Complete $k$-arcs in $\text{PG}(n, q)$, $q$ even

L. Storme* and J.A. Thas

Seminar of Geometry and Combinatorics, State University of Ghent, Krijgslaan 281, B-9000 Gent, Belgium

Received 13 December 1991

Abstract

Storme, L., J.A. Thas, Complete $k$-arcs in $\text{PG}(n, q)$, $q$ even, Discrete Mathematics 106/107 (1992) 455-469.

This paper investigates the completeness of $k$-arcs in $\text{PG}(n, q)$, $q$ even. We determine all values of $k$ for which there exists a complete $k$-arc in $\text{PG}(n, q)$, $q - 2 > n > q - \sqrt{q/2 - \frac{1}{4}}$. This is proven by using the duality principle between $k$-arcs in $\text{PG}(n, q)$ and dual $k$-arcs in $\text{PG}(k - n - 2, q)$ ($k \geq n + 4$). The theorems show that the classification of all complete $k$-arcs in $\text{PG}(n, q)$, $q$ even and $q - 2 > n > q - \sqrt{q/2 - \frac{1}{4}}$, is closely related to the classification of all $(q + 2)$-arcs in $\text{PG}(2, q)$.

1. Introduction

Let $\Sigma = \text{PG}(n, q)$ denote the $n$-dimensional projective space over the field $\text{GF}(q)$. A $k$-arc of points in $\Sigma$, with $k > n + 1$, is a set $K$ of $k$ points with the property that no $n + 1$ points of $K$ lie in a hyperplane.

A normal rational curve in $\text{PG}(n, q)$, $2 \leq n \leq q - 2$, is a $(q + 1)$-arc of $\text{PG}(n, q)$ which is projectively equivalent to the $(q + 1)$-arc $\{(1, t, \ldots, t^n) \mid t \in \text{GF}(q)^+\}$ $(\text{GF}(q)^+ = \text{GF}(q) \cup \{\infty\}; \infty \notin \text{GF}(q); t = \infty \text{ corresponds to } (0, \ldots, 0, 1))$. A normal rational curve of $\text{PG}(2, q)$ (resp. $\text{PG}(3, q)$) is a conic (resp. a twisted cubic).

The study of $k$-arcs in $\text{PG}(n, q)$ is also interesting from a coding-theoretic point of view. The $k$-arcs of $\text{PG}(n, q)$ and the linear MDS codes (maximum distance separable codes) of dimension $n + 1$ and length $k$ over $\text{GF}(q)$ are equivalent objects. Any result on $k$-arcs can be translated into an equivalent theorem on linear MDS codes. The $k$-arcs which are subsets of a normal rational curve correspond to GRS (generalized Reed-Solomon) codes and GDRS (generalized doubly-extended Reed-Solomon) codes.

* Correspondence to: J.A. Thas, Seminar of Geometry and Combinatorics, State University of Ghent, Krijgslaan 281, B-9000 Gent, Belgium.

* Research Assistant of the National Fund for Scientific Research (Belgium).
A point $p$ of $\text{PG}(n, q)$ extends a $k$-arc $K$ of $\text{PG}(n, q)$ to a $(k + 1)$-arc if and only if $K \cup \{p\}$ is a $(k + 1)$-arc. A $k$-arc $K$ of $\text{PG}(n, q)$ is complete if it is not contained in a $(k + 1)$-arc. Otherwise, $K$ is called incomplete.

The following question can then be posed. For which values of $k$ does there exist a complete $k$-arc in $\text{PG}(n, q)$? We present some new answers to this question by using the relation between $k$-arcs in $\text{PG}(n, q)$ and their dual $k$-arcs in $\text{PG}(k - n - 2, q)$.

2. Known results

**Theorem 2.1** (Segre [12]).

(a) In $\text{PG}(2, q)$, $q$ odd, all $k$-arcs satisfy $k \leq q + 1$.

(b) In $\text{PG}(2, q)$, $q$ even, any $(q + 1)$-arc is incomplete and can be extended to a $(q + 2)$-arc in a unique way. The point which extends a $(q + 1)$-arc to a $(q + 2)$-arc is called the nucleus of this $(q + 1)$-arc.

(c) In $\text{PG}(2, q)$, $q$ odd, every $(q + 1)$-arc is a conic.

**Remark 2.2.** Theorem 2.1(c) is not valid in $\text{PG}(2, q)$, $q$ even. In $\text{PG}(2, q)$, $q$ even and $q \geq 16$, at least two types of $(q + 2)$-arcs are known.

Any $(q + 2)$-arc of $\text{PG}(2, q)$, $q = 2^h$ and $h > 1$, is projectively equivalent to a $(q + 2)$-arc $\{(1, t, F(t)) \mid t \in \text{GF}(q)\} \cup \{(0, 1, 0), (0, 0, 1)\}$ where $F$ is a permutation polynomial over $\text{GF}(q)$ of degree at most $q - 2$, satisfying $F(0) = 0$, $F(1) = 1$ and such that for each $s$ in $\text{GF}(q)$

$$F_s(X) = \frac{F(X + s) + F(s)}{X}$$

is a permutation polynomial satisfying $F_s(0) = 0$ [6, p. 174].

A survey of all known types of $(q + 2)$-arcs in $\text{PG}(2, q)$, $q$ even and $q \neq 64$, can be found in [10]; for $q = 64$, see [11].

**Theorem 2.3** (Casse and Glynn [4, 5]). Every $(q + 1)$-arc of $\text{PG}(3, q)$, $q = 2^h$ and $h \geq 3$, is projectively equivalent to a $(q + 1)$-arc $L = \{(1, t, t^e) \mid t \in \text{GF}(q)^+\}$ where $e = 2^v$ and $(v, h) = 1$ ($\text{GF}(q)^+ = \text{GF}(q) \cup \{\infty\}; \infty \notin \text{GF}(q); \infty$ corresponds to $(0, 0, 0, 1)$).

Any $(q + 1)$-arc of $\text{PG}(4, q)$, $q \geq 8$, is a normal rational curve.

**Further theorems 2.4.** A lot of research on $k$-arcs in $\text{PG}(n, q)$ is focussed upon the determination of lower bounds $k_n$ such that if $K$ is a $k$-arc of $\text{PG}(n, q)$ with $k > k_n$, then $K$ is incomplete and can be extended (in a unique way) to a complete $(q + 2)$-arc (when $n = 2$ and $q$ is even) or to a complete $(q + 1)$-arc. Table 1 shows these bounds and also gives the type of the complete $k'$-arc $L$ of which $K$ is a subset. The results of row (i) are valid for all $k$-arcs $K$ in $\text{PG}(n, q)$,
Table 1

<table>
<thead>
<tr>
<th>n</th>
<th>q</th>
<th>( k )</th>
<th>( k' )</th>
<th>( L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) ( \geq 2 )</td>
<td>( q = p^{2h} ) (( p ) prime, ( h \geq 1 ))</td>
<td>( k &gt; q - \sqrt{q/4 + n - \frac{3}{4}} )</td>
<td>( q + 1 )</td>
<td>( {(1, t, \ldots, t^n) \mid t \in GF(q)^*} )</td>
</tr>
<tr>
<td>(2) ( \geq 2 )</td>
<td>( q = p^{2h+1} ) (( p ) prime, ( h \geq 1 ))</td>
<td>( k &gt; q - \sqrt{pq/4 + p/3} + n - 1 )</td>
<td>( q + 1 )</td>
<td>( {(1, t, \ldots, t^n) \mid t \in GF(q)^*} )</td>
</tr>
<tr>
<td>(3) ( \geq 2 )</td>
<td>( q ) prime</td>
<td>( k &gt; \frac{4q}{2q + n - \frac{3}{2}} )</td>
<td>( q + 1 )</td>
<td>( {(1, t, \ldots, t^n) \mid t \in GF(q)^*} )</td>
</tr>
<tr>
<td>(4) ( \geq 2 )</td>
<td>( q = 2^{2h} ) (( h \geq 1 ))</td>
<td>( k &gt; q - \sqrt{q} + 1 )</td>
<td>( q + 2 )</td>
<td>( {(1, t, F(t)) \mid t \in GF(q)^* } \cup {(0, 1, 0), (0, 0, 1)} ) (see 2.2)</td>
</tr>
<tr>
<td>(5) ( \geq 2 )</td>
<td>( q = 2^{2h+1} ) (( h \geq 1 ))</td>
<td>( k &gt; q - \sqrt{2q} + 2 )</td>
<td>( q + 2 )</td>
<td>( {(1, t, F(t)) \mid t \in GF(q)^* } \cup {(0, 1, 0), (0, 0, 1)} ) (see 2.2)</td>
</tr>
<tr>
<td>(6) ( \geq 3 )</td>
<td>( q = 2^q ) (( h \geq 1 ))</td>
<td>( k &gt; q - \sqrt{q/2 + \frac{2}{3}} )</td>
<td>( q + 1 )</td>
<td>( {(1, t, \ldots, t^n) \mid t \in GF(q)^*} ) (( e = 2^e ) and ( (v, h) = 1 ); see 2.3)</td>
</tr>
<tr>
<td>(7) ( \geq 4 )</td>
<td>( q = 2^q ) (( h &gt; 2 ))</td>
<td>( k &gt; q - \sqrt{q/2 + n - \frac{3}{4}} )</td>
<td>( q + 1 )</td>
<td>( {(1, t, \ldots, t^n) \mid t \in GF(q)^*} )</td>
</tr>
</tbody>
</table>

These results are due to: (1) Thas [15, 17]; (2), (5) Voloch [19]; (3) Voloch [18]; (4) Segre [12]; (6). (7) Storme and Thas [13].

where \( n \) and \( q \) are given in columns 2 and 3, and where the value \( k \) is larger than the lower bound in column 4. These \( k \)-arcs can be extended in a unique way to a \( k' \)-arc \( L \) where the value of \( k' \) can be found in column 5. This \( k' \)-arc \( L \) is projectively equivalent to the \( k' \)-arc in column 6.

3. MDS codes [9]

**Theorem 3.1** (the Singleton bound). *Let \( C \) be a linear \([n, k, d]\) code over GF(q), that is, \( C \) is a linear code of dimension \( k \), length \( n \) and with minimum distance \( d \). Then \( d \leq n - k + 1 \).*

**Definition 3.2.** A linear \([n, k, d]\) code \( C \) is called a linear MDS (maximum distance separable) code if and only if \( d = n - k + 1 \).

**Theorem 3.3.** *Let \( C \) be a linear \([n, k, d]\) code over GF(q). Then the following statements are equivalent:*

(a) \( C \) is MDS;  
(b) every \( k \) columns of a generator matrix of \( C \) are linearly independent;  
(c) every \( n - k \) columns of a parity check matrix of \( C \) are linearly independent;  
(d) the dual code \( C^\perp \) is MDS.*
Remark 3.4. Let $C$ be a linear $[n, k, d]$ code over GF($q$), with $k \geq 3$, and suppose that $G$ is a generator matrix of $C$. The preceding theorem then shows that $C$ is an MDS code if and only if the $n$ columns of $G$ constitute a $n$-arc of points in PG($k - 1$, $q$).

Furthermore, a linear $[n, k, d]$ code $C$ over GF($q$), with $n \geq k + 3$, is MDS if and only if the $n$ columns of a parity check matrix $H$ of $C$ constitute a $n$-arc of PG($n - k - 1$, $q$).

4. Duality principle for $k$-arcs

Definition 4.1. Let $K = \{p_1, \ldots, p_k\}$ be a $k$-arc in PG($n$, $q$), $k \geq n + 4$. All the $k$-arcs $\hat{K} = \{q_1, \ldots, q_k\}$ in PG($k - n - 2$, $q$) for which the resp. coordinates $g_{0i}$, $\ldots$, $g_{ni}$ of $p_i$ ($1 \leq i \leq k$), the resp. coordinates $h_{0j}$, $\ldots$, $h_{k-n-2,j}$ of $q_j$ ($1 \leq j \leq k$), and the indices $j$ of the points $q_j$ can be chosen in such a way that the $(n + 1) \times k$ matrix $G = (g_{ij})$ and the $(k - n - 1) \times k$ matrix $H = (h_{ij})$ define dual linear MDS codes of length $k$ over GF($q$), are called the dual $k$-arcs in PG($k - n - 2$, $q$) of $K = \{p_1, \ldots, p_k\}$.

Remark 4.2. It follows from Definition 4.1 that $k$-arcs in PG($n$, $q$), $k \geq n + 4$, correspond to $k$-arcs in PG($k - n - 2$, $q$). Hence, (non-)existence theorems on $k$-arcs in PG($n$, $q$) yield (non-)existence theorems on $k$-arcs in PG($k - n - 2$, $q$).

Each theorem about $k$-arcs in PG($n$, $q$) can be dualized. This means, a theorem about $k$-arcs in PG($n$, $q$) gives a corresponding theorem about $k$-arcs in PG($k - n - 2$, $q$) which can be obtained by using the relationship between linear MDS codes and the dual linear MDS codes (Theorem 3.3).

Remark 4.3 [14]. The relation `$k$-arc $K$ in PG($n$, $q$) and dual $k$-arc $\hat{K}$ in PG($k - n - 2$, $q$) ($k \geq n + 4$)' is not a one-to-one correspondence between the $k$-arcs of PG($n$, $q$) and the $k$-arcs of PG($k - n - 2$, $q$) ($k \geq n + 4$). It defines a one-to-one correspondence between the classes of projectively equivalent $k$-arcs in PG($n$, $q$) and the classes of projectively equivalent $k$-arcs in PG($k - n - 2$, $q$) ($k \geq n + 4$). With a fixed $k$-arc $K$ in PG($n$, $q$) corresponds one class of projectively equivalent dual $k$-arcs in PG($k - n - 2$, $q$).

5. GDRS codes

Definition 5.1 [9]. A linear $[n, k, n - k + 1]$ MDS code $C$ over GF($q$) is called a GDRS (generalized doubly-extended Reed–Solomon) code if and only if $C$ has a generator matrix, the columns of which constitute $n$ points of the normal rational curve $\{(1, t, \ldots, t^{k-1}) \mid t \in GF(q)^{\times}\}$ in PG($k - 1$, $q$).
Theorem 5.2 [9]. The dual code of a GDRS \( [n, k, n - k + 1] \) code \( C \) over \( \text{GF}(q) \) is a GDRS \( [n, n - k, k + 1] \) code \( C^\perp \) over \( \text{GF}(q) \).

Hence, let \( K \) be a s-arc of \( \text{PG}(r, q) \) \( (s \geq r + 4) \) which is contained in a normal rational curve of \( \text{PG}(r, q) \). Then all dual s-arcs of \( K \) are contained in normal rational curves of \( \text{PG}(s - r - 2, q) \).

Remark 5.3. By applying Theorem 5.2 on (1), (2), (3) and (7) of Table 1 the following results hold:

(a) In \( \text{PG}(n, q), q = p^{2h}, h \geq 1, p \text{ prime}, p > 2, k - 4 \geq n > q - \sqrt{q}/4 - \frac{39}{16} \), any \( k \)-arc \( K \) is contained in a unique normal rational curve.

(b) In \( \text{PG}(n, q), q = p^{2h + 1}, h \geq 1, p \text{ prime}, p \neq 2, k - 4 \geq n > q - \sqrt{pq}/4 + \frac{29}{16}p - 3 \), any \( k \)-arc \( K \) is contained in a unique normal rational curve.

(c) In \( \text{PG}(n, q), q \text{ prime}, q \neq 2, k - 4 \geq n > \frac{33}{15}q - \frac{28}{6} \), any \( k \)-arc \( K \) is contained in a unique normal rational curve.

(d) In \( \text{PG}(n, q), q = 2^h, h > 2, k - 6 \geq n > q - \sqrt{q}/2 - \frac{11}{6} \), any \( k \)-arc \( K \) is contained in a unique normal rational curve.

Corollary 5.4. Consider the following projective spaces:

(a) \( \text{PG}(n, q), q = p^{2h}, h \geq 1, p \text{ prime}, p > 2, q - 2 \geq n > q - \sqrt{q}/4 - \frac{39}{16}; \)

(b) \( \text{PG}(n, q), q = p^{2h + 1}, h \geq 1, p \text{ prime}, p \neq 2, q - 2 \geq n > q - \sqrt{pq}/4 + \frac{29}{16}p - 3; \)

(c) \( \text{PG}(n, q), q \text{ prime}, q \neq 2, q - 2 \geq n > \frac{33}{15}q - \frac{28}{6}. \)

In these projective spaces, every complete arc is a \( (q + 1) \)-arc. Moreover, these \( (q + 1) \)-arcs are normal rational curves.

Proof. (a) \( q - 3 \geq n \). A \( (n + 1) \)-arc \( K \) of \( \text{PG}(n, q) \) is projectively equivalent to the \( (n + 1) \)-arc \( \{e_0(1, 0, \ldots, 0), \ldots, e_n(0, \ldots, 0, 1)\} \). The point \( e_{n+1}(1, \ldots, 1) \) extends \( K \) to a \( (n + 2) \)-arc. Any \( (n + 2) \)-arc \( K' \) of \( \text{PG}(n, q) \) is projectively equivalent to \( \{e_0, \ldots, e_{n+1}\} \) and can be extended to a \( (n + 3) \)-arc by the point \( (1, o, \ldots, o^n) \) where \( o \) is a primitive element of \( \text{GF}(q) \). A \( (n + 3) \)-arc \( K'' \) of \( \text{PG}(n, q) \) is contained in a unique normal rational curve \( L \) of \( \text{PG}(n, q) \) [7, p. 229]. Since \( |L| = q + 1 \) and since \( q \geq n + 3, K'' \) is incomplete.

All \( k \)-arcs \( K \) of \( \text{PG}(n, q), k \geq n + 4 \) (satisfies the bounds in (a), (b) or (c)), are contained in a unique normal rational curve (Remark 5.3). So \( k \leq q + 1 \) for all such \( k \)-arcs in \( \text{PG}(n, q) \) and hence every \( (q + 1) \)-arc is a normal rational curve.

(b) \( n = q - 2 \). As in (a), only the \( k \)-arcs \( K \) for which \( k \geq n + 3 \) have to be considered. Hence, \( k \geq q + 1 \). Every \( (q + 1) \)-arc \( K \) of \( \text{PG}(q - 2, q), q > 3, \) is contained in a unique normal rational curve \( L \) of \( \text{PG}(q - 2, q) \) [7, p. 229]. Since \( |L| = q + 1, K \) is equal to \( L \).

Assume that \( K' \) is a \( (q + 2) \)-arc in \( \text{PG}(q - 2, q), q > 3; \) then \( K' \) is a dual arc of a \( (q + 2) \)-arc in \( \text{PG}(2, q) \) (Definition 4.1). There are no \( (q + 2) \)-arcs in \( \text{PG}(2, q) \), \( q \) odd (Theorem 2.1(a)). So, there are no \( (q + 2) \)-arcs in \( \text{PG}(q - 2, q), q \) odd, \( q > 3. \) Hence, every \( (q + 1) \)-arc \( K \) of \( \text{PG}(q - 2, q), q \) odd, \( q > 3, \) is complete. Moreover, \( K \) is a normal rational curve of \( \text{PG}(q - 2, q). \) \( \square \)
6. Completeness and duality

**Theorem 6.1.** Let $K = \{ p_1, \ldots, p_k \}$ be a $k$-arc in $\text{PG}(n, q)$, $k \geq n + 4$, and let $\bar{K} = \{ q_1, \ldots, q_{k} \}$ be a dual $k$-arc of $K$ in $\text{PG}(k - n - 2, q)$.

The $k$-arc $K$ can be extended to a $k'$-arc projectively equivalent to $K'$ in $\text{PG}(n, q)$ ($k' > k$) if and only if there exists a dual $k'$-arc $\bar{K}' = \{ r_1, \ldots, r_{k'} \}$ of $K'$ in $\text{PG}(k' - n - 2, q)$ and $k' - k$ points $r_{k+1}, \ldots, r_{k'}$ of $\bar{K}'$ such that the projection of $\bar{K}' \setminus \{ r_{k+1}, \ldots, r_{k'} \}$ from $\beta = (r_{k+1}, \ldots, r_{k'})$ onto a subspace $\text{PG}(k - n - 2, q)$, skew to $\beta$, is a $k$-arc of $\text{PG}(k - n - 2, q)$ which is projectively equivalent to $K$.

**Proof.** Part 1. Choose the projective reference system in $\text{PG}(n, q)$ in such a way that $p_1(1, 0, \ldots, 0), \ldots, p_{n+1}(0, 0, \ldots, 1)$. Assume that $p_{r+1}(a_0, \ldots, a_r), \ldots, p_{k}(a_0, \ldots, a_{k-n-2})$ extend $K$ to a $k'$-arc $K'$ in $\text{PG}(n, q)$. Suppose that the points $p_{k+1}(a_{0,k-n-1}, \ldots, a_{n,k-n-1}), \ldots, p_{k}(a_{0,k'-n-2}, \ldots, a_{n,k'-n-2})$ extend $K$ to a $k'$-arc $K'$ in $\text{PG}(n, q)$.

Let $G = (I_{n+1}A)$, with $A = (a_{ij})$ ($0 \leq i \leq n, 0 \leq j \leq k - n - 2$) be the matrix whose columns are the coordinate vectors of the points $p_v$, $n + 2 \leq v \leq k$, of $K$. This matrix $G$ generates a $[k, n + 1, k - n]$ MDS code over $\text{GF}(q)$ (Remark 3.4). Then $H = (-ATI_{k-n-1})$, where $A^T$ is the transpose of $A$ and where $I_{k-n-1}$ is the square identity matrix of order $k-n-1$, defines the dual $[k', k-n-1, n+2]$ MDS code [9, p. 5].

So

$$\bar{K} = \{ e_0(1, 0, \ldots, 0), \ldots, e_{k-n-2}(0, \ldots, 0, 1), (a_{00}, \ldots, -a_{0,k-n-2}),$$

$$\ldots, (a_{n0}, \ldots, -a_{n,k-n-2}) \}$$

is a dual $k$-arc of $K$ in $\text{PG}(k - n - 2, q)$ (Definition 4.1).

Part 2. Assume that $K$ can be extended to a $k'$-arc $K'$, $k' > k$, of $\text{PG}(n, q)$. Suppose that the points $p_{k+1}(a_{0,k-n-1}, \ldots, a_{n,k-n-1}), \ldots, p_{k}(a_{0,k'-n-2}, \ldots, a_{n,k'-n-2})$ extend $K$ to a $k'$-arc $K'$ in $\text{PG}(n, q)$.

Let $G_1 = (I_{n+1}A_1)$, with $A_1 = (a_{ij})$ ($0 \leq i \leq n, 0 \leq j \leq k' - n - 2$), be the generator matrix of the corresponding $[k', n + 1, k' - n]$ MDS code $C_1$ (Remark 3.4). As in Part 1, $H_1 = (-A_1^TI_{k-n-1})$ is a parity check matrix of $C_1$.

This means that

$$\bar{K}' = \{ e'_0(1, 0, \ldots, 0), \ldots, e'_{k-n-2}(0, \ldots, 0, 1), (a_{00}, \ldots, -a_{0,k'-n-2}),$$

$$\ldots, (a_{n0}, \ldots, -a_{n,k'-n-2}) \}$$

is a dual $k'$-arc of $K'$ in $\text{PG}(k' - n - 2, q)$ (Definition 4.1).

Project now from $\beta = (e'_{k-n-1}, \ldots, e'_{k'-n-2})$ in $\text{PG}(k' - n - 2, q)$ onto the subspace $\delta: X_{k-n-1} = \cdots = X_{k'-n-2} = 0$ skew to $\beta$. The $k$ points

$$e'_0, \ldots, e'_{k-n-2}, (-a_{00}, \ldots, -a_{0,k'-n-2}), \ldots, (a_{n0}, \ldots, -a_{n,k'-n-2})$$

of $\bar{K}'$ are projected from $\beta$ onto the points

$$e'_0, \ldots, e'_{k-n-2}, (-a_{00}, \ldots, -a_{0,k-n-2}, 0, \ldots, 0), \ldots, (a_{n0}, \ldots, -a_{n,k-n-2}, 0, \ldots, 0).$$
These $k$ points constitute a $k$-arc in $\delta$ which is projectively equivalent to $\hat{K}$.

**Part 3.** Suppose there exists a $k'$-arc $\hat{K}' = \{r_1, \ldots, r_{k'}\}$, $k' > k$, in $PG(k' - n - 2, q)$ and $k' - k$ points $r_{k+1}, \ldots, r_{k'}$ of $\hat{K}'$ such that the $k$-arc $K$ is the projection of $\hat{K}' \setminus \{r_{k+1}, \ldots, r_{k'}\}$ from $\beta = \langle r_{k+1}, \ldots, r_{k'} \rangle$ onto a subspace $\delta$ of dimension $k - n - 2$ skew to $\beta$.

Choose the reference system such that

\[ r_1(-a_{0,0}, \ldots, -a_{0,k'-n-2}), \ldots, r_{n+1}(-a_{n,0}, \ldots, -a_{n,k'-n-2}), \]

\[ r_{n+1}(0, \ldots, 0, 1, 0, \ldots, 0) \]

with the 1 of $r_{n+1}$ in the $(i - 1)$th position ($2 \leq i \leq k' - n$). Project now from $\langle r_{k+1}, \ldots, r_{k'} \rangle$ onto $\delta$: $X_{k-n-1} = \cdots = X_{k'-n-2} = 0$. Then the projection

\[ \hat{K} = \{(-a_{0,0}, \ldots, -a_{0,k-n-2}, 0, \ldots, 0), \ldots, \]

\[ (-a_{n,0}, \ldots, -a_{n,k-n-2}, 0, \ldots, 0), r_{n+2}, \ldots, r_{k}\} \]

of $\hat{K}' \setminus \{r_{k+1}, \ldots, r_{k'}\}$ is a $k$-arc in $\delta$.

As in Part 1,

\[ K' = \{e_0(1, 0, \ldots, 0), \ldots, e_n(0, \ldots, 0, 1), (a_{0,0}, \ldots, a_{n,0}), \ldots, \]

\[ (a_{0,k'-n-2}, \ldots, a_{n,k'-n-2})\} \]

is a dual $k'$-arc of $\hat{K}'$ in $PG(n, q)$. Also,

\[ K = \{e_0, \ldots, e_n, (a_{0,0}, \ldots, a_{n,0}), \ldots, (a_{0,k-n-2}, \ldots, a_{n,k-n-2})\} \]

is a dual $k$-arc of $\hat{K}$ with $K \subseteq K'$. So $K$ can be extended to a $k'$-arc in $PG(n, q)$ having as dual the $k'$-arc $\hat{K}'$. □

**Corollary 6.2.** A $k$-arc $K$ in $PG(n, q)$, $k \geq n + 4$, is complete if and only if for any dual $k$-arc $\hat{K}$ (in $PG(k - n - 2, q)$) of $K$ there is no $k'$-arc $\hat{K}'$ in $PG(k' - n - 2, q)$ $\supseteq PG(k - n - 2, q)$, $k' > k$, such that $\hat{K}$ is the projection of $\hat{K}' \setminus PG(k' - k - 1, q)$ from $PG(k' - k - 1, q)$ onto $PG(k - n - 2, q)$, where $PG(k' - k - 1, q)$ is skew to $PG(k - n - 2, q)$ and generated by $k' - k$ points of $K'$.

**Theorem 6.3.** In $PG(n, q)$, $q$ even, $q = 2^{2h}$, $h \geq 1$, $q - 2 > n > q - \sqrt{q} - 3$ and in $PG(n, q)$, $q$ even, $q = 2^{2h+1}$, $h \geq 1$, $q - 2 > n > q - \sqrt{2q} - 2$, is every $(n+4)$-arc the projection of $L \setminus PG(q - n - 3, q)$ from $PG(q - n - 3, q)$ onto $PG(n, q)$, where $L$ is a $(q + 2)$-arc of $PG(q - 2, q) \supseteq PG(n, q)$, $PG(q - n - 3, q) \cap PG(n, q) = \emptyset$, and $PG(q - n - 3, q)$ is generated by $q - n - 2$ points of $L$.

**Proof.** Part 1. $q = 2^{2h}$, $h \geq 1$. Let $K$ be a $k$-arc in $PG(2, q)$ with $k > q - \sqrt{q} + 1$. It follows from Table 1 (4) that $K$ can be extended in a unique way to a $(q + 2)$-arc $K'$ of $PG(2, q)$. Let $\hat{K}$ be a dual $k$-arc of $K$ in $PG(k - 4, q)$ (Definition 4.1). Since $q + 2 > k > q - \sqrt{q} + 1$, we have $q - 2 > k - 4 > q - \sqrt{q} - 3$. 

Hence, putting \( k = n + 4 \), \( \hat{K} \) is a \((n + 4)\)-arc in \( \text{PG}(n, q) \), where \( q - 2 \geq n > q - \sqrt{q} - 3 \).

Let \( \hat{K}' \) be a dual \((q + 2)\)-arc of \( K' \) in \( \text{PG}(q - 2, q) \). Theorem 6.1 then states that there exist \( q + 2 - k \) points in \( \hat{K}' \) such that if we project from the \((q + 1 - k)\)-dimensional subspace \( \gamma \), generated by these \( q + 2 - k \) points of \( \hat{K}' \), onto a subspace \( \delta = \text{PG}(k - 4, q) \) skew to \( \gamma \), then \( \hat{K}' \) is projected onto a \( k \)-arc in \( \delta \) which is projectively equivalent to \( \hat{K} \).

But every \( k \)-arc \( \hat{K} \) in \( \text{PG}(k - 4, q) \), \( q + 2 \geq k > q - \sqrt{q} + 1 \), is a dual \( k \)-arc of a \( k \)-arc \( K \) in \( \text{PG}(2, q) \), \( q + 2 \geq k > q - \sqrt{q} + 1 \), so this result is valid for all \( k \)-arcs \( \hat{K} \) in \( \text{PG}(k - 4, q) \), \( q + 2 \geq k > q - \sqrt{q} + 1 \). Replacing \( k \) by \( n + 4 \) proves the theorem.

**Part 2.** \( q = 2^{2h+1}, \ h \geq 1 \). Let \( K \) be a \( k \)-arc in \( \text{PG}(2, q) \) with \( q + 2 \geq k > q - \sqrt{2q} + 2 \). This \( k \)-arc \( K \) is contained in a \((q + 2)\)-arc of \( \text{PG}(2, q) \) (Table 1 (5)).

Let \( \hat{K} \) be a dual \( k \)-arc of \( K \) in \( \text{PG}(k - 4, q) \). Then, putting \( k = n + 4 \), \( \hat{K} \) is a \((n + 4)\)-arc in \( \text{PG}(n, q) \), where \( q - 2 \geq n > q - \sqrt{2q} - 2 \).

Now the proof of Part 1 can be copied to show that there exists a \((q + 2)\)-arc \( \hat{K}' \) in \( \text{PG}(q - 2, q) \) and \( q - n - 2 \) points \( r_1, \ldots, r_{q-n-2} \) of \( \hat{K}' \) such that if we project from \( \gamma = \langle r_1, \ldots, r_{q-n-2} \rangle \) onto a subspace \( \text{PG}(n, q) \) skew to \( \gamma \), then \( \hat{K}' \setminus \{r_1, \ldots, r_{q-n-2}\} \) is projected onto a \((n + 4)\)-arc of \( \text{PG}(n, q) \) which is projectively equivalent to \( \hat{K} \).

**Remark 6.4.** Every \((q + 2)\)-arc of \( \text{PG}(q - 2, q) \), \( q \) even, is projectively equivalent to a \((q + 2)\)-arc \( \hat{K} = \{(1, t, \ldots, t^{i-2}) \mid t \in \text{GF}(q)^+\} \cup \{(a_0, \ldots, a_{q-2})\} \) with \( a_2 = 0 \), \( i = 0, \ldots, q/2 - 1 \) and with \( \sum_{i=0}^{q-2} a_i = 1 \). Furthermore, \( \{(1, t, F(t)) \mid t \in \text{GF}(q)\} \cup \{(0, 1, 0), (0, 0, 1)\} \) with \( F(t) = \sum_{i=0}^{q-2} a_{q-2-i}X^{i+1} \) is a dual \((q + 2)\)-arc of \( \hat{K} \) [14].

So by projecting the points not in \( \gamma \) of any such \((q + 2)\)-arc \( \hat{K} \) of \( \text{PG}(q - 2, q) \) from the \((q - n - 3)\)-dimensional subspace \( \gamma \), generated by any \( q - n - 2 \) points of \( \hat{K} \), onto a subspace \( \text{PG}(n, q) \) skew to \( \gamma \), we get all possible types of \((n + 4)\)-arcs in \( \text{PG}(n, q) \) when \( 0 \leq q - n - 2 < \sqrt{q} + 1 \) and \( q = 2^{2h}, \ h \geq 1 \) (resp. \( 0 \leq q - n - 2 < \sqrt{2q} \) and \( q = 2^{2h+1}, \ h \geq 1 \).

**Theorem 6.5.** In \( \text{PG}(n, q) \), \( q \) even, \( q > 2, \ q - 4 \geq n > q - \sqrt{q}/2 - \frac{1}{4} \), is every \((n + 5)\)-arc the projection of \( L \setminus \text{PG}(q - n - 5, q) \) from \( \text{PG}(q - n - 5, q) \) onto \( \text{PG}(n, q) \), where \( L \) is a \((q + 1)\)-arc of \( \text{PG}(q - 4, q) \supset \text{PG}(n, q) \), \( \text{PG}(q - n - 5, q) \cap \text{PG}(n, q) = \emptyset \) and \( \text{PG}(q - n - 5, q) \) is generated by \( q - n - 4 \) points of \( L \).

**Proof.** Let \( K \) be a \( k \)-arc in \( \text{PG}(3, q) \), \( q > 2 \), with \( q + 1 \geq k > q - \sqrt{q}/2 + \frac{3}{4} \). This \( k \)-arc \( K \) is contained in a unique \((q + 1)\)-arc \( L \) of \( \text{PG}(3, q) \) (Table 1 (6)). Let \( \hat{K} \) be a dual \( k \)-arc of \( K \) (Definition 4.1). Then \( \hat{K} \) is a \( k \)-arc of \( \text{PG}(k - 5, q) \) with \( q - 4 \geq k > 5 > q - \sqrt{q}/2 - \frac{1}{4} \). The dual \((q + 1)\)-arcs of \( L \) are \((q + 1)\)-arcs in \( \text{PG}(q - 4, q) \).

We may now conclude from Theorem 6.1 that there exists a \((q + 1)\)-arc \( L' \) in
Theorem 6.6. Any \((q + 1)\)-arc of \(\text{PG}(q - 4, q)\), \(q = 2^h\), \(h \geq 3\), is projectively equivalent to a \((q + 1)\)-arc \(K = \{(1, t, \ldots, t^{q - 2 - e}, t^{q - e}, \ldots, t^{q - 3}) \mid t \in \text{GF}(q)\}\) \(\cup \{r_1, \ldots, r_{q - 2 - e}\}\) where \(e = 2^e\), \((v, h) = 1\) and with \(r_1, \ldots, r_{q - 2 - e}\) having a one in position \(q - 1 - e\) and zeros elsewhere. (Remark: for \(e = 2\), we have \(K = \{(1, t, \ldots, t^{q - 4}) \mid t \in \text{GF}(q)^*\}\).)

Proof. A \((q + 1)\)-arc of \(\text{PG}(q - 4, q)\) is a dual arc of a \((q + 1)\)-arc \(\hat{K}\) in \(\text{PG}(3, q)\) (Definition 4.1). Every \((q + 1)\)-arc of \(\text{PG}(3, q)\), \(q = 2^h\), \(h \geq 3\), is projectively equivalent to a \((q + 1)\)-arc \(L = \{(1, t^r, t^{r+1}) \mid t \in \text{GF}(q)^*\}\) with \(e = 2^e\) and with \((v, h) = 1\) (Theorem 2.3). Since all dual \((q + 1)\)-arcs in \(\text{PG}(q - 4, q)\) of a \((q + 1)\)-arc in \(\text{PG}(3, q)\) are projectively equivalent (Remark 4.3), it is sufficient to find one dual arc of such a \((q + 1)\)-arc \(L\) in order to classify all types of \((q + 1)\)-arcs in \(\text{PG}(q - 4, q)\).

It will now be shown that the matrices

\[
G = \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 & 0 \\
1 & x_1 & \cdots & x_{q-2} & 0 & 0 \\
1 & x_1^2 & \cdots & x_{q-2} & 0 & 0 \\
1 & x_1^{q+1} & \cdots & x_{q-2}^{q+1} & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
g_0 \\
g_1 \\
g_2 \\
g_3
\end{pmatrix}
\]

and

\[
H = \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 & 0 \\
1 & x_1 & \cdots & x_{q-2} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & x_1^{q-2-e} & \cdots & x_{q-2}^{q-2-e} & 0 & 1 \\
1 & x_1^{q-e} & \cdots & x_{q-2}^{q-e} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & x_1^{q-3} & \cdots & x_{q-2}^{q-3} & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
h_0 \\
h_1 \\
h_2 \\
h_{q-2-e} \\
\vdots \\
\vdots \\
h_{q-4}
\end{pmatrix}
\]

where \(x_i = x^i\) (\(1 \leq i \leq q - 2\)) with \(x\) a primitive element of \(\text{GF}(q)\), define dual codes. Let \(g_i = (g_{i_1}, \ldots, g_{i_{q+1}})\) be row \(i\) of \(G\) and let \(h_i = (h_{j_1}, \ldots, h_{j_{q+1}})\) be row \(j\) of \(H\). We will now show that the inner product \(g_i \cdot h_j\) \((0 \leq i \leq 3; 0 \leq j \leq q - 4)\) is either \(q \cdot 1 = 0\) or \(\sum_{z \in \text{GF}(q)^*} z\) where \(r = 1, 2, \ldots, q - 2\) (\(\text{GF}(q)^* = \text{GF}(q) \setminus \{0\}\)).
Since \( \sum_{r \in \mathrm{GF}(q)} z'_r = 0 \) for \( r \not\equiv 0 \pmod{(q - 1)} \) (see Lemma 3.3 of [14]), all inner products \( g_i \cdot h_j \) are zero \( (0 \leq i \leq 3; 0 \leq j \leq q - 4) \). Hence \( G \) and \( H \) define dual codes [9, p. 26]. The columns of \( G \) are the points of a \((q + 1)\)-arc in \( \mathrm{PG}(3, q) \). So \( G \) generates an MDS code (Remark 3.4) and \( H \) generates the dual MDS code (Theorem 3.3). Hence, the columns of \( H \) constitute a dual \((q + 1)\)-arc in \( \mathrm{PG}(q - 4, q) \) of \( L \) (Definition 4.1). Consequently \( \hat{L} = \{(1, t, . . . , t^{q-3-e}, t^{q-3}) \mid t \in \mathrm{GF}(q)\} \cup \{r_{q-2-e}\} \) is a dual \((q + 1)\)-arc of \( L \) in \( \mathrm{PG}(q - 4, q) \). \( \square \)

**Remark 6.7.** Consider in \( \mathrm{PG}(q - 4, q), q = 2^h, h \geq 3 \), all the \((q + 1)\)-arcs \( L = \{(1, t, . . . , t^{q-2-e}, t^{q-3}) \mid t \in \mathrm{GF}(q)\} \cup \{r_{q-2-e}\} \) \( (e = 2^v, (v, h) = 1) \). Then by projecting the points not in \( \gamma \) of any such \((q + 1)\)-arc \( L \) of \( \mathrm{PG}(q - 4, q) \) from the \((q - n - 5)\)-dimensional subspace \( \gamma \), generated by any \( q - n - 4 \) points of \( L \), onto a subspace \( \mathrm{PG}(n, q) \) skew to \( \gamma \), we get all possible types of \((n + 5)\)-arcs in \( \mathrm{PG}(n, q) \) when \( 0 \leq q - n - 4 < \sqrt{q/2} - \frac{1}{4} \).

### 7. Complete \((n + 4)\)-arcs and complete \((n + 5)\)-arcs in \( \mathrm{PG}(n, q) \)

**Theorem 7.1.** There exist complete \((n + 4)\)-arcs in \( \mathrm{PG}(n, q) \), \( q \) even, \( q > 4, q - 4 \geq n > q - \sqrt{q/2} - \frac{1}{4} \). The complete \((n + 4)\)-arcs in these projective spaces \( \mathrm{PG}(n, q) \) are the dual arcs of the \( k \)-arcs \( \hat{K} \) \( (k = n + 4) \) in \( \mathrm{PG}(2, q) \) \( (q \geq k > q - \sqrt{q/2} + \frac{1}{4}) \) which are contained in any \( q \)-arc \( L \) of \( \mathrm{PG}(2, q) \) which is not projectively equivalent to a \( q \)-arc \( \{(1, t) \mid t \in \mathrm{GF}(q)\} \), with \( e = 2^v, q = 2^h, (v, h) = 1 \).

**Proof.** Let \( K \) be a \((n + 4)\)-arc in \( \mathrm{PG}(n, q) \), \( q - 4 \geq n > q - \sqrt{q/2} - \frac{1}{4} \). Assume that \( \hat{K} \) is a dual \( k \)-arc of \( K \) (Definition 4.1). Then \( \hat{K} \) is a \((n + 4)\)-arc in \( \mathrm{PG}(2, q) \). Equivalently, \( \hat{K} \) is a \( k \)-arc in \( \mathrm{PG}(2, q) \), \( q = 2^h, q \geq k > q - \sqrt{q/2} + \frac{1}{4} \) \( (k = n + 4) \). It then follows from Table 1 (4) and (5) that \( \hat{K} \) is contained in a \((q + 2)\)-arc of \( \mathrm{PG}(2, q) \).

The \((n + 4)\)-arc \( K \) is incomplete if and only if there exists a \((k + 1)\)-arc \( \hat{K}' \) in \( \mathrm{PG}(3, q) \) and a point \( r \) of \( \hat{K}' \) such that if we project from \( r \) onto a plane \( \delta \) not containing \( r \), then \( \hat{K}' \setminus \{r\} \) is projected onto a \( k \)-arc of \( \delta \) which is projectively equivalent to \( \hat{K} \) (Theorem 6.1).

The arc \( \hat{K}' \) is a \((k + 1)\)-arc of \( \mathrm{PG}(3, q) \) for which \( q + 1 \geq k + 1 > q - \sqrt{q/2} + 2 \). It then follows from Table 1 (6) that \( \hat{K}' \) is contained in a \((q + 1)\)-arc \( L \) of \( \mathrm{PG}(3, q) \) where \( L \) is projectively equivalent to a \((q + 1)\)-arc \( \{(1, t, t', t^{+1}) \mid t \in \mathrm{GF}(q)^+\} \), with \( e = 2^v, q = 2^h, (v, h) = 1 \).

The \((q + 1)\)-arc \( M = \{(1, t, t', t'^{+1}) \mid t \in \mathrm{GF}(q)^+\} \) of \( \mathrm{PG}(3, q) \) is fixed by a projective group which acts sharply 3-transitive on \( M \) [7, p. 252]. So all points of \( M \) are equivalent with respect to \( M \). Assume that we project from the point
Complete $k$-arcs in $\text{PG}(n, q)$, $q$ even 465

(0, 0, 0, 1) of $M$ onto $X_3 = 0$. Then $M \setminus \{(0, 0, 0, 1)\}$ is projected onto the $q$-arc $\{(1, t, t') \mid t \in \text{GF}(q)\}$ of $X_3 = 0$.

It then follows from the beginning of this proof that the $(n + 4)$-arc $K$ is incomplete if and only if its dual $(n + 4)$-arc $\hat{K}$ is contained in a $q$-arc of $\text{PG}(2, q)$ which is projectively equivalent to a $q$-arc $\{(1, t, t') \mid t \in \text{GF}(q)\}$ ($e = 2^n$, $q = 2^h$, $(v, h) = 1$).

So the dual $(n + 4)$-arcs $K$ in $\text{PG}(n, q)$, $q > 4, q - 4 \geq n > q - \sqrt{q}/2 - \frac{1}{4}$, of $k$-arcs $\hat{K}$ in $\text{PG}(2, q)$ ($k = n + 4, q \geq k > q - \sqrt{q}/2 + \frac{3}{4}$) which are not contained in a $q$-arc of type $\{(1, t, t') \mid t \in \text{GF}(q)\}$ ($e = 2^n$, $q = 2^h$, $(v, h) = 1$) are complete $(n + 4)$-arcs of $\text{PG}(n, q)$.

We prove that such $k$-arcs $\hat{K}$ exist in $\text{PG}(2, q)$. Consider the conic $C = \{(1, t, t') \mid t \in \text{GF}(q)^+\}$ in $\text{PG}(2, q)$. The nucleus $(0, 1, 0)$ extends $C$ to a $(q + 2)$-arc. Consider a $k$-arc $K$ consisting of $k - 1$ points of the conic $C$ and the nucleus $(0, 1, 0)$, and suppose that $q \geq k > q - \sqrt{q}/2 + \frac{3}{4}$. We show that this $k$-arc $K$ is not contained in a $q$-arc $L$ which is projectively equivalent to a $q$-arc $N = \{(1, t, t') \mid t \in \text{GF}(q)\}$ ($e = 2^n$, $q = 2^h$, $(v, h) = 1$). Such a $q$-arc $N$ is fixed by a projective group $G$ defining on $N$ a sharply 2-transitive group $G'$ consisting of the transformations $\alpha : t \rightarrow a + b$ with $a \in \text{GF}(q)^*$ and $b \in \text{GF}(q)$ [6, p. 178].

Assume that $K$ is contained in a $q$-arc $N'$ projectively equivalent to $N$. Consider the permutation group $G''$ of $N'$ which corresponds to $G'$. Let $\beta$ be an element of $G''$ which does not fix $(0, 1, 0)$. Then

$$|(K \setminus \{(0, 1, 0)\}) \cap (\beta(K \setminus \{(0, 1, 0)\}))| \geq 2|\hat{K} \setminus \{(0, 1, 0)\}| - |N'|.$$  

So at least $2(q - \sqrt{q}/2 + \frac{3}{4}) - q = q - \sqrt{q} + \frac{1}{2}$ points of $\hat{K} \setminus \{(0, 1, 0)\}$ must be mapped onto points of $K \setminus \{(0, 1, 0)\}$. These $q - \sqrt{q} + \frac{1}{2}$ points all belong to the conic $C$. Hence $\beta$ must fix the conic $C$ since $q - \sqrt{q} + \frac{1}{2} > 5$. So $\beta$ must fix the nucleus $(0, 1, 0)$ of the conic $C$. We have a contradiction.

Hence $K$ is not contained in a $q$-arc which is projectively equivalent to such a $q$-arc $\{(1, t, t') \mid t \in \text{GF}(q)\}$ ($e = 2^n$, $q = 2^h$, $(v, h) = 1$). It then follows from the beginning of this proof that all dual $k$-arcs $K$ of $\hat{K}$ are complete $(n + 4)$-arcs in $\text{PG}(n, q)$, $q - 4 \geq n > q - \sqrt{q}/2 - \frac{1}{4}$ ($k = n + 4$). So, there exist complete $(n + 4)$-arcs in $\text{PG}(n, q)$, $q - 4 \geq n > q - \sqrt{q}/2 - \frac{1}{4}$, $q > 4$. □

Corollary 7.2. There exist complete 8-arcs in $\text{PG}(4, 8)$.

Proof. We apply Theorem 7.1 for $q = 8$. When $q = 8$, then $q - 4 \geq n > q - \sqrt{q}/2 - \frac{1}{4}$ implies $n = 4$. So $k = n + 4 = 8$.

Hence, there exist complete 8-arcs in $\text{PG}(4, 8)$. □

Theorem 7.3. There exist complete $(n + 5)$-arcs in $\text{PG}(n, q)$, $q$ even, $q \neq 64$, $q - 5 \geq n > q - \sqrt{q}/2 - \frac{1}{4}$. In $\text{PG}(n, 64)$, $n \in \{58, 59\}$, no complete $(n + 5)$-arcs exist.

The complete $(n + 5)$-arcs of $\text{PG}(n, q)$, $q$ even, $q \neq 64$, $q - 5 \geq n > q - \sqrt{q}/2 - \frac{1}{4}$.
are the dual arcs of the $k$-arcs $\hat{K}$ ($k = n + 5$) in $\text{PG}(3, q)$ ($q \geq k > q - \sqrt{q/2 + \frac{2}{q}}$) which are not a subset of a twisted cubic of $\text{PG}(3, q)$.

**Proof.** The inequality $q - 5 > n > q - \sqrt{q/2} - \frac{11}{4}$ implies $q > 32$.

The $(n + 5)$-arcs $K$ of $\text{PG}(n, q)$, $q - 5 \geq n > q - \sqrt{q/2} - \frac{11}{4}$, are dual arcs of $k$-arcs $\hat{K}$ ($k = n + 5$) in $\text{PG}(3, q)$ where $q \geq k > q - \sqrt{q/2} + \frac{2}{q}$. It follows from Table 1 (6) that $\hat{K}$ is contained in a $(q + 1)$-arc of $\text{PG}(3, q)$. This $(q + 1)$-arc $L$ is projectively equivalent to a $(q + 1)$-arc $\{(1, t, t^e, t^{e+1}) \mid t \in \text{GF}(q)^+\}$ with $e = 2^n$, $q = 2^h$, $(v, h) = 1$.

The $(n + 5)$-arc $K$ is incomplete if and only if there exists a $(k + 1)$-arc $\hat{K}'$ in $\text{PG}(4, q)$ and a point $r$ of $\hat{K}'$ such that if we project from $r$ onto a hyperplane $\delta$ not containing $r$, then $\hat{K}' \setminus \{r\}$ is projected from $r$ onto a $k$-arc of $\delta$ which is projectively equivalent to $\hat{K}$, where $\hat{K}$ is a dual $k$-arc of $K$ in $\text{PG}(3, q)$ (Theorem 6.1).

The $(k + 1)$-arc $\hat{K}'$ satisfies $q + 1 \geq k + 1 > q - \sqrt{q/2} + \frac{2}{q}$.

Table 1 (7) then states that $\hat{K}'$ is contained in a normal rational curve of $\text{PG}(4, q)$.

Let $r$ be a point of $\hat{K}'$. If we project from $r$ onto a hyperplane of $\text{PG}(4, q)$ not containing $r$, then $\hat{K}' \setminus \{r\}$ is projected onto a $k$-arc of $\text{PG}(3, q)$ which is a subset of a twisted cubic of $\text{PG}(3, q)$. So, $K$ is incomplete if and only if its dual arc $\hat{K}$ ($k = n + 5$) in $\text{PG}(3, q)$ is a subset of a twisted cubic of $\text{PG}(3, q)$.

In $\text{PG}(3, q)$, $q \geq 37$, $q \neq 64$, different types of $(q + 1)$-arcs exist [7, p. 253]. As every $k$-arc $\hat{K}$ with $k > (q + 4)/2$ of $\text{PG}(3, q)$ is contained in a unique complete arc of $\text{PG}(3, q)$ [1, p. 6], there exist $k$-arcs $\hat{K}$ ($q \geq k > q - \sqrt{q/2} + \frac{2}{q}$) which are not a subset of a twisted cubic. Hence, in $\text{PG}(n, q)$, $q \geq 32$, $q \neq 64$, $q - 5 \geq n > q - \sqrt{q/2} - \frac{11}{4}$, complete $(n + 5)$-arcs exist.

In $\text{PG}(3, 64)$ only one type of 65-arc exists, namely the twisted cubic. So all $k$-arcs of $\text{PG}(3, 64)$, $64 \geq k \geq 63$, are contained in a 64-arc on a twisted cubic. Hence, no complete $(n + 5)$-arcs exist in $\text{PG}(n, 64)$, $59 \geq n \geq 58$.

8. Complete $k$-arcs in $\text{PG}(n, q)$, $q$ even

**Theorem 8.1.** The values of $k$ for which there exists a complete $k$-arc in $\text{PG}(n, q)$, $q = 2^h$, $q \geq 32$, $q \neq 64$, $q - 5 \geq n > q - \sqrt{q/2} - \frac{11}{4}$, are $k = n + 4$, $k = n + 5$ and $k = q + 1$.

Also, for these values of $n$ and $q$, every $(q + 1)$-arc of $\text{PG}(n, q)$ is a normal rational curve.

**Proof.** For $k = n + 1$, $k = n + 2$, $k = n + 3$, see the first part of the proof of Corollary 5.4. It follows from Theorems 7.1 and 7.3 that there exist complete $(n + 4)$-arcs and complete $(n + 5)$-arcs in $\text{PG}(n, q)$, $q \geq 32$, $q \neq 64$, $q - 5 \geq n > q - \sqrt{q/2} - \frac{11}{4}$.

Let $K$ be a $k$-arc of $\text{PG}(n, q)$, $q \geq 32$, $q \neq 64$, $q - 5 \geq n > q - \sqrt{q/2} - \frac{11}{4}$, $k \geq n + 6$ (in particular $k = q + 1$). This $k$-arc can be extended in a unique way to
a normal rational curve of $\text{PG}(n, q)$ (Remark 5.3(d)). Moreover, these normal rational curves are complete [13].

Hence, a $k$-arc $K$ of $\text{PG}(n, q)$, $q \geq 32$, $q \neq 64$, $q - 5 \geq n > q - \sqrt{q/2} - \frac{1}{4}$, $k \geq n + 6$, is complete if and only if $k = q + 1$. 

**Theorem 8.2.** If $K$ is a complete $k$-arc in $\text{PG}(n, 64)$, $n \in \{58, 59\}$, then $K$ is a $(n + 4)$-arc or a $65$-arc; also, every $65$-arc of $\text{PG}(58, 64)$ or $\text{PG}(59, 64)$ is a normal rational curve.

**Proof.** For $k = n + 1$, $k = n + 2$ and $k = n + 3$, see the first part of the proof of Corollary 5.4. In $\text{PG}(n, 64)$, $n \in \{58, 59\}$, there exist complete $(n + 4)$-arcs (Theorem 7.1). Further, there are no complete $(n + 5)$-arcs in $\text{PG}(n, 64)$, $n \in \{58, 59\}$ (Theorem 7.3).

Every $k$-arc $K$ of $\text{PG}(n, 64)$, $n = 58$ or $n = 59$, with $k \geq n + 6$ (in particular $k = q + 1$), is contained in a unique normal rational curve of $\text{PG}(n, 64)$ (Remark 5.3(d)). Moreover, these normal rational curves are complete [13]. So $K$ is complete when $k = 65$. 

**Theorem 8.3.** If $K$ is a complete $k$-arc in $\text{PG}(q - 4, q)$, $q > 4$ and $q$ even, then $K$ is a $q$-arc or a $(q + 1)$-arc; any $(q + 1)$-arc of $\text{PG}(q - 4, q)$ is a dual arc of a $(q + 1)$-arc in $\text{PG}(3, q)$.

**Proof.** The values $k = n + 1$, $k = n + 2$ and $k = n + 3$ ($n = q - 4$) are treated in the same way as in the preceding theorems.

Let $K$ be a $(n + 4)$-arc in $\text{PG}(q - 4, q)$ ($n = q - 4$). Hence $K$ is a $q$-arc in $\text{PG}(q - 4, q)$. It then follows from Theorem 7.1 that there exist complete $q$-arcs in $\text{PG}(q - 4, q)$.

Every $(q + 1)$-arc of $\text{PG}(q - 4, q)$ is complete [13]. So there are no $(q + 2)$-arcs in $\text{PG}(q - 4, q)$. The duality principle (Definition 4.1) shows that a $(q + 1)$-arc of $\text{PG}(q - 4, q)$ is a dual arc of a $(q + 1)$-arc in $\text{PG}(3, q)$. 

**Theorem 8.4.** In $\text{PG}(q - 3, q)$, $q$ even, $q \geq 8$, the only complete arcs are the $(q + 1)$-arcs. These $(q + 1)$-arcs of $\text{PG}(q - 3, q)$ are the dual arcs of the $(q + 1)$-arcs in $\text{PG}(2, q)$.

**Proof.** As in the preceding theorems, only the values $k > n + 4 = q + 1$ ($n = q - 3$) have to be considered.

Let $K$ be a $(q + 2)$-arc in $\text{PG}(q - 3, q)$. Then $K$ has a dual $(q + 2)$-arc $\hat{K}$ in $\text{PG}(3, q)$. No $(q + 2)$-arcs exist in $\text{PG}(3, q)$, $q \geq 8$ [3]. So there are no $(q + 2)$-arcs in $\text{PG}(q - 3, q)$. Hence, in $\text{PG}(q - 3, q)$, $q$ even, $q \geq 8$, the only complete arcs are the $(q + 1)$-arcs and these $(q + 1)$-arcs are the dual $(q + 1)$-arcs of the $(q + 1)$-arcs in $\text{PG}(2, q)$ (Definition 4.1).
Table 2

<table>
<thead>
<tr>
<th>$q$</th>
<th>$n$</th>
<th>$k = n + 4$</th>
<th>$k = n + 5$</th>
<th>$k = q + 1$</th>
<th>$k = q + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q = p^{2h}, h \geq 1, p &gt; 2, p \text{ prime}$</td>
<td>$q - 2 \geq n &gt; q - \sqrt{q/4 - \frac{7n}{16}}$</td>
<td>+</td>
<td></td>
<td></td>
<td>5.4</td>
</tr>
<tr>
<td>$q = p^{2h+1}, h \geq 1, p &gt; 2, p \text{ prime}$</td>
<td>$q - 2 \geq n &gt; q - \sqrt{pq/4 + \frac{7n}{16}} - 3$</td>
<td>+</td>
<td></td>
<td></td>
<td>5.4</td>
</tr>
<tr>
<td>$q$ prime, $q &gt; 2$</td>
<td>$q - 4 \geq n &gt; \frac{44}{23}q - \frac{3n}{q}$</td>
<td>+</td>
<td></td>
<td></td>
<td>5.4</td>
</tr>
<tr>
<td>$q = 2^h, q &gt; 32, q \neq 64$</td>
<td>$q - 3 \geq n &gt; q - \sqrt{q/2 - \frac{1}{4}}$</td>
<td>+</td>
<td>+</td>
<td></td>
<td>8.1</td>
</tr>
<tr>
<td>$q = 64$</td>
<td>$n = 58$ or $n = 59$</td>
<td>+</td>
<td></td>
<td>+</td>
<td>8.2</td>
</tr>
<tr>
<td>$q = 2^h, q &gt; 8$</td>
<td>$q - 4$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>8.3</td>
</tr>
<tr>
<td>$q = 2^h, q = 8$</td>
<td>$q - 3$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>8.4</td>
</tr>
<tr>
<td>$q = 4$</td>
<td>$q - 2$</td>
<td>+</td>
<td></td>
<td></td>
<td>8.5</td>
</tr>
</tbody>
</table>

1 Here $n + 5 = q + 1$.
2 Here $n + 4 = q + 1$.

**Theorem 8.5.** In $\text{PG}(q - 2, q)$, $q$ even and $q > 2$, the only complete arcs are the $(q + 2)$-arcs. These $(q + 2)$-arcs are the dual arcs of the $(q + 2)$-arcs in $\text{PG}(2, q)$.

**Proof.** Every $(n + 3)$-arc of $\text{PG}(n, q)$, $2 \leq n \leq q - 2$, is contained in a unique normal rational curve [7, p. 229]. Hence, every $(q + 1)$-arc of $\text{PG}(q - 2, q)$ is a normal rational curve. A normal rational curve of $\text{PG}(q - 2, q)$, $q$ even, can be extended to a $(q + 2)$-arc in different ways [14, Theorem 3.12]. In $\text{PG}(q - 2, q)$ no $(q + 3)$-arcs exist [16]. So, the only complete arcs in $\text{PG}(q - 2, q)$ are the $(q + 2)$-arcs. They are the dual arcs of the $(q + 2)$-arcs in $\text{PG}(2, q)$ (Definition 4.1). □

**Summary 8.6.** Table 2 presents a summary of the results of Corollary 5.4 and Section 8. In the first two columns the values for $q$ and $n$ are given. Columns 3, 4, 5 and 6 give the values of $k$ for which $k$-arcs are considered in $\text{PG}(n, q)$. A '+' in column 3, 4, 5 or 6 means that there exists a complete $k$-arc in $\text{PG}(n, q)$. The last column shows the section in which the corresponding theorem is stated.

**References**

Complete $k$-arcs in $PG(n, q)$, $q$ even