Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

J. Math. Anal. Appl. 326 (2007) 1370–1378

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Smooth approximation of Lipschitz functions on Riemannian manifolds

D. Azagra *, J. Ferrera, F. López-Mesas, Y. Rangel

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad Complutense, 28040 Madrid, Spain

Received 10 February 2006

Available online 2 May 2006

Submitted by R.M. Aron

Abstract

We show that for every Lipschitz function f defined on a separable Riemannian manifold M (possibly of infinite dimension), for every continuous $\varepsilon : M \to (0, +\infty)$, and for every positive number $r > 0$, there exists a C^∞ smooth Lipschitz function $g : M \to \mathbb{R}$ such that $|f(p) - g(p)| \leqslant \varepsilon(p)$ for every $p \in M$ and $\mathrm{Lip}(g) \leqslant \mathrm{Lip}(f) + r$. Consequently, every separable Riemannian manifold is uniformly bumpable. We also present some applications of this result, such as a general version for separable Riemannian manifolds of Deville–Godefroy–Zizler's smooth variational principle.

Keywords: Lipschitz function; Riemannian manifold; Smooth approximation

1. Introduction and main results

It is well known, and very useful, that every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ can be uniformly approximated by C^∞ smooth Lipschitz functions whose Lipschitz constants are the same as f's. This can be done easily by considering the integral convolutions

$$f_n = \int_{\mathbb{R}^d} f(y)\varphi_n(x - y)\,dy,$$

* Corresponding author.

E-mail addresses: azagra@mat.ucm.es (D. Azagra), ferrera@mat.ucm.es (J. Ferrera), flopez_mesas@mat.ucm.es (F. López-Mesas), yenny_rangel@mat.ucm.es (Y. Rangel).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.03.088

where the φ_n satisfy $\int_{\mathbb{R}^d} \varphi_n = 1$ and $\mathrm{supp}(\varphi_n) \subset B(0, 1/n)$. This method of smooth approximation has many advantages over other standard procedures like smooth partitions of unity, as the integral convolutions preserve many geometrical properties that f may have, such as convexity or Lipschitzness. Indeed, if f is K-Lipschitz then f_n is K-Lipschitz as well.

For finite-dimensional Riemannian manifolds, Greene and Wu [7–9] used a refinement of this integral convolution procedure to get very useful results on smooth approximation of convex or Lipschitz functions defined on Riemannian manifolds (in fact they applied this method to prove several theorems about the structure of complete noncompact manifolds of positive curvature). It turned out, however, that, when one is interested in approximating a convex function h by C^∞ convex functions this method works out (that is, gives convex f_n's) in Riemannian manifolds only when the function h is *strictly convex* (see [8]); and also that, when one needs to perform a C^0-fine approximation of a Lipschitz function f by C^∞ smooth Lipschitz functions, the approximations f_n have Lipschitz constants which are arbitrarily close to the Lipschitz constant of f (but are not equal in general).

Unfortunately, the integral convolution method breaks down in infinite dimensions (due to the lack of a suitable measure like Lebesgue's one), and other methods have to be employed instead. It is well known that C^∞ smooth partitions of unity exist on every Riemannian manifold and can of course be used to get C^0-fine approximation of continuous functions by C^∞ smooth functions. On the other hand, Moulis [11] showed that C^1-fine approximations of C^1 smooth functions by C^∞ smooth functions are also available on infinite-dimensional Riemannian manifolds. We should also mention that infimal convolutions with squared geodesic distances can be used to regularize convex functions on Riemannian manifolds of nonpositive sectional curvature [1].

However, no one seems to have considered the natural question whether every Lipschitz function f defined on an infinite-dimensional Riemannian manifold can be C^0-finely approximated by C^∞ smooth functions g whose Lipschitz constants also approximate the Lipschitz constant of f. We think this is a very interesting question because many functions arising from geometrical problems on Riemannian manifolds are Lipschitz but not C^1 smooth (the distance function to a closed subset of a manifold is a typical instance), so smooth approximations which almost preserve Lipschitz constants can be very helpful in the analysis of such problems.

In fact we were motivated to study this question by an open problem from [2]: whether a version for Riemannian manifolds of the Deville–Godefroy–Zizler smooth variational principle [4,5] holds for every complete separable Riemannian manifold. This is a very interesting problem because the DGZ variational principle is an invaluable tool in the (nonsmooth) analysis of Hamilton–Jacobi equations defined on Riemannian manifolds. In [2] we were able to prove such a variational principle under the assumption that the manifold was *uniformly bumpable* (see Definition 2), but the question whether or not every Riemannian manifold is uniformly bumpable remained open.

In this note, as a consequence of our result on smooth Lipschitz approximation we will answer these two questions in the affirmative: every separable Riemannian manifold M is uniformly bumpable and, consequently, if M is complete, satisfies the DGZ smooth variational principle.

On the other hand, we have been informed that Garrido, Jaramillo and Rangel [6] have recently established an infinite-dimensional version of the Myers–Nakai theorem [12,13] under the assumption that the manifold is uniformly bumpable (and therefore their result holds in fact for every separable Riemannian manifold). This encourages us to expect that the result we present on smooth Lipschitz approximation (as well as the fact that every separable Riemannian manifold is uniformly bumpable) will find more applications beyond the DGZ variational principle or the infinite-dimensional Myers–Nakai theorem.

Let us now state the main result of this note.

Theorem 1. *Let M be a separable Riemannian manifold, let $f: M \to \mathbb{R}$ be a Lipschitz function, let $\varepsilon: M \to (0, +\infty)$ be a continuous function, and $r > 0$ a positive number. Then there exists a C^∞ smooth Lipschitz function $g: M \to \mathbb{R}$ such that $|f(p) - g(p)| \leqslant \varepsilon(p)$ for every $p \in M$, and* $\mathrm{Lip}(g) \leqslant \mathrm{Lip}(f) + r$.

Here $\mathrm{Lip}(f)$ and $\mathrm{Lip}(g)$ stand for the Lipschitz constants of f and g, respectively, that is,

$$\begin{aligned}\mathrm{Lip}(f) &= \inf\bigl\{L \geqslant 0\colon \bigl|f(p) - f(q)\bigr| \leqslant L d(p,q)\bigr\} \\ &= \sup\left\{\frac{|f(p) - f(q)|}{d(p,q)}\colon\ p, q \in M,\ p \neq q\right\}.\end{aligned}$$

(Recall that f is said to be Lipschitz provided $\mathrm{Lip}(f)$ is finite, and if $L \geqslant \mathrm{Lip}(f)$ then we say that L is a Lipschitz constant of f, or that f is L-Lipschitz.)

We should stress that we do not know whether a similar statement holds for infinite-dimensional separable Banach spaces with C^∞ smooth bump functions, and that even in the case when M is a separable infinite-dimensional Hilbert space this result seems to be new.

Theorem 1 will be proved in the next section. Let us now deduce the announced consequences. We first recall the definition of uniform bumpability given in [2].

Definition 2. A Riemannian manifold M is *uniformly bumpable* provided there exist numbers $R > 1$ (possibly large) and $r > 0$ (possibly small) such that for every $p \in M$, $\delta \in (0, r)$ there exists a C^1 smooth function $b: M \to [0, 1]$ such that:

(1) $b(p) = 1$;
(2) $b(q) = 0$ if $d(q, p) \geqslant \delta$;
(3) $\sup_{q \in M} \|db(q)\|_q \leqslant R/\delta$.

We have:

Corollary 3. *All separable Riemannian manifolds are uniformly bumpable. In fact the constant R in Definition 2 can always be chosen to be any number bigger than 1, the number r any positive number, and the function b of class C^∞.*

Proof. Let $R > 1$, $0 < \delta < r$, and $p \in M$ be given, and consider the function $f: M \to [0, 1]$ defined by

$$f(q) = \begin{cases} 1 - \frac{1}{\delta} d(q, p), & \text{if } d(q, p) \leqslant \delta, \\ 0, & \text{if } d(q, p) \geqslant \delta. \end{cases}$$

It is clear that f is $\frac{1}{\delta}$-Lipschitz and satisfies $f(p) = 1$, and $f = 0$ off $B(p, \delta)$. By Theorem 1, for any $\varepsilon > 0$ there exists a C^∞ smooth function $g: M \to \mathbb{R}$ such that $|g(q) - f(q)| \leqslant \varepsilon$ for all $q \in M$ and $\mathrm{Lip}(g) \leqslant \frac{1}{\delta} + \varepsilon$. Now take a C^∞ smooth function $\theta: \mathbb{R} \to [0, 1]$ such that

(i) $\theta(t) = 0$ for $t \leqslant \varepsilon$,
(ii) $\theta(t) = 1$ for $t \geqslant 1 - \varepsilon$, and
(iii) $\mathrm{Lip}(\theta) \leqslant \frac{1+\varepsilon}{1-2\varepsilon}$,

and define $b(q) = \theta(g(q))$ for all $q \in M$. Then it is clear that $b(p) = 1$, $b(q) = 0$ if $d(q, p) \geqslant \delta$, and

$$\sup_{q\in M} \left\| db(q) \right\|_q = \mathrm{Lip}(b) \leqslant \mathrm{Lip}(\theta)\,\mathrm{Lip}(g) \leqslant \frac{1+\varepsilon}{1-2\varepsilon}\left(\frac{1}{\delta} + \varepsilon\right) \leqslant \frac{R}{\delta}$$

if ε is chosen small enough. □

As a consequence, the version of the Deville–Godefroy–Zizler variational principle which was proved in [2] for uniformly bumpable complete Riemannian manifolds is now seen to hold for every complete separable Riemannian manifold.

Corollary 4 *(DGZ smooth variational principle for Riemannian manifolds). Let M be a complete Riemannian manifold modelled on a separable Hilbert space, and let $F : M \to (-\infty, +\infty]$ be a lower semicontinuous function which is bounded below, $F \not\equiv +\infty$. Then, for each $\delta > 0$ there exists a bounded C^1 smooth and Lipschitz function $\varphi : M \to \mathbb{R}$ such that*:

(1) *$F - \varphi$ attains its strong minimum on M*;
(2) *$\|\varphi\|_\infty := \sup_{p\in M} |\varphi(p)| < \delta$, and $\|d\varphi\|_\infty := \sup_{p\in M} \|d\varphi(p)\|_p < \delta$.*

2. Proof of the main theorem

The proof combines all of the most important approximation methods we know of, that is: integral convolutions, partitions of unity, and infimal convolutions. We first obtain, by using exponential charts and infimal convolutions, local C^1 smooth Lipschitz approximations of f, next we regularize these local approximations by resorting to a result of Moulis [11] (which partially relies on the use of integral convolutions on finite-dimensional subspaces of the separable Hilbert space X on which M is modelled), making sure that the Lipschitz estimates are preserved, and finally we glue all the local approximations together with the help of a specially constructed partition of unity.

We begin with the precise statement of Moulis's result.

Theorem 5 *(Moulis). Let G be an open subset of a separable Hilbert space X, let $f : G \to \mathbb{R}$ be a C^1 smooth function, and $\varepsilon : G \to (0, +\infty)$ be a continuous function. Then there exists a C^∞ function $g : G \to \mathbb{R}$ such that $|f(x) - g(x)| \leqslant \varepsilon(x)$ and $\|f'(x) - g'(x)\| \leqslant \varepsilon(x)$ for every $x \in G$.*

For a proof see [11], or, for a version of this theorem that holds in every Banach space with an unconditional basis and a smooth bump function, see [3].

The first step in the proof of Theorem 1 is to prove a weaker statement in the special case when $M = X$, for a constant ε, and assuming f is bounded. This can be done by combining Moulis's theorem with Lasry and Lions's regularization technique of sup–inf convolutions [10].

Theorem 6. *Let $(X, \|\cdot\|)$ be a separable Hilbert space, let $f : X \to \mathbb{R}$ be a bounded and Lipschitz function, and let $\varepsilon > 0$. Then there exists a C^∞ smooth Lipschitz function $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| \leqslant \varepsilon$ for every $x \in X$, and* $\mathrm{Lip}(g) \leqslant \mathrm{Lip}(f) + \varepsilon$.

Proof. Let us denote $K = \mathrm{Lip}(f)$. Because f is Lipschitz and bounded on X, according to the main theorem of [10], the functions

$$x \mapsto (f_\lambda)^\mu(x) := \sup_{z\in X}\inf_{y\in X}\left\{ f(y) + \frac{1}{2\lambda}\|z-y\|^2 - \frac{1}{2\mu}\|x-z\|^2 \right\}$$

are of class $C^{1,1}$ on X and converge to f uniformly on X as $0<\mu<\lambda\to 0$. So let us pick λ and μ with $0<\mu<\lambda$ and small enough so that

$$\left|(f_\lambda)^\mu(x) - f(x)\right| \leqslant \frac{\varepsilon}{2} \tag{1}$$

for all $x\in X$. We first see that $(f_\lambda)^\mu$ is K-Lipschitz on X. This is an immediate consequence of the fact that the operations of inf- and sup-convolutions (with squared norms or with any other kernel) preserve the Lipschitz constants of the functions to be regularized: that is, if $h : X\to\mathbb{R}$ is L-Lipschitz on X then the function

$$h_\lambda(x) = \inf_{y\in X}\left\{ h(y) + \frac{1}{2\lambda}\|x-y\|^2 \right\}$$

is L-Lipschitz on X as well. Indeed, note first that

$$\inf_{y\in X}\left\{ h(y) + \frac{1}{2\lambda}\|x-y\|^2 \right\} = \inf_{y\in X}\left\{ h(x-y) + \frac{1}{2\lambda}\|y\|^2 \right\},$$

so the function h_λ can be redefined as

$$h_\lambda(x') = \inf_{y\in X}\left\{ h(x'-y) + \frac{1}{2\lambda}\|y\|^2 \right\}. \tag{2}$$

The function h_λ is in formula (2) an infimum of L-Lipschitz continuous functions, so it is L-Lipschitz continuous. Obviously, the same is true of the function h^μ defined by

$$h^\mu(x) = \sup_{z\in X}\left\{ h(z) - \frac{1}{2\mu}\|x-z\|^2 \right\}.$$

Therefore the function $(f_\lambda)^\mu$ has the same Lipschitz constant as f, namely K.

Now, since $(f_\lambda)^\mu$ is C^1 smooth, we can use Moulis's theorem to find a C^∞ smooth function $g : X\to\mathbb{R}$ such that

$$\left|g(x) - (f_\lambda)^\mu(x)\right| \leqslant \frac{\varepsilon}{2} \quad \text{and} \quad \left\|g'(x) - \left((f_\lambda)^\mu\right)'(x)\right\| \leqslant \varepsilon \tag{3}$$

for all $x\in X$. By combining (1) and (3) we obtain that $|f(x)-g(x)|\leqslant\varepsilon$ and also

$$\mathrm{Lip}(g) = \sup_{x\in X}\left\|g'(x)\right\| \leqslant \sup_{x\in X}\left\|\left((f_\lambda)^\mu\right)'(x)\right\| + \varepsilon \leqslant K+\varepsilon = \mathrm{Lip}(f)+\varepsilon. \qquad \square$$

In the proof of Theorem 1 we will have to use the fact that a locally K-Lipschitz function defined on a Riemannian manifold is globally K-Lipschitz.

Lemma 7. *Let M be a Riemannian manifold and let $f : M\to\mathbb{R}$ be a function which is locally K-Lipschitz* (*that is, for every $a\in M$ there exists $\delta=\delta(a)>0$ such that $|f(p)-f(q)|\leqslant Kd(p,q)$ for all $p,q\in B(a,\delta)$*)*. Then f is K-Lipschitz on M.*

Proof. This fact is proved, for instance, in [8, Lemma 2] in the setting of finite-dimensional Riemannian manifolds, but it is clear that the same argument is also valid in the infinite-dimensional case. □

Let us start with the proof of Theorem 1. In this proof X will stand for the separable Hilbert space on which the manifold M is modelled, and $B(p,\delta)$ will denote the open ball of center p and radius δ in M, that is $B(p,\delta)=\{q\in M\colon d(q,p)<\delta\}$. We will also put $K=\mathrm{Lip}(f)$ for short.

With no loss of generality, we can assume that $\varepsilon(p)\leqslant r/2$ for all $p\in M$ (if necessary just replace ε with the continuous function $p\mapsto\min\{\varepsilon(p),r/2\}$). Also, let us fix any number $\varepsilon'>0$ small enough so that

$$\big(K(1+\varepsilon')+\varepsilon'\big)(1+\varepsilon')<K+\frac{r}{2}.$$

Now, for every $p\in M$, let us choose $\delta_p>0$ small enough so that the exponential mapping is a bi-Lipschitz C^∞ diffeomorphism of constant $1+\varepsilon'$ from the ball $B(0_p,3\delta_p)\subset TM_p$ onto the ball $B(p,3\delta_p)\subset M$ (see [2, Theorem 2.3]). Moreover, by continuity of f and ε, we can assume that the δ_p also are sufficiently small so that $\varepsilon(q)\geqslant\varepsilon(p)/2$ and $|f(q)-f(p)|\leqslant\varepsilon(p)/2$ for every $q\in B(p,3\delta_p)$.

Since M is separable we can take a sequence (p_n) of points in M such that

$$M=\bigcup_{n=1}^{\infty}B(p_n,\delta_n),$$

where we denote $\delta_n=\delta_{p_n}$, and also $\varepsilon_n=\varepsilon(p_n)$. Now, for each $n\in\mathbb{N}$ define a function $f_n:B(0_{p_n},3\delta_n)\to\mathbb{R}$ by

$$f_n(x)=f\big(\exp_{p_n}(x)\big),$$

which is $K(1+\varepsilon')$-Lipschitz. We can extend f_n to all of TM_{p_n} by defining

$$\hat{f}_n(x)=\inf_{y\in B(0_{p_n},3\delta_{p_n})}\big\{f_n(y)+K(1+\varepsilon')\|x-y\|_p\big\}.$$

It is well known and very easy to show that $\hat{f}_n$ is a Lipschitz extension of f_n to all of TM_{p_n}, with the same Lipschitz constant $K(1+\varepsilon')$. The function $\hat{f}_n$ is bounded on bounded sets (because it is Lipschitz) but is not bounded on all of TM_{p_n}. Nevertheless we can modify $\hat{f}_n$ outside the ball $B(0_{p_n},4\delta_n)$ so as to make it bounded on all of TM_{p_n}. For instance, put $C=\sup\{|\hat{f}_n(x)|+1\colon x\in B(0_{p_n},4\delta_n)\}$, and define $\tilde{f}_n:TM_{p_n}\to\mathbb{R}$ by

$$\tilde{f}_n(x)=\begin{cases}-C, & \text{if } \hat{f}_n(x)\leqslant -C,\\ \hat{f}_n(x), & \text{if } -C\leqslant\hat{f}_n(x)\leqslant C,\\ +C, & \text{if } C\leqslant\hat{f}_n(x).\end{cases}$$

It is clear that $\tilde{f}_n$ is bounded on all of TM_{p_n} and has the same Lipschitz constant as $\hat{f}_n$, which is less than or equal to $K(1+\varepsilon')$. That is, $\tilde{f}_n$ is a bounded $K(1+\varepsilon')$-Lipschitz extension of f_n to TM_{p_n}.

Next we are going to construct a C^∞ smooth partition of unity subordinated to the covering $\{B(p_n,2\delta_n)\}_{n\in\mathbb{N}}$ of M and to estimate the Lipschitz constant of each of the functions of this partition of unity. Let us take a C^∞ smooth function $\theta_n : \mathbb{R} \to [0, 1]$ such that $\theta_n = 1$ on $(-\infty, \delta_n]$ and $\theta_n = 0$ on $[2\delta_n, +\infty)$, and define

$$\varphi_n(p) = \begin{cases} \theta_n(\|\exp_{p_n}^{-1}(p)\|_{p_n}), & \text{if } p \in B(p_n, 3\delta_n), \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that each of the functions $\varphi_n : M \to \mathbb{R}$ is C^∞ smooth and Lipschitz, and satisfies $\varphi_n = 1$ on the ball $B(p_n, \delta_n)$, and $\varphi_n = 0$ on $M \setminus B(p_n, 2\delta_n)$.

Let us define the functions $\psi_k =: M \to [0, 1]$ by

$$\psi_k = \varphi_k \prod_{j<k} (1 - \varphi_j).$$

It is clear that ψ_k is C_k-Lipschitz, where

$$C_k := \sum_{j \leqslant k} \mathrm{Lip}(\varphi_j),$$

and it is easy to see that

(1) for each $p \in M$, if $k = k(p) = \min\{j: \ p \in B(p_j, \delta_j)\}$ then, because $1 - \psi_k = 0$ on $B(p_k, \delta_k)$, we have that $B(p_k, \delta_k)$ is an open neighborhood of p that meets only finitely many of the supports of the functions ψ_ℓ. Indeed, $\mathrm{supp}(\psi_\ell) \cap B(p_k, \delta_k) = \emptyset$ for all $\ell > k$, and $\mathrm{supp}(\psi_k) \subset B(p_k, 2\delta_k)$;
(2) $\sum_k \psi_k = 1$;

that is, $\{\psi_n\}_{n\in\mathbb{N}}$ is a C^∞ smooth partition of unity subordinated to the covering $\{B(p_n, 2\delta_n)\}_{n\in\mathbb{N}}$ of M.

Now, according to Theorem 6 we can find a C^∞ smooth function $g_n : TM_{p_n} \to \mathbb{R}$ such that

$$\left|g_n(x) - \tilde{f}_n(x)\right| \leqslant \frac{\varepsilon_n}{2^{n+2}(C_n + 1)} \tag{4}$$

for all $x \in TM_{p_n}$, and

$$\mathrm{Lip}(g_n) \leqslant \mathrm{Lip}(\tilde{f}_n) + \varepsilon' \leqslant K(1 + \varepsilon') + \varepsilon'. \tag{5}$$

We are ready to define our approximation $g : M \to \mathbb{R}$ by

$$g(p) = \sum_n \psi_n(p) g_n\bigl(\exp_{p_n}^{-1}(p)\bigr)$$

for any $p \in M$. Observe that if $p \in B(p_n, 3\delta_n)$, because $\exp_{p_n}$ is a C^∞ diffeomorphism from $B(0_{p_n}, 3\delta_n)$ onto $B(p_n, 3\delta_n)$, the expression $\psi_n(p) g_n(\exp_{p_n}^{-1}(p))$ is well defined and is C^∞ smooth on $B(p_n, 3\delta_n)$. On the other hand, if $p \notin B(p_n, 2\delta_n) \supset \mathrm{supp}(\psi_n)$ then $\psi_n(p) = 0$. So we will agree that, for any $p \notin B(p_n, 3\delta_n)$, the expressions $\psi_n(p) g_n(\exp_{p_n}^{-1}(p))$ and $g_n(\exp_{p_n}^{-1}(p))$ both mean zero (whether or not $\exp_{p_n}^{-1}(p)$ makes sense in this case). With these conventions, since the ψ_n form a C^∞ smooth partition of unity it follows that g is well defined and is C^∞ smooth on M.

Let us see that g and $\mathrm{Lip}(g)$ approximate f and $\mathrm{Lip}(f)$, respectively, as required.

Fix any $p \in M$, and let $k = k(p)$ be as in (1) above, so that we have $\psi_\ell = 0$ on $B(p_k, \delta_k)$ for all $\ell > k$, and let us estimate $|f - g|$. To simplify the notation let us denote $x_m = \exp_{p_m}^{-1}(p) \in TM_{p_m}$ (and observe that this expression may well make no sense for many m's, but in such cases the corresponding $g_m(x_m)$ have been defined to be zero; in the following estimation we will also understand that $\tilde{f}_m(x_m)$ means zero if p is outside the ball $B(p_n, 3\delta_n)$). We have

$$\begin{aligned}
|g(p) - f(p)| &= \left| \sum_{m \leqslant k} \psi_m(p) g_m\bigl(\exp_{p_m}^{-1}(p)\bigr) - f(p) \right| \\
&= \left| \sum_{m \leqslant k} \psi_m(p)\bigl[g_m(x_m) - f(p)\bigr] \right| = \left| \sum_{m \leqslant k} \psi_m(p)\bigl[g_m(x_m) - \tilde{f}_m(x_m)\bigr] \right| \\
&\leqslant \sum_{m \leqslant k} \psi_m(p) \frac{\varepsilon_m}{2^{m+2}(C_m+1)} \leqslant \sum_{m \leqslant k} \psi_m(p) \frac{\varepsilon_m}{2} \leqslant \sum_{m \leqslant k} \psi_m(p)\varepsilon(p) \\
&= \sum_m \psi_m(p)\varepsilon(p) = \varepsilon(p).
\end{aligned}$$

Finally, let us check that $\mathrm{Lip}(g) \leqslant K + r$. Since g is defined on a Riemannian manifold, according to Lemma 7, it is enough to show that g is locally $(K+r)$-Lipschitz. Take a point $a \in M$, and define $k = k(a) = \min\{j\colon\ a \in B(p_j, \delta_j)\}$, so that $\mathrm{supp}(\psi_\ell) \cap B(p_k, \delta_k) = \emptyset$ for all $\ell > k$. Let also

$$\delta_a = \min\bigl\{\delta_1, \ldots, \delta_k, \delta_k - d(a, p_k)\bigr\}$$

and

$$F_{p,q} = \bigl\{m \in \{1, \ldots, k\}\colon\ B(p_m, 2\delta_m) \cap \{p, q\} \neq \emptyset\bigr\}.$$

We have that, if $p, q \in B(a, \delta_a)$, then:

(i) For every $m \in \{1, \ldots, k\}$, we have that $p \in B(p_m, 3\delta_m)$ whenever $q \in B(p_m, 2\delta_m)$; and symmetrically $q \in B(p_m, 3\delta_m)$ whenever $p \in B(p_m, 2\delta_m)$. Consequently, for every $m \in F_{p,q}$ we have that $p, q \in B(p_m, 3\delta_m)$; in particular, if $m \in F_{p,q}$, then $x_m := \exp_{p_m}^{-1}(p)$ and $y_m := \exp_{p_m}^{-1}(q)$ are well defined, and we have

$$\bigl|g_m(x_m) - g_m(y_m)\bigr| \leqslant (K + r/2) d(p,q). \tag{6}$$

Indeed, using (5) above and the choice of ε',

$$\begin{aligned}
\bigl|g_m(x_m) - g_m(y_m)\bigr| &\leqslant \bigl(K(1+\varepsilon') + \varepsilon'\bigr) \bigl\|\exp_{p_m}^{-1}(p) - \exp_{p_m}^{-1}(q)\bigr\|_{p_m} \\
&\leqslant \bigl(K(1+\varepsilon') + \varepsilon'\bigr)(1+\varepsilon') d(p,q) \\
&\leqslant (K + r/2) d(p,q).
\end{aligned}$$

(ii) If $m \in \mathbb{N} \setminus F_{p,q}$ then $\psi_m(p) = 0 = \psi_m(q)$ (because $\mathrm{supp}(\psi_m) \subset B(p_m, 2\delta_m)$ and $\mathrm{supp}(\psi_\ell) \cap B(p_k, \delta_k) = \emptyset$ for all $\ell > k$).

Hence we have that, for $p, q \in B(a, \delta_a)$ (with the notation $x_m = \exp_{p_m}^{-1}(p)$ and $y_m = \exp_{p_m}^{-1}(q)$),

$$g(p) = \sum_{m \in F_{p,q}} g_m(x_m)\psi_m(p), \qquad g(q) = \sum_{m \in F_{p,q}} g_m(y_m)\psi_m(q),$$

$$1 = \sum_{m \in F_{p,q}} \psi_m(p) = \sum_{m \in F_{p,q}} \psi_m(q), \quad \text{and}$$

$$\bigl|g_m(x_m) - g_m(y_m)\bigr| \leqslant \bigl(K(1+\varepsilon') + \varepsilon'\bigr)(1+\varepsilon') d(p,q) \quad \text{whenever } m \in F_{p,q}.$$

Let us fix $p, q \in B(a, \delta_a)$. Since $\sum_{m \in F_{p,q}} f(p)(\psi_m(p) - \psi_m(q)) = 0$, we have

$$\begin{aligned} g(p) - g(q) &= \sum_{m \in F_{p,q}} g_m(x_m)\psi_m(p) - \sum_{m \in F_{p,q}} g_m(y_m)\psi_m(q) \\ &= \sum_{m \in F_{p,q}} \big(g_m(x_m) - f(p)\big)\big(\psi_m(p) - \psi_m(q)\big) \\ &\quad + \sum_{m \in F_{p,q}} \big(g_m(x_m) - g_m(y_m)\big)\psi_m(q). \end{aligned}$$

Therefore, using (4), (6), and the fact that ψ_m is C_m-Lipschitz continuous,

$$\begin{aligned} &\big|g(p) - g(q)\big| \\ &\quad \leqslant \sum_{m \in F_{p,q}} \big|g_m(x_m) - f(p)\big| \cdot \big|\psi_m(p) - \psi_m(q)\big| + \sum_{m \in F_{p,q}} \big|g_m(x_m) - g_m(y_m)\big|\psi_m(q) \\ &\quad \leqslant \sum_{m \leqslant k} \frac{\varepsilon_m}{2^{m+2}(C_m + 1)} C_m d(p,q) + \sum_{m \leqslant k} (K + r/2) d(p,q)\psi_m(q) \\ &\quad \leqslant \sum_{m \leqslant k} \frac{\varepsilon(a)}{2^{m+1}} d(p,q) + (K + r/2)d(p,q) \leqslant (K + r)d(p,q), \end{aligned}$$

because $\sum_{m \leqslant k} \frac{\varepsilon(a)}{2^{m+1}} \leqslant \varepsilon(a) \leqslant r/2$. This shows that g is locally $(\mathrm{Lip}(f) + r)$-Lipschitz and concludes the proof of Theorem 1.

References

[1] D. Azagra, J. Ferrera, Inf-convolution and regularization of convex functions on Riemannian manifolds of nonpositive curvature, Rev. Mat. Complut., in press.

[2] D. Azagra, J. Ferrera, F. López-Mesas, Nonsmooth analysis and Hamilton–Jacobi equations on Riemannian manifolds, J. Funct. Anal. 220 (2) (2005) 304–361.

[3] D. Azagra, R. Fry, J. Gómez, J.A. Jaramillo, M. Lovo, C^1-fine approximation of functions on Banach spaces with unconditional basis, Quart. J. Math. 56 (1) (2005) 13–20.

[4] R. Deville, G. Godefroy, V. Zizler, A smooth variational principle with applications to Hamilton–Jacobi equations in infinite dimensions, J. Funct. Anal. 111 (1) (1993) 197–212.

[5] R. Deville, G. Godefroy, V. Zizler, Smoothness and Renormings in Banach Spaces, Pitman Monographs Surveys Pure Appl. Math., vol. 64, 1993.

[6] M.I. Garrido, J.A. Jaramillo, Y. Rangel, Algebras of differentiable functions on infinite-dimensional Riemannian manifolds, preprint.

[7] R.E. Greene, H. Wu, On the subharmonicity and plurisubharmonicity of geodesically convex functions, Indiana Univ. Math. J. 22 (1972/1973) 641–653.

[8] R.E. Greene, H. Wu, C^∞ convex functions and manifolds of positive curvature, Acta Math. 137 (3–4) (1976) 209–245.

[9] R.E. Greene, H. Wu, C^∞ approximations of convex, subharmonic, and plurisubharmonic functions, Ann. Sci. École Norm. Sup. (4) 12 (1) (1979) 47–84.

[10] J.-M. Lasry, P.-L. Lions, A remark on regularization in Hilbert spaces, Israel J. Math. 55 (3) (1986) 257–266.

[11] N. Moulis, Approximation de fonctions différentiables sur certains espaces de Banach, Ann. Inst. Fourier (Grenoble) 21 (4) (1971) 293–345.

[12] S.B. Myers, Algebras of differentiable functions, Proc. Amer. Math. Soc. 5 (1954) 917–922.

[13] M. Nakai, Algebras of some differentiable functions on Riemannian manifolds, Japan. J. Math. 29 (1959) 60–67.