View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Linear Algebra and its Applications 414 (2006) 517–532
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Rigid systems of second-order linear differential equations

Mª. Isabel García-Planas [a], M. Dolors Magret [a], Vladimir V. Sergeichuk [b,*], Nadya A. Zharko [c]

[a] Universitat Politècnica de Catalunya, Minería 1, Esc. C, 1-3, 08038 Barcelona, Spain
[b] Institute of Mathematics, Tereshchenkivska St. 3, Kiev, Ukraine
[c] Mech.-Math. Faculty, Kiev National University, Vladimirskaya 64, Kiev, Ukraine

Received 9 June 2005; accepted 27 October 2005
Available online 20 December 2005
Submitted by R.A. Brualdi

Abstract

We say that a system of differential equations

$$\ddot{x}(t) = A\dot{x}(t) + Bx(t) + Cu(t), \quad A, B \in \mathbb{C}^{m\times m}, \quad C \in \mathbb{C}^{m\times n},$$

is rigid if it can be reduced by substitutions

$$x(t) = Sy(t), \quad u(t) = U\dot{y}(t) + Vy(t) + Pv(t),$$

with nonsingular S and P to each system obtained from it by a small enough perturbation of its matrices A, B, C. We prove that there exists a rigid system for given m and n if and only if $m < n(1+\sqrt{5})/2$, and describe all rigid systems.
© 2005 Elsevier Inc. All rights reserved.

AMS classification: 15A21; 34D10; 93B10

Keywords: Differential equations; Rigid systems; Perturbations; Normal forms

* Corresponding author.
E-mail addresses: maria.isabel.garcia@upc.edu (Mª.I. García-Planas), m.dolors.magret@upc.edu (M.D. Magret), sergeich@imath.kiev.ua (V.V. Sergeichuk), n.zharko@mail.ru (N.A. Zharko).

0024-3795/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2005.10.037

1. Introduction

We consider a system of differential equations

$$\ddot{x}(t) = A\dot{x}(t) + Bx(t) + Cu(t), \quad A, B \in \mathbb{C}^{m\times m}, \quad C \in \mathbb{C}^{m\times n}, \tag{1}$$

in which $x(t)$ is the unknown vector function, $u(t)$ is a vector function, and $\dot{x}(t) = \mathrm{d}x(t)/\mathrm{d}t$. Any substitution

$$x(t) = Sy(t),$$
$$u(t) = U\dot{y}(t) + Vy(t) + Pv(t),$$

with nonsingular S and P transforms it to the system

$$\ddot{y}(t) = S^{-1}(AS + CU)\dot{y}(t) + S^{-1}(BS + CV)y(t) + S^{-1}CPv(t),$$

which has the form (1) and is given by the matrices

$$A' = S^{-1}(AS + CU), \quad B' = S^{-1}(BS + CV), \quad C' = S^{-1}CP.$$

In partitioned matrix notation

$$[C' \ \ B' \ \ A'] = S^{-1}[C \ \ B \ \ A]\begin{bmatrix} P & V & U \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix}. \tag{2}$$

Definition 1. By an $m \times (n, m, m)$ triple we mean a triple of $m \times n$, $m \times m$, and $m \times m$ matrices. Two such triples (C, B, A) and (C', B', A') are said to be *feedback similar* if they satisfy (2) for some V, U, and nonsingular P and S. (The term "feedback similarity" comes from systems theory.)

Every transformation of feedback similarity on a triple (C, B, A) can be realized by a sequence of the following operations:

(i) A simultaneous elementary row operation on C, B, and A, and then the inverse column operation on B and the inverse column operation on A.
(ii) An elementary column operation on C.
(iii) Adding any constant multiple of a column of C to a column of B or A.

The matrices A, B, and C are written in the block matrix $[C \ \ B \ \ A]$ in the reverse order to ensure that all admissible additions of columns are performed from a left block to a right block as is customary in matrix problems (see, for instance, [5] or [12]).

Related matrix problems are considered by systems theorists [6–8].

The canonical form problem for a matrix triple (C, B, A) up to feedback similarity is hopeless even if $C = 0$ since then the pair (B, A) reduces by simultaneous similarity transformations, and the problem of classifying pairs of matrices up to simultaneous similarity contains both the problem of classifying *any* system of linear operators and the problem of classifying representations of *any* finite-dimensional algebra [2]. Classification problems that contain the problem of classifying matrix pairs of up to simultaneous similarity are called *wild*.

Nevertheless, using Belitskii's algorithm [1,12] one can reduce any given triple $\mathscr{T} = (C, B, A)$ by transformations (i)–(iii) to some *canonical* triple $\mathscr{T}_{\mathrm{can}}$; this means that $\mathscr{T}_{\mathrm{can}}$ is feedback similar to $\mathscr{T}$ and two triples $\mathscr{T}$ and $\mathscr{T}'$ are reduced by Belitskii's algorithm to the same triple

$\mathscr{T}_{\text{can}} = \mathscr{T}'_{\text{can}}$, an if and only if $\mathscr{T}$ and $\mathscr{T}'$ are feedback similar. (Of course, an explicit description of *all* canonical matrices does not exist since the matrix problem is wild.)

A canonical form problem simplifies if the matrices are considered up to arbitrarily small perturbations (this case is important for applications in which one has matrices that arise from physical measurements since then their entries are known only approximately). For instance, a square matrix A reduces to a diagonal matrix D by an arbitrarily small perturbation (making its eigenvalues pairwise distinct) and similarity transformations. The matrix D is determined by A up to small perturbations of diagonal entries.

In Lemma 8 we give a normal form of $m \times (n, m, m)$ triples for arbitrarily small perturbations and feedback similarity. A canonical form of such triples if n divides m is obtained in Theorem 10.

By analogy with quiver representations [4, p. 203], we say that a matrix t-tuple $\mathscr{A}$ is *rigid* with respect to some equivalence relation on the set of t-tuples of the same size if there is a neighborhood of $\mathscr{A}$ consisting of t-tuples that are equivalent to $\mathscr{A}$. For instance, the matrices I, $[I\ 0]$, and $[I\ 0]^{\mathrm{T}}$ are rigid with respect to elementary transformations, but each matrix is not rigid with respect to similarity transformations.

In Theorem 11 we prove that there exists an $m \times (n, m, m)$ triple that is rigid with respect to feedback similarity if and only if

$$m < \frac{1+\sqrt{5}}{2}n. \tag{3}$$

We also construct such a rigid triple $\mathscr{T}_{mn}$ for each m and n satisfying (3) and prove that each $m \times (n, m, m)$ triple reduces to $\mathscr{T}_{mn}$ by an arbitrarily small perturbation and a feedback similarity transformation (so $\mathscr{T}_{mn}$ can be considered as a canonical triple for arbitrarily small perturbations and feedback similarity). All triples that reduce to $\mathscr{T}_{mn}$ by feedback similarity transformations form an open and everywhere dense set in the space of all $m \times (n, m, m)$ triples; moreover, this set consists of all rigid triples of this size.

The mentioned results about triples will be obtained in Section 4.

In Section 3 we consider analogous problems for systems of first-order linear differential equations. Such a system is given by a matrix pair; the results of Section 3 are used in Section 4.

In Section 2 we prove a technical lemma.

2. Perturbations

The *norm* of a complex matrix $A = [a_{ij}]$ is the nonnegative real number

$$\|A\| = \sqrt{\sum |a_{ij}|^2}.$$

For each $m \times (n, m, m)$ triple $\mathscr{P} = (C, B, A)$, we denote

$$\|\mathscr{P}\| := \|C\| + \|B\| + \|A\|$$

and define the block matrix

$$[\mathscr{P}] := [C \quad B \quad A].$$

We say that a matrix triple $\widetilde{\mathscr{T}}$ is obtained from $\mathscr{T}$ by a sequence of perturbations and feedback similarity transformations if there is a sequence of triples

$$\mathscr{T} = \mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3, \ldots, \mathscr{T}_{l+1} = \widetilde{\mathscr{F}},$$

in which

$$\mathscr{T}_2 = S_1^{-1}[\mathscr{T} + \Delta\mathscr{T}_1]R_1, \quad \mathscr{T}_3 = S_2^{-1}[\mathscr{T}_2 + \Delta\mathscr{T}_2]R_2, \ldots \tag{4}$$

($\Delta\mathscr{T}_1, \Delta\mathscr{T}_2, \ldots$ are triples), and all R_i have the form (2):

$$R_i = \begin{bmatrix} P_i & V_i & U_i \\ 0 & S_i & 0 \\ 0 & 0 & S_i \end{bmatrix}.$$

Lemma 2. *Let ε be any positive number and let a triple $\widetilde{\mathscr{T}}$ be obtained from a triple $\mathscr{T}$ by a sequence* (4) *of perturbations and feedback similarity transformations satisfying*

$$\|\Delta\mathscr{T}_1\| < \frac{\varepsilon}{2}, \quad \|\Delta\mathscr{T}_{i+1}\| < \frac{\varepsilon}{2^{i+1}\|\widetilde{S}_i\|\|\widetilde{R}_i^{-1}\|} \quad (i = 1, 2, \ldots, l),$$

where

$$\widetilde{S}_i := S_1 S_2 \cdots S_i, \quad \widetilde{R}_i := R_1 R_2 \cdots R_i.$$

Then $\widetilde{\mathscr{T}}$ is feedback similar to some triple $\mathscr{T} + \nabla\mathscr{T}$, $\|\nabla\mathscr{T}\| < \varepsilon$.

Proof. If $l = 2$, then by (4)

$$\begin{aligned}[\widetilde{\mathscr{F}}] = [\mathscr{T}_3] &= S_2^{-1}[\mathscr{T}_2 + \Delta\mathscr{T}_2]R_2 = S_2^{-1}(S_1^{-1}[\mathscr{T} + \Delta\mathscr{T}_1]R_1 + [\Delta\mathscr{T}_2])R_2 \\ &= (S_1 S_2)^{-1}([\mathscr{T}] + [\Delta\mathscr{T}_1] + S_1[\Delta\mathscr{T}_2]R_1^{-1})R_1 R_2.\end{aligned}$$

Analogously, for any l

$$[\widetilde{\mathscr{F}}] = [\mathscr{T}_{l+1}] = \widetilde{S}_l^{-1}[\mathscr{T} + \nabla\mathscr{T}]\widetilde{R}_l,$$

where

$$[\nabla\mathscr{T}] := [\Delta\mathscr{T}_1] + \widetilde{S}_1[\Delta\mathscr{T}_2]\widetilde{R}_1^{-1} + \cdots + \widetilde{S}_{l-1}[\Delta\mathscr{T}_l]\widetilde{R}_{l-1}^{-1}.$$

Then

$$\begin{aligned}\|\nabla\mathscr{T}\| &\leqslant \|\Delta\mathscr{T}_1\| + \|\widetilde{S}_1\| \cdot \|\Delta\mathscr{T}_2\| \cdot \|\widetilde{R}_1^{-1}\| + \cdots + \|\widetilde{S}_{l-1}\| \cdot \|\Delta\mathscr{T}_l\| \cdot \|\widetilde{R}_{l-1}^{-1}\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \cdots + \frac{\varepsilon}{2^l} < \varepsilon. \qquad \square\end{aligned}$$

Corollary 3. *Let a matrix triple $\mathscr{T}$ reduce to a triple from some set $\mathscr{S}$ by a sequence of arbitrarily small perturbations and feedback similarity transformations. Then $\mathscr{T}$ is transformed by some arbitrarily small perturbation to a triple that is feedback similar to a triple in $\mathscr{S}$.*

3. Feedback similarity of matrix pairs

In this preliminary section we consider problems studied in Section 4 in much simpler case: for systems of first-order linear differential equations

$$\dot{x}(t) = Ax(t) + Bu(t), \quad A \in \mathbb{C}^{m\times m}, \quad B \in \mathbb{C}^{m\times n}. \tag{5}$$

Any substitution

$$\begin{aligned} x(t) &= Sy(t), \\ u(t) &= Uy(t) + Pv(t) \end{aligned}$$

with nonsingular S and P transforms it to the system

$$\dot{y}(t) = S^{-1}(AS + BU)y(t) + S^{-1}BPv(t)$$

of the form (5), whose matrices A' and B' can be calculated as follows:

$$[B' \;\; A'] = S^{-1}[B \;\; A]\begin{bmatrix} P & U \\ 0 & S \end{bmatrix}. \tag{6}$$

In systems theory, (5) is called the standard linear system (without output), A is called the system matrix, B is called the input matrix, $u(t)$ is the input to the system at time t (it is the way that the external world affects the system), and $x(t)$ is the state of system at time t (it is the memory of the net effect of past inputs). The system (5) is said to be *controllable* if the spectrum of $A + BU$ can be placed arbitrarily by choice of U, this holds if and only if rank $[B \;\; AB \;\; \ldots \;\; A^m B] = m$.

Definition 4. By an $m \times (n, m)$ pair we mean a pair of $m \times n$ and $m \times m$ matrices. Two such pairs (B, A) and (B', A') are said to be *feedback similar* if they satisfy (6) for some U and nonsingular P and S.

Every feedback similarity transformation on (B, A) can be realized by a sequence of the following operations:

(i′) A simultaneous elementary row operation on both matrices, and then the inverse column operation on A.
(ii′) An elementary column operation on B.
(iii′) Adding any constant multiple of a column of B to a column of A.

In the next section we will reduce a triple (C, B, A) to canonical form for arbitrarily small perturbations and feedback similarity using results of this section as follows. First we reduce its subpair (C, B) to the pair $(C_{\text{can}}, B_{\text{can}})$ defined in (14), which is canonical with respect to arbitrarily small perturbations and feedback similarity; respectively, (C, B, A) reduces to some triple $(C_{\text{can}}, B_{\text{can}}, A')$. Then we reduce A' to canonical form for arbitrarily small perturbations and those feedback similarity transformations on $(C_{\text{can}}, B_{\text{can}}, A')$ that preserve $(C_{\text{can}}, B_{\text{can}})$, these transformations are described in Theorem 6(b).

In the next lemma we recall a known canonical form of pairs for feedback similarity. In the case of controllable systems, it is known as the Brunovsky canonical form [3]. It can be deduced from the canonical form of matrix pencils [7, Proposition 3.3]. A much more general canonical matrix problem was solved in [11, Section 2].

Denote by 0_{mn} the $m \times n$ zero matrix; $0_m := 0_{mm}$. It is agreed that there exists exactly one matrix of size $0 \times n$ and there exists exactly one matrix of size $n \times 0$ for every nonnegative integer n; they give the linear mappings $\mathbb{C}^n \to 0$ and $0 \to \mathbb{C}^n$ and are considered as zero matrices 0_{0n} and 0_{n0}. For any p-by-q matrix M_{pq}, we have

$$M_{pq} \oplus 0_{m0} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{m0} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{p0} \\ 0_{mq} & 0_{m0} \end{bmatrix} = \begin{bmatrix} M_{pq} \\ 0_{mq} \end{bmatrix}$$

and

$$M_{pq} \oplus 0_{0n} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \\ 0_{0q} & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \end{bmatrix}.$$

Denote

$$J_k(\lambda) := \begin{bmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{bmatrix} \quad (k \text{ by } k),$$

$$F_{rl} := [I_r \ \ 0_{r,l-r}], \quad G_{rl} := [0_{r,l-r} \ \ I_r] \quad (0 \leqslant r \leqslant l); \tag{7}$$

in particular, $F_{0n} = G_{0n} = 0_{0n}$. The *direct sum* of matrix t-tuples is denned as follows:

$$(A_1, \ldots, A_t) \oplus (B_1, \ldots, B_t) := (A_1 \oplus B_1, \ldots, A_t \oplus B_t).$$

Lemma 5. *Every $m \times (n, m)$ pair (B, A) is feedback similar to a direct sum of pairs of the form*

$$([1\,0 \ldots 0]^{\mathrm{T}}, J_k(0)), \quad (0_{k0}, J_k(\lambda)), \quad (0_{01}, 0_0), \tag{8}$$

This sum is determined by (B, A) uniquely up to permutation of summands.

Proof. Let (B, A) be an $m \times (n, m)$ pair. If $B = 0$, then

$$(B, A) = (0_{m0}, A) \oplus (0_{0n}, 0_{00}) = (0_{m0}, A) \oplus (0_{01}, 0_{00}) \oplus \cdots \oplus (0_{01}, 0_{00}).$$

The summand $(0_{m0}, A)$ is feedback similar to a direct sum of pairs of the form $(0_{k0}, J_k(\lambda))$.

Suppose $B \neq 0$. Then (B, A) reduces by transformations (i′) and (ii′) to the form

$$\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, C\right) = \left(\begin{bmatrix} I_r \\ 0 \end{bmatrix}, C\right) \oplus (0_{01}, 0_0) \oplus \cdots \oplus (0_{01}, 0_0).$$

The first summand reduces by transformations (iii′) to the form

$$\mathscr{H}(M, N) := \left(\begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} 0_r & 0_{r,m-r} \\ M & N \end{bmatrix}\right). \tag{9}$$

If (M, N) is feedback similar to (M', N'), that is,

$$S[M' \ \ N'] = [M \ \ N] \begin{bmatrix} P & U \\ 0 & S \end{bmatrix}, \tag{10}$$

then

$$\begin{bmatrix} P & U \\ 0 & S \end{bmatrix} \begin{bmatrix} I_k & 0 & 0 \\ 0 & M' & N' \end{bmatrix} = \begin{bmatrix} I_k & 0 & 0 \\ 0 & M & N \end{bmatrix} \begin{bmatrix} P & UM' & UN' \\ 0 & P & U \\ 0 & 0 & S \end{bmatrix} \tag{11}$$

and so $\mathscr{H}(M, N)$ is feedback similar to $\mathscr{H}(M', N')$. Using induction in $m + n$, we may assume that (M, N) is feedback similar to a direct sum of pairs $\mathscr{P}_i$ of the form (8). Then $\mathscr{H}(M, N)$ is feedback similar to the direct sum of the pairs

$$\mathscr{H}(\mathscr{P}_i) = \begin{cases} ([1\ 0\ \cdots\ 0]^{\mathrm{T}}, J_{k+1}(0)) & \text{if } \mathscr{P}_i = ([1\ 0\ \cdots\ 0]^{\mathrm{T}}, J_k(0)), \\ \mathscr{P}_i & \text{if } \mathscr{P}_i = (0_{k0}, J_k(\lambda)), \\ (I_1, 0_1) & \text{if } \mathscr{P}_i = (0_{01}, 0_0). \end{cases}$$

The uniqueness of this decomposition follows, for instance, from [12, Theorem 2.2], in which the uniqueness of decompositions into indecomposables is proved for all linear matrix problems. □

For each $m \times (n, m)$ pair $\mathscr{P} = (B, A)$, we define the block matrix

$$[\mathscr{P}] := [B \quad A]. \tag{12}$$

Consider a category, whose objects are $m \times (n, m)$ pairs and each morphism from (B, A) to (B', A') is a matrix triple (P, U, S) such that

$$[B \quad A]\begin{bmatrix} P & U \\ 0 & S \end{bmatrix} = S[B' \quad A'].$$

By (6), two pairs are isomorphic in this category if and only if they are feedback similar. The next theorem gives canonical pairs for arbitrarily small perturbations and feedback similarity and calculates their endomorphism rings in this category.

Theorem 6. (a) (Canonical pairs) *In the space* $\mathbb{C}^{m\times(n,m)}$ *of* $m \times (n, m)$ *matrix pairs*, $n \geqslant 1$, *all pairs that are feedback similar to*

$$\mathscr{F}_{mn} := \begin{cases} (F_{mn}, 0_m) & \text{if } m \leqslant n, \\ \left(\begin{bmatrix} I_n \\ 0_{m-n,n} \end{bmatrix}, \begin{bmatrix} 0_{nm} \\ F_{m-n,m} \end{bmatrix}\right) & \text{if } m > n \end{cases} \tag{13}$$

($F_{m-n,m}$ *if is defined in* (7)) *form an open and everywhere dense set*, *which is also the set of all* $m \times (n, m)$ *pairs that are rigid with respect to feedback similarity*.

Alternatively, *instead of* $\mathscr{F}_{mn}$ *one can take*

$$\mathscr{H}_{mn} := \begin{cases} (G_{mn}, 0_m) & \text{if } m \leqslant n, \\ \left(\begin{bmatrix} I_n \\ 0_{m-n,n} \end{bmatrix}, \begin{bmatrix} 0_{nm} \\ H_{m-n,m} \end{bmatrix}\right) & \text{if } m > n, \end{cases} \tag{14}$$

where

$$H_{m-n,m} := \begin{bmatrix} I_{(\alpha-1)n} & 0 & 0 \\ 0 & G_{\beta n} & 0_\beta \end{bmatrix}, \tag{15}$$

$G_{\beta n}$ *is defined in* (7), *and* α *and* β *are nonnegative integers defined as follows*:

$$m = \alpha n + \beta, \quad 0 < \beta \leqslant n. \tag{16}$$

(b) (Endomorphisms of canonical pairs) *The equality*

$$[\mathscr{H}_{mn}]\begin{bmatrix} P & U \\ 0 & S \end{bmatrix} = S[\mathscr{H}_{mn}], \quad P \in \mathbb{C}^{n\times n}, \ S \in \mathbb{C}^{m\times m}, \ U \in \mathbb{C}^{n\times m} \tag{17}$$

(*see* (6) *and* (12)) *holds if and only if for some* $S_1 \in \mathbb{C}^{(n-\beta)\times(n-\beta)}$, $S_3 \in \mathbb{C}^{\beta\times\beta}$, *and* $S_2, S_4 \in \mathbb{C}^{(n-\beta)\times\beta}$ *we have*

$$\begin{bmatrix} P & U \\ 0 & S \end{bmatrix} = R_{\alpha+2}(S_1, S_2, S_3, S_4), \quad S = R_{\alpha+1}(S_1, S_2, S_3, S_4), \tag{18}$$

where

$$R_\gamma(S_1,S_2,S_3,S_4) := \begin{array}{c} \begin{array}{ccccccccc} \overbrace{\quad}^{n-\beta} & \overbrace{\quad}^{\beta} & & & & & \overbrace{\quad}^{n-\beta} & \overbrace{\quad}^{\beta} & \overbrace{\quad}^{\beta} \end{array} \\ \left[\begin{array}{cc|cc|cc|cc|c} S_1 & S_2 & 0 & S_4 & & & & & \\ & S_3 & 0 & 0 & & & & & \\ \hline & & \ddots & & \ddots & & & & \\ & & & \ddots & & \ddots & & & \\ \hline & & & & S_1 & S_2 & 0 & S_4 & \\ & & & & & S_3 & 0 & 0 & \\ \hline & & & & & & S_1 & S_2 & S_4 \\ & & & & & & & S_3 & 0 \\ \hline & & & & & & & & S_3 \end{array}\right] \end{array} \tag{19}$$

(γ *is the number of diagonal blocks* S_3; *unspecified blocks are zero*).

In this statement one can replace $\mathscr{H}_{mn}$ *by* $\mathscr{F}_{mn}$, *which is simpler, but then* $R_\gamma(S_1,\dots,S_4)$ *must be replaced by*

$$\left[\begin{array}{cc|cc|cc|c} S_1 & 0 & 0 & 0 & & & \\ S_2 & S_3 & S_4 & 0 & & & \\ \hline & & \ddots & & \ddots & & \\ & & & \ddots & & \ddots & \\ \hline & & & & S_1 & 0 & 0 \\ & & & & S_2 & S_3 & S_4 \\ \hline & & & & & & S_3 \end{array}\right], \tag{20}$$

which is not block-triangular.

Proof. (a) Let $(B,A) \in \mathbb{C}^{m\times(n,m)}$, $n \geqslant 1$. First, we make rank $B = \min(m,n)$ by an arbitrarily small perturbation and reduce B to the form

$$\begin{cases} F_{mn} = [I_m \ \ 0] & \text{if } m \leqslant n, \\ \begin{bmatrix} I_n \\ 0 \end{bmatrix} & \text{if } m > n, \end{cases}$$

using transformations (i′) and (ii′). Then we reduce the pair by transformations (iii′) to the form $(F_{mn}, 0_m)$ if $m \leqslant n$ or to the form (9) with $r = n$ if $m > n$.

If $m > n$, using induction in m we can assume that (M,N) reduces by an arbitrary small perturbation to some $(M+\Delta M, N+\Delta N)$ being feedback similar to $\mathscr{F}_{m-n,n}$, defined in (13). By (10) and (11), $\mathscr{H}(M+\Delta M, N+\Delta N)$ is feedback similar to $\mathscr{F}_{mn}$. Reasoning as in Corollary 3, we can prove that (B,A) is transformed by an arbitrarily small perturbation to a pair that is feedback similar to $\mathscr{F}_{mn}$. Hence, the set $\mathscr{S}$ of pairs that are feedback similar to $\mathscr{F}_{mn}$ is everywhere dense in $\mathbb{C}^{m\times(n,m)}$. Since $\mathscr{F}_{mn}$ is rigid, there exists its neighborhood V in $\mathbb{C}^{m\times(n,m)}$ such that $V \subset \mathscr{S}$. For any pair $\mathscr{P} \in \mathscr{S}$, there is a transformation of feedback similarity that transforms $\mathscr{F}_{mn}$ to $\mathscr{P}$; it also transforms V to some neighborhood W of $\mathscr{P}$. Since each pair in V is feedback similar to $\mathscr{F}_{mn}$, each pair in W is also feedback similar to $\mathscr{F}_{mn}$, hence $W \subset \mathscr{S}$. Therefore, each pair $\mathscr{P} \in \mathscr{S}$ possesses a neighborhood that is contained in $\mathscr{S}$, and so the set $\mathscr{S}$ is open in $\mathbb{C}^{m\times(n,m)}$.

If $m > n$, then the pair $\mathscr{F}_{mn}$ reduces to $\mathscr{H}_{mn}$ in (14) by those permutations of rows and columns that are special cases of transformations (i′) and (ii′).

(b) Assume first that (17) holds. If $m \leqslant n$, then $[\mathscr{H}_{mn}] = [G_{mn}\ \ 0_m] = [[0_{m,n-m}\ \ I_m]\ \ 0_m]$, and so (17) ensures

$$\begin{bmatrix} P & U \\ 0 & S \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} * & * \\ 0 & S \end{bmatrix} & \begin{bmatrix} * \\ 0 \end{bmatrix} \\ 0 & s \end{bmatrix} = R_2(*, *, S, *).$$

We have (18) since $R_1(*, *, S, *) = S$ and by (16) $\alpha = 0$.

Let $m > n$. Equating the corresponding vertical strips in (17), we obtain

$$\begin{bmatrix} P \\ 0_{m-n,n} \end{bmatrix} = S \begin{bmatrix} I_n \\ 0_{m-n,n} \end{bmatrix}, \quad \begin{bmatrix} U \\ H_{m-n,m} S \end{bmatrix} = S \begin{bmatrix} 0_{nm} \\ H_{m-n,m} \end{bmatrix}. \tag{21}$$

Let us prove that $S = R_{\alpha+1}(S_1, S_2, S_3, S_4)$ for some $S_1, \ldots, S_4$. Partition S into blocks

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1,\alpha+1} \\ \cdots & \cdots & \cdots \\ S_{\alpha+1,1} & \cdots & S_{\alpha+1,\alpha+1} \end{bmatrix},$$

with $n \times n, \ldots, n \times n, \beta \times \beta$ diagonal blocks. By the first equality in (21),

$$P = S_{11}, \quad S_{21} = \cdots = S_{\alpha 1} = 0, \quad S_{\alpha+1,1} = 0. \tag{22}$$

Since

$$H_{m-n,m} = \begin{bmatrix} I_n & & 0 & 0_{n\beta} \\ & \ddots & & \vdots \\ & & I_n & 0_{n\beta} \\ 0 & & G_{\beta n} & 0_\beta \end{bmatrix}, \quad G_{\beta n} = [0_{\beta,n-\beta}\ I_\beta],$$

by the second equality in (21) we have

$$U = [S_{12}\ \ \cdots\ \ S_{1\alpha}\ \ S_{1,\alpha+1} G_{\beta n}\ 0] \tag{23}$$

and

$$\begin{bmatrix} S_{11} & \cdots & S_{1,\alpha-1} & S_{1\alpha} & S_{1,\alpha+1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ S_{\alpha-1,1} & \cdots & S_{\alpha-1,\alpha-1} & S_{\alpha-1,\alpha} & S_{\alpha-1,\alpha+1} \\ G_{\beta n} S_{\alpha 1} & \cdots & G_{\beta n} S_{\alpha,\alpha-1} & G_{\beta n} S_{\alpha\alpha} & G_{\beta n} S_{\alpha,\alpha+1} \end{bmatrix}$$
$$= \begin{bmatrix} S_{22} & \cdots & S_{2\alpha} & S_{2,\alpha+1} G_{\beta n} & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ S_{\alpha 2} & \cdots & S_{\alpha\alpha} & S_{\alpha,\alpha+1} G_{\beta n} & 0 \\ S_{\alpha+1,2} & \cdots & S_{\alpha+1,\alpha} & S_{\alpha+1,\alpha+1} G_{\beta n} & 0 \end{bmatrix}.$$

Let us equate the entries of these matrices along each line that is parallel with the main diagonal:

(a) The equalities

$$\begin{aligned} & & S_{1,\alpha+1} &= 0, \\ S_{1\alpha} &= S_{2,\alpha+1} G_{\beta n}, & S_{2,\alpha+1} &= 0, \\ S_{1,\alpha-1} = S_{2,\alpha} &= S_{3,\alpha+1} G_{\beta n}, & S_{3,\alpha+1} &= 0, \\ & & &\vdots \\ S_{13} = S_{24} = \cdots = S_{\alpha-2,\alpha} &= S_{\alpha-1,\alpha+1} G_{\beta n}, & S_{\alpha-1,\alpha+1} &= 0 \end{aligned}$$

imply $S_{ij} = 0$ if $j - i \geqslant 2$.

(b) The equalities

$$S_{12} = S_{23} = \cdots = S_{\alpha-1,\alpha} = S_{\alpha,\alpha+1} G_{\beta n}, \quad G_{\beta n} S_{\alpha,\alpha+1} = 0$$

imply

$$S_{12} = S_{23} = \cdots = S_{\alpha-1,\alpha} = \begin{bmatrix} 0 & S_4 \\ 0 & 0 \end{bmatrix}, \quad S_{\alpha,\alpha+1} = \begin{bmatrix} S_4 \\ 0 \end{bmatrix}$$

for some $(n - \beta) \times \beta$ matrix S_4.

(c) The equalities

$$S_{11} = S_{22} = \cdots = S_{\alpha\alpha}, \quad G_{\beta n} S_{\alpha\alpha} = S_{\alpha+1,\alpha+1} G_{\beta n}$$

imply

$$S_{11} = S_{22} = \cdots = S_{\alpha\alpha} = \begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix}, \quad S_3 := S_{\alpha+1,\alpha+1}.$$

(d) The equalities

$$\begin{aligned} S_{21} = S_{32} = \cdots = S_{\alpha,\alpha-1}, &\quad G_{\beta n} S_{\alpha,\alpha-1} = S_{\alpha+1,\alpha}, \\ S_{31} = \cdots = S_{\alpha,\alpha-2}, &\quad G_{\beta n} S_{\alpha,\alpha-2} = S_{\alpha+1,\alpha-1}, \\ &\quad \vdots \\ S_{\alpha-1,1} = S_{\alpha 2}, &\quad G_{\beta n} S_{\alpha 2} = S_{\alpha+1,3}, \\ &\quad G_{\beta n} S_{\alpha 1} = S_{\alpha+1,2} \end{aligned}$$

and (22) imply $S_{i,j} = 0$ if $i > j$.

This proves the first equality in (18). The second equality in (18) follows from (23) and the first equality in (22).

Conversely, the equalities (18) ensure (17). For example, if $\alpha = 1$, then (17) takes the form

$$\begin{aligned} &\begin{bmatrix} I_n & 0_n & 0 \\ 0 & [0 \;\; I_\beta] & 0_\beta \end{bmatrix} \begin{bmatrix} \begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix} & \begin{bmatrix} 0 & S_4 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix} & \begin{bmatrix} S_4 \\ 0 \end{bmatrix} \\ 0 & 0 & S_3 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix} & \begin{bmatrix} S_4 \\ 0 \end{bmatrix} \\ 0 & S_3 \end{bmatrix} \begin{bmatrix} I_n & 0_n & 0 \\ 0 & [0 \; I_\beta] & 0_\beta \end{bmatrix}. \quad \square \end{aligned} \tag{24}$$

Remark 7. The condition $n \geqslant 1$ in Theorem 6(a) is essential: each $m \times (0, m)$ pair is transformed by an arbitrarily small perturbation to a pair that is feedback similar to $(0_{m0}, \operatorname{diag}(\lambda_1, \ldots, \lambda_m))$ with distinct $\lambda_1, \ldots, \lambda_m$ determined up to small perturbations.

4. Feedback similarity of triples

The next lemma is proved by using several steps of Belitskii's algorithm [1,12] and arbitrarily small perturbations.

Lemma 8. *Every $m \times (n, m, m)$ triple (C, B, A), $n \geqslant 1$, is transformed by an arbitrarily small perturbation to a triple that is feedback similar to*

$$\begin{cases} (G_{mn}, 0, 0) & \text{if } m \leqslant n, \\ \mathscr{K}(N) := \left(\begin{bmatrix} I_n \\ 0_{m-n,n} \end{bmatrix}, \begin{bmatrix} 0_{nm} \\ H_{m-n,m} \end{bmatrix}, \begin{bmatrix} 0_{nm} \\ N \end{bmatrix} \right) & \text{if } m > n, \end{cases} \tag{25}$$

where N is some $(m-n) \times m$ matrix and $H_{m-n,m}$ is defined in (15).

Two triples $\mathscr{K}(N)$ and $\mathscr{K}(N')$ are feedback similar if and only if

$$N' = R_\alpha(S_1, S_2, S_3, S_4)^{-1} \cdot N \cdot R_{\alpha+1}(S_1, S_2, S_3, S_4) \tag{26}$$

(*see* (18)) *for some S_2, $S_4 \in \mathbb{C}^{(n-\beta)\times\beta}$ and nonsingular matrices $S_1 \in \mathbb{C}^{(n-\beta)\times(n-\beta)}$ and $S_3 \in \mathbb{C}^{\beta\times\beta}$.*

Proof. Let (C, B, A) be an $m \times (n, m, m)$ triple, $n \geqslant 1$. By Theorem 6(a), there is an arbitrarily small perturbation of (C, B) such that the obtained pair $(C + \Delta C, B + \Delta B)$ is feedback similar to the pair $\mathscr{H}_{mn}$ in (14), and then $(C + \Delta C, B + \Delta B, A)$ is feedback similar to (25).

Let $N, N' \in \mathbb{C}^{(m-n)\times m}$. Suppose first that $\mathscr{K}(N)$ and $\mathscr{K}(N')$ are feedback similar. By (2),

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & H_{m-n,m} & N \end{bmatrix} \begin{bmatrix} P & U & V \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} = S \begin{bmatrix} I_n & 0 & 0 \\ 0 & H_{m-n,m} & N' \end{bmatrix} \tag{27}$$

for some U, V and nonsingular P and S. Then (17) holds, which ensures (18). Equating the last vertical strips of the matrices in (27) gives

$$\begin{bmatrix} V \\ NS \end{bmatrix} = S \begin{bmatrix} 0_{nm} \\ N' \end{bmatrix},$$

which defines V and ensures (26).

Conversely, if (26) holds, then by analogy with (24) we have (27) for

$$P = \begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & S_4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad V = UN'$$

and $S = R_{\alpha+1}(S_1, S_2, S_3, S_4)$. Hence, $\mathscr{K}(N)$ and $\mathscr{K}(N')$ are feedback similar. □

Remark 9. Instead of G_{mn} and $H_{m-n,m}$ in (25), one may take F_{mn} and $F_{m-n,m}$ replacing in (26) the matrix $R_\gamma(S_1, \ldots, S_4)$ defined in (19) with (20). We prefer (25) since the matrix (19) is upper block-triangular and we can reduce N to Belitskii's canonical form [1,12] by transformations (26) preserving the other blocks of $\mathscr{K}(N)$. Examples of this reduction are given in Theorems 10 and 11.

Theorem 10. *Each $m \times (1, m, m)$ triple (C, B, A), $m \geqslant 2$, reduces by an arbitrarily small perturbation to a triple that is feedback similar to*

$$\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \cdots & 0 & 0 \\ * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \end{bmatrix} \right)$$

(*the stars denote unspecified entries*). *This triple is determined by (C, B, A) uniquely up to small perturbations of the entries denoted by stars.*

In greater generality, each $\alpha n \times (n, \alpha n, \alpha n)$ triple (C, B, A), $\alpha \geqslant 2$, reduces by an arbitrarily small perturbation to a triple that is feedback similar to a triple of the form

$$\left(\begin{bmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & \cdots & 0 & 0 \\ I_n & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & \cdots & 0 & 0 \\ N_{11} & \cdots & N_{1,\alpha-1} & N_{1\alpha} \\ \vdots & \ddots & \vdots & \vdots \\ N_{\alpha-1,1} & \cdots & N_{\alpha-1,\alpha-1} & N_{\alpha-1,\alpha} \end{bmatrix}\right), \tag{28}$$

in which all blocks are n-by-n,

$$N_{11} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \quad (\lambda_1, \ldots, \lambda_n \textit{ are distinct}),$$

$$N_{12} = \begin{bmatrix} * & 1 & \cdots & 1 \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix} \quad (\textit{the stars denote unspecified entries})$$

and the other N_{ij} are arbitrary. The triple (28) is determined by (C, B, A) uniquely up to small perturbations of $\lambda_1, \ldots, \lambda_n$ in N_{11}, of the entries denoted by stars in N_{12}, and of the entries in the other N_{ij}.

Proof. Let (C, B, A) be $\alpha n \times (n, \alpha n, \alpha n)$, $\alpha \geqslant 2$. By Lemma 8, (C, B, A) reduces by an arbitrarily small perturbation and a feedback similarity transformation to a triple of the form (28), in which N_{ij} are n-by-n. We can reduce $N := [N_{ij}]$ by transformations (26) preserving the other blocks of the triple (28). We have

$$R_\gamma(S_1, S_2, S_3, S_4) = S_3 \oplus \cdots \oplus S_3 \quad (\gamma \text{ summands})$$

since $\beta = n$ in (16) and so S_1 is 0×0 in (19). Hence we can reduce all N_{ij} by simultaneous similarity transformations

$$N'_{ij} = S_3 N_{ij} S_3^{-1}, \quad 1 \leqslant i \leqslant \alpha - 1, \;\; 1 \leqslant j \leqslant \alpha. \tag{29}$$

By an arbitrarily small perturbation and some transformation (29) we reduce N_{11} to a diagonal matrix with distinct diagonal entries. To preserve N_{11} we must reduce the other blocks N_{ij} by transformations (29) with diagonal S_3. Using an arbitrarily small perturbation we make nonzero the $(1, 2), \ldots, (1, n)$ entries of the first row of N_{12} and reduce them to 1 by transformations (29) with diagonal S_3. Each transformation (29) that preserves N_{11} and the $(1, 2), \ldots, (1, n)$ entries of N_{12} is the identity transformation, so we can reduce the other entries of N_{ij} only by arbitrarily small perturbations. □

For every $p \times (q, p, p)$ triple (C, B, A), we define the $(2p + q) \times (p + q, 2p + q, 2p + q)$ triple

$$\mathscr{L}(C, B, A) := \left(\begin{bmatrix} I_{p+q} \\ 0_{p,p+q} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0_{pq} & I_p & 0_p \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ C & B & A \end{bmatrix}\right).$$

Put

$$\mathscr{L}^{(i)}(C, B, A) := \underbrace{\mathscr{L} \ldots \mathscr{L}}_{i\text{-times}}(C, B, A), \quad i = 0, 1, 2, \ldots$$

Theorem 11. *Let m and n be natural numbers, and let* $\mathbb{C}^{m\times(n,m,m)}$ *denote the space of all* $m \times (n, m, m)$ *triples.*

(a) *If* $m < n(1+\sqrt{5})/2$, *then there is exactly one* $m \times (n, m, m)$ *triple of the form* $\mathscr{L}^{(l)}(F_{pq}, 0_p, 0_p)$. *All triples that are feedback similar to it form an open and everywhere dense set in* $\mathbb{C}^{m\times(n,m,m)}$ *which coincides with the set of all* $m \times (n, m, m)$ *triples that are rigid with respect to feedback similarity.*

(b) *If* $m > n(1+\sqrt{5})/2$, *then all* $m \times (n, m, m)$ *triples are not rigid with respect to feedback similarity.*

Proof. Let m and n be natural numbers, and let (C, B, A) be $m \times (n, m, m)$. We say that a triple $\mathscr{T}$ *reduces* to a triple $\mathscr{T}'$ if $\mathscr{T}$ reduces to $\mathscr{T}'$ by an arbitrarily small perturbation and a feedback similarity transformation.

(a) Suppose first that

$$m < n(1+\sqrt{5})/2 \approx 1.618n \tag{30}$$

and prove by induction on $m - n$ that (C, B, A) reduces to some $\mathscr{L}^{(l)}(F_{pq}, 0_p, 0_p)$.

The base of induction is trivial: if $m \leqslant n$, then by Lemma 8 (C, B, A) reduces to $(F_{mn}, 0_m, 0_m)$. This triple is rigid for feedback similarity and is feedback similar to each rigid $m \times (n, m, m)$ triple.

Let $m > n$. Then by (16) and (30) we have $\alpha = 1$ and $\beta = m - n$. According to Lemma 8, (C, B, A) reduces to some triple

$$\begin{aligned}\mathscr{K}([C'\ B'\ C']) &= \left(\begin{bmatrix} I_n \\ 0_{\beta n}\end{bmatrix}, \begin{bmatrix} 0_{n,n-\beta} & 0_{n\beta} & 0_{n\beta} \\ 0_{\beta,n-\beta} & I_\beta & 0_\beta \end{bmatrix}, \begin{bmatrix} 0_{n,n-\beta} & 0_{n\beta} & 0_{n\beta} \\ C' & B' & C' \end{bmatrix}\right) \\ &= \mathscr{L}(C', B', A'),\end{aligned}$$

in which (C', B', A') is $m' \times (n' \times m' \times m')$ and

$$m' := m - n, \quad n' := -m + 2n. \tag{31}$$

By Lemma 8, $\mathscr{K}([C'\ \ B'\ \ C'])$ is feedback similar to $\mathscr{K}([C_1'\ \ B_1'\ \ C_1'])$ if and only if there exist U, V, and nonsingular P and S such that

$$[C_1'\ \ B_1'\ \ A_1'] = S^{-1}[C'\ \ B'\ \ A'] \begin{bmatrix} P & U & V \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} \tag{32}$$

(the last matrix is $R_2(P, U, S, V)$ defined in (19)). Therefore, $\mathscr{L}(C', B', A')$ is feedback similar to $\mathscr{L}(C_1', B_1', A_1')$ if and only if (C', B', A') is feedback similar to (C_1', B_1', A_1').

The numbers m' and n' are natural: $m' > 0$ since $m > n$, and $n' = 2n - m > 0$ since $1.7n - m > 0$ by (30). Furthermore, $m' < n'(1+\sqrt{5})/2$ because by (30)

$$\frac{m_1}{n_1} = \frac{m-n}{-m+2n} < \frac{n(1+\sqrt{5})/2 - n}{-n(1+\sqrt{5})/2 + 2n} = \frac{-1+\sqrt{5}}{3-\sqrt{5}} = \frac{1+\sqrt{5}}{2}.$$

Since $(m - n) - (m' - n') = n' > 0$, the induction hypothesis ensures that (C', B', A') reduces to $\mathscr{L}^{(l-1)}(F_{pq}, 0_p, 0_p)$ for some p, q, and l that are uniquely determined by m' and n'. Therefore, (C, B, A) reduces to $\mathscr{K}([C'B'C']) = \mathscr{L}(C', B', A')$, which reduces to $\mathscr{L}^{(l)}(F_{pq,}0_p, 0_p)$ that is uniquely determined by m and n.

We have proved that all $m \times (n, m, m)$ triples reduce to the same triple $\mathscr{L} := \mathscr{L}^{(l)}(F_{pq}, 0_p, 0_p)$, and so the set $\mathscr{S}$ of all triples that are feedback similar to $\mathscr{L}$ is everywhere dense. Since $\mathscr{L}$ is rigid with respect to feedback similarity, there exists its neighborhood V that is contained in $\mathscr{S}$. For any triple $\mathscr{T} \in \mathscr{S}$, there is a transformation of feedback similarity that transforms $\mathscr{L}$ to $\mathscr{T}$; it also transforms V to some neighborhood W of $\mathscr{T}$. Since each triple in V is feedback similar to $\mathscr{L}$, each triple in W is also feedback similar to $\mathscr{L}$, hence $W \subset \mathscr{S}$. Therefore, each triple $\mathscr{T} \in \mathscr{S}$ possesses a neighborhood that is contained in $\mathscr{S}$, and so the set $\mathscr{S}$ is open.

(b) Let

$$m \geqslant n(1 + \sqrt{5})/2. \tag{33}$$

Since (C, B, A) is fixed, the equality (2) defines the mapping

$$f : \mathscr{U} \to \mathbb{C}^{m\times(n,m,m)}, \quad (S, P, U, V) \mapsto (C', B', A'),$$

where

$$\mathscr{U} := \{(S, P, U, V) \in \mathbb{C}^{m\times m} \times \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m} \times \mathbb{C}^{n\times m} \mid \det(S)\det(P) \neq 0\}.$$

This mapping is rational since by (2) the entries of C', B', and A' are polynomials (in entries of S, P, U, and V) divided by $\det(S)$. Its image is the set of all triples that are feedback similar to (C, B, A).

Suppose that (C, B, A) is rigid. Then the image of f contains a neighborhood of (C, B, A), hence $\mathbb{C}^{m\times(n,m,m)} \setminus \operatorname{Im}(f)$ cannot be dense in $\mathbb{C}^{m\times(n,m,m)}$ and so $\dim(\mathscr{U}) \geqslant \dim(\mathbb{C}^{m\times(n,m,m)})$ by [9, Section 3, Proposition 1.2]. This means that $m^2 + n^2 + 2mn \geqslant mn + 2m^2$,

$$(m/n)^2 - m/n - 1 \leqslant 0, \quad m/n < (1 + \sqrt{5})/2,$$

which contradicts to (33). Therefore, there are no rigid triples of this size. □

For each $m \times (n, m, m)$ triple $\mathscr{T} = (C, B, A)$, we define the $m \times (n + 2m)$ polynomial matrix

$$\mathscr{T}(x, y) = [C \;\; xI_m + B \;\; yI_m + A].$$

The next lemma is trivial, but it can be useful.

Lemma 12. *Two matrix triples $\mathscr{T}$ and $\mathscr{T}'$ are feedback similar if and only if the corresponding polynomial matrices $\mathscr{T}(x, y)$ and $\mathscr{T}'(x, y)$ are strictly equivalent; this means that*

$$S\mathscr{T}'(x, y) = \mathscr{T}(x, y)R \tag{34}$$

for some nonsingular complex matrices S and R.

Proof. Let $\mathscr{T} = (C, B, A)$ and $\mathscr{T}' = (C', B', A')$ be $m \times (n, m, m)$.

If $\mathscr{T}$ and $\mathscr{T}'$ are feedback similar, then there exists a nonsingular matrix

$$R = \begin{bmatrix} P & U & V \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} \tag{35}$$

such that

$$S[C' \;\; B' \;\; A'] = [C \;\; B \;\; A]R.$$

Since

$$\mathscr{T}(x, y) = [C \;\; B \;\; A] + x[0 \;\; I \;\; 0] + y[0 \;\; 0 \;\; I]$$

and

$$S[0\ \ I_m\ \ 0] = [0\ \ I_m\ \ 0]R, \quad S[0\ \ 0\ \ I_m] = [0\ \ 0\ \ I_m]R,$$

we have (34).

Conversely, let (34) hold. This polynomial equality breaks into three scalar equalities:

$$S[C'\ \ B'\ \ A'] = [C\ \ B\ \ A]R,$$
$$S[0\ \ I_m\ \ 0] = [0\ \ I_m\ \ 0]R, \quad S[0\ \ 0\ \ I_m] = [0\ \ 0\ \ I_m]R.$$

By the last two equalities, the matrix R has the form (35). So by the first equality $\mathscr{T}$ and $\mathscr{T}'$ are feedback similar. □

Remark 13. The authors are grateful to the reviewer for suggestions and the following commentaries: we study the orbits of the action (2) of the product of two groups on the space of matrix triples, which can be identified with $\mathbb{C}^{m(n+2m)}$. Namely, from the left one has the action of $\mathrm{GL}(m;\mathbb{C})$, and from the right one has the action of the 3×3 block upper triangular subgroup of $\mathrm{GL}(n+2m;\mathbb{C})$ with (2, 3) block is equal to zero. It is known that each orbit under such an action is a smooth irreducible semi-affine variety V, i.e. its closure is an affine irreducible variety $\overline{V}$, and $V=\overline{V}\backslash W$, where W is a strict subvariety of $\overline{V}$. Moreover, all singular points of $\overline{V}$ are contained in W. The orbits of the maximal dimension d are called the "generic" orbits. Theorem 10 gives the unique canonical form of a generic orbit. The parameter space of such orbits is $m(n+2m)-d$ dimensional. The notion of rigid system is equivalent to the assumption of the existence of orbits of dimension $m(n+2m)$. Since such an orbit V is an irreducible semi-affine variety, it follows that $\overline{V}=\mathbb{C}^{m(n+2m)}$. Hence there is only one orbit like that as Theorem 11 claims.

Acknowledgments

The authors would like to thank Professor Diederich Hinrichsen for very valuable suggestions on two manuscript versions of this article.

References

[1] G. Belitskii, Normal forms in matrix spaces, Integral Equations Operator Theory 38 (3) (2000) 251–283.
[2] G.R. Belitskii, V.V. Sergeichuk, Complexity of matrix problems, Linear Algebra Appl. 361 (2003) 203–222.
[3] P. Brunovsky, A classification of linear controllable systems, Kybernetika 3 (1970) 173–187.
[4] H. Derksen, J. Weyman, Quiver representations, Notices AMS 52 (2) (2005) 200–206.
[5] P. Gabriel, A.V. Roiter, Representations of finite-dimensional algebras, Encyclopaedia of Math Sci, Algebra VIII, vol. 73, Springer, 1992.
[6] H. Glüsing-Lüerssen, Feedback canonical form for singular systems, Int. J. Control 52 (2) (1990) 347–376.
[7] D. Hinrichsen, J. O'Halloran, A pencil approach to high gain feedback and generalized state space systems, Kybernetika 31 (1995) 109–139.
[8] D. Hinrichsen, J. O'Halloran, Limits of generalized state space systems under proportional and derivative feedback, Math. Control Signals Syst. 10 (1997) 97–124.
[9] M.W. Hirsch, Differential topology, Graduate Texts in Mathematics, Springer, 1976.
[10] J.J. Loiseau, K. Özcaldiran, M. Malabre, N. Karcanias, Feedback canonical forms of singular systems, Kybernetika 27 (1991) 289–305.

[11] L.A. Nazarova, A.V. Roiter, V.V. Sergeichuk, V.M. Bondarenko, Application of modules over a dyad for the classification of finite *p*-groups possessing an abelian subgroup of index *p* and of pairs of mutually annihilating operators, J. Soviet Math. 3 (5) (1975) 636–654.
[12] V.V. Sergeichuk, Canonical matrices for linear matrix problems, Linear Algebra Appl. 317 (2000) 53–102.