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ON THE MORDELL–WEIL RANK OF AN ABELIAN VARIETY OVER A NUMBER FIELD

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Let K be a number field and A an abelian variety over K. The K-rational points of A are known to constitute a finitely generated abelian group (Mordell–Weil theorem). The problem studied in this paper is to find an explicit upper bound for the rank r of its free part in terms of other invariants of A/K. This is achieved by a close inspection of the classical proof of the so-called 'weak Mordell–Weil theorem'.

1. Introduction

Let K be a number field and A an abelian variety over K. The K-rational points of A are known to constitute a finitely generated abelian group (Mordell-Weil theorem) and it is an interesting question to give an explicit upper bound for the rank r of its free part in terms of other invariants of A/K.

In case A is an elliptic curve and $K=\mathbb{Q}$ there are already some theorems in this direction. For example, Tate proved the following (cf. [2, Chapter 6]):

"Let E be an elliptic curve over $\mathbb{Q}$ given by an equation $y^2=x^3+ax^2+bx$ with $a, b\in\mathbb{Z}$. Then $r\leq s+t+1$ where s and t are the numbers of prime divisors of b and a^2-4b respectively. (Note that the discriminant of this model of E is $2^4b^2(a^2-4b)$.)"

A somewhat sharper bound for elliptic curves over $\mathbb{Q}$ having $\mathbb{Q}$-rational (not necessarily 2-) torsion points can be found in [5], and for elliptic curves over $\mathbb{Q}$ having no rational 2-torsion points a similar bound is obtained in [1].

Under the assumption of very powerful conjectures (Birch and Swinnerton–Dyer, Taniyama–Weil and the generalized Riemann hypothesis), Mestre proves in [6] and

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[7] that for elliptic curves E over $\mathbb{Q}$, the asymptotic equality $r = O(\log N/\log\log N)$ (with N the conductor of $E/\mathbb{Q}$) holds. It seems to be beyond our present knowledge to obtain such a result without using these hypotheses. Mestre also mentions in the introduction of [6], that for elliptic curves $E/\mathbb{Q}$ one can prove $r = O(\log N)$ without using any conjectures. This type of bound is true for abelian varieties over number fields in general; our aim in this paper is to prove this fact.

More precisely our theorem is the following:

Theorem 1. *Let K be a number field and A an abelian variety over K. Write $d = [K : \mathbb{Q}]$, $g = \dim(A)$ and $r = \operatorname{rank}(A(K))$. Denote by $\mathcal{N}_{A/K}$ the conductor of A/K and by $N_{K/\mathbb{Q}}$ the norm with respect to $K/\mathbb{Q}$. Then one has*

$$r \leq C_1 \log |N_{K/\mathbb{Q}} \mathcal{N}_{A/K}| + C_2 \tag{1}$$

where C_1 is a constant depending only on $[K : \mathbb{Q}]$ and on $\dim A$, and C_2 is another constant depending not only on $\dim A$ and on $[K : \mathbb{Q}]$ but also on the discriminant of $K/\mathbb{Q}$. In particular, one can take for C_1 and C_2 the values

$$C_1 = 2g \prod_{i=0}^{2g-1} (2^{2g} - 2^i)\left(1 + \frac{d}{\log 2}\left(d - 1 + \frac{\prod_{i=0}^{2g-1}(2^{2g} - 2^i) d \log d}{\log 2}\right)\right)$$

and

$$C_2 = 2g\left(2d - 1 + d \prod_{i=0}^{2g-1} (2^{2g} - 2^i) + \frac{d \log |\Delta_{K/\mathbb{Q}}| \prod_{i=0}^{2g-1} (2^{2g} - 2^i)^2}{\log 2}\right).$$

The proof will be a refinement of that of the (weak) Mordell–Weil theorem, as is already remarked in [6] for the case $K = \mathbb{Q}$, $\dim A = 1$.

2. A result from algebraic number theory

In this section we will prove the following:

Proposition 1. *Let K be a number field and L a finite extension of K. Denote by $\Delta_{L/K}$ the discriminant of this extension. Then for every prime ideal $\mathfrak{p}$ of K one has*

$$v_{\mathfrak{p}}(\Delta_{L/K}) \leq [L : K] - 1 + [L : \mathbb{Q}] \log([L : K])/\log p. \tag{2}$$

where $v_{\mathfrak{p}}$ is the valuation at $\mathfrak{p}$ and p is the characteristic of the residue class field at $\mathfrak{p}$.

This is [10, Corollaire on p. 128]; see also [9, Chapter III, the end of §6]. The proof of it follows immediately from the corresponding local result:

Proposition 2. *Let $L_1 \subseteq L_2$ be a finite extension of local fields. Write, as usual, f for the degree of the corresponding extension of the residue class fields and e for the ramification index. Suppose that these residue class fields have characteristic $p > 0$. Denote by v_1 the discrete valuation on L_1. Then*

$$v_1(\Delta_{L_2/L_1}) \leq f(e-1+ev_1(e)).$$

To prove this, write the extension as $L_1 \subseteq L_3 \subseteq L_2$, with L_3/L_1 unramified of degree f and L_2/L_3 totally ramified of degree e. One has $\Delta_{L_2/L_1} = \Delta_{L_2/L_3}^f$. Let v_i be the extention of v_1 to L_i for $i=2,3$ and $\mathfrak{D}$ the different of L_2/L_3. Then

$$v_1(\Delta_{L_2/L_1}) = fv_3(\Delta_{L_2/L_3}) = fv_2(\mathfrak{D}).$$

To compute $v_2(\mathfrak{D})$, take a uniformizing element π of L_2. This element π satisfies an Eisenstein equation $f(\pi)=0$ for a polynomial f of degree e with coefficients in L_3. The ideal $\mathfrak{D}$ is generated by $f'(\pi)$ and it is not hard to check that

$$v_2(f'(\pi)) \leq e-1+v_3(e).$$

From this the proposition easily follows.

3. The main theorem

We will first give a corollary of Proposition 1.

Proposition 3. *For an abelian variety A defined over a number field K, let L be the field obtained by adjoining the coordinates of all m-torsion points of A to K. Denote by $\Delta_{L/K}$ the discriminant of L/K and by $\mathcal{N}_{A/K}$ the conductor of A/K. Then*

$$\Delta_{L/K} \mid (m\mathcal{N}_{A/K})^c \tag{3}$$

for a constant c depending only on m, $[K:\mathbb{Q}]$ and $\dim A$.

Proof. By [11], a prime dividing $\Delta_{L/K}$ divides either m or $\mathcal{N}_{A/K}$. Since $[L:K] \leq \#\mathrm{GL}_{2g}(\mathbb{Z}/m\mathbb{Z})$ (where $g=\dim A$), the proposition immediately follows from Proposition 1.

The following is essentially [12, Exercise 8.1].

Theorem 2. *K and A being as in Proposition* 1, *suppose that all m-torsion points of A are rational over K. Denote by $A(K)$ the group of K-rational points of A. For a finite abelian group G we let $\varrho(G)$ be the minimal number of generators of G. Then the following inequality holds*:

$$\varrho(A(K)/mA(K)) \leq 2g\#S + 2g\varrho(\mathcal{H}_K[m]). \tag{4}$$

Here $g=\dim A$, S is the set consisting of archimedian primes, primes where A has bad reduction, and primes dividing m, $\mathcal{H}_K$ is the ideal class group of K, $\mathcal{H}_K[m]$ is its m-primary part.

Proof. Let $\bar{K}$ be an algebraic closure of K and G the Galois group of $\bar{K}/K$.

Multiplication by m yields an exact sequence of G-modules

$$0 \to A[m] \to A(\bar{K}) \to A(\bar{K}) \to 0$$

($A[m]$ is the group of m-torsion points of A). Using Galois cohomology we get an injective map

$$A(K)/mA(K) \to H^1(G, A[m]).$$

By the assumption it follows that $H^1(G, A[m]) = \mathrm{Hom}(G, A[m])$. Now denote by L the Galois extension of K obtained by adjoining to K the coordinates of all points $P \in A(\bar{K})$ such that $mP \in A(K)$. In fact from its definition it follows that the image of the map above consists of homomorphisms which are trivial on $\mathrm{Gal}(\bar{K}/L)$, hence we obtain an induced map

$$A(K)/mA(K) \to \mathrm{Hom}(G_{L/K}, A[m]),$$

where $G_{L/K}$ denotes the Galois group of L/K.

It is known that L has the following properties [8, Appendix II]:

(1) L/K is abelian and of exponent m.

(2) L/K is unramified outside S.

Such L/K is finite by Kummer theory and this fact is proven for example in [12, VIII, Proposition 1.6]. Close examination of this proof enables one to give an effective upper bound for $[L : K]$ as follows:

From each m-primary cyclic component of $\mathcal{H}_K$, take a generating ideal class and add a prime ideal representing it to S. We thus get a set of primes S' such that $\# S' = \# S + \varrho(\mathcal{H}_K[m])$ and that the ring of S'-integers $\mathcal{O}_{S'}$ has no m-torsion in its ideal class group. Let L' be the maximal abelian extension of K which is of exponent m and which is unramified outside S'. Then $L' = K(\sqrt[m]{a}; a \in \mathcal{O}_{S'}^*/\mathcal{O}_{S'}^{*m})$ and by Dirichlet's unit theorem (suitably modified version, see e.g. [4, V, § 1]), we find $\# S'$ cyclic components in $\mathcal{O}_{S'}^*/\mathcal{O}_{S'}^{*m}$.

So $G_{L/K}$ is a quotient of a subgroup of $(\mathbb{Z}/m\mathbb{Z})^{\# S'}$, thereby proving Theorem 2.

We are now ready to prove our main theorem.

Proof of Theorem 1. Let L be the extension of K generated by the 2-torsion points of A. We apply Theorem 2 to A/L and $m = 2$. Clearly $\mathrm{rank}\, A(K) \leq \mathrm{rank}\, A(L)$. For $S = S(A/L)$ as in Theorem 2 one has

$$\# S \leq [L : K] \log |N_{K/\mathbb{Q}} \mathcal{N}_{A/K}| + 2[L : \mathbb{Q}] \leq C(\log |N_{K/\mathbb{Q}} \mathcal{N}_{A/K}| + 2[K : \mathbb{Q}])$$

with $C = [L : K] \leq \# \mathrm{GL}_{2g}(\mathbb{Z}/2\mathbb{Z})$ bounded solely in terms of $g = \dim A$.

On the other hand, every ideal class of L contains an ideal $\mathcal{I}$ with $N_{L/\mathbb{Q}} \mathcal{I} \leq C' \sqrt{|\Delta_{L/\mathbb{Q}}|}$ for a constant C' depending only on $[K : \mathbb{Q}]$ and $\dim A$ (for an exact form of C', compare [3, V, Theorem 4]). Starting with a rational integer $a \leq C' \sqrt{|\Delta_{L/\mathbb{Q}}|}$, one finds there are at most $[L : \mathbb{Q}]^2 \log a$ prime ideals, so at most $a^{[L : \mathbb{Q}]}$ ideals which divide a. We thus get an inequality

$$\varrho(\mathscr{H}_L[2]) \leq [L : \mathbb{Q}]\log |\Delta_{L/\mathbb{Q}}|/\log 2 + 2^2 \log C'.$$

But $|\Delta_{L/\mathbb{Q}}| = N_{K/\mathbb{Q}}(\Delta_{L/K})|\Delta_{K/\mathbb{Q}}|^{[L:K]}$ and by Proposition 3 it follows that $\Delta_{L/K} \mid (2\mathscr{N}_{A/K})^{\text{some constant}}$. Taking the logarithm, we obtain the desired inequality. It is a routine computation to arrive at the explicit constants mentioned in the statement of Theorem 1.

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References

[1] A. Brumer and K. Kramer, The rank of elliptic curves, Duke Math. J. 44 (1977) 715–743.
[2] D. Husemöller, Elliptic Curves, Graduate Texts in Mathematics 111 (Springer, Berlin, 1987).
[3] S. Lang, Algebraic Number Theory (Addison–Wesley, Reading, MA, 1970).
[4] S. Lang, Fundamentals of Diophantine Geometry (Springer, Berlin, 1983).
[5] B. Mazur, Rational points of abelian varieties with values in towers of number fields, Invent. Math. 25 (1974) 1–61.
[6] J.F. Mestre, Courbes elliptiques et formules explicites, in: Sém. de Théorie des Nombres, Paris, 1981/82, Progress in Mathematics 38 (Birkhäuser, Basel) 179–188.
[7] J.F. Mestre, Formules explicites et minorations de conducteurs de variétés algébriques, Compositio Math. 58 (1986) 209–232.
[8] D. Mumford, Abelian Varieties, (Oxford University Press, Oxford, 2nd ed., 1974).
[9] J.P. Serre, Corps Locaux (Hermann, Paris, 1968).
[10] J.P. Serre, Quelques applications du théorème de densité de Chebotarev, Publ. Math. I.H.E.S. 54 (1981) 123–201.
[11] J.P. Serre and J. Tate, Good reduction of abelian varieties, Ann. of Math. 88 (1968) 492–517.
[12] J.H. Silverman, The Arithmetic of Elliptic Curves, Graduate Texts in Mathematics 106 (Springer, Berlin, 1986).