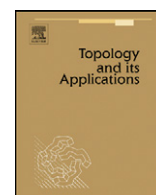


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Weakly infinite dimensional subsets of $\mathbb{R}^{\mathbb{N}}$

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ABSTRACT

The Continuum Hypothesis implies an Erdős–Sierpiński like duality between the ideal of first category subsets of $\mathbb{R}^{\mathbb{N}}$, and the ideal of countable dimensional subsets of $\mathbb{R}^{\mathbb{N}}$. The algebraic sum of a Hurewicz subset – a dimension theoretic analogue of Sierpiński sets and Lusin sets – of $\mathbb{R}^{\mathbb{N}}$ with any compactly.

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1. Introduction

The main purpose of this paper is threefold: We point out strong analogies between the well-known class of countable dimensional subsets of the Hilbert cube, and the classes of Lebesgue measure zero subsets of the real line and the class of first category subsets of complete separable metric spaces. A classical theorem of Erdős and Sierpiński gives, under the Continuum Hypothesis, some explanation of this analogy. We show also that the dimension theoretic analogue of Lusin sets and Sierpiński sets has some of the deeper properties shared by Lusin sets and Sierpiński sets. And we give some information about a class of weakly infinite dimensional spaces which is emerging as an important class in dimension theory. In particular we show, using the Continuum Hypothesis, that the most restrictive of these classes contains metric spaces which are not countable dimensional.

$\mathbb{R}^{\mathbb{N}}$ denotes the Tychonoff product of countably many copies of the real line, \mathbb{R} . The symbol $[0, 1]$ denotes the unit interval and the subspace $[0, 1]^{\mathbb{N}}$ of $\mathbb{R}^{\mathbb{N}}$ denotes the Hilbert cube. The symbol \mathbb{P} denotes the set of irrational numbers, viewed as a subspace of the real line.

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2. The countable dimensional subsets of $\mathbb{R}^{\mathbb{N}}$

Hurewicz [8] defined a subset of a topological space to be *countable dimensional* if it is a union of countably many zero-dimensional subsets of the space, and proved that the Hilbert cube, thus also $\mathbb{R}^{\mathbb{N}}$ is not countable dimensional. Thus, the collection CD of countable dimensional subsets of $\mathbb{R}^{\mathbb{N}}$ is a σ -ideal. Since each metrizable space of cardinality less than 2^{\aleph_0} is zero-dimensional, the cofinality of CD, denoted $\text{cof}(\text{CD})$, is 2^{\aleph_0} . Smirnov and independently Nagami also showed that every separable metric space is a union of \aleph_1 zero-dimensional subsets, and thus the covering number of CD, denoted $\text{cov}(\text{CD})$, is \aleph_1 . This is unlike the σ -ideal \mathcal{M} of first category subsets of \mathbb{R} , and the σ -ideal \mathcal{N} of Lebesgue measure zero subsets of \mathbb{R} , where the cofinality and the covering number are not decided by ZFC.

If one assumes the Continuum Hypothesis (CH), these differences vanish and one can give an explanation for the unusually strong analogies between the theory of the countable dimensional subspaces of $\mathbb{R}^{\mathbb{N}}$ and the theory of the first category subsets of $\mathbb{R}^{\mathbb{N}}$: These two collections are dual in the sense of Erdős and Sierpiński. Chapter 19 of [16] gives a nice exposition of the following classical theorem:

Theorem 1 (Erdős–Sierpiński Duality Theorem). *Let X be a set of cardinality \aleph_1 . Let \mathcal{I} and \mathcal{J} be σ -complete ideals on X such that:*

- (1) $X = A \cup B$ where A and B a disjoint sets with $A \in \mathcal{I}$ and $B \in \mathcal{J}$.
- (2) $\text{cof}(\mathcal{I}) = \aleph_1 = \text{cof}(\mathcal{J})$.
- (3) For each $I \in \mathcal{I}$ there is an $S \subset X \setminus I$ such that $|S| = \aleph_1$ and $S \in \mathcal{I}$.
- (4) For each $J \in \mathcal{J}$ there is an $S \subset X \setminus J$ such that $|S| = \aleph_1$ and $S \in \mathcal{J}$.

Then there is a bijective function $f : X \rightarrow X$ which is its own inverse, such that for each set $E \subseteq X$ we have

$$E \in \mathcal{I} \iff f[E] \in \mathcal{J}.$$

A classical theorem of Tumarkin [26] is an important tool in proving the duality between countable dimensional sets and first category sets of $\mathbb{R}^{\mathbb{N}}$ under CH:

Theorem 2 (Tumarkin). *In a separable metric space each n -dimensional set is contained in an n -dimensional G_δ set.*

Corollary 3. *Assume the Continuum Hypothesis. There is a bijection $f : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}$ such that $f = f^{-1}$ and for each set $E \subset \mathbb{R}^{\mathbb{N}}$, E is countable dimensional if, and only if, $f[E]$ is first category.*

Proof. In Theorem 1 take \mathcal{I} to be CD, the ideal of countable dimensional subsets of $\mathbb{R}^{\mathbb{N}}$, and \mathcal{J} to be \mathcal{M} , the ideal of first category subsets of $\mathbb{R}^{\mathbb{N}}$. By Hurewicz's theorem that $\mathbb{R}^{\mathbb{N}}$ is not countable dimensional, CD is a σ -complete ideal on $\mathbb{R}^{\mathbb{N}}$. By the Baire category theorem \mathcal{M} is a σ -complete ideal on $\mathbb{R}^{\mathbb{N}}$.

Since $\mathbb{R}^{\mathbb{N}}$ is separable, let D be a countable dense set. Then D is zero-dimensional. By Theorem 2 there is a zero-dimensional G_δ set $A \supset D$. But then $B = \mathbb{R}^{\mathbb{N}} \setminus A$ is a first category set, and so (1) of Theorem 1 is satisfied.

Theorem 2 also implies that each countable dimensional subset of $\mathbb{R}^{\mathbb{N}}$ is contained in a countable union of zero-dimensional G_δ sets, and so CD has a cofinal family of cardinality 2^{\aleph_0} . Since \mathcal{M} has a cofinal family of cardinality 2^{\aleph_0} , CH implies that (2) of Theorem 1 holds. Evidently (3) and (4) also hold. \square

3. Hurewicz sets

Some strong analogies between the theory of the ideal of countable dimensional subsets of $\mathbb{R}^{\mathbb{N}}$ and the theories of the ideal \mathcal{N} of Lebesgue measure zero subsets of the real line, and the ideal \mathcal{M} of first category subsets of uncountable complete separable metric spaces emerge when we consider the following features: Call a subset of $\mathbb{R}^{\mathbb{N}}$ a *Hurewicz set* if it is uncountable, but its intersection with every zero-dimensional subset of $\mathbb{R}^{\mathbb{N}}$ is countable. Hurewicz sets are first category: Take a countable dense set $D \subset \mathbb{R}^{\mathbb{N}}$. Since D is zero-dimensional it is, by Theorem 2, contained in a zero-dimensional dense G_δ set E . Since a Hurewicz set meets E in only countably many points the Hurewicz set is first category. We shall see below that Hurewicz sets are first category in a strong sense.

The notion of a Hurewicz set is analogous to the notions of a Lusin set and of a Sierpiński set: A subset of \mathbb{R} is a *Lusin set* if it is uncountable but its intersection with any first category subset of \mathbb{R} is countable. A subset of the real line is a *Sierpiński set* if it is uncountable but its intersection with each Lebesgue measure zero set is countable.

Mahlo [14] and Lusin [13] independently showed that the Continuum Hypothesis implies the existence of a Lusin set of cardinality 2^{\aleph_0} , and Sierpiński [24] showed that the Continuum Hypothesis implies the existence of a Sierpiński set of cardinality 2^{\aleph_0} . It is known that the existence of a Lusin set of cardinality 2^{\aleph_0} does not imply the Continuum Hypothesis, and that the existence of a Sierpiński set of cardinality 2^{\aleph_0} does not imply the Continuum Hypothesis. But Rothberger [23] proved that the simultaneous existence of both a Lusin set of cardinality 2^{\aleph_0} and a Sierpiński set of cardinality 2^{\aleph_0} is equivalent to $2^{\aleph_0} = \aleph_1$. W. Hurewicz [9] showed that the existence of a Hurewicz set is equivalent to the Continuum Hypothesis.

Some deeper properties of the Lusin sets and the Sierpiński sets are tied up with the algebraic properties of the real line viewed as a topological group.

Theorem 4 (Galvin–Mycielski–Solovay). *If $L \subset \mathbb{R}$ is a Lusin set, then for each first category set $M \subset \mathbb{R}$, $L + M \neq \mathbb{R}$.*

Galvin, Mycielski and Solovay [6] proved a significantly stronger theorem: Strong measure zero sets are characterized as sets whose translates by first category sets do not cover the real line. Theorem 4 follows from this since Sierpiński proved that Lusin sets are strong measure zero. Pawlikowski [18] proved the counterpart for Sierpiński sets:

Theorem 5 (Pawlikowski). *If $S \subset \mathbb{R}$ is a Sierpiński set, then for each Lebesgue measure zero set $N \subset \mathbb{R}$, $S + N \neq \mathbb{R}$.*

We expect that a similar statement is true about Hurewicz sets in the topological group $(\mathbb{R}^{\mathbb{N}}, +)$:

Conjecture 1. *If $H \subset \mathbb{R}^{\mathbb{N}}$ is a Hurewicz set, then for each countable dimensional set $C \subset \mathbb{R}^{\mathbb{N}}$, $H + C \neq \mathbb{R}^{\mathbb{N}}$.*

We have a partial result on this conjecture: Call a subset of $\mathbb{R}^{\mathbb{N}}$ *strongly countable dimensional* if it is a union of countably many closed, finite dimensional subsets. This notion was introduced by Nagata [15] and Smirnov [25]. Every strongly countable dimensional set is countable dimensional, but not conversely. Thus, the σ -ideal SCD generated by the strongly countable dimensional subsets of $\mathbb{R}^{\mathbb{N}}$ is a proper subideal of CD. Indeed, every strongly countable dimensional set is of first category in $\mathbb{R}^{\mathbb{N}}$, and thus SCD is also a subideal of the σ -ideal of first category subsets of $\mathbb{R}^{\mathbb{N}}$. Call a subset of $\mathbb{R}^{\mathbb{N}}$ *compactly countable dimensional* if it is a union of countably many compact finite dimensional sets. The σ -ideal KCD generated by the compact finite dimensional sets is a proper subideal of SCD. We shall show

Theorem 6. *If H is a Hurewicz subset of $\mathbb{R}^{\mathbb{N}}$, then for every compactly countable dimensional set $C \subset \mathbb{R}^{\mathbb{N}}$, $H + C$ is a first category subset of $\mathbb{R}^{\mathbb{N}}$.*

Since the union of countably many first category sets is first category, Theorem 6 follows directly from Lemma 8 below. This lemma uses another important classical result:

Theorem 7 (Hurewicz, Tumarkin). *In a separable metric space the union of countably many closed, n -dimensional sets, is an n -dimensional set.*

Lemma 8. *If $H \subseteq \mathbb{R}^{\mathbb{N}}$ is a Hurewicz set and $C \subset \mathbb{R}^{\mathbb{N}}$ is a compact n -dimensional set, then $H + C$ is a first category subset of $\mathbb{R}^{\mathbb{N}}$.*

Proof. Let $D \subset \mathbb{R}^{\mathbb{N}}$ be a countable dense set. Then $D - C$ is a union of countably many closed n -dimensional sets, so by Theorem 7 it is n -dimensional. By Theorem 2 choose open sets $U_1 \supset U_2 \supset \dots \supset U_k \supset \dots \supset D - C$ such that $\bigcap_{k=1}^{\infty} U_k$ is n -dimensional. Then put $Y = \{x \in \mathbb{R}^{\mathbb{N}} : x - C \subseteq \bigcap_{k=1}^{\infty} U_k\}$. Then Y is dense in $\mathbb{R}^{\mathbb{N}}$. Note that Y is a G_{δ} set: Consider a k , and an $x \in Y$. Since C is compact there is an open neighborhood $V_k(x)$ of x such that $V_k(x) - C \subset U_k$. But then $V_k = \bigcup_{x \in Y} V_k(x)$ is an open set containing Y , and $V_k - C \subset U_k$. It follows that $Y = \bigcap_{k < \infty} V_k$.

Next, put

$$X = H \cap \left(\bigcap_{k=1}^{\infty} U_k \right).$$

Since H is a Hurewicz set and $\bigcap_{k=1}^{\infty} U_k$ is n -dimensional, X is a countable set. Define $Z = Y \setminus (X + C)$. Since $X + C$ is a union of countably many closed, n -dimensional sets, it is an n -dimensional F_{σ} set and thus first category in $\mathbb{R}^{\mathbb{N}}$. Consequently Z is co-meager. But

$$Z = \{x \in Y : (x - C) \cap H = \emptyset\} \subseteq Y \setminus (H + C).$$

Thus, $H + C$ is meager. \square

One can show that if there are Hurewicz sets, then there are ones satisfying the property in Conjecture 1. There are several well-studied classes of weakly infinite dimensional subsets of $\mathbb{R}^{\mathbb{N}}$ which are not countable dimensional. In Example 1 we give, under CH, an example of a set S in a very restrictive class of weakly infinite dimensional spaces (but not countable dimensional), and a Hurewicz set H , such that $S + H = \mathbb{R}^{\mathbb{N}}$. The following is an unresolved weaker instance of Conjecture 1:

Problem 1. Is it true that if C is a strongly countable dimensional set in $\mathbb{R}^{\mathbb{N}}$ and H is a Hurewicz set, then $H + C \neq \mathbb{R}^{\mathbb{N}}$?

4. New classes of weakly infinite dimensional spaces

Many examples in infinitary dimension theory belong to classes of weakly infinite dimensional spaces introduced in [2]. Our examples in connection with Conjecture 1 are also in these classes. These classes are defined by applying selection principles to specific types of open covers. We now introduce these covers and selection principles.

Classes of open covers

For a given space X the symbol \mathcal{O} denotes the collection of all open covers of X . The following special types of open covers are relevant to our discussion:

- \mathcal{O}_{fd} : This is the collection of all open covers \mathcal{U} such that there is for each finite dimensional $F \subset X$ a $U \in \mathcal{U}$ with $F \subseteq U$, but X is not a member of \mathcal{U} . (Thus, we are assuming X is not finite dimensional.)
- \mathcal{O}_{cfd} : This is the collection of all open covers \mathcal{U} such that there is for each *closed* finite dimensional $F \subset X$ a $U \in \mathcal{U}$ with $F \subseteq U$, but X is not a member of \mathcal{U} . (Thus, we are assuming X is not finite dimensional.)
- \mathcal{O}_{kfd} : This is the collection of all open covers \mathcal{U} such that there is for each *compact* finite dimensional $F \subset X$ a $U \in \mathcal{U}$ with $F \subseteq U$, but X is not a member of \mathcal{U} . (Thus, we are assuming X is not compact and finite dimensional.)
- \mathcal{O}_k : This is the collection of all open covers \mathcal{U} such that there is for each *compact* $F \subset X$ a $U \in \mathcal{U}$ with $F \subseteq U$, but X is not a member of \mathcal{U} . (Thus, we are assuming X is not compact.)
- Ω : This is the collection of ω covers of X . An open cover \mathcal{U} of X is an ω -cover if X is not a member of \mathcal{U} , but for each finite subset F of X there is a $U \in \mathcal{U}$ with $F \subseteq U$.
- Γ : This is the collection of γ covers of X . An open cover \mathcal{U} of X is a γ -cover if it is infinite and every infinite subset of \mathcal{U} is a cover of X .
- \mathcal{O}^{gp} : This is the collection of groupable open covers: An open cover \mathcal{U} of a space X is *groupable* if there is a partition $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ into finite sets \mathcal{F}_n that are disjoint from each other, such that for each $x \in X$ there is an N with $x \in \bigcup \mathcal{F}_n$ for all $n \geq N$.

We have the following inclusion relations among these classes of open covers:

$$\mathcal{O}_{fd} \subset \mathcal{O}_{cfd} \subset \mathcal{O}_{kfd} \subset \Omega \subset \mathcal{O}; \quad \mathcal{O}_k \subset \mathcal{O}_{kfd}; \quad \Gamma \subset \Omega.$$

It is also worth noting that if X is a separable metric space then each $\mathcal{U} \in \mathcal{O}_{kfd}$ has a countable subset $\mathcal{V} \in \mathcal{O}_{kfd}$. The same is true about \mathcal{O}_k .

Three selection principles

Let \mathcal{A} and \mathcal{B} be families of sets. The selection principle, $S_1(\mathcal{A}, \mathcal{B})$ states:

For each sequence $(A_n: n \in \mathbb{N})$ of elements of \mathcal{A} there is a sequence $(B_n: n \in \mathbb{N})$ such that for all n we have $B_n \in A_n$, and $\{B_n: n \in \mathbb{N}\} \in \mathcal{B}$.

The selection principle $S_{fin}(\mathcal{A}, \mathcal{B})$ states:

There is for each sequence $(A_n: n \in \mathbb{N})$ of members of \mathcal{A} a sequence $(B_n: n \in \mathbb{N})$ of finite sets such that for each n we have $B_n \subset A_n$ and $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

The selection principle $S_c(\mathcal{A}, \mathcal{B})$ states:

For each sequence $(A_n: n \in \mathbb{N})$ of elements of \mathcal{A} , there is a sequence $(B_n: n \in \mathbb{N})$ where each B_n is a refinement of the collection of sets A_n , each B_n is a pairwise disjoint family, and $\bigcup_{n \in \mathbb{N}} B_n$ is a member of \mathcal{B} .

In our context the families \mathcal{A} and \mathcal{B} will be classes of open covers of a topological space. The selection principle $S_1(\mathcal{A}, \mathcal{B})$ has the following monotonicity properties: If $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{B} \subset \mathcal{D}$, then we have the following implications:

$$S_1(\mathcal{C}, \mathcal{B}) \Rightarrow S_1(\mathcal{A}, \mathcal{B}); \quad S_1(\mathcal{A}, \mathcal{B}) \Rightarrow S_1(\mathcal{A}, \mathcal{D}).$$

Replacing S_1 with S_{fin} or S_c in these implications give corresponding facts for the other two selection principles.

Special instances of these three selection principles appear in classical literature. For example: $S_1(\mathcal{O}, \mathcal{O})$ is known as Rothberger's property and was introduced in the 1938 paper [22]. $S_{fin}(\mathcal{O}, \mathcal{O})$ is known as Menger's property and was introduced in the 1925 paper [7]. $S_c(\mathcal{O}, \mathcal{O})$ is known as property C, and was introduced in the 1978 paper [1]. $S_c(\mathcal{O}, \mathcal{O})$ is a selective version of Bing's property of screenability [3], thus also known as *selective screenability*. Selective screenability is an important property in infinitary dimension theory. Let \mathcal{O}_2 denote the collection of open covers \mathcal{U} with $|\mathcal{U}| \leq 2$. Then $S_c(\mathcal{O}_2, \mathcal{O})$ is equivalent to Alexandroff's notion of *weakly infinite dimensional*. A space which is not weakly infinite dimensional is said to be *strongly infinite dimensional*.

The class $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$

Metrisable spaces with property $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$ seem important. First: Theorem 17 of [2] shows that for separable metric spaces $S_1(\mathcal{O}_{\text{fd}}, \mathcal{O}) \Rightarrow S_c(\mathcal{O}, \mathcal{O})$. Monotonicity properties of the selection principles give: $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O}) \Rightarrow S_1(\mathcal{O}_{\text{ofd}}, \mathcal{O}) \Rightarrow S_1(\mathcal{O}_{\text{fd}}, \mathcal{O})$. Thus for metrisable spaces $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$ implies the important property $S_c(\mathcal{O}, \mathcal{O})$. Second: The following lemma shows that these spaces have the classical Menger property $S_{\text{fin}}(\mathcal{O}, \mathcal{O})$:

Lemma 9. $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O}) \Rightarrow S_1(\mathcal{O}_k, \mathcal{O}) \Rightarrow S_{\text{fin}}(\mathcal{O}, \mathcal{O})$.

Third: Several central examples from the theory of infinite dimensional spaces are in this class. For example: R. Pol's example in [20] which shows that a weakly infinite dimensional space need not be countable dimensional is in the class $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$.

The preservation of infinitary dimension properties by products is not well understood yet. Whenever a new class of infinite dimensionality is defined it is of interest to know how this class behaves under various mathematical constructions, like products. We now make some remarks about products by spaces in the class $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$, using recent examples constructed by E. Pol and R. Pol in [19]:

Proposition 10. Assume Martin's Axiom.

- (1) Finite powers of sets in $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$ need not be in $S_c(\mathcal{O}, \mathcal{O})$.
- (2) The product of a space in $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$ with the space of irrational numbers need not be weakly infinite dimensional.

To see this consider the following: In the proof of Proposition 4.1 of [19] E. Pol and R. Pol construct, using Martin's Axiom, spaces E_0 and E_1 which have $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$. To see that these two spaces have this property we consider elements of the argument given in part (6) of the proof of [19, Proposition 4.1]: For $i \in \{0, 1\}$ the spaces E_i are of the form $E \cup S_i$ where:

- S_i is the union of countably many compact, finite dimensional subspaces, and
- for each open set $W \supset S_i$ in E_i , $|E_i \setminus W| < 2^{\aleph_0}$.

Consider E_i . Let $(\mathcal{U}_n: n < \infty)$ be a sequence in \mathcal{O}_{kfd} . Write S_i as a union of compact finite dimensional subspaces $S_1^i \subset S_2^i \subset \dots \subset S_n^i \subset \dots$. For each n choose a $U_{2,n-1} \in \mathcal{U}_{2,n-1}$ with $S_n^i \subseteq U_{2,n-1}$. Then $W = \bigcup_{n < \infty} U_{2,n-1}$ is an open set containing S_i , and so $E_i \setminus W$ has cardinality less than 2^{\aleph_0} . But Martin's Axiom implies that a separable metric space of cardinality less than 2^{\aleph_0} has Rothberger's property $S_1(\mathcal{O}, \mathcal{O})$ (Theorem 5 of [12]). Thus, choose for each n a $U_{2,n} \in \mathcal{U}_{2,n}$ such that $E_i \setminus W \subseteq \bigcup_{n < \infty} U_{2,n}$.

It follows that the space X constructed there as the free union of E_0 and E_1 still has $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$. Recall that a metric space (X, d) has the Haver property if there is for each sequence $(\epsilon_n: n < \infty)$ of positive real numbers a sequence $(\mathcal{U}_n: n < \infty)$ such that each \mathcal{V}_n is a pairwise disjoint family of open sets, each of d -diameter less than ϵ_n , and $\bigcup_{n < \infty} \mathcal{U}_n$ is a cover of X . Metrisable spaces with property $S_c(\mathcal{O}, \mathcal{O})$ have the Haver property in all equivalent metrics. Since [19] shows that X^2 has the Menger property but under some metric X^2 does not have the Haver property, it follows that X^2 does not have property $S_c(\mathcal{O}, \mathcal{O})$.

To see the second item: In Theorem 6.1 of [19] it is pointed out that the product of the space E_i with the subspace of irrational numbers of the real line is not even $S_c(\mathcal{O}_2, \mathcal{O})$, that is, is strongly infinite dimensional.

The class $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{Q})$

$S_1(\mathcal{O}_k, \mathcal{Q})$ was considered in Section 2 of [17]. Theorem 2.1 of [17] implies:

Lemma 11 (Pansera, Pavlovic). A space has $S_1(\mathcal{O}_k, \mathcal{Q})$ if, and only if, it has $S_1(\mathcal{O}_k, \mathcal{O})$ in all finite powers.

Proposition 12. If a space has property $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{Q})$ then it has property $S_{\text{fin}}(\mathcal{Q}, \mathcal{Q})$.

Proof. Let X be a space with $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{Q})$. By monotonicity properties of $S_1(\cdot, \cdot)$ X then has $S_1(\mathcal{O}_k, \mathcal{Q})$. Then by Lemma 11 X^n has $S_1(\mathcal{O}_k, \mathcal{O})$ for all finite n . By Lemma 9 X^n has $S_{\text{fin}}(\mathcal{O}, \mathcal{O})$ for all n . By Theorem 3.9 of [10] this is equivalent to: X has $S_{\text{fin}}(\mathcal{Q}, \mathcal{Q})$. \square

Using standard techniques one can prove: If a space has the property $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{Q})$, then it has the property $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$ in all finite powers. We don't know if the converse is true:

Problem 2. For metrisable space X is it true that if each X^n has property $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$ then X has $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{Q})$?

It is likely that the spaces E_0 and E_1 of [19] mentioned above are in $S_1(\mathcal{O}_{\text{kfd}}, \Omega)$, but we have not examined this. The space X which is the free union of E_0 and E_1 is an example of a space in $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$ but not in $S_1(\mathcal{O}_{\text{kfd}}, \Omega)$. Thus, under Martin's Axiom, $S_1(\mathcal{O}_{\text{kfd}}, \Omega)$ is a proper subclass of $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$. Below we construct under CH an example (Example 1) of a space which has the property $S_1(\mathcal{O}_{\text{kfd}}, \Omega)$ and which is not countable dimensional.

The class $S_1(\mathcal{O}_{\text{kfd}}, \Gamma)$

$S_1(\mathcal{O}_k, \Gamma)$ was explicitly considered in [4, Section 3], and in [5, Section 3], where the second implication of Lemma 13 is noted.

Lemma 13. *Property $S_1(\mathcal{O}_{\text{kfd}}, \Gamma) \Rightarrow S_1(\mathcal{O}_k, \Gamma) \Rightarrow S_{\text{fin}}(\Omega, \mathcal{O}^{\text{gp}})$.*

Theorem 12 of [4] shows that each σ -compact space has the property $S_1(\mathcal{O}_k, \Gamma)$. This is not true for $S_1(\mathcal{O}_{\text{kfd}}, \Gamma)$: The Hilbert cube is not weakly infinite dimensional, but is a σ -compact space. Combining [5, Theorem 10] and [5, Proposition 13] gives the following result:

Lemma 14 (Di Maio, Kočinac, Meccariello). *If X has $S_1(\mathcal{O}_k, \Gamma)$, then for all n also X^n has this property.*

The Hurewicz covering property is a strengthening of the Menger property $S_{\text{fin}}(\mathcal{O}, \mathcal{O})$ and is defined as follows: Topological space X has the *Hurewicz covering property* if: For each sequence $(\mathcal{U}_n: n < \infty)$ of open covers of X there is a sequence $(\mathcal{V}_n: n < \infty)$ of finite sets such that for each n , $\mathcal{V}_n \subseteq \mathcal{U}_n$, and for each $x \in X$, for all but finitely many n , $x \in \bigcup \mathcal{V}_n$. It was shown in [11] that the Hurewicz covering property is equivalent to the selection principle $S_{\text{fin}}(\Omega, \mathcal{O}^{\text{gp}})$.

Corollary 15. *If a space X has property $S_1(\mathcal{O}_{\text{kfd}}, \Gamma)$ then it has $S_{\text{fin}}(\Omega, \mathcal{O}^{\text{gp}})$ in all finite powers.*

Using the technique in the proof of [2, Theorem 17] and using the techniques in [2, Lemma 6] one obtains the following two results:

Theorem 16. $S_1(\mathcal{O}_{\text{fd}}, \mathcal{O}^{\text{gp}}) \Rightarrow S_c(\mathcal{O}, \mathcal{O}^{\text{gp}})$.

Lemma 17. *If a space X has $S_1(\mathcal{O}_{\text{kfd}}, \Gamma)$, then it has $S_c(\mathcal{O}, \mathcal{O})$ in all finite powers.*

Proof. Let X be a space with property $S_1(\mathcal{O}_{\text{kfd}}, \Gamma)$. By Lemma 13 it has property $S_1(\mathcal{O}_k, \Gamma)$, and by Lemma 14 it has $S_1(\mathcal{O}_k, \Gamma)$ in all finite powers, and thus the Hurewicz property in all finite powers. Then by Corollary 13 of [2], all finite powers of X have the property $S_c(\mathcal{O}, \mathcal{O})$. \square

Let X be an infinite dimensional space which satisfies $S_1(\mathcal{O}_{\text{kfd}}, \Gamma)$. By [2] and Lemma 13, all finite powers of X have the Haver property, and indeed, the product of X with any space having the Haver property again has the Haver property. It follows that $X \times \mathbb{P}$ has the Haver property. We don't know if this product must be selectively screenable:

Problem 3. *If X has $S_1(\mathcal{O}_{\text{kfd}}, \Gamma)$ then does $X \times \mathbb{P}$ have $S_c(\mathcal{O}, \mathcal{O})$?*

5. Examples

We now describe two infinite dimensional examples that will be used to demonstrate a number of facts in connection with the preceding sections. In both cases we are interested in showing properties of Hurewicz sets, and thus we are confined to assuming the Continuum Hypothesis.

A ZFC + CH example of $S_1(\mathcal{O}_{\text{kfd}}, \Omega)$

Write: $\mathbb{R}_n := \{x \in \mathbb{R}^{\mathbb{N}}: (\forall j > n) (x(j) = 0)\}$ and $\mathbb{R}_{\infty} := \bigcup_{n=1}^{\infty} \mathbb{R}_n$. Standard arguments prove Lemma 18 and Corollary 19 below.

Lemma 18. *Assume $G \supset \mathbb{R}_{\infty}$ is an open subset of $\mathbb{R}^{\mathbb{N}}$ and I_1, \dots, I_n are closed intervals of positive length. For each $f \in \mathbb{R}^{\mathbb{N}}$ there are an $m > n$ and closed intervals I_{n+1}, \dots, I_m of positive length such that*

$$I_1 \times \dots \times I_m \times \mathbb{R}^{\mathbb{N}} \subseteq G - f.$$

Corollary 19. *If $(G_n: n < \infty)$ is a sequence of G_{δ} subset of $\mathbb{R}^{\mathbb{N}}$ such that each G_n contains \mathbb{R}_{∞} , and if $(f_n: n < \infty)$ is sequence of elements of $\mathbb{R}^{\mathbb{N}}$, then $\bigcap_{n=1}^{\infty} (G_n - f_n)$ contains a Hilbert cube.*

Example 1 (CH). There are subsets X and Y of $\mathbb{R}^{\mathbb{N}}$ such that:

- (1) X and Y satisfy $S_1(\mathcal{O}_{\text{kfd}}, \Omega)$;
- (2) $X \setminus \mathbb{R}_{\infty}$ and $Y \setminus \mathbb{R}_{\infty}$ are Hurewicz sets;
- (3) X and Y have $S_1(\mathcal{O}_{\text{kfd}}, \Omega)$ and $X \times Y$ not;
- (4) $X \cup Y$ is $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$, and not $S_1(\mathcal{O}_{\text{kfd}}, \Omega)$.

Fix the following enumerations:

- $(f_{\alpha}: \alpha < 2^{\aleph_0})$, the list of all elements of $\mathbb{R}^{\mathbb{N}}$;
- $(F_{\alpha}: \alpha < 2^{\aleph_0})$, the list of all finite dimensional G_{δ} subsets of $\mathbb{R}^{\mathbb{N}}$;
- $(L_{\alpha}: \alpha < 2^{\aleph_0})$, the list of all G_{δ} subsets of $\mathbb{R}^{\mathbb{N}}$ containing \mathbb{R}_{∞} ;
- $((U_n^{\alpha}: n < \omega): \alpha < 2^{\aleph_0})$ where for each α each element of \mathbb{R}_{∞} is in U_n^{α} for all but finitely many n , and each U_n^{α} is open in $\mathbb{R}^{\mathbb{N}}$.

For each $\alpha < 2^{\aleph_0}$ the set $T_{\alpha} = \bigcap_{n < \infty} (\bigcup_{m \geq n} U_m^{\alpha})$ is a G_{δ} set containing \mathbb{R}_{∞} .

Since $G_0 = T_0 \cap L_0$ is a dense G_{δ} set which contains \mathbb{R}_{∞} . Thus $G_0 \cap (G_0 - f_0)$ contains, by Corollary 19, a homeomorphic copy of the Hilbert cube. Choose

$$x_0 \in (G_0 \cap (G_0 - f_0)) \setminus ((\mathbb{R}_{\infty} \cup F_0) \cup ((\mathbb{R}_{\infty} \cup F_0) - f_0))$$

and fix $y_0 \in G_0$ with $x_0 = y_0 - f_0$. Then define $S_0^1(0) = \{n: x_0 \in U_n^0\}$ and $S_0^2(0) = \{n: y_0 \in U_n^0\}$.

Assume we have $0 < \alpha < \omega_1$, and that we have selected for each $\gamma < \alpha$ an x_{γ} and a y_{γ} and defined sets $S_{\nu}^1(\gamma)$ and $S_{\nu}^2(\gamma)$, $\nu \leq \gamma$ such that:

- (1) $\delta < \gamma < \alpha \Rightarrow S_{\gamma}^1(\delta) = S_{\gamma}^2(\delta) = \omega$;
- (2) $\gamma \leq \nu < \delta < \alpha \Rightarrow$
 - (a) $S_{\gamma}^1(\delta) \subseteq^* S_{\nu}^1(\delta)$; $S_{\gamma}^2(\delta) \subseteq^* S_{\nu}^2(\delta)$;
 - (b) $S_{\gamma}^1(\delta), S_{\gamma}^1(\nu) \subseteq S_{\gamma}^1(0)$; $S_{\gamma}^2(\delta), S_{\gamma}^2(\nu) \subseteq S_{\gamma}^2(0)$ are all infinite;
- (3) $\gamma \leq \delta < \alpha \Rightarrow \{n: x_{\delta} \in U_n^{\gamma}\} \supseteq S_{\gamma}^1(\delta)$; $\{n: y_{\delta} \in U_n^{\gamma}\} \supseteq S_{\gamma}^2(\delta)$;
- (4) $\gamma \leq \delta < \alpha \Rightarrow \{x_{\delta}, y_{\delta}\} \subseteq G_{\gamma} \setminus (\mathbb{R}_{\infty} \cup (\bigcup_{\nu \leq \delta} F_{\nu})) \cup \{x_{\nu}: \nu < \delta\} \cup \{y_{\nu}: \nu < \delta\}$;
- (5) $\delta < \alpha \Rightarrow x_{\delta} = y_{\delta} - f_{\delta}$.

Towards selecting x_{α} and y_{α} we consider two cases:

Case 1. $\alpha = \beta + 1$.

For $\delta < \alpha$ define $S_{\alpha}^1(\delta) = \omega = S_{\alpha}^2(\delta)$. For $\gamma \leq \beta$ we define

$$T_{\gamma}^1 = \bigcap_{n=1}^{\infty} \left(\bigcup_{m \geq n, m \in S_{\gamma}^1(\beta)} U_m^{\gamma} \right),$$

$$T_{\gamma}^2 = \bigcap_{n=1}^{\infty} \left(\bigcup_{m \geq n, m \in S_{\gamma}^2(\beta)} U_m^{\gamma} \right),$$

and for α we define

$$T_{\alpha}^1 = T_{\alpha}^2 = \bigcap_{n=1}^{\infty} \left(\bigcup_{m \geq n} U_m^{\alpha} \right)$$

and

$$G_{\alpha} = \bigcap_{\gamma \leq \alpha} (L_{\gamma} \cap T_{\gamma}^1 \cap T_{\gamma}^2).$$

Then G_{α} is a G_{δ} set containing \mathbb{R}_{∞} and so by Corollary 19 the set $G_{\alpha} \cap (G_{\alpha} - f_{\alpha})$ contains a Hilbert cube. But the set

$$B_{\alpha} = \left(\mathbb{R}_{\infty} \cup \left(\bigcup_{\gamma \leq \alpha} F_{\gamma} \right) \cup \{x_{\gamma}: \gamma < \alpha\} \cup \{y_{\gamma}: \gamma < \alpha\} \right) - (\{f_{\gamma}: \gamma \leq \alpha\} \cup \{\mathbf{0}\})$$

is countable dimensional and thus does not contain a Hilbert cube. Choose $x_\alpha \in (G_\alpha \cap (G_\alpha - f_\alpha)) \setminus B_\alpha$ and fix $y_\alpha \in G_\alpha$ with $x_\alpha = y_\alpha - f_\alpha$.

Then (4) holds at α . Since for $\gamma < \alpha$ we have $x_\alpha \in T_\gamma^1$ and $y_\alpha \in T_\gamma^2$ the sets $S_\gamma^1(\alpha) = \{n \in S_\gamma^1(\beta) : x_\alpha \in U_n^\gamma\}$ and $S_\gamma^2(\alpha) = \{n \in S_\gamma^2(\beta) : y_\alpha \in U_n^\gamma\}$ are infinite. Similarly the sets $S_\alpha^1(\alpha) = \{n : x_\alpha \in U_n^\alpha\}$ and $S_\alpha^2(\alpha) = \{n : y_\alpha \in U_n^\alpha\}$ are infinite.

Case 2. α is a limit ordinal.

For $\gamma < \alpha$ choose infinite sets F_γ^1 and F_γ^2 so that for all $\beta < \alpha$, $F_\gamma^1 \subseteq^* S_\gamma^1(\beta)$ and $F_\gamma^2 \subseteq^* S_\gamma^2(\beta)$. For $\gamma \leq \beta$ we define

$$T_\gamma^1 = \bigcap_{n=1}^\infty \left(\bigcup_{m \geq n, m \in F_\gamma^1} U_m^\gamma \right),$$

$$T_\gamma^2 = \bigcap_{n=1}^\infty \left(\bigcup_{m \geq n, m \in F_\gamma^2} U_m^\gamma \right),$$

and for α we define

$$T_\alpha^1 = T_\alpha^2 = \bigcap_{n=1}^\infty \left(\bigcup_{m \geq n} U_m^\alpha \right)$$

and

$$G_\alpha = \bigcap_{\gamma \leq \alpha} (L_\gamma \cap T_\gamma^1 \cap T_\gamma^2).$$

Since G_α is a G_δ set containing \mathbb{R}_∞ , Corollary 19 implies that the set $G_\alpha \cap (G_\alpha - f_\alpha)$ contains a Hilbert cube. The set

$$B_\alpha = \left(\mathbb{R}_\infty \cup \left(\bigcup_{\gamma \leq \alpha} F_\gamma \right) \cup \{x_\gamma : \gamma < \alpha\} \cup \{y_\gamma : \gamma < \alpha\} \right) - (\{f_\gamma : \gamma \leq \alpha\} \cup \{\underline{0}\})$$

is countable dimensional and does not contain a Hilbert cube.

Choose $x_\alpha \in (G_\alpha \cap (G_\alpha - f_\alpha)) \setminus B_\alpha$ and fix $y_\alpha \in G_\alpha$ with $x_\alpha = y_\alpha - f_\alpha$.

Then (4) holds at α . Since for $\gamma < \alpha$ we have $x_\alpha \in T_\gamma^1$ and $y_\alpha \in T_\gamma^2$ the sets $S_\gamma^1(\alpha) = \{n \in F_\gamma^1 : x_\alpha \in U_n^\gamma\}$ and $S_\gamma^2(\alpha) = \{n \in F_\gamma^2 : y_\alpha \in U_n^\gamma\}$ are infinite. Similarly the sets $S_\alpha^1(\alpha) = \{n : x_\alpha \in U_n^\alpha\}$ and $S_\alpha^2(\alpha) = \{n : y_\alpha \in U_n^\alpha\}$ are infinite.

This completes the description of the process for choosing x_α and y_α for $\alpha < \omega_1$. Finally define $X = \mathbb{R}_\infty \cup \{x_\alpha : \alpha < \omega_1\}$ and $Y = \mathbb{R}_\infty \cup \{y_\alpha : \alpha < \omega_1\}$.

Claim 1. If \mathcal{U} is a collection of open subsets of $\mathbb{R}^\mathbb{N}$ which contains an infinite subset \mathcal{V} , each infinite subset of which covers \mathbb{R}_∞ , then there are countable sets A_X and A_Y such that \mathcal{V} is an ω -cover of $X \setminus A_X$ and of $Y \setminus A_Y$.

Proof. For let \mathcal{V} be such a subfamily of \mathcal{U} . We may assume that \mathcal{V} is countable. Thus, we may assume that for some α , fixed from now on, \mathcal{V} is $(U_n^\alpha : n < \omega)$, as enumerated before the construction of X and Y . Put $A_X = \{x_\gamma : \gamma \leq \alpha\}$ and $A_Y = \{y_\gamma : \gamma \leq \alpha\}$.

Consider any finite subset $F \subset X \setminus A_X$. We may write $F = \{x_{\gamma_1}, \dots, x_{\gamma_n}\}$ where we have $\alpha < \gamma_1 < \dots < \gamma_n$. From the construction of X and Y we have $S_\alpha^1(\gamma_n) \subseteq^* \dots \subseteq^* S_\alpha^1(\gamma_1)$. Choose N large enough that

$$S_\alpha^1(\gamma_n) \setminus N \subseteq \dots \subseteq S_\alpha^1(\gamma_1).$$

Then for $k \in S_\alpha^1(\gamma_n)$ we have $F \subset U_k^\alpha$. A similar argument applies to $Y \setminus A_Y$. \square

Claim 2. X and Y both satisfy the selection principle $S_1(\mathcal{O}_{\text{kfd}}, \Omega)$.

Proof. We prove it for X . The proof for Y is similar. Let $(\mathcal{U}_n : n < \infty)$ be a sequence of families of open sets of $\mathbb{R}^\mathbb{N}$ such that each is an element of \mathcal{O}_{kfd} for X . Write $\mathbb{N} = \bigcup_{k < \infty} Z_k$ where each Z_k is infinite and $Z_k \cap Z_m = \emptyset$ whenever we have $k \neq m$.

Starting with Z_1 , choose for each $n \in Z_1$ a $U_n \in \mathcal{U}_1$ such that every infinite subset of $\{U_n : n \in Z_1\}$ covers \mathbb{R}_∞ . Choose a countable set $X_1 \subset X$ such that $\{U_n : n \in Z_1\}$ is an ω -cover of $X \setminus X_1$.

Continuing with Z_2 , choose for each $n \in Z_2$ a $U_n \in \mathcal{U}_2$ such that every infinite subset of $\{U_n : n \in Z_2\}$ covers $X_1 \cup \mathbb{R}_\infty$. Choose a countable set $X_2 \subset X$ such that $\{U_n : n \in Z_2\}$ is an ω -cover of $X \setminus X_2$. Note that we may assume $X_2 \cap X_1 = \emptyset$.

With X_1, \dots, X_n selected countable subsets of X , and for each $j \leq n$, $U_k \in \mathcal{U}_k$ selected for $k \in Z_j$ so that $(U_k : k \in Z_j)$ is an ω -cover of $X \setminus X_j$, and every infinite subset of $(U_k : k \in Z_j)$ covers $X_1 \cup \dots \cup X_{j-1} \cup \mathbb{R}_\infty$. Choose $U_k \in \mathcal{U}_k$ for $k \in Z_{n+1}$ so

that each infinite subset of $\{U_k: k \in \mathbb{Z}_{n+1}\}$ is a cover of $\mathbb{R}_\infty \cup X_1 \cup \dots \cup X_n$ and by Claim 1 choose a countable $X_{n+1} \subset \mathbb{R}^\mathbb{N}$ such that $\{U_k: k \in \mathbb{Z}_{n+1}\}$ is an ω -cover of $X \setminus X_{n+1}$.

Thus we obtain $U_k: k \in \mathbb{N}$ and countable subsets X_n of X , disjoint from each other, such that for each n , $\{U_k: k \in \mathbb{N}\}$ is an ω -cover of $X \setminus X_n$. Since for each finite subset F of X there is an n with $F \cap X_n = \emptyset$, it follows that $\{U_k: k < \infty\}$ is an ω -cover for X . \square

Claim 3. The sets $X \setminus \mathbb{R}_\infty$ and $Y \setminus \mathbb{R}_\infty$ both are Hurewicz sets.

Proof. For consider any zero-dimensional subset of $\mathbb{R}^\mathbb{N}$. It is contained in a zero-dimensional set of form F_γ , and $F_\gamma \cap (X \setminus \mathbb{R}_\infty) \subset \{x_\nu: \nu < \gamma\}$, a countable set. \square

Claim 4. $X \setminus \mathbb{R}_\infty - Y \setminus \mathbb{R}_\infty = \mathbb{R}^\mathbb{N}$.

Proof. This follows from (5) in the construction of X and Y . \square

Remarks. (1) X and Y are Menger in all finite powers.

Proof. Claim 2 and Corollary 12. \square

(2) There is a Hurewicz set H and an $S_1(\mathcal{O}_{\text{kfd}}, \Omega)$ set S with $H + S = \mathbb{R}^\mathbb{N}$.

Proof. Put $H = -(Y \setminus \mathbb{R}_\infty)$ and $S = X$. Apply Claims 2–4. \square

(3) $S_1(\mathcal{O}_{\text{kfd}}, \Omega)$ is not preserved by finite products.

Proof. Since the map $(x, y) \mapsto x - y$ is continuous, if $X \times Y$ is Menger, so is $X - Y = \mathbb{R}^\mathbb{N}$. But the latter is not Menger. Thus $X \times Y$ is not Menger. By Lemma 9 $X \times Y$ is not $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$. \square

(4) $S_1(\mathcal{O}_{\text{kfd}}, \Omega)$ is not preserved by finite unions.

Proof. For the same reason, $X \cup Y$ is not Menger in all finite powers, even though $X \cup Y$ is Menger. It follows that $X \cup Y$ has the property $S_1(\mathcal{O}_{\text{kfd}}, \mathcal{O})$, but not the property $S_1(\mathcal{O}_{\text{kfd}}, \Omega)$, and also not the property $S_1(\mathcal{O}_k, \Omega)$. \square

A ZFC + CH example of $S_1(\mathcal{O}_{\text{kfd}}, \Gamma)$

The point with the following example is to demonstrate that spaces in this class need not be countable dimensional. We do not know if non-countable dimensional examples of metric spaces in this class can be obtained without resorting to hypotheses beyond the standard axioms of Zermelo–Fraenkel set theory.

Put $J = \{0\} \cup [\frac{1}{2}, 1]$. Define $J_m = \{x \in J^\mathbb{N}: (\forall n > m) (x(n) = 0)\}$ and $J_\infty = \bigcup_{m \in \mathbb{N}} J_m$. Since each J_m is compact and m -dimensional, J_∞ is compactly countable dimensional and is a dense subset of $J^\mathbb{N}$.

For X any subset of \mathbb{N} , define: $J(X) = \{x \in J^\mathbb{N}: \{n: x(n) \neq 0\} \subseteq X\}$. When X is infinite, define $X^* = \{Y \subseteq \mathbb{N}: Y \text{ is infinite and } Y \setminus X \text{ is finite}\}$, and $J^*(X) = \bigcup\{J(Y): Y \in X^*\}$.

Lemma 20. Let an infinite set $X \subseteq \mathbb{N}$, a countable set $\mathcal{C} \subset J^\mathbb{N}$, as well as a family \mathcal{J} of open subsets of $J^\mathbb{N}$ be given so that relative to $\mathcal{C} \cup J_\infty$, $\mathcal{J} \in \mathcal{O}_{\text{kfd}}$. Then there are an infinite $Y \subseteq X$, and a sequence $D_1, D_2, \dots, D_n, \dots$ in \mathcal{J} such that $J_\infty \cup \mathcal{C} \cup J^*(Y) \subseteq \bigcup_{m \in \mathbb{N}} (\bigcap_{n \geq m} D_n)$.

Proof. Enumerate \mathcal{C} bijectively as $(c_n: n < \infty)$. Choose $k_0 = \min(X)$. Then $J(\{1, \dots, k_0\}) = J^{k_0} \times \{0\}^\mathbb{N} \subset J_\infty$ is compact and finite dimensional. Since \mathcal{J} is in \mathcal{O}_{kfd} , choose $D_1 \in \mathcal{J}$ with $J_0 \cup \{c_0\} \cup J(\{1, \dots, k_0\}) \subset D_1$.

For each $x \in J(\{1, \dots, k_0\})$ choose a basic open set B_x of the form $I_1(x) \times \dots \times I_{k_0}(x) \times \{0\}^{k(x)} \times J^\mathbb{N} \subset D_1$. Since $J(\{1, \dots, k_0\})$ is compact, choose a finite set $\{x_1, \dots, x_m\} \in J(\{1, \dots, k_0\})$ for which

$$J(\{1, \dots, k_0\}) \subseteq \bigcup_{i=1}^m B_{x_i} \subseteq D_1.$$

Choose $k_1 \in X$ with $k_1 > \max\{k_0 + k(x_i): 1 \leq i \leq m\}$. Then we have

$$J^{k_0} \times \{0\}^{k_1 - k_0} \times J^\mathbb{N} \subseteq D_1.$$

This specifies k_0, k_1 and D_1 .

Now $J(\{1, \dots, k_1\}) = J^{k_1} \times \{0\}^{\mathbb{N}} \subset J_{\infty}$ is a compact and finite dimensional subset of J_{∞} . Since \mathcal{J} is in \mathcal{O}_{kfd} for $\mathcal{C} \cup J_{\infty}$, choose a $D_2 \in \mathcal{J}$ with $J_1 \cup \{c_0, c_1\} \cup J(\{1, \dots, k_1\}) \subseteq D_2$. For each $x \in J(\{1, \dots, k_1\})$ choose a basic open set B_x of the form $I_1(x) \times \dots \times I_{k_1}(x) \times \{0\}^{k(x)} \times J^{\mathbb{N}} \subset D_2$. Since $J(\{1, \dots, k_1\})$ is compact, choose a finite set $\{x_1, \dots, x_m\} \in J(\{1, \dots, k_1\})$ for which

$$J(\{1, \dots, k_1\}) \subseteq \bigcup_{i=1}^m B_{x_i} \subseteq D_2.$$

Choose $k_2 \in X$ with $k_2 > \max\{k_1 + k(x_i) : 1 \leq i \leq m\}$. Then we have

$$J^{k_1} \times \{0\}^{k_2-k_1} \times J^{\mathbb{N}} \subseteq D_2.$$

This specifies k_0, k_1 and k_2 , as well as D_1 and D_2 . Continuing in this way we find sequences

- (1) $k_0 < k_1 < \dots < k_n < \dots$ in X , and
- (2) $D_1, D_2, \dots, D_n, \dots \in \mathcal{J}$

such that for $n \geq 1$ we have

$$J_n \cup \{c_0, \dots, c_n\} \cup J(\{1, \dots, k_n\}) \subseteq D_{n+1} \quad \text{and} \quad J^{k_n} \times \{0\}^{k_{n+1}-k_n} \times J^{\mathbb{N}} \subseteq D_{n+1}.$$

Put $Y = \{k_n : 0 \leq n < \infty\}$, an infinite subset of X . Consider any infinite set $Z \subset \mathbb{N}$ with $Z \setminus Y$ finite. Choose k_n so large that

$$Z \setminus \{1, \dots, k_n\} \subseteq Y.$$

Now $J(Z) \subseteq J^{k_m} \times \{0\}^{k_{m+1}-k_m} \times J^{\mathbb{N}} \subseteq D_m$ for all $m > n$, and so

$$J(Z) \subseteq \bigcap_{m>n} D_m.$$

It follows that $J_{\infty} \cup \mathcal{C} \cup J^*(Y) \subseteq \bigcup_{n<\infty} (\bigcap_{m>n} D_m)$. \square

For each $Z \in Y^*$ the space $J(Z)$ has a subspace homeomorphic to a Hilbert cube, and so $J(Z)$ is strongly infinite dimensional. Moreover, $J^*(Z) \subseteq J^*(Y)$.

Now we construct the example:

Example 2 (CH). There is a set $X \subset [0, 1]^{\mathbb{N}}$ which is not countable dimensional, but it has the property $S_1(\mathcal{O}_{\text{kfd}}, \Gamma)$.

Proof. Let $(H_{\alpha} : \alpha < \omega_1)$ enumerate the finite dimensional G_{δ} subsets of $J^{\mathbb{N}}$. Also, let $(\mathcal{J}_{\alpha} : \alpha < \omega_1)$ enumerate all countable families of open sets, and let $(f_{\alpha} : \alpha < \omega_1)$ enumerate ${}^{\mathbb{N}}\mathbb{N}$.

Now using CH recursively choose $X_{\alpha} \subset \mathbb{N}$ infinite and $x_{\alpha} \in J(X_{\alpha})$ such that:

- (1) $\alpha < \beta \Rightarrow X_{\beta} \subseteq^* X_{\alpha}$,
- (2) if \mathcal{J}_{α} is an \mathcal{O}_{kfd} -cover of $J_{\infty} \cup \{x_{\beta} : \beta < \alpha\}$, then $X_{\alpha} \in [\mathbb{N}]^{\aleph_0}$ is chosen such that for each $\beta < \alpha$, $X_{\alpha} \subseteq^* X_{\beta}$ and for some sequence $(D_n : n \in \mathbb{N})$ in \mathcal{J}_{α} we have $J_{\infty} \cup \{x_{\beta} : \beta < \alpha\} \cup J^*(X_{\alpha}) \subseteq \bigcup_{n \in \mathbb{N}} (\bigcap_{m \geq n} D_m)$, and
- (3) for each β , $x_{\beta} \notin \bigcup_{\gamma \leq \beta} H_{\gamma}$.

Here is how we accomplish (2): First, choose an infinite set $X \subset \mathbb{N}$ such that for all $\beta < \alpha$, $X \subseteq^* X_{\beta}$. Since \mathcal{J}_{α} is an \mathcal{O}_{kfd} -cover for $J_{\infty} \cup \{x_{\beta} : \beta < \alpha\}$, choose by Lemma 20 an $X_{\alpha} \in [X]^{\aleph_0}$ and a sequence $(D_n : n \in \mathbb{N})$ from \mathcal{J}_{α} such that $J^*(X_{\alpha}) \subseteq \bigcup_{n \in \mathbb{N}} (\bigcap_{m \geq n} D_m)$. Then we have

$$J_{\infty} \cup J^*(X_{\alpha}) \cup \{x_{\beta} : \beta < \alpha\} \subseteq \bigcup_{n \in \mathbb{N}} \left(\bigcap_{m \geq n} D_m \right).$$

Then as $J(X_{\alpha})$ contains a homeomorphic copy of the Hilbert cube, choose $x_{\alpha} \in J(X_{\alpha}) \setminus \bigcup_{\beta \leq \alpha} H_{\beta}$.

Put $Y = \{x_{\alpha} : \alpha < \omega_1\}$ and put $X = J_{\infty} \cup Y$.

Claim 1. Y is a Hurewicz set.

For let F be a finite dimensional subset of the Hilbert cube. Choose $\alpha < \omega_1$ so that $F \subset H_{\alpha}$. Then, as $Y \cap H_{\alpha} \subset \{x_{\beta} : \beta < \alpha\}$, it follows that $Y \cap F$ is countable. Thus Y has a countable intersection with countable dimensional sets.

Claim 2. Each open cover of X which is in \mathcal{O}_{kfd} contains a subset which is a γ -cover for X .

For let \mathcal{U} be in \mathcal{O}_{kfd} for X . Choose an α with $\mathcal{U} = \mathcal{J}_\alpha$. Now at stage α we chose X_α and a sequence $(D_n: n \in \mathbb{N})$ from \mathcal{J}_α so that $J^*(X_\alpha) \cup J_\infty \cup \{x_\beta: \beta < \alpha\} \subseteq \bigcup_{n \in \mathbb{N}} (\bigcap_{m \geq n} D_m)$. Since for all $\gamma > \alpha$ we have $X_\gamma \subset^* X_\alpha$, it follows that for all $\gamma > \alpha$ we have $J^*(X_\gamma) \subseteq J^*(X_\alpha)$. But then $X \subseteq \bigcup_{n \in \mathbb{N}} (\bigcap_{m \geq n} D_m)$.

Claim 3. X has $S_1(\mathcal{O}_{\text{kfd}}, \Gamma)$.

For let $(\mathcal{U}_n: n < \infty)$ be a sequence from \mathcal{O}_{kfd} for X . Choose an infinite discrete subset $(d_n: n < \infty)$ from X . For each n , put $\mathcal{V}_n = \{U \setminus \{d_n\}: U = V_1 \cap \dots \cap V_n \text{ and for } i \leq n, V_i \in \mathcal{U}_i\} \setminus \{\emptyset\}$. Then put $\mathcal{U} = \bigcup_{n < \infty} \mathcal{V}_n$. Still \mathcal{U} is in \mathcal{O}_{kfd} for X . By Claim 2 choose $D_m \in \mathcal{U}, m < \infty$, such that $X \subseteq \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} D_n$. Choose n_1 so large that $d_1 \in D_m$ for all $m \geq n_1$. Then choose $n_2 > n_1$ so large that for an $m_1 < n_2, D_{n_1} \in \mathcal{V}_{m_1}$ and $\{d_1, \dots, d_{\max\{n_1, m_1\}}\} \subseteq D_k$ for all $k \geq n_2$. Choose $n_3 > n_2$ so large that for an $m_2 < n_3$ we have $D_{n_2} \in \mathcal{V}_{m_2}$, and for all $k \geq n_3$ we have $\{d_j: j \leq \max\{n_2, m_2\}\} \subseteq D_k$, and so on.

We have $m_1 < m_2 < \dots < m_k$, and $X \subseteq \bigcup_{k < \infty} \bigcap_{j \geq k} D_{n_j}$, and each D_{n_j} is of the form $\bigcap_{i \leq m_j} V_i^j$. For $i \leq m_1$ put $U_i = V_i^1$, and for $k \geq 1$ and for $m_k < i \leq m_{k+1}$ put $U_i = V_i^{k+1}$. Then we have for each i that $U_i \in \mathcal{U}_i$, and $\{U_i: i < \infty\}$ is a γ -cover of X .

By Claim 1 X is not countable dimensional. \square

Remarks. When we choose X_α we may assume the enumeration function of X_α dominates f_α . This guarantees that the subspace Y of X does not have the Menger property.

By Corollary 15 X has the Hurewicz covering property in all finite powers. By Corollary 13 of [2] X has the property $S_c(\mathcal{O}, \mathcal{O})$ in all finite powers.

In Theorem 15 of [4] it was shown that a Tychonoff space X has the property $S_1(\mathcal{O}_k, \Gamma)$ if, and only if, the space $C(X)$ consisting of continuous real-valued functions on X has the following property: For every sequence $(A_n: n < \infty)$ of subsets of X , each having the function $f \in \overline{A_n}$ in the compact-open topology on $C(X)$, there is a sequence $(f_n: n \in \mathbb{N})$ in $C(X)$ such that for each $n, f_n \in A_n$, and the sequence $(f_n: n < \infty)$ converges to f in the point-open topology of $C(X)$. Since $S_1(\mathcal{O}_{\text{kfd}}, \Gamma)$ implies $S_1(\mathcal{O}_k, \Gamma)$, it follows that for the space X constructed in Example 2, the function space $C(X)$ has the corresponding property.

6. Added in proof

Shortly after we submitted our paper Roman Pol informed us that a certain modification of our construction of Example 2 answers our Problem 3. With Pol's kind permission we include his remarks here:

With CH assumed and with the notation established in Example 2 do the following: In addition to listing the $H_\alpha, \mathcal{J}_\alpha$ and f_α in the beginning of the construction, also list all open sets in $J^\mathbb{N}$ as $(G_\alpha: \alpha < \omega_1)$.

At stage α , when picking the point x_α we proceed as follows: if $J(X_\alpha) \setminus (G_\alpha \cup (\bigcup_{\beta \leq \alpha} H_\beta))$ is nonempty, x_α is chosen from this set. Otherwise, x_α is an arbitrary point in $J(X_\alpha) \setminus (\bigcup_{\beta \leq \alpha} H_\beta)$ if this is a subset of G_α .

Claim 4. $Y = \{x_\alpha: \alpha < \omega_1\}$ is not in $S_c(\mathcal{O}_2, \mathcal{O})$.

To this end repeat an argument from pp. 90, 91 of [21]: For $x \in J^\mathbb{N}$ write $\text{supp}(x) = \{n: x(n) \neq 0\}$. For each i , let $U_{i,0}$ (respectively, $U_{i,1}$) be an open set in $J^\mathbb{N}$ consisting of points x such that for some $j, |\text{supp}(x) \cap \{1, 2, \dots, j\}| = i$, and $0 < x(j) < 5/6$ (respectively, $x(j) > 4/6$). A key observation (which justifies (18) on p. 90) is the following. Let $X = \{n_1, n_2, \dots\}$ with $n_1 < n_2 < \dots$. Then the trace of $U_{i,0}$ (respectively, $U_{i,1}$) on $J(X)$ consists of the points x in $J(X)$ with $0 < x(n_i) < 5/6$ (respectively, $x(n_i) > 4/6$).

Then for each i set $\mathcal{U}_i = \{U_{i,0}, U_{i,1}\}$.

Subclaim 4.1. If X is any infinite subset of \mathbb{N} , the traces of \mathcal{U}_i on $J(X)$ form a sequence in $J(X)$ witnessing that $J(X)$ does not have $S_c(\mathcal{O}_2, \mathcal{O})$.

For consider the closed subspace C of $J(X)$ defined as follows: For $n \in X$ put $S_n = [\frac{1}{2}, 1]$, and for $n \notin X$ put $S_n = \{0\}$. Then $C = \prod_{n=1}^\infty S_n$ is homeomorphic to $[\frac{1}{2}, 1]^X$.

For each i , define:

$$A_{i0} = \left\{ x \in J(X): |\text{supp}(x) \cap \{1, \dots, j\}| = i \text{ and } x(j) = \frac{1}{2} \right\}$$

and

$$A_{i1} = \{x \in J(X): |\text{supp}(x) \cap \{1, \dots, j\}| = i \text{ and } x(j) = 1\}.$$

Then A_{i_0} and A_{i_1} are disjoint closed subsets of $J(X)$ and are essentially opposite faces of the Hilbert cube $[\frac{1}{2}, 1]^X$. Moreover, $A_{ij} \subset U_{ij}$ and $A_{ij} \cap \bar{U}_{i(1-j)} = \emptyset$ for $j \in \{0, 1\}$. Then the argument used for the Hilbert cube can be adapted and applied to the sequence of opposite pairs of faces A_{i_0} and A_{i_1} to show that the sequence $(\mathcal{U}_n: n < \infty)$ does not have a sequence of disjoint refinements that cover $J(X)$. Observe that if C is a countable dimensional subset of $J(X)$, then the same is true about $J(X) \setminus C$.

Now, return to the set Y : Assume, aiming at a contradiction, that Y is an $S_c(\mathcal{O}_2, \mathcal{O})$ -space. Let \mathcal{V}_i be a disjoint open collection in $J^{\mathbb{N}}$ such that \mathcal{V}_i refines \mathcal{U}_i and the union G of all elements of all \mathcal{V}_i contains Y . Then $G = G_\alpha$ for some $\alpha < \omega_1$ and since x_α is in G_α , by the rules for choosing x_α we have that for some countable dimensional set C , $J(X_\alpha) \setminus C$ is contained in G_α . Therefore, the traces of V_i on $J(X_\alpha) \setminus C$ provide a forbidden sequence of open disjoint refinements of the traces of U_i on $J(X_\alpha) \setminus C$, a contradiction.

Claim 5. *With $X = J_\infty \cup Y$, the product $X \times \mathbb{P}$ is not in $S_c(\mathcal{O}_2, \mathcal{O})$.*

Indeed, let $u: J \rightarrow \{0, 1\}$ take 0 to 0 and the interval $[\frac{1}{2}, 1]$ to 1, and let $f: J^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ be the product map. Then u and f are continuous maps, implying f (the graph of f) is a closed subset of $J^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$.

Since the set $\mathbb{P} = \{x \in \{0, 1\}^{\mathbb{N}}: |\{n: x(n) = 1\}| = \aleph_0\}$ is homeomorphic to the irrationals, and $f^{-1}(\mathbb{P}) = J^{\mathbb{N}} \setminus J_\infty \supseteq Y$, we have $f \cap (X \times \mathbb{P}) = f \cap (Y \times \mathbb{P}) = \{(y, f(y)): y \in Y\}$ is a closed subset of $X \times \mathbb{P}$. The map $g: Y \rightarrow f \cap (Y \times \mathbb{P})$ defined by $g(y) = (y, f(y))$ is one-to-one, continuous, and its inverse is the projection onto the first coordinate, which also is continuous, and therefore g shows that Y is homeomorphic to $f \cap (Y \times \mathbb{P})$. By Claim 4 the closed subset $f \cap (X \times \mathbb{P})$ of $X \times \mathbb{P}$ is not $S_c(\mathcal{O}_2, \mathcal{O})$, and thus $X \times \mathbb{P}$ is not $S_c(\mathcal{O}_2, \mathcal{O})$.

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