Maximal and Primitive Elements in Weyl Modules for Type $A_2$

Nanhua Xi

Institute of Mathematics, Academia Sinica, Beijing 100080, People’s Republic of China, and Department of Mathematics, University of California at Riverside, Riverside, California 92507

E-mail: nanhua@math08.math.ac.cn, nanhua@math.ucr.edu

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The submodule structure of Weyl modules has been determined in [DS, I, K] and by Cline (unpublished). The main technique used in [DS] is an analysis of the intertwining homomorphisms between certain Weyl modules. The approach in [I, K] is essentially a direct calculation. Also, Jantzen’s translation principle plays an important role in these works. For a few special cases, Irving gives some maximal and primitive elements in Verma modules of the Frobenius kernel and Weyl modules (see [I]). In this paper we will find out all maximal and primitive elements in Verma modules of the quantized Frobenius kernel and Weyl modules for type $A_2$; see Sections 2–3. Thus we understand the structure of the Verma modules and the Weyl modules more explicitly. Our method is based on the main results in [X1] and some ideas in [I] and is rather simple. In this paper we only work with quantized enveloping algebras at roots of 1 (Lusztig version); for hyperalgebras the approach is completely similar. The method also works for type $B_2$ (see [X2]). The author also did some computations for type $A_3$ using the method; although the size of the computation is large, it still does not seem difficult to work out the structure of Weyl modules for type $A_3$.

1. MAXIMAL AND PRIMITIVE ELEMENTS

In this section we fix notation and give the definition for maximal and primitive elements. We refer to [L1–L4, X1] for additional information.
1.1. Let $U_\xi$ be a quantized enveloping algebra (over $\mathbb{Q}(\xi)$) at a root $\xi$ of 1. We assume that the rank of the associated Cartan matrix is $n$ and the order of $\xi \geq 3$. As usual, the generators of $U_\xi$ are denoted by $E_i^{(a)}$, $F_i^{(a)}$, $K_i$, $K_i^{-1}$, etc. Let $\mathfrak{u}$ be the Frobenius kernel and $\tilde{\mathfrak{u}}$ the subalgebra of $U_\xi$ generated by all elements in $\mathfrak{u}$ and in the zero part of $U_\xi$. For $\lambda \in \mathbb{Z}^n$ and a $U_\xi$-module (or $\tilde{\mathfrak{u}}$-module $M$) we denote by $M_\lambda$ the $\lambda$-weight space of $M$. A nonzero element in $M_\lambda$ will be called a vector of weight $\lambda$ or a weight vector. Let $m$ be a weight vector of a $U_\xi$-module (resp. $\tilde{\mathfrak{u}}$-module) $M$. We call $m$ maximal if $E_i^{(a)}m = 0$ for all $i$ and $a \geq 1$ (resp. $E_i^{(a)}m = 0$ for all root vectors $E_i$ in the positive part of $\tilde{\mathfrak{u}}$). We call $m$ a primitive element if there exist two submodules $M_2 \subset M_1$ of $M$ such that $m \in M_1$ and the image in $M_1/M_2$ of $m$ is maximal. Obviously, maximal elements are primitive. We have:

(a) Let $m \in M$ be a weight vector and let $P_m$ be the submodule of $M$ generated by $m$. Then $m$ is primitive if and only if the image in $P_m/P_m$ of $m$ is maximal for some submodule $P_m$ of $P$.

Proof. Assume $m$ is primitive. Then we can find submodules $M_2 \subset M_1$ of $M$ such that $m \in M_1$ and the image in $M_1/M_2$ of $m$ is maximal. Let $P_m = P_1 \cap M_2$. Then the image in $P_m/P_m$ of $m$ is maximal. The other direction is obvious.

(b) If $m$ is a primitive element of weight $\lambda$, then $L(\lambda)$ (or $\tilde{L}(\lambda)$) is a composition factor of $M$ (depending on whether $M$ is a $U_\xi$-module or a $\tilde{\mathfrak{u}}$-module). Here and later we write $L(\lambda)$ (resp. $L(\lambda)$) for an irreducible $U_\xi$-module (resp. $\tilde{\mathfrak{u}}$-module) of highest weight $\lambda$.

Proof. Let $P_1, P_2$ be as in (a) and let $Q_1$ be a maximal submodule of $P_1$ which includes $P_2$. Then $N = P_1/Q_1$ is irreducible and the image in $N$ of $m$ is maximal. Therefore $N$ is necessarily isomorphic to $L(\lambda)$ (or $\tilde{L}(\lambda)$).

(c) Let $M$ and $N$ be modules and $\phi : M \rightarrow N$ a homomorphism. Let $m$ be a weight vector in $M$. If $\phi(m)$ is a primitive element of $N$, then $m$ is a primitive element of $M$.

Proof. Let $P_1$ and $Q_1$ be the submodules of $M$ and $N$ generated by $m$ and $\phi(m)$, respectively. Then $\phi$ induces a surjective homomorphism $\phi_1 : P_1 \rightarrow Q_1$. If $\phi(m) = \phi_1(m)$ is primitive, by (a), we can find a submodule $Q_2$ of $Q_1$ such that the image in $Q_1/Q_2$ of $\phi_1(m)$ is maximal. Let $P_2 = \phi_1^{-1}(Q_2)$. Then $\phi_1$ induces an isomorphism $\tilde{\phi}_1 : P_1/P_2 \rightarrow Q_1/Q_2$. Therefore the image in $P_1/P_2$ of $m$ is maximal. In particular, $m$ is primitive.

(d) Let $M, N, \phi, m$ be as in (c) and assume $\phi(m) \neq 0$. If $m$ is a primitive element of $M$, then either $\phi(m)$ is a primitive element of $N$ or $\phi(P_1) = \phi(P_2)$, where $P_1$ is the submodule of $M$ generated by $m$ and $P_2$ is any submodule of $P_1$ such that the image in $P_1/P_2$ of $m$ is maximal.
Proof. If $\phi(P_1) \neq \phi(P_2)$, then $Q = \phi(P_1)/\phi(P_2) \neq 0$. In this case $\phi(m)$ is in $\phi(P_1)$ but not in $\phi(P_2)$, and the image in $Q$ of $\phi(m)$ is maximal, hence $\phi(m)$ is primitive.

If $\phi(P_1) = \phi(P_2)$ for any submodule $P_2$ of $P_1$ such that the image in $P_1/P_2$ of $m$ is maximal, then $\phi(m)$ is not primitive. Otherwise, let $P_2$ be as in the proof of (c). Then the image in $P_1/P_2$ of $m$ is maximal and $\phi(P_1) \neq \phi(P_2)$; this contradicts the assumption.

(e) Let $M, N, \phi, m$ be as in (c) and assume $\phi(m) \neq 0$. If $m$ is a maximal element of $M$, then $\phi(m)$ is a maximal element of $N$.

Proof. This is obvious.

We shall denote by $Z(\lambda)$ the Verma module of $\tilde{U}$ with highest weight $\lambda$ and denote by $1_\lambda$ a nonzero element in $Z(\lambda)_L$. Recall that to define $U_L$ we need to choose $d_i \in \{1, 2, 3\}$ such that $(d_i^a_i)$ is symmetric, where $(a_i^a_i)$ is the concerned $n \times n$ Cartan matrix. Let $l_i$ be the order of $\xi^{2d_i}$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ we set $\lambda^i = (l_i\lambda_1, \ldots, l_i\lambda_n)$. We call $\lambda$ $1$-restricted if $0 \leq \lambda_i \leq l_i - 1$ for all $i$. Denote by $Z_{\lambda^i}^{\mathbb{N}}$ the set of all $1$-restricted elements in $\mathbb{Z}^n$. The following fact is well known.

(f) Let $\lambda' \in Z_{\lambda^i}^{\mathbb{N}}$, $\lambda'' \in \mathbb{Z}^n$. Set $\lambda = \lambda' + \lambda'' \in \mathbb{Z}^n$. Then $F_i\lambda^i + 1\lambda''_\lambda$ is maximal if $\lambda' \neq \lambda''$. The following fact is well known.

2. STRUCTURE OF $\tilde{Z}(\lambda)$ FOR TYPE $A_2$

2.1. From now on we assume that $U_L$ is of type $A_2$. In this section we determine the maximal and primitive elements in $\tilde{Z}(\lambda)$ (or equivalently in any highest weight module of $\tilde{U}$) and the submodule structure of $\tilde{Z}(\lambda)$, see Theorems 2.2–2.7. For completeness, we give the definition of $U_L$ and $\tilde{Z}(\lambda)$.

Let $U$ be the associative algebra $\mathbb{Q}(v)$ ($v$ an indeterminate) generated by $E_i, F_i, K_i, K_i^{-1}$ ($i = 1, 2$) with relations

$$K_1K_2 = K_2K_1, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1$$

$$E_iF_j - F_jE_i = \delta_{ij} \frac{K_j - K_j^{-1}}{v - v^{-1}}$$

$$E_i^2E_j - (v + v^{-1})E_iE_iE_j + E_jE_i^2 = 0 \quad \text{if } i, j \text{ are different}$$

$$F_i^2F_j - (v + v^{-1})F_iF_j + F_jF_i^2 = 0 \quad \text{if } i, j \text{ are different}.$$
\[ \prod_{h=1}^{a}((v^h - v^{-h})/(v - v^{-1})) \] if \( a \geq 1 \) and \([0] = 1\). Note that

\[
\begin{bmatrix} K_i, c \\ a \end{bmatrix} = \prod_{h=1}^{a} \frac{v^{c-h+1}K_i - v^{-c-h+1}K_i}{v^h - v^{-h}}
\]

is in \( U' \) for all \( c \in \mathbb{Z}, a \in \mathbb{N} \). We understand that \([K_i, c] = 1\) if \( a = 0\) also.

\[ F_{i}^{(a)} = (F_{i}F_{i} - vF_{i}F_{i})^{a}/[a] \] and \[ F_{i}^{(a)} = (F_{i}F_{i} - vF_{i}F_{i})^{a}/[a] \] are in \( U' \) for all \( a \in \mathbb{N} \). Regard \( Q(\xi) \) as an \( \mathbb{A} \)-algebra by specializing \( v \) to \( \xi \). Then \( U_{\xi} = U' \otimes_{\mathbb{A}} Q(\xi) \).

For convenience, the images in \( U'_{\xi} \) of \( E_{i}^{(a)}, F_{i}^{(a)}, F_{i}^{(a)}_{12}, F_{i}^{(a)}_{12}, K_{i}, K_{i}^{-1} / [K_{i}, c] \), etc., will be denoted by the same notation, respectively. Let \( l' = l_1 = l_2 \). In \( U_{\xi} \) we have \( F_{i}^{l'} = F_{i}^{l'} = 0 \). The Frobenius kernel \( u \) of \( U_{\xi} \) is the subalgebra of \( U'_{\xi} \) generated by all \( E_{i}, F_{i}, K_{i}, K_{i}^{-1}, i = 1, 2 \). Its negative part \( u^- \) is generated by all \( F_{i} \). Note that \( F_{i}^{l'}, F_{i}^{(a)} \) are in \( u^- \) if \( 0 \leq a \leq l' - 1 \). The subalgebra \( u \) of \( U_{\xi} \) is generated by all \( E_{i}, F_{i}, K_{i}, K_{i}^{-1} / [K_{i}, c] \), \( i = 1, 2; c \in \mathbb{Z}, a \in \mathbb{N} \).

For \( \lambda = (\lambda_{1}, \lambda_{2}) \in \mathbb{Z}^{2} \), we denote by \( I_{\lambda} \) the left ideal of \( u \) generated all \( E_{i}, K_{i} - \xi^{[\lambda]} [K_{i}, c] - h_{-}^{[\lambda]} e_{i} \). (We denote by \( [\lambda]_{\xi} \) the value at \( \xi \) of \( \prod_{h=1}^{a}((v^{h-1} - v^{-h+1})/(v - v^{-1})) \) for any \( b \in \mathbb{Z} \) and \( a \in \mathbb{N} \).) The Verma module \( Z(\lambda) \) of \( u \) is defined as \( u/I_{\lambda} \). Recall that we denote by \( 1_{\lambda} \) a nonzero vector in \( Z(\lambda) \). The following result is a special case of [X1, 4.2 (ii)]

(a) Assume \( 0 \leq a, b \leq l' - 1 \), \( c, d \in \mathbb{Z} \), and let \( \mu = (l'c - 1 + a, l'd - 1 + b) \). Then the element \( F_{i}^{(a)} F_{i}^{(a+b)} F_{i}^{(b)} 1_{\mu} = F_{i}^{(a+b)} F_{i}^{(b)} F_{i}^{(a)} 1_{\mu} \) is maximal in \( Z(\mu) \) and generates the unique irreducible submodule of \( Z(\mu) \). The irreducible submodule is isomorphic to \( L(l'c - 1 - b, l'd - 1 - a) \).

We shall need a few formulas, which are due to Lusztig.

(b) In \( U'_{\xi} \) we have

\[
F_{i}^{(a)} F_{i}^{(b)} = \begin{bmatrix} a + b \\ a \end{bmatrix} F_{i}^{(a+b)}
\]

\[
F_{i}^{(a)} F_{i}^{(b)} = \sum_{0 \leq r \leq a, b} \xi^{(a-r)b(r)} F_{i}^{(b-r)} F_{i}^{(b-r)} F_{i}^{(r)}
\]

\[
F_{i}^{(a)} F_{i}^{(b)} = \sum_{0 \leq r \leq a, b} \xi^{(a-r)b(r)} F_{i}^{(b-r)} F_{i}^{(b-r)} F_{i}^{(r)}
\]
(c) Assume $0 \leq a_0, b_0 \leq l' - 1$, and $a_1, b_1 \in \mathbb{Z}$. We have

$$\begin{bmatrix} a_0 + a_1l' \\ b_0 + b_1l' \end{bmatrix} = \xi (a_0 b_1 - a_1 b_0) l' + (a_1 + 1) b_1 l' \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \left( \begin{array}{c} a_1 \\ b_1 \end{array} \right),$$

where $(\cdot)_n$ is the ordinary binomial coefficient.

Let $\alpha_1 = (2, -1), \alpha_2 = (-1, 2) \in \mathbb{Z}^2$. The set of positive roots is $R^+ = \{ \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \}$. Let $W$ be the Weyl group generated by the simple reflections $s_i$ corresponding to $\alpha_i$. Let $\lambda = (a, b) \in \mathbb{Z}^2_{\geq 0}$ such that $\langle \lambda + \rho, \beta \rangle \neq l'$ for all $\beta \in R^+$. We consider the $W$-orbit of $\lambda$ (dot action), which consists of the following 6 elements,

$$\lambda, s_1 \lambda = \lambda - (a + 1) \alpha_1, \quad s_2 \lambda = \lambda - (b + 1) \alpha_2,$$

$$s_2 s_1 \lambda = \lambda - (a + 1) \alpha_1 - (a + b + 2) \alpha_2,$$

$$s_1 s_2 \lambda = \lambda - (a + b + 2) \alpha_1 - (b + 1) \alpha_2,$$

$$s_4 s_2 s_1 \lambda = \lambda - (a + b + 2) \alpha_1 - (a + b + 2) \alpha_2.$$

**Theorem 2.2.** Let $\lambda' = (a, b) \in \mathbb{Z}^2_{\geq 0}, \lambda'' \in \mathbb{Z}^2$. Set $\lambda = \lambda' + l \lambda'' \in \mathbb{Z}^2$. Assume that $a + b < l'$. Then

(i) The following 6 elements are maximal in $\tilde{Z}(\lambda)$:

$$\tilde{1}_\lambda, \quad F_1^{(a + 1) \lambda}, \quad F_1^{(b + 1) \lambda}, \quad F_1^{(a + b + 2)} F_1^{(a + 1) \lambda}, \quad F_2^{(a + b + 2)} F_2^{(b + 1) \lambda}, \quad F_2^{(a + 1)} F_2^{(a + b + 2)} F_2^{(b + 1) \lambda}.$$

(ii) The following 3 elements are primitive elements in $\tilde{Z}(\lambda)$ but not maximal:

$$\left( F_1^{(a + b + 2)} F_2^{(b + 1) \lambda} - \xi F_2^{(a + b + 2)} F_1^{(a + b + 2)} F_1^{(a + 1) \lambda}, \right.$$

$$\left. F_2^{(a + b + 2)} F_1^{(b + 1) \lambda} - \xi F_1^{(a + b + 2)} F_2^{(a + b + 2)} F_2^{(a + 1) \lambda}, \right.$$
(iii) The maximal and primitive elements in (i)–(ii) provide nine composite factors of \( \hat{Z}(\lambda) \), which are

\[ \begin{align*}
& \tilde{L}(\lambda), \quad \tilde{L}(\lambda - (a + 1)\alpha_1), \quad \tilde{L}(\lambda - (b + 1)\alpha_2), \\
& \tilde{L}(\lambda - (a + 1)\alpha_1 - (a + b + 2)\alpha_2), \\
& \tilde{L}(\lambda - (a + b + 2)\alpha_1 - (b + 1)\alpha_2), \\
& \tilde{L}(\lambda - (a + b + 2)\alpha_1 - (a + b + 2)\alpha_2), \\
& \tilde{L}(\lambda - (a + b + 2)\alpha_1 - (l' + b + 1)\alpha_2), \\
& \tilde{L}(\lambda - (l' + a + 1)\alpha_1 - (a + b + 2)\alpha_2), \quad \tilde{L}(\lambda - l'\alpha_1 - l\alpha_2).
\end{align*} \]

Moreover, \( \hat{Z}(\lambda) \) has only the nine composition factors.

(iv) The submodule lattice of \( \hat{Z}(\lambda) \) is (see [DS, I, K])

\[ \begin{align*}
& \tilde{L}(\lambda) \quad \tilde{L}(\lambda - (a + 1)\alpha_1) \quad \tilde{L}(\lambda - (b + 1)\alpha_2) \\
& \tilde{L}(\lambda - \gamma_1) \quad \tilde{L}(\lambda - \gamma_2) \quad \tilde{L}(\lambda - \gamma_3) \quad \tilde{L}(\lambda - \gamma_4) \quad \tilde{L}(\lambda - \gamma_5) \\
& \tilde{L}(\lambda - (a + b + 2)\alpha_1 - (a + b + 2)\alpha_2)
\end{align*} \]

where \( \gamma_1 = (l' + a + 1)\alpha_1 + (a + b + 2)\alpha_2, \quad \gamma_2 = (a + 1)\alpha_1 + (a + b + 2)\alpha_2, \quad \gamma_3 = l'\alpha_1 + l\alpha_2, \quad \gamma_4 = (a + b + 2)\alpha_1 + (b + 1)\alpha_2, \quad \gamma_5 = (a + b + 2)\alpha_1 + (l' + b + 1)\alpha_2. \)

Proof. According to Subsection 1.1(e)–(f), we see (i) is true. Using Subsections 1.1(e)–(f) and 2.1(a) we get

1. Let \( \mu = \lambda + (l' - a - 1)\alpha_1 \). The following elements are maximal in \( \hat{Z}(\mu) \):

\[ \begin{align*}
& 1_\mu, \quad F_1^{(l' - a - 1)}1_\mu, \quad F_2^{(a + b + 2)}1_\mu, \quad F_1^{(b + 1)}F_1^{(l' - a - 1)}1_\mu, \\
& F_1^{(b + 1)}F_2^{(a + b + 2)}1_\mu, \quad F_2^{(l' - a - 1)}F_1^{(r + b + 1)}F_2^{(a + b + 2)}1_\mu.
\end{align*} \]
(2) Let $\mu = \lambda + (l' - b - 1)\alpha_2$. The following elements are maximal in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \quad F_1^{(a+b+2)}\tilde{1}_\mu, \quad F_2^{(r-b-1)}\tilde{1}_\mu, \quad F_2^{(a+1)}F_1^{(a+b+2)}\tilde{1}_\mu, \quad F_2^{(b+1)}F_2^{(r-b-1)}\tilde{1}_\mu, \quad F_2^{(r-a-1)}F_2^{(f-a+1)}F_1^{(a+b+2)}\tilde{1}_\mu.$$

(3) Let $\mu = \lambda + (l' - a - 1)\alpha_1 + (l' - a - b - 2)\alpha_2$. The following elements are maximal in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \quad F_1^{(b+1)}\tilde{1}_\mu, \quad F_2^{(r-a-b-2)}\tilde{1}_\mu, \quad F_2^{(r-a-1)}F_2^{(b+1)}\tilde{1}_\mu, \quad F_1^{(r-a-1)}F_2^{(r-a-b-2)}\tilde{1}_\mu.$$

(4) Let $\mu = \lambda + (l' - a - b - 2)\alpha_1 + (l' - b - 1)\alpha_2$. The following elements are maximal in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \quad F_1^{(r-a-b-2)}\tilde{1}_\mu, \quad F_2^{(a+1)}\tilde{1}_\mu, \quad F_2^{(r-a-1)}F_2^{(r-a-b-2)}\tilde{1}_\mu.$$

(5) Let $\mu = \lambda + (l' - a - b - 2)\alpha_1 + (l' - a - b - 2)\alpha_2$. The following elements are maximal in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \quad F_1^{(r-a-1)}\tilde{1}_\mu, \quad F_2^{(r-a-1)}\tilde{1}_\mu, \quad F_2^{(r-a-b-2)}\tilde{1}_\mu, \quad F_1^{(r-a-1)}F_2^{(r-a-b-2)}\tilde{1}_\mu.$$

Now we consider the homomorphism,

$$\tilde{Z}(\lambda) \to \tilde{Z}(\lambda + (l' - a - 1)\alpha_1), \quad \tilde{1}_\lambda \to s = F_1^{(r-a-1)}\tilde{1}_\lambda + (l' - a - 1)\alpha_1.$$

Let

$$x = (F_1^{(a+b+2)}F_2^{(r)} - \xi F_1^{(a+b+2)}F_2^{(a+b+2)})F_2^{(b+1)}$$

$$\in \mathfrak{u}^+. $$

Note that

$$F_1^{(a+b+2)}F_2^{(r+b+1)} = F_1^{(a+b+2)}F_2^{(r+b+1)}F_1^{(r-a-1)}\tilde{1}_\lambda + (l' - a - 1)\alpha_1 = \xi F_1^{(r+b+1)}xx.$$

Using Subsection 1.1(c) we see that $\tilde{x}\tilde{1}_\lambda$ is a primitive element of weight $\lambda - (a + b + 2)\alpha_1 - (l' + b + 1)\alpha_2$. It is easy to see that $\tilde{x}\tilde{1}_\lambda$ is not a maximal element, namely $E_2x\tilde{1}_\lambda \neq 0$. 
Similarly we consider the homomorphism,

\[ \tilde{Z}(\lambda) \to \tilde{Z}(\lambda + (l' - b - 1)\alpha_2) \]

Let

\[ y = \left( F_2^{(a + b + 2)} F_1^{(a + 1)} - \xi F_2^{(a + b + 2)} F_1^{(a + 1)} F_2^{(a + b + 2)} \right) F_1^{(a + 1)} \in u^- . \]

Note that

\[ F_2^{(a + b + 2)} F_1^{(a + 1)} = F_2^{(a + b + 2)} F_1^{(a + 1)} F_2^{(a + b + 2)} \tilde{\zeta}_{\lambda + (l' - b - 1)\alpha_2} = \xi F_2^{(a + 1)} y . \]

Using Subsection 1.1(c) we see that \( y \tilde{\zeta}_{\lambda} \) is a primitive element of weight \( \lambda - (l' + a + 1)\alpha_1 - (a + b + 2)\alpha_2 \). It is easy to see that \( y \tilde{\zeta}_{\lambda} \) is not a maximal element.

Since \( F_2^{(l' - a - b - 2)} F_1^{(l' - b - 1)} \in F_1^{(a + 1)} u^- \) (see Subsection 2.1(b)), we have a surjective homomorphism

\[ \tilde{\zeta}_{\lambda} F_1^{(a + 1)} \to \tilde{\zeta}_{\lambda} F_2^{(l' - a - b - 2)} F_1^{(l' - b - 1)} \tilde{\zeta}_{\lambda + (l - a - b - 2)\alpha_1 + (l - a - b - 2)\alpha_2} . \]

Let \( z' = ( F_1^{(l' - a - 1)} F_2^{(l')} - \xi F_2^{(l' - a - 1)} F_1^{(l')} F_2^{(a + 1)} ) \in u^- . \) Then

\[ F_2^{(l' - a - 1)} F_2^{(l' - a - b - 2)} F_1^{(l' - b - 1)} = F_2^{(l' - a - b - 2)} z' F_2^{(l' - a - b - 2)} F_1^{(l' - b - 1)} . \]

Let \( z = z' F_1^{(a + 1)} = F_1^{(l' - a - 1)} F_2^{(l')} F_1^{(a + 1)} . \) A corollary to Subsection 1.1(c) we see \( z \tilde{\zeta}_{\lambda} \) is a primitive element of weight \( \lambda - l\alpha_1 - l\alpha_2 . \) We also can choose \( z \) to be \( F_2^{(l' - b - 1)} F_1^{(l')} F_2^{(a + 1)} \) by consider the homomorphism

\[ \tilde{\zeta}_{\lambda} F_1^{(a + 1)} \to \tilde{\zeta}_{\lambda} F_2^{(l' - a - b - 2)} F_2^{(l' - a - b - 2)} \tilde{\zeta}_{\lambda + (l - a - b - 2)\alpha_1 + (l - a - b - 2)\alpha_2} . \]

Since all the weights of \( x \tilde{\zeta}_{\lambda}, y \tilde{\zeta}_{\lambda}, z \tilde{\zeta}_{\lambda} \) are smaller than \( \lambda - (a + b + 2)\alpha_1 - (a + b + 2)\alpha_2 \) and \( F_2^{(a + 1)} F_1^{(a + b + 2)} F_2^{(b + 1)} \tilde{\zeta}_{\lambda} \) generates the unique irreducible submodule of \( \tilde{Z}(\lambda) \), we see there are no maximal elements in \( \tilde{Z}(\lambda) \) which have the same weight with \( x \tilde{\zeta}_{\lambda} \) or \( y \tilde{\zeta}_{\lambda} \) or \( z \tilde{\zeta}_{\lambda} \).

We have proved (ii).

A corollary to Subsection 1.1(b) we see \( \tilde{Z}(\lambda) \) has the nine composition factors. Note that the dimensions of irreducible \( \tilde{u} \)-modules are known. Comparing the dimensions we know \( \tilde{Z}(\lambda) \) has only the above composition factors. Thus (iii) is proved.

Part (iv) is due to [D S, I, K], which also can be checked directly by using (i)–(ii).
The theorem is proved.

**Theorem 2.3.** Let \( \lambda' = (a, b) \in \mathbb{Z}^2 \), \( \lambda'' \in \mathbb{Z}^2 \). Set \( \lambda = \lambda' + 1 \lambda'' \in \mathbb{Z}^2 \). Assume that \( 0 < a + 1, b + 1 < l' \) and \( a + b + 2 > l' \). Then

(i) The following 7 elements are maximal in \( \tilde{Z}(\lambda) \),

\[
\tilde{1}_\lambda, \quad F_1^{(a+1)} \tilde{1}_\lambda, \quad F_2^{(b+1)} \tilde{1}_\lambda, \quad F_2^{(a+b+2-l')F_1^{(a+1)}} \tilde{1}_\lambda, \\
F_1^{(a+b+2-l')F_2^{(b+1)}} \tilde{1}_\lambda, \quad F_2^{(a+1)}F_1^{(a+b+2)}F_2^{(b+1)} \tilde{1}_\lambda, \\
\sum_{0 \leq r \leq a+b+2-l'} \left[ \frac{b+1-r}{l' - a - 1} \right]^{-1} e^{(a+b+2-l'-r)(b+1-r)} \\
\times F_2^{(a+b+2-l'-r)}F_1^{(a+b+2-l'-r)} \tilde{1}_\lambda.
\]

(ii) The following 2 elements are primitive elements in \( \tilde{Z}(\lambda) \) but not maximal:

\[
(F_2^{(a+1)}F_1^{(b')}) - e^{F_2^{(a+1)}F_1^{(b')}F_1^{(a+1)}} \tilde{1}_\lambda, \\
(F_2^{(b+1)}F_1^{(b')}) - e^{F_2^{(b+1)}F_1^{(b')}F_2^{(b+1)}} \tilde{1}_\lambda.
\]

Moreover there are no maximal elements in \( \tilde{Z}(\lambda) \) which have the same weight with any of above two elements.

(iii) The maximal and primitive elements in (i)–(ii) provide nine composition factors of \( Z(\lambda) \) which are

\[
\tilde{L}(\lambda), \quad \tilde{L}(\lambda - (a+1) \alpha_1), \quad \tilde{L}(\lambda - (b+1) \alpha_2), \\
\tilde{L}(\lambda - (a+1) \alpha_1 - (a+b+2-l') \alpha_2), \\
\tilde{L}(\lambda - (a+b+2-l') \alpha_1 - (b+1) \alpha_2) \\
\tilde{L}(\lambda - (a+b+2) \alpha_1 - (a+b+2) \alpha_2), \\
\tilde{L}(\lambda - (a+b+2-l') \alpha_1 - (a+b+2-l') \alpha_2) \\
\tilde{L}(\lambda - (a+1) \alpha_1 - l' \alpha_2), \quad \tilde{L}(\lambda - l' \alpha_1 - (b+1) \alpha_2).
\]
Moreover, \( \tilde{Z}(\lambda) \) has only nine composition factors.

(iv) The submodule lattice of \( \tilde{Z}(\lambda) \) is (see [DS, 1, K])

\[
\begin{array}{c}
\tilde{L}(\lambda) \\
\tilde{L}(\lambda - \gamma_1) \quad \tilde{L}(\lambda - \gamma_2) \quad \tilde{L}(\lambda - \gamma_3) \\
\tilde{L}(\lambda - (a + 1)\alpha_1 - (a + b + 2 - \ell')\alpha_2) \\
\tilde{L}(\lambda - (a + b + 2 - \ell')\alpha_1 - (b + 1)\alpha_2)
\end{array}
\]

where \( \gamma_1 = (a + 1)\alpha_1, \gamma_2 = (a + 1)\alpha_1 + \ell'\alpha_2, \gamma_3 = (a + b + 2 - \ell')\alpha_1 + (a + b + 2 - \ell')\alpha_2, \gamma_4 = \ell'\alpha_1 + (b + 1)\alpha_2, \gamma_5 = (b + 1)\alpha_2. \)

Proof. According to Subsections 1.1(e)–(f) and 2.1(a) we see the first 6 elements in (i) are maximal.

Consider the homomorphism

\[
\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + (\ell' - a - 1)\alpha_3), \quad \tilde{1}_{\lambda} \rightarrow s = F_1^{(\ell' - a - 1)}1_{\lambda + (\ell' - a - 1)\alpha_1}.
\]

Let

\[
x = \sum_{r=0}^{a+b+2-\ell'} \left[ \frac{b + 1 - r}{\ell' - a - 1} \right]^{-1} \xi^{(a+b+2-\ell'-r)(b+1-r)} F_1^{(\ell' - a - 1)} 1_{\lambda + (\ell' - a - 1)\alpha_1} \times F_2^{(\ell' - a - 1)} 1_{\lambda + (\ell' - a - 1)\alpha_1} \in u^-.
\]

Note that (see Subsection 2.1(b))

\[
F_1^{(b+1)} F_2^{(a+b+2-\ell')} 1_{\lambda + (\ell' - a - 1)\alpha_1} = x F_1^{(\ell' - a - 1)} 1_{\lambda + (\ell' - a - 1)\alpha_1}.
\]

Using Subsection 1.1(c) we see that \( x1_{\lambda} \) is a primitive element of weight \( \lambda - (a + b + 2 - \ell')\alpha_1 - (a + b + 2 - \ell')\alpha_2. \) We will prove that \( x1_{\lambda} \) is maximal after establishing (iv).
Now we consider the homomorphism
\[ \tilde{Z}(\lambda) \to \tilde{Z}(\lambda + (2l' - a - b - 2)\alpha_1 + (l' - b - 1)\alpha_2), \]
\[ \tilde{1}_\lambda \to t = \xi^{(a+1)F_2}F_1^{(a+1)F_1^{(2l'-a-b-2)\alpha_1 + (l'-b-1)\alpha_2}}. \]

Let
\[ y = (F_1^{(a+1)F_2} - \xi^{F_1^{(2l'-a-b-2)\alpha_1 + (l'-b-1)\alpha_2}}) \in u^- \cdot \]

Note that
\[ yt = \xi^{(l'-b-1)F_1^{(a+1)F_2}F_1^{(2l'-a-b-2)\alpha_1 + (l'-b-1)\alpha_2}} \]
is a maximal element in the image of \( \tilde{Z}(\lambda) \). Using Subsection 1.1(c) we see that \( y \tilde{1}_\lambda \) is a primitive element of weight \( \lambda - (a + 1)\alpha_1 - l'\alpha_2 \). We have \( E_2y \tilde{1}_\lambda \neq 0 \).

Similarly, by considering the homomorphism
\[ \tilde{Z}(\lambda) \to \tilde{Z}(\lambda + (l' - a - 1)\alpha_1 + (2l' - a - b - 2)\alpha_2), \]
\[ \tilde{1}_\lambda \to t = \xi^{(l'-a-1)F_1^{(a+1)F_2}F_1^{(2l'-a-b-2)\alpha_1 + (l'-a-1)\alpha_2}}. \]
we see that \( z \tilde{1}_\lambda \) is a primitive element. We have \( E_2z \tilde{1}_\lambda \neq 0 \). We will argue in the end of the proof for the second assertion of (ii).

The nine maximal and primitive elements provide nine composition factors listed in (ii). Comparing the dimensions we see \( \tilde{Z}(\lambda) \) has only the nine composition factors.

By a direct computation we get (iv).

Now we can see easily that the 7th element in (i) is maximal. A simple computation shows that \( E_i y \tilde{1}_\lambda \) and \( E_i z \tilde{1}_\lambda \) \( (i = 1, 2) \) are contained in the submodule of \( \tilde{Z}(\lambda) \) generated by \( \tilde{1}_\lambda \) and \( F_2^{(a+b+2-l')}F_1 \). Thus \( \lambda - (a + b + 2 - l')\alpha_2 - (a + b + 2 - l')\alpha_2 \) is a maximal weight of the maximal submodule of \( \tilde{Z}(\lambda) \), so the 7th element in (i) is maximal.

Now we prove the second assertion of (ii). Assume \( m \) is a maximal element in \( Z(\lambda) \) of weight \( \tau = \lambda - (a + 1)\alpha_1 - l'\alpha_2 \). Since \( Z(\lambda) \) has only one composition factor of highest weight \( \tau \), by Subsections 1.1(b) and 1.1(e), we see \( m \in \mu y \tilde{1}_\lambda \). Let \( m_1 = F_1^{(a+b+2-l')}F_1^{(a+1)\tilde{1}_\lambda} \) and \( m_2 = F_2^{(a+b+2-l')}F_2^{(a+1)\tilde{1}_\lambda} \). By (iv) we see \( m \not\in \mu m_1 + \mu m_2 \). It is easy to check that the \( \tau \)-weight space of \( \mu m_2 \) is zero. Thus we have \( m = hy \tilde{1}_\lambda + r \) for some \( r \in \mu m_2 \) and some nonzero \( h \in Q(\xi) \). Applying \( \xi^{F_2^{(2l'-a-b-2)}} \) to both sides of \( m = hy \tilde{1}_\lambda + r \) we see that \( m_1 \not\in \mu m_2 \), this is not true.
Similarly we see there are no maximal elements in \( Z(\lambda) \) of weight \( \lambda - l\alpha_1 - (b + 1)\alpha_2 \).
The theorem is proved.

Completely as the arguments for Theorems 2.2–2.3 we get the following results.

**Theorem 2.4.** Let \( \lambda' = (a, b) \in \mathbb{Z}_{*, 1}^2 \), \( \lambda'' \in \mathbb{Z}^2 \). Set \( \lambda = \lambda' + \lambda'' \in \mathbb{Z}^2 \). Assume that \( 0 < a + 1, b + 1 < l' \) and \( a + b + 2 = l' \). Then

(i) The following 4 elements are maximal in \( \tilde{Z}(\lambda) \):

\[
\tilde{1}_\lambda, \quad F_1^{(a+1)} \tilde{1}_\lambda, \quad F_2^{(b+1)} \tilde{1}_\lambda, \quad F_2^{(a+1)} F_1^{(b+1)} \tilde{1}_\lambda.
\]

(ii) The maximal elements in (i) provide four composition factors of \( \tilde{Z}(\lambda) \), which are \( \tilde{L}(\lambda), \tilde{L}(\lambda) - (a + 1) \alpha_1, \tilde{L}(\lambda) - (b + 1) \alpha_2, \tilde{L}(\lambda - l' \alpha_1 - l' \alpha_2) \). Moreover, \( \tilde{Z}(\lambda) \) has only the four composition factors.

(iii) The submodule lattice of \( \tilde{Z}(\lambda) \) is as follows (see [DS, I, K]),

\[
\begin{array}{ccc}
\tilde{L}(\lambda) & \tilde{L}(\lambda) - (a + 1) \alpha_1 & \tilde{L}(\lambda) - (b + 1) \alpha_2 \\
\tilde{L}(\lambda - l' \alpha_1 - l' \alpha_2) & & \\
\end{array}
\]

**Theorem 2.5.** Let \( \lambda' = (l' - 1, b) \in \mathbb{Z}_{*, 1}^2 \), \( \lambda'' \in \mathbb{Z}^2 \). Set \( \lambda = \lambda' + \lambda'' \in \mathbb{Z}^2 \). Assume that \( 0 < b + 1 < l' \). Then

(i) The following 3 elements are maximal in \( \tilde{Z}(\lambda) \):

\[
\tilde{1}_\lambda, \quad F_2^{(b+1)} \tilde{1}_\lambda, \quad F_1^{(b+1)} F_2^{(b+1)} \tilde{1}_\lambda.
\]

(ii) The element \( x = (F_2^{(b+1)} F_1^{(l')} - \xi (b+1) F_2^{(b+1)} F_1^{(b+1)}) \tilde{1}_\lambda \) is primitive. There are no maximal elements in \( \tilde{Z}(\lambda) \) of weight \( \lambda - l' \alpha_1 - (b + 1) \alpha_2 \).

(iii) The maximal and primitive elements in (i)–(ii) provide four composition factors of \( \tilde{Z}(\lambda) \) which are \( \tilde{L}(\lambda), \tilde{L}(\lambda) - (b + 1 - \alpha_2), \tilde{L}(\lambda) - (b + 1) \alpha_1 - (b + 1) \alpha_2, \tilde{L}(\lambda - l' \alpha_1 - (b + 1) \alpha_2) \). Moreover, \( \tilde{Z}(\lambda) \) has only the four composition factors.
(iv) The submodule lattice of $\tilde{Z}(\lambda)$ is as follows (see [D S, I, K]),

\[ \begin{array}{c}
\tilde{L}(\lambda) \\
\tilde{L}(\lambda - l'\alpha_1) \\
\tilde{L}(\lambda - (b + 1)\alpha_2) \\
\tilde{L}(\lambda - (b + 1)\alpha_2 - (b + 1)\alpha_2) \\
\end{array} \]

**Theorem 2.6.** Let $\lambda' = (a, l' - 1) \in \mathbb{Z}^2$, $\lambda'' \in \mathbb{Z}^2$. Set $\lambda = \lambda' + \mathbf{1}\lambda'' \in \mathbb{Z}^2$. Assume that $0 < a + 1 < l'$. Then

(i) The following 3 elements are maximal in $\tilde{Z}(\lambda)$:

\[ \tilde{1}_\lambda, \quad F_1^{(a+1)}F_1^{(1)}, \quad F_2^{(a+1)}F_2^{(1)} \tilde{1}_\lambda. \]

(ii) The element $x = (F_1^{(a+1)}E_1^{(1)} - \xi^{l'(a+1)}E_2^{(1)}E_2^{(a+1)})\tilde{1}_\lambda$ is primitive. There are no maximal elements in $\tilde{Z}(\lambda)$ of weight $\lambda - (a + 1)\alpha_1 - l'\alpha_2$.

(iii) The maximal and primitive elements in (i) provide four composition factors of $\tilde{Z}(\lambda)$, which are $\tilde{L}(\lambda)$, $\tilde{L}(\lambda - (a + 1)\alpha_1)$, $\tilde{L}(\lambda - (a + 1)\alpha_1 - (a + 1)\alpha_2)$, $\tilde{L}(\lambda - (a + 1)\alpha_1 - l'\alpha_2)$. Moreover, $\tilde{Z}(\lambda)$ has only the four composition factors.

(iv) The submodule lattice of $\tilde{Z}(\lambda)$ is as follows (see [D S, I, K]),

\[ \begin{array}{c}
\tilde{L}(\lambda) \\
\tilde{L}(\lambda - (b + 1)\alpha_2) \\
\tilde{L}(\lambda - (a + 1)\alpha_1 - l'\alpha_2) \\
\tilde{L}(\lambda - (a + 1)\alpha_1 - (a + 1)\alpha_2) \\
\end{array} \]

**Theorem 2.7.** Let $\lambda' = (l' - 1, l' - 1) \in \mathbb{Z}^2$, $\lambda'' \in \mathbb{Z}^2$. Set $\lambda = \lambda' + \mathbf{1}\lambda'' \in \mathbb{Z}^2$. Then $\tilde{Z}(\lambda)$ is irreducible. (see [X 1]).

3. STRUCTURE OF WEYL MODULES FOR TYPE $A_2$

For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$, we denote by $I_1$ the left ideal of $U_q^+$ generated by all $E_i^{(1)} (a \geq 1)$, $F_i^{(a)} (a_i \geq \lambda_i + 1)$, $K_i - \xi^{\lambda_i} [K_i - \xi^{\lambda_i + 1}]$. The Weyl
module \( V(\lambda) \) of \( U_\xi \) is defined to be \( U_\xi/I_\xi \); its dimension is \((\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2)/2\). Let \( v_\alpha \) be a nonzero element in \( V(\lambda)_\alpha \). A similar discussion as in Section 2 leads to the following results.

**Theorem 3.1.** Let \( \lambda' = (a, b) \in \mathbb{Z}_+^2 \), \( \lambda'' = (c, d) \in \mathbb{Z}_+^2 \). Set \( \lambda = \lambda' + 1 \lambda'' \in \mathbb{Z}_+^2 \). Assume that \( a + b + 2 < l' \) and \( 1 \leq c, d \). Then

(i) The following 6 elements are maximal in \( V(\lambda) \):

\[
\begin{align*}
  v_{\lambda}, & \quad F_1^{(a+1)}v_{\lambda}, \quad F_2^{(b+1)}v_{\lambda}, \quad F_2^{(a+b+2)}F_1^{(a+1)}v_{\lambda}, \\
  F_1^{(a+b+2)}F_2^{(b+1)}v_{\lambda}, & \quad F_2^{(a+1)}F_1^{(a+b+2)}F_2^{(b+1)}v_{\lambda}.
\end{align*}
\]

(ii) The following 3 elements are primitive elements in \( V(\lambda) \) but not maximal:

\[
\begin{align*}
  (dF_2^{(a+b+2)}F_1^{(l')}) - (d-1)F_2^{(l')(a+b+2)}F_1^{(a+1)}v_{\lambda} & \quad (d \geq 2) \\
  (cF_2^{(a+b+2)}F_1^{(l')}) - (c-1)F_2^{(l')(a+b+2)}F_1^{(a+1)}v_{\lambda} & \quad (c \geq 2)
\end{align*}
\]

(We also can choose the third one to be \( F_2^{(l'-b-1)}F_1^{(l')}v_{\lambda} \).) Moreover there are no maximal elements in \( V(\lambda) \) which have the same weight with any of the above three elements.

(iii) The maximal and primitive elements in (i)–(ii) provide the following composition factors of \( V(\lambda) \),

\[
\begin{align*}
  L(\lambda), & \quad L(\lambda - (a + 1) \alpha_1), \quad L(\lambda - (b + 1) \alpha_2), \\
  L(\lambda - (a + 1) \alpha_1 - (a + b + 2) \alpha_2), & \quad L(\lambda - (a + b + 2) \alpha_1 - (b + 1) \alpha_2) \\
  L(\lambda - (a + b + 2) \alpha_1 - (a + b + 2) \alpha_2), & \quad L(\lambda - (a + b + 2) \alpha_1 - (l' + b + 1) \alpha_2) \quad (d \geq 2), \\
  L(\lambda - (l' + a + 1) \alpha_1 - (a + b + 2) \alpha_2) & \quad (c \geq 2), \\
  L(\lambda - l' \alpha_1 - l' \alpha_2).
\end{align*}
\]

Moreover, \( V(\lambda) \) has only the above composition factors.
(iv) If $c, d \geq 2$, the submodule lattice of $V(\lambda)$ is as follows (see [DS, I, K]),

\[
L(\lambda) \quad \xrightarrow{\gamma_1 = (l' + a + 1)\alpha_1 + (a + b + 2)\alpha_2} \quad L(\lambda - (a + 1)\alpha_1) \quad \xrightarrow{\gamma_2 = (a + b + 2)\alpha_2} \quad L(\lambda - (b + 1)\alpha_2)
\]

where $\gamma_1 = (l' + a + 1)\alpha_1 + (a + b + 2)\alpha_2$, $\gamma_2 = (a + b + 2)\alpha_2$, $\gamma_3 = l'\alpha_1 + l'\alpha_2$, $\gamma_4 = (a + b + 2)\alpha_1 + (b + 1)\alpha_2$, $\gamma_5 = (a + b + 2)\alpha_1 + (b' + b + 1)\alpha_2$. When $c = 1, d \geq 2$ (resp. $c \geq 2, d = 1$; or $c = d = 1$), the submodule lattice of $V(\lambda)$ is obtained from the above lattice by deleting $L(\lambda - \gamma_1)$ and the two segments connecting $L(\lambda - \gamma_2)$ (resp. $L(\lambda - \gamma_5)$ and the two segments connecting $L(\lambda - \gamma_5)$; or $L(\lambda - \gamma_1)$, $L(\lambda - \gamma_5)$ and the four segments connecting any of the two modules).

**Theorem 3.2.** Let $\lambda' = (a, b) \in \mathbb{Z}_{+}^2$, $\lambda'' = (c, d) \in \mathbb{Z}_{+}^2$. Set $\lambda = \lambda' + \lambda'' \in \mathbb{Z}_{+}^2$. Assume that $0 < a + 1, b + 1 < l', a + b + 2 > l''$, and $1 \leq c, d$. Then

(i) The following 7 elements are maximal in $V(\lambda)$:

\[
u_{\lambda}, \quad F_1^{(a+1)}u_{\lambda}, \quad F_2^{(b+1)}u_{\lambda}, \quad F_1^{(a+b+2-l')}F_2^{(b+1)}u_{\lambda}, \quad F_1^{(a+b+2)}F_2^{(b+1)}u_{\lambda}, \quad \sum_{0 \leq r \leq a+b+2-l'} \left[ \frac{l' - a - 1}{l' - a - 1} \right]^{-1} \times F_1^{(a+b+2-l'-r)}F_2^{(a+b+2-l'-r)}u_{\lambda}.
\]
(ii) The following 2 elements are primitive elements in $V(\lambda)$ but not maximal:

$$
((d + 1)F_1^{(a + 1)}F_2^{(F)}) - d \xi^{(a + 1)}F_1^{(a + 1)}v_\lambda,
$$

$$
((c + 1)F_2^{(b + 1)}F_1^{(F)}) - c \xi^{(b + 1)}F_1^{(b + 1)}v_\lambda.
$$

Moreover there are no maximal elements in $V(\lambda)$ which have the same weight with any of above two elements.

(iii) The maximal and primitive elements in (i)-(ii) provide nine composition factors of $V(\lambda)$, which are

$$
L(\lambda), \quad L(\lambda - (a + 1)\alpha_1), \quad L(\lambda - (b + 1)\alpha_2),
$$

$$
L(\lambda - (a + 1)\alpha_1 - (a + b + 2 - l')\alpha_2),
$$

$$
L(\lambda - (a + b + 2 - l')\alpha_1 - (b + 1)\alpha_2)
$$

Moreover, $V(\lambda)$ has only the nine composition factors.

(iv) The submodule lattice of $V(\lambda)$ is (see [DS, 1, K])

![Diagram](image.png)

where $\gamma = (a + 1)\alpha_1$, $\gamma_2 = (a + 1)\alpha_1 + l'\alpha_2$, $\gamma_3 = (a + b + 2 - l')\alpha_1 + (a + b + 2 - l')\alpha_2$, $\gamma_4 = l'\alpha_1 + (b + 1)\alpha_2$, $\gamma_5 = (b + 1)\alpha_2$. 
THEOREM 3.3. Let $\lambda' = (a, b) \in \mathbb{Z}_{i,1}^2$, $\lambda'' = (c, d) \in \mathbb{Z}_i^2$. Set $\lambda = \lambda' + 1\lambda'' \in \mathbb{Z}_i^2$. Assume that $0 < a + 1, b + 1 < l', a + b + 2 = l'$ and $1 \leq c, d$. Then

(i) The following 4 elements are maximal in $V(\lambda)$:

$$v_{\lambda'}, F_1^{(a+1)}v_{\lambda'}, F_2^{(b+1)}v_{\lambda'}, F_2^{(a+1)}F_2^{(b+1)}v_{\lambda'}.$$

(ii) The maximal elements in (i) provide four composition factors of $V(\lambda)$, which are $L(\lambda), L(\lambda - (a + 1)\alpha_1), L(\lambda - (b + 1)\alpha_2), L(\lambda - l'\alpha_1 - l'\alpha_2)$. Moreover, $V(\lambda)$ has only the four composition factors.

(iii) The submodule lattice of $V(\lambda)$ is as follows (see [DS, 1, K]),

$$L(\lambda)$$

$$L(\lambda - (a + 1)\alpha_1) \quad L(\lambda - (b + 1)\alpha_2)$$

$$L(\lambda - l'\alpha_1) \quad L(\lambda - l'\alpha_2)$$

THEOREM 3.4. Let $\lambda' = (l' - 1, b) \in \mathbb{Z}_{i,1}^2$, $\lambda' = (c, d) \in \mathbb{Z}_i^2$. Set $\lambda = \lambda' + 1\lambda'' \in \mathbb{Z}_i^2$. Assume that $0 < b + 1 < l'$ and $1 \leq c, d$. Then

(i) The following 3 elements are maximal in $V(\lambda)$:

$$v_{\lambda'}, F_2^{(b+1)}v_{\lambda'}, F_2^{(a+1)}F_2^{(b+1)}v_{\lambda'}.$$

(ii) The element $x = ((c + 1)F_2^{(b+1)}F_2^{(b)} - c\xi F_2^{(b+1)}F_2^{(b+1)})v_{\lambda'}$ is primitive. There are no maximal elements in $V(\lambda)$ of weight $\lambda - l'\alpha_1 - (b + 1)\alpha_2$.

(iii) The maximal and primitive elements in (i)–(ii) provide four composition factors of $V(\lambda)$, which are $L(\lambda), L(\lambda - (b + 1)\alpha_2), L(\lambda - (b + 1)\alpha_1 - (b + 1)\alpha_2), L(\lambda - l'\alpha_1 - (b + 1)\alpha_2)$. Moreover, $V(\lambda)$ has only the four composition factors.
The submodule lattice of \( V(\lambda) \) is as follows (see \([DS, I, K]\)),

\[
L(\lambda) \quad \begin{aligned}
L(\lambda - l' \alpha_1 - (b + 1) \alpha_2) & \quad L(\lambda - (b + 1) \alpha_2) \\
L(\lambda - (b + 1) \alpha_2) & \quad L(\lambda - (a + 1) \alpha_1 - (a + 1) \alpha_2)
\end{aligned}
\]

**Theorem 3.5.** Let \( \lambda' = (a, l' - 1) \in \mathbb{Z}_+^2 \), \( \lambda'' = (c, d) \in \mathbb{Z}_+^2 \). Set \( \lambda = \lambda' + 1 \lambda'' \in \mathbb{Z}_+^2 \). Assume that \( 0 < a + 1 < l' \) and \( 1 \leq c, d \). Then

(i) The following 3 elements are maximal in \( V(\lambda) \):

\[
v_{\lambda''}, \quad F_1^{(a + 1)} v_{\lambda'}, \quad F_2^{(a + 1)} F_1^{(a + 1)} v_{\lambda}.
\]

(ii) The element \( x = ((d + 1) F_1^{(a + 1)} F_2^{(a + 1)} - d F_2^{(a + 1)} F_2^{(a + 1)} v_{\lambda'}) \) is primitive. There are no maximal elements in \( V(\lambda) \) of weight \( \lambda - (a + 1) \alpha_1 - l' \alpha_2 \).

(iii) The maximal and primitive elements in (i)–(ii) provide four composition factors of \( V(\lambda) \), which are \( L(\lambda), L(\lambda - (a + 1) \alpha_1), L(\lambda - (a + 1) \alpha_1 - (a + 1) \alpha_2), L(\lambda - (a + 1) \alpha_1 - l' \alpha_2) \). Moreover, \( V(\lambda) \) has only the four composition factors.

(iv) The submodule lattice of \( V(\lambda) \) is as follows (see \([DS, I, K]\)),

\[
L(\lambda) \quad \begin{aligned}
L(\lambda - (a + 1) \alpha_1) & \quad L(\lambda - (a + 1) \alpha_1 - l' \alpha_2) \\
L(\lambda - (a + 1) \alpha_1 - (a + 1) \alpha_2) & \quad L(\lambda - (a + 1) \alpha_1 - (a + 1) \alpha_2)
\end{aligned}
\]

**Theorem 3.6.** Let \( \lambda' = (l' - 1, l' - 1) \in \mathbb{Z}_+^2 \), \( \lambda'' = (c, 0) \in \mathbb{Z}_+^2 \). Set \( \lambda = \lambda' + 1 \lambda'' \in \mathbb{Z}_+^2 \). Then \( V(\lambda) \) is irreducible (see \([X1]\)).

**Theorem 3.7.** Let \( \lambda = (a, b) \in \mathbb{Z}_+^2 \), \( \lambda'' = (c, 0) \in \mathbb{Z}_+^2 \). Set \( \lambda = \lambda' + 1 \lambda'' \in \mathbb{Z}_+^2 \). Assume that \( a + b + 2 < l' \) and \( 1 \leq c \). Then

(i) The following \( v_{\lambda''}, F_1^{(a + 1)} v_{\lambda''}, F_2^{(a + b + 2)} F_1^{(a + b + 1)} v_{\lambda} \) (\( c \geq 2 \)) are maximal in \( V(\lambda) \).

(ii) The maximal elements in (i) provide the following composition factors of \( V(\lambda) \): \( L(\lambda), L(\lambda - (a + 1) \alpha_1), L(\lambda - (l' + a + 1) \alpha_1 - (a + b \alpha_2). \)
+ 2)\alpha_2$ (c ≥ 2). Moreover, $V(\lambda)$ has only these composition factors. The submodule lattice of $V(\lambda)$ is trivial.

**Theorem 3.7.** Let $\lambda' = (a, b) \in \mathbb{Z}_{+}^{2}$, $\lambda'' = (0, d) \in \mathbb{Z}_{+}^{2}$. Set $\lambda = \lambda' + 1 \lambda'' \in \mathbb{Z}_{+}^{2}$. Assume that $a + b + 2 < l'$ and $1 \leq d$. Then

(i) The following $v_{\lambda'}, F_{2}^{(b+1)}v_{\lambda'}, F_{1}^{(a+b+2)}F_{2}^{(l'+1)}v_{\lambda'} (d \geq 2)$ are maximal in $V(\lambda)$.

(ii) The maximal elements in (i) provide the following composition factors of $V(\lambda)$, $L(\lambda), L(\lambda - (b + 1)\alpha_2), L(\lambda - (a + b + 2)\alpha_1 - (l' + b + 1)\alpha_2) (d \geq 2)$. Moreover, $V(\lambda)$ has only these composition factors. The submodule lattice of $V(\lambda)$ is trivial.

**Theorem 3.8.** Let $\lambda' = (a, b) \in \mathbb{Z}_{+}^{2}, \lambda'' = (c, 0) \in \mathbb{Z}_{+}^{2}$. Set $\lambda = \lambda' + 1 \lambda'' \in \mathbb{Z}_{+}^{2}$. Assume that $0 < a + 1, b + 1 < l', a + b + 2 > l'$, and $1 \leq c$. Then

(i) The following 4 elements are maximal in $V(\lambda)$,

\[ v_{\lambda'}, F_{1}^{(a+1)}v_{\lambda'}, F_{2}^{(b+2-r)}F_{1}^{(a+1)}v_{\lambda'}, \]

\[ \sum_{0 \leq r \leq a+b+2-l'} \left[ b + 1 - r \right]^{-1} \left[ l' - a - 1 \right]^{\xi} \times \xi^{(a+b+2-r)(b+1-r)}F_{2}^{(a+b+2-r)}F_{1}^{(a+b+2-r)}v_{\lambda'}. \]

(ii) The element $F_{2}^{(b+1)}F_{1}^{(l')}v_{\lambda'}$ is a primitive element in $V(\lambda)$ but not maximal. Moreover there are no maximal elements in $V(\lambda)$ of weight $\lambda - l'\alpha_1 - (b + 1)\alpha_2$.

(iii) The maximal and positive elements in (i)–(ii) provide five composition factors of $V(\lambda)$, which are

\[ L(\lambda), L(\lambda - (a + 1)\alpha_1), \]

\[ L(\lambda - (a + 1)\alpha_1 - (a + b + 2 - l')\alpha_2), \]

\[ L(\lambda - (a + b + 2 - l')\alpha_1 - (a + b + 2 - l')\alpha_2), \]

\[ L(\lambda - l'\alpha_1 - (b + 1)\alpha_2). \]

Moreover, $V(\lambda)$ has only the five composition factors.

(iv) The submodule lattice of $V(\lambda)$ is as follows (see [DS, I, K]),

\[
\begin{array}{c}
L(\lambda) \\
L(\lambda - \gamma_2) \\
L(\lambda - \gamma_3) \\
L(\lambda - (a + 1)\alpha_1 - (a + b + 2 - l')\alpha_2) \\
L(\lambda - \gamma_4)
\end{array}
\]

\[
\begin{array}{c}
L(\lambda - \gamma_2) \\
\downarrow \\
L(\lambda - \gamma_3)
\end{array}
\]

\[
\begin{array}{c}
L(\lambda - (a + 1)\alpha_1 - (a + b + 2 - l')\alpha_2) \\
\downarrow \\
L(\lambda - \gamma_4)
\end{array}
\]
where $\gamma_2 = (a + 1)\alpha_1$, $\gamma_3 = (a + b + 2 - l')\alpha_1 + (a + b + 2 - l')\alpha_2$, $\gamma_4 = l\alpha_1 + (b + 1)\alpha_2$.

**Theorem 3.8.** Let $\lambda' = (a, b) \in \mathbb{Z}^*_+, \lambda' = (c, 0) \in \mathbb{Z}^*_+. Set \lambda = \lambda' + \lambda'' \in \mathbb{Z}^*_+. Assume that $0 < a + 1, b + 1 < l', a + b + 2 > l'$, and $1 \leq d$. Then

(i) The following 4 elements are maximal in $V(\lambda)$,
$$v_\lambda, \quad F_2^{(b+1)}v_\lambda, \quad F_1^{(a+b+2-r)}F_2^{(b+1)}v_\lambda,$$
$$\sum_{0 \leq r \leq a+b+2-l'} \left[ b + 1 - r \right]^{-1} \left[ l' - a - 1 \right] \xi^{(a+b+2-r)(b+1-r)}F_2^{(a+b+2-r)}F_1^{(a+b+2-r)}v_\lambda.$$

(ii) The element $F_1^{(a+1)}F_2^{(r)}v_\lambda$ is a primitive element in $V(\lambda)$ but not maximal. Moreover there are no maximal elements in $V(\lambda)$ of weight $\lambda - (a + 1)\alpha_1 - l\alpha_2$.

(iii) The maximal and primitive elements in (i) provide five composition factors $V(\lambda)$, which are
$$L(\lambda), \quad L(\lambda - (b + 1)\alpha_2),$$
$$L(\lambda - (a + b + 2 - l')\alpha_1 - (b + 1)\alpha_2),$$
$$L(\lambda - (a + b + 2 - l')\alpha_1 - (a + b + 2 - l')\alpha_2),$$
$$L(\lambda - (a + 1)\alpha_1 - l\alpha_2).$$
Moreover, $V(\lambda)$ has only the five composition factors.

(iv) The submodule lattice of $V(\lambda)$ is as follows (see [DS, 1, K]),

$$\begin{array}{ccc}
L(\lambda) & / & L(\lambda - \gamma_2) \\
/ & / & L(\lambda - \gamma_3) \\
L(\lambda - (a + b + 2 - l')\alpha_1 - (b + 1)\alpha_2) & / & L(\lambda - \gamma_4)
\end{array}$$

where $\gamma_2 = (a + 1)\alpha_1 + l\alpha_2$, $\gamma_3 = (a + b + 2 - l')\alpha_1 + (a + b + 2 - l')\alpha_2$, $\gamma_4 = (b + 1)\alpha_2$.

**Theorem 3.9.** Let $\lambda' = (a, b) \in \mathbb{Z}^*_+, \lambda' = (c, 0) \in \mathbb{Z}^*_+. Set \lambda = \lambda' + \lambda'' \in \mathbb{Z}^*_+. Assume that $0 < a + 1, b + 1 < l', a + b + 2 > l'$, and $1 \leq c$. Then

(i) The two elements $v_\lambda, F_1^{(a+1)}v_\lambda$ are maximal in $V(\lambda)$.

(ii) The maximal elements in (i) provide two composition factors of $V(\lambda)$, which are $L(\lambda), L(\lambda - (a + 1)\alpha_2)$. Moreover, $V(\lambda)$ has only the two composition factors and the submodule lattice of $V(\lambda)$ is trivial.
**Theorem 3.9.** Let \( \lambda' = (a, b) \in \mathbb{Z}_+^2, \lambda'' = (0, d) \in \mathbb{Z}_+^2 \). Set \( \lambda = \lambda' + 1 \lambda'' \in \mathbb{Z}_+^2 \). Assume that \( 0 < a + 1, b + 1 < l', a + b + 2 = l' \), and \( 1 \leq d \). Then

(i) The two elements \( v_\lambda, F_2^{(b+1)}v_\lambda \) are maximal in \( V(\lambda) \).

(ii) The maximal elements in (i) provide two composition factors of \( V(\lambda) \), which are \( L(\lambda), L(\lambda - (b + 1) \alpha_2) \). Moreover, \( V(\lambda) \) has only the two composition factors and the submodule lattice of \( V(\lambda) \) is trivial.

**Theorem 3.10.** Let \( \lambda' = (l' - 1, b) \in \mathbb{Z}_+^2, \lambda'' = (c, 0) \in \mathbb{Z}_+^2 \). Set \( \lambda = \lambda' + 1 \lambda'' \in \mathbb{Z}_+^2 \). Assume that \( 0 < b + 1 < l' \) and \( 1 \leq c \). Then

(i) The two elements \( v_\lambda, F_2^{(b+1)}F_1^{(b')}v_\lambda \) are maximal in \( V(\lambda) \).

(ii) The maximal elements in (i) provide two composition factors of \( V(\lambda) \), which are \( L(\lambda), L(\lambda - (b + 1) \alpha_2) \). Moreover, \( V(\lambda) \) has only the two composition factors and the submodule lattice of \( V(\lambda) \) is trivial.

**Theorem 3.10'.** Let \( \lambda' = (l' - 1, b) \in \mathbb{Z}_+^2, \lambda'' = (c, 0) \in \mathbb{Z}_+^2 \). Set \( \lambda = \lambda' + 1 \lambda'' \in \mathbb{Z}_+^2 \). Assume that \( 0 < b + 1 < l' \) and \( 1 \leq c \). Then

(i) The three elements \( v_\lambda, F_2^{(b+1)}v_\lambda, F_2^{(b+1)}F_2^{(b+1)}v_\lambda \) are maximal in \( V(\lambda) \).

(ii) The maximal elements in (i) provide three composition factors of \( V(\lambda) \), which are \( L(\lambda), L(\lambda - (a + 1) \alpha_1), L(\lambda - (a + 1) \alpha_1 - (b + 1) \alpha_2) \). Moreover, \( V(\lambda) \) has only the three composition factors and the submodule lattice of \( V(\lambda) \) is trivial.

**Theorem 3.11.** Let \( \lambda' = (a, l' - 1) \in \mathbb{Z}_+^2, \lambda'' = (c, 0) \in \mathbb{Z}_+^2 \). Set \( \lambda = \lambda' + 1 \lambda'' \in \mathbb{Z}_+^2 \). Assume that \( 0 < a + 1 < l' \) and \( 1 \leq c \). Then

(i) The following 3 elements are maximal in \( V(\lambda) \):

\[
v_\lambda, F_2^{(a+1)}v_\lambda, F_2^{(a+1)}F_2^{(a+1)}v_\lambda.
\]

(ii) The maximal elements in (i) provide three composition factors of \( V(\lambda) \), which are \( L(\lambda), L(\lambda - (a + 1) \alpha_1), L(\lambda - (a + 1) \alpha_1 - (a + 1) \alpha_2) \). Moreover, \( V(\lambda) \) has only the three composition factors and the module structure is trivial.

**Theorem 3.11'.** Let \( \lambda' = (a, l' - 1) \in \mathbb{Z}_+^2, \lambda'' = (c, 0) \in \mathbb{Z}_+^2 \). Set \( \lambda = \lambda' + 1 \lambda'' \in \mathbb{Z}_+^2 \). Assume that \( 0 < a + 1 < l' \) and \( 1 \leq d \). Then

(i) The two elements \( v_\lambda, F_2^{(a+1)}F_2^{(b')}v_\lambda \) are maximal in \( V(\lambda) \).

(ii) The maximal elements in (i) provide two composition factors of \( V(\lambda) \), which are \( L(\lambda), L(\lambda - (a + 1) \alpha_1 - l' \alpha_2) \). Moreover, \( V(\lambda) \) has only the two composition factors and the submodule lattice is trivial.
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