k \oplus \operatorname{Ker} P_k^{\times} = \mathbb{C}^r \quad (k = 0, 1, \ldots).$

Moreover, assume that (3.6) *or* (3.7) *hold for one dichotomy $(I - Q_k^{\times})_{k=0}^{\infty}$ of* (3.4) *with dual sequence of projections $(P_k^{\times})_{k=0}^{\infty}$, and define projections ρ_k and τ_k in $\mathbb{C}^r$ via*

$$\operatorname{Ker} \rho_k = \operatorname{Im} P_k, \quad \operatorname{Im} \rho_k = \operatorname{Ker} P_k^{\times} \quad (k = 0, 1, \ldots)$$

$$\operatorname{Ker} \tau_k = \operatorname{Im} Q_k, \quad \operatorname{Im} \tau_k = \operatorname{Ker} Q_k^{\times} \quad (k = 0, 1, \ldots).$$

Then the canonical upper lower factorization $T = T_- T_+$ holds, where T_-, T_+, T_-^{-1}, T_+^{-1} are, respectively, the input-output operators of the following systems

$$\Sigma_- \begin{cases} A_{k+1}x_{k+1} = G_k x_k + (I - \rho_k)B_k u_k & (k = 0, 1, \ldots), \\ y_k = -C_{k+1}x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_0 \in L, \end{cases}$$

$$\Sigma_+ \begin{cases} A_{k+1}x_{k+1} = G_k x_k + B_k u_k & (k = 0, 1, \ldots), \\ y_k = -C_{k+1}\tau_{k+1}x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_0 \in L, \end{cases}$$

$$\Sigma_-^\times \begin{cases} A_{k+1}^\times x_{k+1} = G_k x_k + B_k u_k & (k = 0, 1, \ldots), \\ y_k = C_{k+1}(I - \tau_{k+1})x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_0 \in L, \end{cases}$$

and

$$\Sigma_+^\times \begin{cases} A_{k+1}^\times x_{k+1} = G_k x_k + \rho_k B_k u_k & (k = 0, 1, \ldots), \\ y_k = C_{k+1} x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_0 \in L, \end{cases}$$

where $A_{k+1}^\times = A_{k+1} - B_k C_{k+1}$ $(k = 0, 1, \ldots)$. *Finally, the entries of the operators* $T_- = [t_{ij}^-]_{ij=0}^\infty$, $T_+ = [t_{ij}^+]_{ij=0}^\infty$, $T_-^{-1} = [\gamma_{ij}^-]_{ij=0}^\infty$, $T_+^{-1} = [\gamma_{ij}^+]_{ij=0}^\infty$ *are given by the following formulas*

$$t_{ij}^- = C_{i+1}\Lambda_{i+1,j}^-(G_j\,|_{\operatorname{Im} Q_j})^{-1}(I_r - \rho_j)B_j \quad (i < j),$$

$$t_{ij}^+ = \delta_{i,j} I_m - C_{i+1}\tau_{i+1}\Lambda_{i+1,j+1}^+(A_{j+1}\,|_{\operatorname{Ker} Q_{j+1}})^{-1}(I_r - P_j)B_j \quad (i \geq j),$$

$$\gamma_{ij}^- = -C_{i+1}(I_r - \tau_{i+1})\Lambda_{i+1,j}^{\times -}(G_j\,|_{\operatorname{Im} Q_j^\times})^{-1}P_j^\times B_j \quad (i < j),$$

$$\gamma_{ij}^+ = \delta_{i,j} I_m + C_{i+1}\Lambda_{i+1,j+1}^{\times +}(A_{j+1}^\times\,|_{\operatorname{Ker} Q_{j+1}^\times})^{-1}\rho_j B_j \quad (i \geq j),$$

and $t_{ij}^+ = \gamma_{ij}^+ = 0$ $(i < j)$, *and* $t_{ij}^- = \gamma_{ij}^- = \delta_{ij} I_m$ $(i \geq j)$, *where* $\Lambda_{i+1,j+1}^+$ *and* $\Lambda_{i+1,j}^-$ (*respectively* $\Lambda_{i+1,j+1}^{\times +}$ *and* $\Lambda_{i+1,j}^{\times -}$) *are the forward and backward evolution operators of the system* (3.2) (*respectively* (3.4)) *corresponding to the dichotomy* $(I - Q_k)_{k=0}^\infty$ (*respectively* $(I - Q_k^\times)_{k=0}^\infty$).

Remark. Note that by Theorem 1.1, the subspaces $\operatorname{Ker} Q_k^\times$ and $\operatorname{Ker} P_k^\times$ $(k = 0, 1, \ldots)$ do not depend on the particular choice of the dichotomy $(I - Q_k^\times)_{k=0}^\infty$ of the system (3.4). Hence, if (3.6) or (3.7) hold for one dichotomy of (3.4) then they hold for each dichotomy of (3.4). For the same reason, the projections ρ_k and τ_k $(k = 0, 1, \ldots)$, and consequently also the systems Σ_-, Σ_+, $\Sigma_-^\times$, and $\Sigma_+^\times$ are the same for all the dichotomies $(I - Q_k^\times)_{k=0}^\infty$ of the system (3.4).

In contrast with the Toeplitz case, in the time-varying case the invertibility of T is not equivalent to the existence of a canonical upper lower factorization.

If $T = T_\Phi$ is a Toeplitz operator in l_m^2 with a rational matrix-valued symbol Φ, then one may represent Φ as in (0.3) and view T_Φ as the input-output operator of the system (0.5). Theorem 3.1 applies to this representation of T_Φ as an input-output operator. Using the description given in Lemma 1.5 of dichotomies of time invariant systems, it follows that condition (i) and either direct sum condition in (ii) of Theorem 0.1 are equivalent to condition (II) or condition (III) above. This proves the first part of Theorem 0.1. The explicit description of the right canonical factorization of Φ given in the second part of Theorem 0.1 follows from: (a) the connections between the right canonical factorization of Φ and the canonical upper lower factorization of the operator T_Φ, given in the beginning of this section, (b) the explicit formulas of the systems $\Sigma_\pm$ and $\Sigma_\pm^\times$ given in Theorem 3.1, and (c) the representation of Toeplitz operators by input-output systems, given by (0.5).

On the other hand the proof of Theorem 3.1 above is obtained from the Banach space version of Theorem 0.1. Before the statement of the latter result some preliminary definitions are in order. Let X be a Banach space and $\mathcal{L}(X)$ the space of bounded linear operators acting in X. We consider an operator-valued function $\Phi : \mathbb{T} \mapsto \mathcal{L}(X);\ t \mapsto \Phi(e^{it})$. A *right canonical factorization of* Φ *relative to the unit circle* is a factorization of the type

$$\Phi(\zeta) = \Phi_-(\zeta)\Phi_+(\zeta) \quad (\zeta \in \mathbb{T}),$$

where Φ_+ and $\tilde{\Phi}_-$, $\tilde{\Phi}_-(\zeta) = \Phi_-(\zeta^{-1})$, are $\mathcal{L}(X)$-valued functions which are analytic on the open unit disc $\mathbb{D}$, continuous in the uniform operator topology on $\mathbb{D} \cup \mathbb{T}$, and are invertible on $\mathbb{D} \cup \mathbb{T}$.

We are interested in operator-valued functions in realized form and begin by describing this notion. Let X, X_1 and X_2 be Banach spaces, and $G, A : X_1 \mapsto X_2$, $B : X \mapsto X_2$ and $C : X_1 \mapsto X$ linear bounded operators such that $\zeta G - A$ is invertible for each $\zeta \in \mathbb{T}$. Then the formula

$$\Phi(\zeta) = I + C(\zeta G - A)^{-1}B \quad (\zeta \in \mathbb{T}), \tag{3.8}$$

defines an operator-valued function Φ, and in this case we say that Φ is given in *realized form* by the formula (3.8). The following result is the Banach space analogue of Theorem 0.1.

Theorem 3.2. *Let* Φ *be an operator-valued function given in the realized form* (3.8). *Put* $A^\times = A - BC$. *Then* Φ *admits a right canonical factorization relative to* $\mathbb{T}$ *if and only if the following two conditions hold true*:

(i) $\zeta G - A^\times$ *is invertible for each* $\zeta \in \mathbb{T}$,

(ii) $X_1 = \operatorname{Im} Q \oplus \operatorname{Ker} Q^\times$ *and* $X_2 = \operatorname{Im} P \oplus \operatorname{Ker} P^\times$.

Here

$$\begin{cases} Q = \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{T}} (\zeta G - A)^{-1} G \, d\zeta, & P = \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{T}} G(\zeta G - A)^{-1} \, d\zeta, \\ Q^\times = \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{T}} (\zeta G - A^\times)^{-1} G \, d\zeta, & P^\times = \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{T}} G(\zeta G - A^\times)^{-1} \, d\zeta. \end{cases} \tag{3.9}$$

In that case a right canonical factorization $\Phi(\zeta) = \Phi_-(\zeta)\Phi_+(\zeta)$ *of* Φ *relative to* $\mathbb{T}$ *is obtained by taking*

$$\begin{cases} \Phi_-(\zeta) = I + C(\zeta G - A)^{-1}(I - \rho)B, & (\zeta \in \mathbb{T}), \\ \Phi_+(\zeta) = I + C\tau(\zeta G - A)^{-1}B, & (\zeta \in \mathbb{T}), \\ \Phi_-(\zeta)^{-1} = I - C(I - \tau)(\zeta G - A^\times)^{-1}B, & (\zeta \in \mathbb{T}), \\ \Phi_+(\zeta)^{-1} = I - C(\zeta G - A^\times)^{-1}\rho B, & (\zeta \in \mathbb{T}). \end{cases} \tag{3.10}$$

Here τ *is the projection of* X_1 *along* $\operatorname{Im} Q$ *onto* $\operatorname{Ker} Q^\times$ *and* ρ *is the projection of* X_2 *along* $\operatorname{Im} P$ *onto* $\operatorname{Ker} P^\times$. *Furthermore, the two equalities in* (ii) *are not independent; in fact, the first equality in* (ii) *implies the second and conversely.*

This theorem has been proved in the finite dimensional case in [GK]. Most of the proof in [GK] carries over without any change to the infinite dimensional case and will not be repeated here. The remaining details to be changed can be proved without much difficulty in the same spirit as in [GK]. Let us also remark that in [GK] the more general case of a Cauchy contour is considered, however we shall not need this here.

Before the proof of Theorem 3.1, we describe the connection between input-output operators of systems and operator-valued functions in realized form. As above, let $T = [t_{ij}]_{ij=0}^{\infty}$ be the input-output operator of the system Σ and let $\zeta \in \mathbb{T}$ be arbitrary. It follows easily from the definition of the dichotomy that since $(I - Q_k)_{k=0}^{\infty}$ is a dichotomy for the system (3.2), $(I - Q_k)_{k=0}^{\infty}$ is also a dichotomy for the system

$$A_{k+1}x_{k+1} = \zeta G_k x_k \quad (\zeta \in \mathbb{T};\ k = 0, 1, \ldots).$$

Hence, for each $\zeta \in \mathbb{T}$ the system

$$\Sigma(\zeta) \begin{cases} A_{k+1}x_{k+1} = \zeta G_k x_k + B_k u_k & (k = 0, 1, \ldots), \\ y_k = -C_{k+1}x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_0 \in L, \end{cases}$$

admits a well defined input-output operator, which we denote by

$$T(\zeta) = [t_{ij}(\zeta)]_{ij=0}^{\infty} \quad (\zeta \in \mathbb{T}).$$

The entries $t_{ij}(\zeta)$ of $T(\zeta)$ are obtained by the formulas (2.2) after replacing G_k by ζG_k. This shows immediately that $t_{ij}(\zeta) = t_{ij}\zeta^{i-j}$. Hence,

$$(3.11) \qquad T(\zeta) = [t_{ij}\zeta^{i-j}]_{ij=0}^{\infty} \quad (\zeta \in \mathbb{T}).$$

One can approach the function $T(\zeta)$ from a different direction as follows. Define the space $l^2_{r,L}$ and the operators G and $A : l^2_{r,L} \to l^2_r$ as in (1.17)–(1.19). Define also operators B and C via

$$(3.12) \qquad \begin{cases} B : l^2_m \to l^2_r, & B(x_0, x_1, \ldots) = (B_0 x_0, B_1 x_1, \ldots), \\ C : l^2_{r,L} \to l^2_m, & C(x_0, x_1, \ldots) = (C_1 x_1, C_2 x_2, \ldots). \end{cases}$$

Then, it follows immediately from the definition of the input-output operator given in Section 2 that $G - A$ is invertible and

$$(3.13) \qquad T = I + C(G - A)^{-1}B.$$

By applying these considerations to the system $\Sigma(\zeta)$ $(\zeta \in \mathbb{T})$ and its input-output operator $T(\zeta)$, it follows that $\zeta G - A$ is invertible and

$$(3.14) \qquad T(\zeta) = I + C(\zeta G - A)^{-1}B \quad (\zeta \in \mathbb{T}).$$

In view of this equality we call the operator-valued function $T(\zeta)$ the *transfer function* of the system Σ. Equality (3.14) gives a representation of the transfer function in realized form. The interplay between the representations (3.11) and (3.14) of the transfer function is essential in what follows. For future reference we note here that (3.13) and (3.14) imply the following relation between the input-output operator T of the system Σ and the corresponding transfer function $T(\zeta)$

(3.15) $\quad T = T(1).$

Proof of Theorem 3.1. The proof is divided into four parts.

Part (a). Here we prove that condition (I) implies condition (II). Assume that the input-output operator T of the system Σ admits a canonical upper lower factorization $T = T_-T_+$. Denote the entries of the operators T, T_-, T_+, T_-^{-1} and T_+^{-1} by $T = [t_{ij}]_{ij=0}^{\infty}$, $T_- = [t_{ij}^-]_{ij=0}^{\infty}$, $T_+ = [t_{ij}^+]_{ij=0}^{\infty}$, $T_-^{-1} = [\gamma_{ij}^-]_{ij=0}^{\infty}$, and $T_+^{-1} = [\gamma_{ij}^+]_{ij=0}^{\infty}$.

The equality $T_- = TT_+^{-1}$ and the fact that T_+^{-1} is lower triangular imply that

$$t_{ij}^- = \Sigma_{k=j}^{\infty} t_{ik}\gamma_{kj}^+.$$

Recall that by the inequality (2.3) we have

$$\|t_{ik}\| \leq Ma^{|i-k|},$$

where $M > 0$ and $0 < a < 1$. Combining this inequality with the preceding equality, and taking into account that $t_{ij}^- = 0$ for $i > j$, we obtain

(3.16) $\quad \|t_{ij}^-\| \leq \Sigma_{k=j}^{\infty} Ma^{|i-k|}\|(T_+)^{-1}\| = M_1a^{|i-j|} \quad (i,j = 0, 1, \ldots),$

where $M_1 = M\|T_+^{-1}\|(1-a)^{-1}$. Similarly, we have

(3.17) $\quad \|t_{ij}^+\| \leq M_2a^{|i-j|} \quad (i,j = 0, 1, \ldots),$

where $M_2 = M\,\|T_-^{-1}\|(1-a)^{-1}$.

Now Theorem 6.1 of [BGK2] shows that the inverse operator $T^{-1} = [\gamma_{ij}]_{i,j=0}^{\infty}$ is also an input-output operator. Hence, there are positive numbers M_3 and a_1, with $a_1 < 1$ such that

$$\|\gamma_{ik}\| \leq M_3a_1^{|i-k|}.$$

As in the previous paragraph, this inequality and the factorization $T^{*-1} = T_-^{*-1}T_+^{*-1}$lead to

(3.18) $\quad \|\gamma_{ij}^+\|, \|\gamma_{ij}^-\| \leq M_4\,a_1^{|i-j|} \quad (i,j = 0, 1, \ldots),$

where M_4 is a positive number.

Let

(3.19) $\quad T(\zeta) = [t_{ij}\zeta^{i-j}]_{ij=0}^{\infty} = I + C(\zeta G - A)^{-1}B \quad (\zeta \in \mathbb{T})$

be the transfer function of the system Σ. We define operator-valued functions by the following formulas

$$T_-(\zeta) = [t_{ij}^-\zeta^{i-j}]_{ij=0}^{\infty}, \quad T_-^{\times}(\zeta) = [\gamma_{ij}^-\zeta^{i-j}]_{ij=0}^{\infty} \quad (|\zeta| \geq 1),$$

and

$$T_+(\zeta) = [t_{ij}^+\zeta^{i-j}]_{ij=0}^{\infty}, \quad T_+^{\times}(\zeta) = [\gamma_{ij}^+\zeta^{i-j}]_{ij=0}^{\infty} \quad (|\zeta| \leq 1).$$

By the inequalities (3.16)–(3.18) and the triangularity properties of $T_-^{\pm 1}$ and $T_+^{\pm 1}$, it follows that $T_\pm(\zeta)$ and $T_\pm^\times(\zeta)$ are continuous operator-valued functions on their respective domains of definitions and are analytic in the interiors of these domains (and at ∞ for $T_-(\zeta)$ and $T_-^\times(\zeta)$). Moreover, after writing the factorization $T = T_- T_+$ entrywise it follows easily from the above definitions that

$$(3.20) \qquad T(\zeta) = T_-(\zeta)T_+(\zeta) \quad (\zeta \in \mathbb{T}).$$

Similarly, the equalities $T_+T_+^{-1} = T_+^{-1}T_+ = I$, $T_-T_-^{-1} = T_-^{-1}T_- = I$, the triangularity of T_+ and T_-, and the above definitions imply

$$T_-^\times(\zeta^{-1}) = T_-^{-1}(\zeta^{-1}) \quad \text{and} \quad T_+^\times(\zeta) = T_+^{-1}(\zeta) \quad (|\zeta| \le 1).$$

These equalities and the above properties of $T_\pm(\zeta)$ and $T_\pm^\times(\zeta)$ show that (3.20) is a right canonical factorization for $T(\zeta)$.

Since $T(\zeta)$ admits a right canonical factorization, and in view of the realized form in (3.19) for $T(\zeta)$, we may apply Theorem 3.2. It first follows that the operator $G - A^\times$ is invertible. By Theorem 1.4 this means that the system (3.4) admits a dichotomy $(I - Q_k^\times)_{k=0}^\infty$ with $\operatorname{Im} Q_0^\times = L$. In view of equality (3.3) we also have $\operatorname{Im} Q_0 = L$, and therefore,

$$(3.21) \qquad \operatorname{Im} Q_0 \oplus \operatorname{Ker} Q_0^\times = \mathbb{C}^r.$$

Further, the first direct sum in condition (ii) of Theorem 3.2 leads to

$$(3.22) \qquad \operatorname{Im}\left[\frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta G - A)^{-1} G \, d\zeta\right] \oplus \operatorname{Ker}\left[\frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta G - A^\times)^{-1} G \, d\zeta\right] = l_{r,L}^2.$$

On the other hand, the integral formulas for the dichotomy appearing in Theorem 1.4 imply that

$$\frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta G - A)^{-1} G \, d\zeta = \operatorname{diag}(I\,|_L, Q_1, Q_2, \ldots),$$

and

$$\frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta G - A^\times)^{-1} G \, d\zeta = \operatorname{diag}(I\,|_L, Q_1^\times, Q_2^\times, \ldots).$$

These diagonal representations and the direct sum condition (3.22) imply that the following direct sum condition holds uniformly

$$\operatorname{Im} Q_k \oplus \operatorname{Ker} Q_k^\times = \mathbb{C}^r \quad (k = 1, 2, \ldots).$$

Combining this with (3.21) we obtain condition (II) of Theorem 3.1.

Part (b). Here we prove that condition (II) implies condition (III). Assume that (II) holds. Then the associated system (3.4) admits a dichotomy $(I - \bar{Q}_k^\times)_{k=0}^\infty$ such that the following direct sum holds uniformly

$$(3.23) \qquad \operatorname{Im} Q_k \oplus \operatorname{Ker} \bar{Q}_k^\times = \mathbb{C}^r \quad (k = 0, 1, \ldots).$$

In particular, we have

$$\operatorname{Im} Q_0 \oplus \operatorname{Ker} \bar{Q}_0^\times = \mathbb{C}^r,$$

and hence by (3.3)

$$L \oplus \operatorname{Ker} \bar{Q}_0^\times = \mathbb{C}^r.$$

Thus, by Theorem 1.2 the associated system (3.4) admits a dichotomy $(I - Q_k^\times)_{k=0}^\infty$ such that

$$\operatorname{Im} Q_0^\times = L.$$

Theorem 1.4 now shows that the operator-valued function $\zeta G - A^\times$ is invertible for each $\zeta \in \mathbb{T}$. Thus, condition (i) in Theorem 3.2 holds.

Now recall that by the characterization of the sequence of kernels of a dichotomy given in Theorem 1.1, we have

$$\operatorname{Ker} Q_k^\times = \operatorname{Ker} \bar{Q}_k^\times \quad (k = 0, 1, \ldots).$$

Hence, by (3.23) the following direct sum holds uniformly

$$\operatorname{Im} Q_k \oplus \operatorname{Ker} Q_k^\times = \mathbb{C}^r \quad (k = 0, 1, \ldots).$$

In a similar way as in Part (a), this uniform direct sum and the first formula in Theorem 1.4 imply that

$$\operatorname{Im} Q \oplus \operatorname{Ker} Q^\times = l_{r,L}^2,$$

where Q and $Q^\times$ are as in Theorem 3.2. Thus the first part of condition (ii) in Theorem 3.2 holds. By the last sentence in Theorem 3.2 we obtain

$$\operatorname{Im} P \oplus \operatorname{Ker} P^\times = l_r^2.$$

Translating this via the second formula in Theorem 1.4, we obtain that the following direct sum holds uniformly

$$\operatorname{Im} P_k \oplus \operatorname{Ker} P_k^\times = \mathbb{C}^r \quad (k = 0, 1, \ldots),$$

where $(P_k)_{k=0}^\infty$ is the dual sequence of $(I - Q_k)_{k=0}^\infty$. Hence, condition (III) of Theorem 3.1 holds.

Part (c). Here we prove that condition (III) implies condition (I), and the existence of the upper lower canonical factorization $T = T_- T_+$, where T_-, T_+, T_-^{-1} and T_+^{-1} are the respective input-output operators of the systems Σ_-, Σ_+, $\Sigma_-^\times$, and $\Sigma_+^\times$ appearing in the statement of Theorem 3.1.

Assume that (III) holds, and let $(I - Q_k^\times)_{k=0}^\infty$ be a dichotomy for the associated system (3.4) whose dual sequence of projections $(P_k^\times)_{k=0}^\infty$ satisfies the following uniform direct sum condition

$$\operatorname{Im} P_k \oplus \operatorname{Ker} P_k^\times = \mathbb{C}^r \quad (k = 0, 1, \ldots). \tag{3.24}$$

In particular, we have $\operatorname{Im} P_0 \oplus \operatorname{Ker} P_0^\times = \mathbb{C}^r$. Hence, Theorem 6.1 of [BGK2] shows that T is invertible and that $\operatorname{Im} Q_0 \oplus \operatorname{Ker} Q_0^\times = \mathbb{C}^r$.

As in Part (b), by (3.3) the last equality leads to $L \oplus \operatorname{Ker} Q_0^\times = \mathbb{C}^r$, and therefore Theorem 1.2 shows that the system (3.4) admits a dichotomy $(I - \bar{Q}_k^\times)_{k=0}^\infty$ with $\operatorname{Im} \bar{Q}_0^\times = L$. Theorem 1.4 then implies that $\zeta G - A^\times$ is invertible for each $\zeta \in \mathbb{T}$. Now define two projections P and $P^\times$ as in Theorem 3.2. By Theorem 1.4, we have

$$P = \operatorname{diag}(P_0, P_1, \dots), \quad P^\times = \operatorname{diag}(\bar{P}_0^\times, \bar{P}_1^\times, \dots),$$

where $(\bar{P}_k^\times)_{k=0}^\infty$ is the dual sequence of the dichotomy $(I - \bar{Q}_k^\times)_{k=0}^\infty$. On the other hand, Theorem 1.1 shows that

$$\operatorname{Ker} \bar{P}_k^\times = \operatorname{Ker} P_k^\times \quad (k = 0, 1, \dots),$$

and hence, the uniform direct sum condition (3.24) implies that

$$\operatorname{Im} P \oplus \operatorname{Ker} P^\times = l_{r,L}^2.$$

Hence, we may apply Theorem 3.2 to the effect that the transfer function

$$T(\zeta) = [t_{ij}\zeta^{i-j}]_{ij=0}^\infty = I + C(\zeta G - A)^{-1}B \quad (\zeta \in \mathbb{T}) \tag{3.25}$$

of the system Σ admits a right canonical factorization

$$T(\zeta) = \Phi_-(\zeta)\Phi_+(\zeta) \quad (\zeta \in \mathbb{T}), \tag{3.26}$$

where the factors $\Phi_-(\zeta)$ and $\Phi_+(\zeta)$ are given by the formulas (3.10).

Let τ and ρ be the projections as defined in the statement of Theorem 3.2. It follows from the formulas in Theorem 1.4 that τ is the projection in $l_{r,L}^2$ along $L \oplus \operatorname{Im} Q_1 \oplus \operatorname{Im} Q_2 \oplus \cdots$ onto $0 \oplus \operatorname{Ker} Q_1 \oplus \operatorname{Ker} Q_2 \oplus \cdots$, and that ρ is the projection in l_r^2 along $\operatorname{Im} P_0 \oplus \operatorname{Im} P_1 \oplus \operatorname{Im} P_2 \oplus \cdots$ onto $\operatorname{Ker} P_0 \oplus \operatorname{Ker} P_1 \oplus \operatorname{Ker} P_2 \oplus \cdots$. Further, let ρ_k and τ_k be the projections in $\mathbb{C}^r$ as defined in the statement of Theorem 3.1 ($k = 0, 1, \dots$). The above description of the kernel and the image of τ and ρ implies that

$$\tau = \operatorname{diag}(0|_L, \tau_1, \tau_2, \dots) : l_{r,L}^2 \to l_{r,L}^2$$

and

$$\rho = \operatorname{diag}(\rho_0, \rho_1, \rho_2, \dots) : l_r^2 \to l_r^2.$$

Thus, it follows from the formulas (3.12) for the operators B and C that ρB and $C\tau$ have the following form

$$\begin{cases} \rho B : l_m^2 \to l_r^2, & B(x_0, x_1, \dots) = (\rho_0 B_0 x_0, \rho_1 B_1 x_1, \dots), \\ C\tau : l_{r,L}^2 \to l_m^2, & C(x_0, x_1, \dots) = (C_1 \tau_1 x_1, C_2 \tau_2 x_2, \dots), \end{cases}$$

and similar formulas hold for $(I - \rho)B$ and $C(I - \tau)$. After comparing the above formulas with (3.12), and comparing (3.13) with (3.10), it is apparent that the functions $\Phi_-(\zeta)$, $\Phi_+(\zeta)$, $\Phi_-(\zeta)^{-1}$ and $\Phi_+(\zeta)^{-1}$ are the transfer functions of the systems Σ_-, Σ_+, $\Sigma_-^\times$ and $\Sigma_+^\times$ appearing in the statement of Theorem 3.1, respectively. Consequently, the general form (3.11) of the entries of a transfer function shows that these transfer functions have the form

$$(3.27)\quad \begin{cases} \Phi_-(\zeta) = [t_{ij}^- \zeta^{i-j}]_{ij=0}^{\infty} & (\zeta \in \mathbb{T}), \\ \Phi_+(\zeta) = [t_{ij}^+ \zeta^{i-j}]_{ij=0}^{\infty} & (\zeta \in \mathbb{T}), \\ \Phi_-(\zeta)^{-1} = [\gamma_{ij}^- \zeta^{i-j}]_{ij=0}^{\infty} & (\zeta \in \mathbb{T}), \\ \Phi_+(\zeta)^{-1} = [\gamma_{ij}^+ \zeta^{i-j}]_{ij=0}^{\infty} & (\zeta \in \mathbb{T}), \end{cases}$$

for some suitable $m \times m$ matrices $t_{ij}^{\pm}$ and $\gamma_{ij}^{\pm}$. However, the canonical factorization (3.26) implies by definition that $\Phi_+(\zeta)$ and $\Phi_+^{-1}(\zeta)$ admit analytic continuations in the disc$\{\zeta : |\zeta| < 1\}$, while $\Phi_-(\zeta)$ and $\Phi_-^{-1}(\zeta)$ admit analytic continuations in the region $\{\zeta : |\zeta| > 1\} \cup \infty$. Hence, the representation (3.27) shows that

$$(3.28)\quad \begin{cases} t_{ij}^- = \gamma_{ij}^- = 0 & (i > j), \\ t_{ij}^+ = \gamma_{ij}^+ = 0 & (i < j). \end{cases}$$

Denote by T_-, T_+, $T_-^\times$ and $T_+^\times$ the respective input-output operators of the systems Σ_-, Σ_+, $\Sigma_-^\times$ and $\Sigma_+^\times$ appearing in the statement of Theorem 3.1. Since $\Phi_-(\zeta)$, $\Phi_+(\zeta)$, $\Phi_-(\zeta)^{-1}$ and $\Phi_+(\zeta)^{-1}$ are the transfer functions of these systems, it follows from (3.15) that

$$(3.29)\quad \begin{cases} T_- = \Phi_-(1) = [t_{ij}^-]_{ij=0}^{\infty}, \\ T_+ = \Phi_+(1) = [t_{ij}^+]_{ij=0}^{\infty}, \\ T_-^\times = \Phi_-(1)^{-1} = [\gamma_{ij}^-]_{ij=0}^{\infty}, \\ T_+^\times = \Phi_+(1)^{-1} = [\gamma_{ij}^+]_{ij=0}^{\infty}, \end{cases}$$

where we used (3.27). These equalities show that T_- and T_+ are invertible with

$$(3.30)\quad T_-^{-1} = T_-^\times = [\gamma_{ij}^-]_{ij=0}^{\infty} \quad \text{and} \quad T_+^{-1} = T_+^\times = [\gamma_{ij}^+]_{ij=0}^{\infty}.$$

In particular, it follows from these equalities and (3.28) and (3.29) that the operators T_- and T_-^{-1} are upper triangular and the operators T_+ and T_+^{-1} are lower triangular. Furthermore, the factorization (3.26) and the equalities (3.29) imply that T admits the following factorization

$$(3.31)\quad T = T_- T_+.$$

By the last mentioned invertibility and triangularity properties of T_- and T_+, this factorization is a canonical upper lower factorization for T. Thus condition (I) holds. Finally, T_- and T_+ are the input-output operators of the systems Σ_- and Σ_+, respectively, and by (3.30) T_-^{-1} and T_+^{-1} are the input-output operators of the systems $\Sigma_-^\times$ and $\Sigma_+^\times$, respectively.

Part (d). We complete the proof of Theorem 3.1 by deriving the formulas in the last part of its statement.

We first derive these formulas in the case when $(I - Q_k^\times)_{k=0}^{\infty}$ is the unique dichotomy of (3.4) satisfying Im $Q_0^\times = L$, and $(P_k^\times)_{k=0}^{\infty}$ is its dual sequence of projections.

Let $T_- = (t_{ij}^-)_{ij=0}^\infty$ and $T_+ = (t_{ij}^+)_{ij=0}^\infty$ be the input-output operators of the systems Σ_- and Σ_+, respectively. Then by Part (c), the canonical upper lower factorization $T = T_- T_+$ holds, and the operators $T_-^{-1} = (\gamma_{ij}^-)_{ij=0}^\infty$ and $T_+^{-1} = (\gamma_{ij}^+)_{ij=0}^\infty$ are the input-output operators of the systems $\Sigma_-^\times$ and $\Sigma_+^\times$, respectively. Since $(I - Q_k^\times)_{k=0}^\infty$ is the unique dichotomy of (3.4) satisfying $\operatorname{Im} Q_0^\times = L$, the entries of these operators may be computed from the formula (2.2). Taking into account the fact that T_- and T_-^{-1} are upper triangular and T_+ and T_+^{-1} are lower triangular the following formulas follow from (2.2)

$$t_{ij}^- = \begin{cases} \delta_{i,j}[I_m - C_{i+1}(A_{i+1}|_{\operatorname{Ker} Q_{i+1}})^{-1}(I_r - P_i)(I_r - \rho_i)B_i] & (i \geq j), \\ C_{i+1}\Lambda_{i+1,j}^-(G_j|_{\operatorname{Im} Q_j})^{-1}P_j(I_r - \rho_j)B_j & (i < j), \end{cases}$$

$$t_{ij}^+ = \begin{cases} \delta_{i,j}I_m - C_{i+1}\tau_{i+1}\Lambda_{i+1,j+1}^+(A_{j+1}|_{\operatorname{Ker} Q_{j+1}})^{-1}(I_r - P_j)B_j & (i \geq j), \\ 0 & (i < j), \end{cases}$$

$$\gamma_{ij}^- = \begin{cases} \delta_{i,j}[I_m + C_{i+1}(I_r - \tau_{i+1})(A_{i+1}^\times|_{\operatorname{Ker} Q_{i+1}^\times})^{-1}(I_r - P_i^\times)B_i] & (i \geq j), \\ -C_{i+1}(I_r - \tau_{i+1})\Lambda_{i+1,j}^{\times -}(G_j|_{\operatorname{Im} Q_j^\times})^{-1}P_j^\times B_j & (i < j), \end{cases}$$

and

$$\gamma_{ij}^+ = \begin{cases} \delta_{i,j}I_m + C_{i+1}\Lambda_{i+1,j+1}^{\times +}(A_{j+1}^\times|_{\operatorname{Ker} Q_{j+1}^\times})^{-1}(I_r - P_i^\times)\rho_j B_j & (i \geq j), \\ 0 & (i < j). \end{cases}$$

Here, $\Lambda_{i+1,j+1}^+$ and $\Lambda_{i+1,j}^-$ (respectively $\Lambda_{i+1,j+1}^{\times +}$ and $\Lambda_{i+1,j}^{\times -}$) are the forward and backward evolution operators of the system (3.2) (respectively (3.4)) relative to the dichotomy $(I - Q_k)_{k=0}^\infty$ (respectively $(I - Q_k^\times)_{k=0}^\infty$).

Now recall that by the definition of ρ_k, $\operatorname{Ker} \rho_k = \operatorname{Im} P_k$ and $\operatorname{Im} \rho_k = \operatorname{Ker} P_k^\times$ $(k = 0, 1, \ldots)$. Hence,

(3.32) $\quad P_k(I_r - \rho_k) = I_r - \rho_k, \quad (I_r - P_k^\times)\rho_k = \rho_k \quad (k = 0, 1, \ldots).$

By the left hand side of this equality we have

(3.33) $\quad (I_r - P_k)(I_r - \rho_k) = 0 \quad (k = 0, 1, \ldots).$

Inserting this equality in the above formula for t_{ij}^- we obtain

(3.34) $\quad t_{ii}^- = I_m \quad (i = 0, 1, \ldots).$

Since both $T_- = [t_{ij}^-]_{ij=0}^\infty$ and its inverse $T_-^{-1} = [\gamma_{ij}^-]_{ij=0}^\infty$ are upper triangular, this leads to

(3.35) $\quad \gamma_{ii}^- = I_m \quad (i = 0, 1, \ldots).$

After inserting (3.34) and (3.35) in the above formulas for the entries t_{ij}^-, t_{ij}^+, γ_{ij}^- and γ_{ij}^+, and further simplifying via (3.32), we obtain the formulas in the last part of the statement of Theorem 3.1.

We still have to show that the formulas for t_{ij}^-, t_{ij}^+, γ_{ij}^- and γ_{ij}^+ in the statement of Theorem 3.1 are applicable with any choice of dichotomy of the system (3.4). Note that the dichotomy $(I - Q_k)_{k=0}^{\infty}$ of the original system (3.2) is uniquely determined by the condition (3.3), and so is its dual sequence $(P_k)_{k=0}^{\infty}$. Hence, the transformations $(A_{k+1}|_{\operatorname{Ker} Q_{k+1}})^{-1}$, $(G_k|_{\operatorname{Im} Q_k})^{-1}$, $\Lambda_{i+1,j+1}^{+}$ and $\Lambda_{i+1,j}^{-}$, are uniquely determined. In addition, it follows from Theorem 1.1 that $\operatorname{Ker} Q_k^{\times}$ and $\operatorname{Ker} P_k^{\times}$ are the same for all the dichotomies of the system (3.4). Therefore, the transformations $(A_{k+1}^{\times}|_{\operatorname{Ker} Q_{k+1}^{\times}})^{-1}$ and $\Lambda_{i+1,j+1}^{\times+}$ are also uniquely determined. Moreover, by the remark following the statement of Theorem 3.1, the projections ρ_k and τ_k are also the same for all the dichotomies of (3.4). Inspecting the formulas in Theorem 3.1, it follows from these remarks that the formulas for t_{ij}^-, t_{ij}^+ and γ_{ij}^+ have the same value for all the dichotomies of (3.4). Thus, there remains to be shown that the expression for γ_{ij}^- given in Theorem 3.1 is valid for all choices of dichotomies of the system (3.4). We may clearly assume that $i < j$, and hence, we must prove that the expression

$$(3.36) \qquad -C_{i+1}(I_r - \tau_{i+1})\,\Lambda_{i+1,j}^{\times-}(G_j|_{\operatorname{Im} Q_j^{\times}})^{-1}\,P_j^{\times} B_j \quad (i \le j),$$

is the same for all dichotomies of (3.4).

As above, let $(I - Q_k^{\times})_{k=0}^{\infty}$ be the unique dichotomy of (3.4) satisfying $\operatorname{Im} Q_0^{\times} = L$, $(P_k^{\times})_{k=0}^{\infty}$ be its dual sequence of projections, and $\Lambda_{i+1,j+1}^{\times+}$ and $\Lambda_{i+1,j}^{\times+}$ be the forward and backward evolution operators. In addition, let $(I - \bar{Q}_k^{\times})_{k=0}^{\infty}$ be an arbitrary dichotomy of (3.4) with dual sequence $(\bar{P}_k^{\times})_{k=0}^{\infty}$ and forward and backward evolution operators $\bar{\Lambda}_{i+1,j+1}^{\times+}$ and $\bar{\Lambda}_{i+1,j}^{\times-}$.

The following formula follows from Lemma 2.6 of [BGK2].

$$(3.37) \qquad \bar{\Lambda}_{i+1,j}^{\times-}(G_j|_{\operatorname{Im} \bar{Q}_j^{\times}})^{-1} = \bar{Q}_{i+1}^{\times}\,\Lambda_{i+1,j}^{\times-}(G_j|_{\operatorname{Im} Q_j^{\times}})^{-1}\,(P_j^{\times}|_{\operatorname{Im} \bar{P}_j^{\times}}) \quad (i < j).$$

In addition, by the definition of τ_{i+1} the equality $\operatorname{Im} \tau_{i+1} = \operatorname{Ker} Q_{i+1}^{\times}$ holds. By Theorem 1.1, this leads to $\operatorname{Im} \tau_{i+1} = \operatorname{Ker} \bar{Q}_{i+1}^{\times}$. Hence,

$$(I_r - \tau_{i+1})(I_r - \bar{Q}_{i+1}^{\times}) = 0 \quad (i = 0, 1, \dots),$$

and therefore,

$$(I_r - \tau_{i+1}) = (I_r - \tau_{i+1})\,\bar{Q}_{i+1}^{\times} \quad (i = 0, 1, \dots).$$

These equalities and (3.37) imply

$$(3.38) \qquad \begin{cases} (I_r - \tau_{i+1})\,\bar{\Lambda}_{i+1,j}^{\times-}(G_j|_{\operatorname{Im} \bar{Q}_j^{\times}})^{-1} \\ = (I_r - \tau_{i+1})\,\bar{Q}_{i+1}\,\Lambda_{i+1,j}^{\times-}(G_j|_{\operatorname{Im} Q_j^{\times}})^{-1}\,(P_j^{\times}|_{\operatorname{Im} \bar{P}_j^{\times}}) \quad (i < j) \\ = (I_r - \tau_{i+1})\,\Lambda_{i+1,j}^{\times-}(G_j|_{\operatorname{Im} Q_j^{\times}})^{-1}\,(P_j^{\times}|_{\operatorname{Im} \bar{P}_j^{\times}}). \end{cases}$$

Now recall that by Theorem 1.1, $\operatorname{Ker} \bar{P}_j^{\times} = \operatorname{Ker} P_j^{\times} (j = 0, 1, \dots)$. Hence, $P_j^{\times}(I_r - \bar{P}_j^{\times}) = 0$, and therefore

$$P_j^{\times}\bar{P}_j^{\times} = P_j^{\times} \quad (j = 0, 1, \dots).$$

This leads to $(P_j^\times|_{\mathrm{Im}\,\bar{P}_j^\times})\bar{P}_j^\times = P_j^\times \bar{P}_j^\times = P_j^\times$ $(j = 0, 1, \dots)$. Therefore, (3.38) implies that

$$\begin{cases} (I_r - \tau_{i+1})\,\bar{\Lambda}_{i+1,j}^{\times -}(G_j|_{\mathrm{Im}\,\bar{Q}_j^\times})^{-1}\bar{P}_j^\times \\ = (I_r - \tau_{i+1})\,\Lambda_{i+1,j}^{\times -}(G_j|_{\mathrm{Im}\,Q_j^\times})^{-1}(P_j^\times|_{\mathrm{Im}\,\bar{P}_j^\times})\bar{P}_j^\times \quad (i < j) \\ = (I_r - \tau_{i+1})\,\Lambda_{i+1,j}^{\times -}(G_j|_{\mathrm{Im}\,Q_j^\times})^{-1}P_j^\times. \end{cases}$$

Hence, the expression in (3.36) is the same for all the dichotomies of the associated system (3.4). □

Remark. The property $t_{ii}^- = I_m$ $(i = 0, 1, \dots)$ of the factor $T_- = [t_{ij}^-]_{ij=0}^\infty$ given by equality (3.34) above, reflects the property

(3.39) $\quad \Phi_-(\infty) = I$

of the factorization $\Phi(\zeta) = \Phi_-(\zeta)\,\Phi_+(\zeta)$ $(\zeta \in \mathbb{T})$ given in Theorem 3.2. We now show that (3.39) holds true in general, using the notation of [GK]. By equality (5.20) of [GK] we have

(3.40) $\quad \Phi_-(\zeta) = I + C_1(\zeta G_{11} - A_{11})^{-1}B_1 \quad (\zeta \in \mathbb{T}),$

where G_{11} and A_{11}, B_1 and C_1 are suitable operators whose relevant property here is that the pencil

$$\zeta G_{11} - A_{11}$$

is $\mathbb{D}_- \cup \mathbb{T}$ regular, where $\mathbb{D}_- = \{\zeta : |\zeta| > 1\} \cup \{\infty\}$. By definition, this implies in particular that G_{11} is invertible, and therefore equality (3.39) follows immediately from the representation (3.40).

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