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Flavor instabilities in the neutrino line model

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A B S T R A C T

A dense neutrino medium can experience collective flavor oscillations through nonlinear neutrino–neutrino refraction. To make this multi-dimensional flavor transport problem more tractable, all existing studies have assumed certain symmetries (e.g., the spatial homogeneity and directional isotropy in the early universe) to reduce the dimensionality of the problem. In this work we show that, if both the directional and spatial symmetries are not enforced in the neutrino line model, collective oscillations can develop in the physical regimes where the symmetry-preserving oscillation modes are stable. Our results suggest that collective neutrino oscillations in real astrophysical environments (such as core-collapse supernovae and black-hole accretion discs) can be qualitatively different from the predictions based on existing models in which spatial and directional symmetries are artificially imposed.

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1. Introduction

Neutrinos are influential in many hot and dense astrophysical environments where they are copiously produced. For example, 99% of the immense power of a core-collapse supernova (SN) is carried away by $\sim 10^{58}$ neutrinos within just ~ 10 seconds (see, e.g., [1] for a review). Through the reactions

$$\nu_e + n \rightleftharpoons p + e^-, \qquad \bar{\nu}_e + p \rightleftharpoons n + e^+ \qquad (1)$$

electron-flavor neutrinos extract energy from and deposit energy into the environment and change the n-to-p ratio of the baryonic matter.

It has been firmly established by various experiments that neutrinos can oscillate among different flavors or weak interaction states during propagation. Most of the neutrino mixing parameters have been determined, although it is still unknown whether the neutrino has a normal mass hierarchy (NH) or an inverted one (IH), i.e. whether $|\nu_3\rangle$ is the most massive of the three neutrino mass eigenstates $|\nu_i\rangle$ $(i = 1, 2, 3)$ or not (see, e.g., [2] for a review). The Mikheyev–Smirnov–Wolfenstein (MSW) [3,4] flavor transformation of neutrinos in ordinary matter is also well understood. However, because of its nonlinearity, our understanding of neutrino oscillations in dense neutrino media (such as the one surrounding the

proto-neutron star in SN) is still elementary and requires more investigation.

In the absence of collisions the flavor evolution of neutrinos is described by the Liouville equations [5–7]

$$\partial_t \rho + \hat{\mathbf{v}} \cdot \nabla \rho = -i[H_0 + H_{\nu\nu}, \rho], \qquad (2a)$$

$$\partial_t \bar{\rho} + \hat{\mathbf{v}} \cdot \nabla \bar{\rho} = -i[\bar{H}_0 + H_{\nu\nu}, \bar{\rho}], \qquad (2b)$$

where $\hat{\mathbf{v}}$ is the velocity of the neutrino, $\rho(t, \mathbf{x}, \mathbf{p})$ is the (Wigner-transformed) neutrino flavor density matrix which is a function of time t, position $\mathbf{x}$ and neutrino momentum $\mathbf{p}$, H_0 is the Hamiltonian in the absence of ambient neutrinos, and $H_{\nu\nu}$ is the neutrino(-neutrino coupling) potential. Throughout this letter the physical quantities with bars such as $\bar{\rho}$ and $\bar{H}_0$ are for antineutrinos. We assume that neutrinos are relativistic so that $|\hat{\mathbf{v}}| = 1$ and neutrino energy $E = |\mathbf{p}|$. The difficulty of solving Eq. (2) stems from the neutrino potential which couples neutrinos and antineutrinos of different momenta in the neutrino medium: [8–10]

$$H_{\nu\nu} = \sqrt{2}G_F \int \frac{d^3 p'}{(2\pi)^3}(1 - \hat{\mathbf{v}} \cdot \hat{\mathbf{v}}')[\rho(t, \mathbf{x}, \mathbf{p}') - \bar{\rho}(t, \mathbf{x}, \mathbf{p}')], \qquad (3)$$

where $G_F \approx (293 \text{ GeV})^{-2}$ is the Fermi coupling constant. When the neutrino potential is not negligible, neutrinos in a dense medium can oscillate in a collective manner (see, e.g., [11] for a review). In many cases collective oscillations cause neutrinos of different flavors to swap their spectra in certain energy ranges, a phenomenon dubbed as "stepwise spectral swap" or "spectral split" (e.g., [12–15]).

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Eq. (2) poses a difficult 7-dimensional problem with 1 temporal dimension, 3 spatial dimensions and 3 momentum dimensions. All existing work on collective neutrino oscillations has assumed certain directional symmetries in momentum space and/or spatial symmetries in position space to make this problem more tractable. For example, the spatial spherical symmetry and the directional axial symmetry (about the radial direction) are generally assumed for SN (e.g., [12–22]), and the spatial homogeneity and directional isotropy for the early universe (e.g., [23,24]). However, these spatial and directional symmetries are not necessarily preserved in collective neutrino oscillations. Imposing these symmetries may lead to results that are qualitatively different from those in real physical environments. It was recently shown that collective oscillations can break the directional axial symmetry in SN spontaneously [25,26], which obviously will not occur if this symmetry is artificially enforced. Similar result is also found in the neutrino media with an initial (approximate) isotropy [27]. A recent numeric study shows that the spatial homogeneity can be also broken in a toy model with 1 temporal and 1 spatial dimensions [28].

In this letter we propose to study the neutrino Line model with 2 spatial dimensions. The results derived from this simple model can provide valuable insights of the collective flavor transformation in the neutrino gas models of multiple spatial dimensions.

2. Neutrino Line model

We consider the time-independent (neutrino) Line model in which neutrinos are constantly emitted from the x axis or the (neutrino) Line and propagate inside the x–z plane. For simplicity, we assume that every point on the Line emits only neutrinos and antineutrinos of single energy E with intensities j_ν and $j_{\bar\nu}$, respectively, and in only two directions

$$\hat{\mathbf{v}}_\zeta = [u_\zeta, 0, v_z] \qquad (\zeta = L, R), \tag{4}$$

where $0 < v_z < 1$ and $u_R = -u_L = \sqrt{1 - v_z^2}$. The Line model has 2 spatial dimensions (x, z) and 2 momentum dimensions (E, ζ) (because an antineutrino of energy E can be treated as a neutrino of energy $-E$ for the purpose of neutrino oscillations).

We will consider the scenario of two flavor mixing, e.g., between ν_e and ν_τ, in vacuum. In the mass basis

$$\mathsf{H}_0 = -\bar{\mathsf{H}}_0 = -\frac{\omega\eta}{2}\sigma_3, \tag{5}$$

where $\omega > 0$ is the vacuum neutrino oscillation frequency, $\eta = +1$ and -1 for NH and IH, respectively, and σ_3 is the third Pauli matrix.

We define reduced neutrino density matrices $\varrho \propto \rho$ and $\bar\varrho \propto \bar\rho$ which are normalized by condition

$$\mathrm{tr}\varrho = \mathrm{tr}\bar\varrho = 1. \tag{6}$$

The equations of motion for ϱ and $\bar\varrho$ in the Line model are

$$i\hat{\mathbf{v}}_\zeta \cdot \nabla \varrho_\zeta(x, z) = [\mathsf{H}_0 + \mathsf{H}_{\nu\nu,\zeta}(x, z), \varrho_\zeta(x, z)], \tag{7a}$$

$$i\hat{\mathbf{v}}_\zeta \cdot \nabla \bar\varrho_\zeta(x, z) = [-\mathsf{H}_0 + \mathsf{H}_{\nu\nu,\zeta}(x, z), \bar\varrho_\zeta(x, z)]. \tag{7b}$$

The neutrino potential in the above equation is

$$\mathsf{H}_{\nu\nu,\zeta}(x, z) = \mu[\varrho_{\tilde\zeta}(x, z) - \alpha\bar\varrho_{\tilde\zeta}(x, z)], \tag{8}$$

where

$$\mu = \sqrt{2}(1 - \hat{\mathbf{v}}_L \cdot \hat{\mathbf{v}}_R)G_F j_\nu \tag{9}$$

is the strength of the neutrino–neutrino coupling, $\tilde\zeta = R, L$ are the opposites of ζ, and $\alpha = j_{\bar\nu}/j_\nu$.

To facilitate numerical calculations we further impose a periodic condition on the x–z plane such that

$$\varrho_\zeta(x, z) = \varrho_\zeta(x + L, z), \qquad \bar\varrho_\zeta(x, z) = \bar\varrho_\zeta(x + L, z), \tag{10}$$

where L is the size of the periodic box. We define neutrino density matrices in the Fourier space as

$$\varrho_{\zeta,m}(z) = \frac{1}{L}\int_0^L e^{-imk_0 x}\varrho_\zeta(x, z)\,\mathrm{d}x \tag{11}$$

such that

$$\varrho_\zeta(x, z) = \sum_m e^{imk_0 x}\varrho_{\zeta,m}(z), \tag{12}$$

where m is an integer, and $k_0 = 2\pi/L$. We also define $\bar\varrho_{\zeta,m}(z)$ for the antineutrino in a similar way. Using Eq. (7) and

$$\hat{\mathbf{v}}_\zeta \cdot \nabla\varrho_\zeta = \sum_m e^{imk_0 x}[v_z\varrho'_{\zeta,m} + imk_0 u_\zeta\varrho_{\zeta,m}] \tag{13}$$

we obtain the equations of motion in the Fourier space:

$$iv_z\varrho'_{\zeta,m} = mk_0 u_\zeta\varrho_{\zeta,m} + [\eta\omega\sigma_3/2, \varrho_{\zeta,m}]$$
$$+ \mu\sum_{m'}[\varrho_{\tilde\zeta,m'} - \alpha\bar\varrho_{\tilde\zeta,m'}, \varrho_{\zeta,m-m'}], \tag{14a}$$

$$iv_z\bar\varrho'_{\zeta,m} = mk_0 u_\zeta\bar\varrho_{\zeta,m} + [-\eta\omega\sigma_3/2, \bar\varrho_{\zeta,m}]$$
$$+ \mu\sum_{m'}[\varrho_{\tilde\zeta,m'} - \alpha\bar\varrho_{\tilde\zeta,m'}, \bar\varrho_{\zeta,m-m'}], \tag{14b}$$

where $\varrho'_{\zeta,m} = \mathrm{d}\varrho_{\zeta,m}/\mathrm{d}z$.

3. Flavor instabilities

It is instructive to first review the flavor instability in the bipolar model with 1 spatial (or temporal) dimension and 1 momentum dimension [29–32]. This model can be obtained from the Line model by imposing the translation symmetry along the x axis and the left–right symmetry ($L \leftrightarrow R$) between the two angle directions. For the bipolar model Eq. (7) has solution

$$\varrho_\zeta(x, z) = e^{-i\omega z\eta\sigma_3/2v_z}\varrho(0)e^{i\omega z\eta\sigma_3/2v_z}, \tag{15a}$$

$$\bar\varrho_\zeta(x, z) = e^{-i(-\omega)z\eta\sigma_3/2v_z}\bar\varrho(0)e^{i(-\omega)z\eta\sigma_3/2v_z} \tag{15b}$$

in the absence of ambient neutrinos (i.e. $\mu = 0$), where $\varrho(0)$ and $\bar\varrho(0)$ are the neutrino density matrices at $z = 0$. In this vacuum oscillation solution an antineutrino behaves as a neutrino with a negative oscillation frequency $-\omega$ or negative energy $-E$. Inside the neutrino medium, however, ϱ and $\bar\varrho$ can oscillate with the same frequency Ω under suitable conditions such that

$$\varrho_\zeta(x, z) = e^{-i\Omega z\eta\sigma_3/2v_z}\varrho(0)e^{i\Omega z\eta\sigma_3/2v_z}, \tag{16a}$$

$$\bar\varrho_\zeta(x, z) = e^{-i\Omega z\eta\sigma_3/2v_z}\bar\varrho(0)e^{i\Omega z\eta\sigma_3/2v_z}, \tag{16b}$$

where Ω is a function of μ, α and ω [32]. This solution is equivalent to the precession motion of a pendulum in flavor space [31]. Similar solutions can exist for the scenarios with a continuous energy distribution of neutrinos [33]. When the neutrino mixing angle θ is small, the collective oscillation solution in Eq. (16) does not result in significant neutrino oscillations unless $\kappa = \mathrm{Im}(\Omega) > 0$. A positive κ indicates that the flavor pendulum cannot precess stably and must experience nutation in flavor space. This flavor instability can lead to collective neutrino oscillations with observable effects.

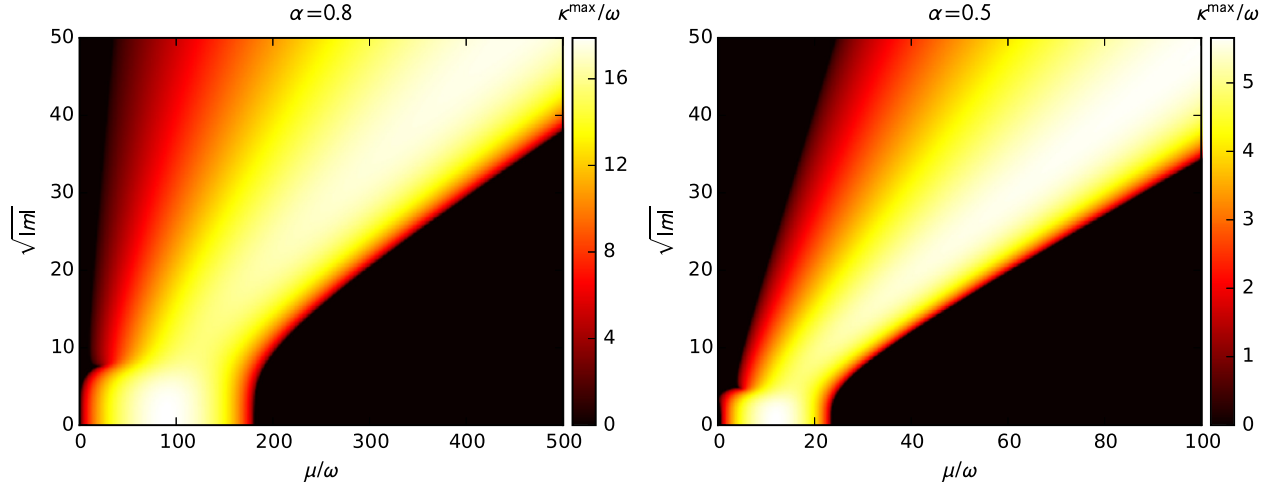For the Line model the collective solution should take the form

Fig. 1. The flavor stability of the two-dimensional $(x$–$z)$, two-angle, mono-energetic neutrino gas in the parameter space of neutrino self-coupling strength μ, which is proportional to the neutrino number density, and moment index m. The color scale of the plots represents $\kappa_m^{\max}(\mu)$, the largest exponential growth rate of the corresponding collective mode of neutrino oscillations in terms of z for given μ and m. Both μ and $\kappa_m^{\max}(\mu)$ are measured in terms of the oscillation frequency ω of the neutrino in vacuum. Collective oscillation modes with $m = 0$ preserve the translation symmetry along the x direction, but those with $m \neq 0$ break this symmetry spontaneously. The larger the value of $|m|$, the smaller scales are the flavor structures in neutrino fluxes. The ratio of the number flux of antineutrinos to that of neutrinos is $\alpha = 0.8$ in the left panel and 0.5 in the right panel. In both panels, the size of the periodic box of the neutrino sources on the x-axis is $L = 20\pi/\omega$, and the propagation directions of the neutrinos are given by unit vectors $[v_x, v_z] = [\pm\sqrt{3}/2, 1/2]$ which make $60°$ angle with the z axis. The results are independent of the neutrino mass hierarchy.

$$\varrho_\zeta(x, z) = e^{-i\Omega z \eta \sigma_3/2v_z}\varrho(x_0, 0)e^{i\Omega z \eta \sigma_3/2v_z}, \tag{17a}$$

$$\bar\varrho_\zeta(x, z) = e^{-i\Omega z \eta \sigma_3/2v_z}\bar\varrho(x_0, 0)e^{i\Omega z \eta \sigma_3/2v_z}, \tag{17b}$$

where $(x_0 = x - u_\zeta z/v_z, z_0 = 0)$ is the coordinate of the emission point of the neutrino which propagates in direction $\hat{\mathbf{v}}_\zeta$ and passes through (x, z). To study the flavor stability in the Line model we assume that the neutrino mixing angle $\theta \ll 1$ and that neutrinos and antineutrinos are in almost pure electron flavor:

$$\varrho \approx \begin{bmatrix} 1 & \epsilon \\ \epsilon^* & 0 \end{bmatrix}, \qquad \bar\varrho \approx \begin{bmatrix} 1 & \bar\epsilon \\ \bar\epsilon^* & 0 \end{bmatrix}, \tag{18}$$

where $|\epsilon| \sim |\bar\epsilon| \ll 1$. The flavor instability of the neutrino medium can be studied by using the method of linearized stability analysis [34]. For this purpose we define

$$\begin{aligned}
D_m^\pm &= \frac{1}{2}(\varrho_{L,m} \pm \varrho_{R,m}) - \frac{\alpha}{2}(\bar\varrho_{L,m} \pm \bar\varrho_{R,m}) \\
&\approx \begin{bmatrix} (1-\alpha)\,\delta_{m,0} & D_m^\pm \\ (D_{-m}^\pm)^* & 0 \end{bmatrix},
\end{aligned} \tag{19a}$$

$$\begin{aligned}
S_m^\pm &= \frac{1}{2}(\varrho_{L,m} \pm \varrho_{R,m}) + \frac{\alpha}{2}(\bar\varrho_{L,m} \pm \bar\varrho_{R,m}) \\
&\approx \begin{bmatrix} (1+\alpha)\,\delta_{m,0} & S_m^\pm \\ (S_{-m}^\pm)^* & 0 \end{bmatrix}.
\end{aligned} \tag{19b}$$

Keeping only the terms up to $\mathcal{O}(|\epsilon|)$ in Eq. (14) we obtain

$$i\frac{d}{dz}\mathbf{W}_m(z) = \Lambda_m \cdot \mathbf{W}_m(z), \tag{20}$$

where $\mathbf{W}_m(z) = [D_m^+, S_m^+, D_m^-, S_m^-]^T$, and

$$\Lambda_m = v_z^{-1}\begin{bmatrix}
0 & -\eta\omega & mq & 0 \\
-\eta\omega - \mu_+ & \mu_- & 0 & mq \\
mq & 0 & 2\mu_- & -\eta\omega \\
0 & mq & -\eta\omega + \mu_+ & \mu_-
\end{bmatrix} \tag{21}$$

with $q = (2\pi/L)\sqrt{1 - v_z^2}$ and $\mu_\pm = (1 \pm \alpha)\mu$.

Eq. (20) shows that in the Line model different Fourier modes are decoupled in the linear regime. In general the real matrix Λ_m has four eigenvalues $\Omega_m^{(i)}$ $(i = 1, 2, 3, 4)$. If $\Omega_m^{(i)}$ is complex, then

its complex conjugate is also an eigenvalue of Λ_m. Because the characteristic polynomial of Λ_m contains only even powers of m, the eigenvalues of Λ_{-m} are the same as those of Λ_m. Further, the set of values of $\kappa_m^{(i)} = \text{Im}[\Omega_m^{(i)}]$ for given m and μ is independent of the neutrino mass hierarchy in the Line model because $\Lambda_m \to 2(\mu_-/v_z)I - \Lambda_{-m}$ under transformation

$$\eta \to -\eta, \qquad (D_m^\pm, S_m^\pm) \to (D_m^\mp, S_m^\mp).$$

In Fig. 1 we plot $\kappa_m^{\max}(\mu)$, the largest of all $\kappa_m^{(i)}(\mu)$, as functions of μ and m for the cases with $v_z = 0.5$, $L = 20\pi/\omega$ and $\alpha = 0.8$ and 0.5, respectively. Fig. 1 shows that collective oscillation modes of different $|m|$ values are unstable in different physical regimes in the Line model. As $|m|$ increases, the flavor unstable region shifts to larger μ, and its width also increases. In addition, as the asymmetry in the number fluxes of neutrinos and antineutrinos decreases, the flavor unstable regions move to larger μ, and the collective oscillation modes develop faster (because $\kappa_m^{\max}$ are larger).

4. Discussion

In the Line model we have assumed two initial approximate symmetries: the (spatial) translation symmetry along the x axis in position space and the (directional) left–right symmetry between the two neutrino beams in momentum space. As mentioned previously, when both symmetries are strictly imposed, the Line model reduces to the bipolar model. Albeit a simple model, the bipolar model has shed important insights on the early simulations of supernova neutrino oscillations in the (neutrino) Bulb model [31,32], which seems perplexing at the first sight [12,13]. For example, because only the homogeneous and (left–right) symmetric mode $[D_0^+, S_0^+]$ can exist in the bipolar model, Eq. (21) shows that collective oscillations can develop only in IH and in the regime

$$\frac{2\omega}{(1 + \sqrt{\alpha})^2} < \mu < \frac{2\omega}{(1 - \sqrt{\alpha})^2}. \tag{22}$$

Indeed, unless the MSW transformation has significantly changed neutrino energy spectra, collective neutrino oscillations in the Bulb model with bipolar-like neutrino fluxes (i.e. dominated by the ν_e

and $\bar{\nu}_e$) occur only in IH and within certain radius range [12]. We note that, like the bipolar model, the Bulb model also has a symmetry in position space and one in momentum space: the spatial spherical symmetry around the center of the proto-neutron star and the directional axial symmetry about the radial direction. This similarity is the reason why the bipolar and Bulb models can produce qualitatively similar results although their geometries are quite different.

The two-beam model proposed in [35] is one step away from the bipolar model where the left–right symmetry in momentum space is not enforced. Eq. (21) shows that, because of the availability of the anti-symmetric mode $[D_0^-, S_0^-]$, collective oscillations can occur also in NH and in the same regime described by Eq. (22). The two-beam model was helpful in understanding collective neutrino oscillations in the extended Bulb model in which the axial symmetry in momentum space is not imposed and the axial-symmetry-breaking modes become unstable in NH for bipolar-like neutrino fluxes [25,36]. Again, the two-beam model and the extend Bulb model can produce qualitatively similar results because a directional symmetry in momentum space is broken in both models.

The toy model studied in [28] is equivalent to the Line model with the left–right symmetry and $\alpha = 1$. In this toy model only the symmetric modes are available. Eq. (21) shows that the inhomogeneous (i.e. with $m \neq 0$) symmetric modes $[D_{m\neq0}^+, S_{m\neq0}^+]$ in this model are unstable in the same regime as the homogeneous symmetric mode does in the bipolar model. Indeed, the numerical calculations in [28] show that a slight matter inhomogeneity can lead to collective oscillations that break the spatial symmetry.

In the Line model neither the spatial symmetry or the directional symmetry is imposed. Because the inhomogeneous symmetric and anti-symmetric modes are coupled even in the linear regime, both the spatial and directional symmetries are generally broken in the Line model, and collective oscillations can occur in both NH and IH. What is more intriguing is that the inhomogeneous modes can be unstable in the regimes of higher neutrino fluxes than the homogeneous modes.

As in the case of bipolar and two-beam models, the results obtained in the Line model may provide valuable insights of collective neutrino oscillations in multi-dimensional models for core-collapse supernovae or black-hole accretion discs (e.g., [37,38]). It seems very likely that the spherical symmetry about the center of the supernova or the axial symmetry about the central axis of the black-hole accretion disc can be broken by collective neutrino oscillations even if such spatial symmetries hold approximately at first. The results of the Line model also suggest that collective neutrino oscillations can occur in astrophysical environments in the regions of higher neutrino fluxes than what is predicted by symmetry-preserving models. This can lead a larger impact on nucleosynthesis in these environments than previously expected (e.g., [17–19,39]).

In the Line model it seems that the central value μ of the flavor unstable region increases linearly with $\sqrt{|m|}$ when $|m|$ is sufficiently large. In a real physical system there must exists a cutoff value m_{max} so that the inhomogeneous modes with $|m| > m_{max}$ are suppressed. After all, Eq. (7) becomes invalid for too small length scales because it is based on the assumption of coherent forward scattering of neutrinos by the medium.

Although we have assumed zero matter density in our discussion, the results also apply to the scenario with large matter density. In the latter case the co-rotating frame technique can be used to "remove" the effects of the matter density on collective neutrino oscillations [30].

5. Conclusions

We have shown that collective neutrino oscillations can break both the spatial and directional symmetries in the neutrino Line model. We found that inhomogeneous neutrino oscillation modes can become unstable in the regimes where the homogeneous modes are stable. Our results suggest that collective neutrino oscillations in real astrophysical environments can be qualitatively different from the predictions based on the models with artificially imposed spatial and directional symmetries.

Our results also suggest that collective oscillations in a multi-dimensional neutrino gas model can be highly inhomogeneous. The large inhomogeneity can pose a great challenge to such studies in both computing resources and numerical modeling.

The Line model which we have studied here has only two neutrino beams from each emission point. It will be interesting to see whether the inhomogeneous modes can be suppressed in the regions of very high matter or neutrino density due to the multi-angle suppression when a continuous angle distribution of neutrino fluxes are employed [40,41].

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