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Local sections of Serre fibrations with 3-manifold fibers

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ABSTRACT

It was proved by H. Whitney in 1933 that a Serre fibration of compact metric spaces admits a global section provided every fiber is homeomorphic to the unit interval $[0, 1]$. An extension of the Whitney's theorem to the case when all fibers are homeomorphic to some fixed compact two-dimensional manifold was proved by the authors (Brodsky et al. (2008) [2]). The main result of this paper proves the existence of local sections in a Serre fibration with all fibers homeomorphic to some fixed compact three-dimensional manifold.

1. Introduction

The following problem is one of the central problems in geometric topology [3]. Let $p : E \to B$ be a Serre fibration of separable metric spaces. Assume that the space B is locally n-connected and all fibers of p are homeomorphic to some fixed n-dimensional manifold M^n. Is p a locally trivial fibration?

In case $n = 1$ an affirmative answer to this problem follows from results of H. Whitney [12].

Conjecture (Shchepin). *A Serre fibration with a locally arcwise connected metric base is locally trivial if every fiber of this fibration is homeomorphic to some fixed manifold M^n of dimension $n \leqslant 4$.*

In dimension $n = 1$ the Shchepin's conjecture is proved even for non-compact fibers [9]. Shchepin proved that positive solution of this conjecture in dimension n implies positive solutions of both CE-problem and homeomorphism group problem in dimension n [10,3]. Since CE-problem was solved in a negative way by A.N. Dranishnikov, there are dimensional restrictions in Shchepin's conjecture.

Under the assumption of the base B of the Serre fibration $p : E \to B$ being finite dimensional the Shchepin's conjecture is proved in dimensions $n = 2$ [7] and $n = 3$ [6]. An interesting result is obtained by S. Ferry proving that p is a Hurewicz fibration [5].

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The first step toward proving the Shchepin's conjecture in dimension $n = 2$ over infinite dimensional base is made in [1,2] where existence of local and global sections of the fibration is proved provided the base space is an *ANR*. The following theorem is the main result of this paper.

Theorem 3.3. *Let $p : E \to B$ be a Serre fibration of LC^2-compacta with all fibers homeomorphic to some fixed compact three-dimensional manifold. If $B \in ANR$, then any section of p over closed subset $A \subset B$ can be extended to a section of p over some neighborhood of A.*

Our strategy of constructing a section of a Serre fibration is as follows (definitions are given in Section 2). We consider the inverse (multivalued) mapping p^{-1} and find its compact submapping admitting continuous approximations. Then we take very close continuous approximation and use it to find again a compact submapping with small diameters of fibers admitting continuous approximations. When we continue this process we get a sequence of compact submappings with diameters of fibers tending to zero. This sequence will converge to the desired singlevalued submapping of p^{-1} (section of p).

The major difference of the proof in this paper from the one in [2] comes from the fact that any open subset of Euclidean plane is aspheric (all homotopy groups vanish in dimensions $\geqslant 2$) which is far from being true in Euclidean 3-space. In order to apply our technique to 3-dimensional manifolds we introduce a new property called hereditarily coconnected asphericity.

2. Preliminaries on spaces and multivalued mappings

Let us recall some definitions and introduce our notations. All spaces will be separable metrizable. We equip the product $X \times Y$ with the metric

$$\mathrm{dist}_{X\times Y}\big((x, y), (x', y')\big) = \mathrm{dist}_X(x, x') + \mathrm{dist}_Y(y, y').$$

By $O(x, \varepsilon)$ we denote the open ε-neighborhood of the point x.

A multivalued mapping $F : A \to Y$ is called *submapping* (or *selection*) of multivalued mapping $G : X \to Y$ if A is a subspace of X and $F(x) \subset G(x)$ for every $x \in A$. The *gauge* of a multivalued mapping $F : X \to Y$ is defined as $\mathrm{cal}(F) = \sup\{\mathrm{diam}\, F(x) \mid x \in X\}$. The *graph* of multivalued mapping $F : X \to Y$ is the subset $\Gamma_F = \{(x, y) \in X \times Y \mid y \in F(x)\}$ of the product $X \times Y$. For arbitrary subset $\mathcal{U} \subset X \times Y$ denote by $\mathcal{U}(x)$ the subset $pr_Y(\mathcal{U} \cap (\{x\} \times Y))$ of Y. Then for the graph Γ_F we have $\Gamma_F(x) = F(x)$.

A multivalued mapping $G : X \to Y$ is called *complete* if there exists a G_δ-set $S \subset X \times Y$ containing the graph Γ_G such that all sets $\{x\} \times G(x)$ are closed in S. Notice that any compact-valued mapping is complete. A multivalued mapping $F : X \to Y$ is called *upper semicontinuous* if for any open set $U \subset Y$ the set $\{x \in X \mid F(x) \subset U\}$ is open in X. A *compact* mapping is an upper semicontinuous multivalued mapping with compact images of points.

Let Z be a space. A sequence $\{Z_k\}$ (finite or infinite) of subspaces

$$Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z$$

is called a *filtration* of Z. The number of elements of the filtration is called *length* of the filtration. Given a filtration $\{Z_k\}$ of Z, its *subfiltration* $\{Z'_k\}$ is a filtration of Z such that $Z'_k \subset Z_k$ for every k. A sequence of multivalued mappings $\{F_k : X \to Y\}$ is called a *filtration of multivalued mapping* $F : X \to Y$ if for any $x \in X$ the sequence $\{F_k(x)\}$ is a filtration of $F(x)$. We say that a filtration of multivalued mappings $G_i : X \to Y$ is *compact* if every mapping G_i is compact.

A pair of spaces $V \subset U$ is called *n-aspheric* if every continuous mapping of the n-sphere into V is homotopic to a constant mapping in U. We assume that -1-*asphericity* of the pair $V \subset U$ means exactly that U is non-empty. A pair of compacta $K \subset K'$ is called *approximately n-aspheric* if for some embedding of K' into *ANR*-space for any neighborhood U of the set K' there is a neighborhood V of the set K such that the pair $V \subset U$ is n-aspheric. A compact space K is called *approximately aspheric* if the pair $K \subset K$ is approximately n-aspheric for every $n \geqslant 2$.

A pair of spaces $V \subset U$ is called *polyhedrally n-connected* if for any finite n-dimensional polyhedron M and its closed subpolyhedron A any mapping of A in V can be extended to a map of M into U.

Definition 2.1. We say that a subset A of a space Z is *coconnected* if the complement $Z \setminus A$ is connected.

A pair $V \subset U$ of proper subsets of a space Z is called *coconnected* if $Z \setminus U$ lies in a connected component of $Z \setminus V$.

Definition 2.2. We call a space Z *hereditarily coconnectedly aspheric* if any non-separating compactum $K \subset Z$ is approximately aspheric.

A space Z is said to be *locally hereditarily coconnectedly aspheric* if any point $z \in Z$ has a hereditarily coconnectedly aspheric neighborhood.

Remark 2.3. An important example of hereditarily coconnectedly aspheric space is Euclidean 3-space [3, Lemma 2.4]. Therefore, any 3-dimensional manifold is locally hereditarily coconnectedly aspheric.

Now we consider different properties of pairs of spaces and define the corresponding local properties for spaces and multivalued maps. We follow definitions and notations from [4].

Definition 2.4. An ordering α of the subsets of a space Y is *proper* provided:

(a) If $W\alpha V$, then $W \subset V$;
(b) If $W \subset V$, and $V\alpha R$, then $W\alpha R$;
(c) If $W\alpha V$, and $V \subset R$, then $W\alpha R$.

We are going to use the following proper orderings:

(1) $V\alpha U$ means U is non-empty and $V \subset U$;
(2) $V\alpha U$ means the pair $V \subset U$ is k-aspheric for every $k \leqslant n$;
(3) $V\alpha U$ means the pair $V \subset U$ is polyhedrally n-connected;
(4) $V\alpha U$ means the pair $V \subset U$ is coconnected;
(5) $V\alpha U$ means V is hereditarily coconnectedly aspheric and $V \subset U$.

Definition 2.5. Let α be a proper ordering. A space Y is *locally of type* α if, whenever $y \in Y$ and V is a neighborhood of y, then there is a neighborhood W of y such that $W\alpha V$.

Let us introduce terminology for spaces which are locally of type α for the examples of proper orderings α described above:

(1) any space is locally of type α;
(2) X is locally of type α means X is *locally n-connected* (notation: $X \in LC^n$);
(3) X is locally of type α means X is *locally polyhedrally n-connected*;
(4) X is locally of type α means X is *locally coconnected*;
(5) X is locally of type α means X is *locally hereditarily coconnectedly aspheric*.

We use the word "equi" for local properties of multivalued maps.

Definition 2.6. Let α be a proper ordering. A multivalued mapping $F: X \to Y$ is *equi locally of type* α if for any points $x \in X$ and $y \in F(x)$ and for any neighborhood V of y in Y there exist neighborhoods W of y in Y and U of x in X such that $(W \cap F(x'))\alpha(V \cap F(x'))$ provided $x' \in U$.

Let us introduce terminology for multivalued mappings which are equi locally of type α for the examples of proper orderings α described above:

(1) F is equi locally of type α means F is *lower semicontinuous*;
(2) F is locally of type α means F is *equi locally n-connected* (briefly, F is equi-LC^n);
(3) F is locally of type α means F is *equi locally polyhedrally n-connected*;
(4) F is locally of type α means F is *equi locally coconnected*;
(5) F is locally of type α means F is *equi locally hereditarily coconnectedly aspheric*.

Since -1-*asphericity* of a pair $V \subset U$ means U is non-empty, then equi local n-connectedness of a multivalued map implies its lower semicontinuity.

The following lemma is easy to prove [2, Lemma 2.7].

Lemma 2.7. *Any equi-LC^n multivalued mapping is equi locally polyhedrally $(n+1)$-connected.*

Corollary 2.8. *Any LC^n space X is locally polyhedrally $(n+1)$-connected.*

Proof. Consider a multivalued mapping from the one-point space onto X and apply Lemma 2.7. □

The following lemma is easy to prove. We will use it with different properties α in Section 3.

Lemma 2.9. *Let α be a proper ordering. Suppose that a multivalued mapping $F: X \to Y$ is equi locally of type α and contains a compact submapping $\Psi: A \to Y$ defined on a compact subspace A of X. Then for any $\varepsilon > 0$ there exists a positive number δ such that for every point $(x, y) \in O(\Gamma_\Psi, \delta) \subset X \times Y$ we have $(O(y,\delta) \cap F(x))\alpha(O(y,\varepsilon) \cap F(x))$.*

A filtration $\{F_i\}$ of multivalued maps is called *equi locally connected* if for any i the mapping F_i is equi locally i-connected. A filtration of multivalued maps $\{F_i\}$ is called *polyhedrally connected* if every pair $F_{i-1}(x) \subset F_i(x)$ is polyhedrally i-connected.

A filtration of compact mappings $\{F_m : X \to Y\}$ is called *approximately connected* if for any point $x \in X$ and for any k the pair $F_k(x) \subset F_{k+1}(x)$ is approximately k-aspheric.

The following lemma is a weak form of Compact Filtration lemma from [11].

Lemma 2.10. *Any polyhedrally connected equi locally connected finite filtration of complete mappings of a compact space contains a compact approximately connected subfiltration of the same length.*

Definition 2.11. A multivalued mapping $F : X \to Y$ *admits continuous approximations* if every neighborhood of the graph Γ_F in $X \times Y$ contains a graph of some single-valued continuous map $f : X \to Y$.

Theorem 2.12. *([2, Theorem 3.14]) Suppose that a compact multivalued mapping of separable metric ANRs $F : X \to Y$ admits a compact approximately connected filtration of infinite length. Then for any compact space $K \subset X$ every neighborhood of the graph $\Gamma_F(K)$ contains the graph of a single-valued and continuous mapping $f : K \to Y$.*

If a pair $G_0 \subset G_1$ of proper subsets of a space Z is coconnected, then we can define an operation of G_1-*coconnectification* on subsets of G_0 as follows: For a subset $F_0 \subset G_0$ its G_1-coconnectification is the union of F_0 and all components of $Z \setminus F_0$ which do not intersect $Z \setminus G_1$. Clearly, the G_1-coconnectification of F_0 is the minimal subset $F_1 \subset G_1$ containing F_0 such that $Z \setminus F_1$ is connected and contains $Z \setminus G_1$.

Lemma 2.13. *Suppose that Z is a locally 0-connected space and $G_0 \subset G_1$ is a coconnected pair of proper subsets of Z. Let F_0 be a subset of G_0. If F_0 is closed in Z then the G_1-coconnectification F_1 of F_0 is also closed in Z.*

Proof. Suppose that $\{z_n\}$ is a sequence of points in F_1 converging to a point $z \in Z \setminus F_1$. If infinitely many points z_n belong to F_0, then $z \in F_0$ by closedness of F_0. So, we assume that $z_n \notin F_0$ for all n. Then for any n the points z and z_n belong to different connected components of $Z \setminus F_0$. Use local 0-connectedness of Z to find a path w_n from z to z_n such that the diameter of w_n tends to 0 as $n \to \infty$. Since each path w_n must intersect F_0, the point z is a limit point of F_0. Contradiction. $\square$

If a multivalued mapping $F : X \to Y$ contains proper submappings G_0 and G_1 such that for any $x \in X$ the pair $G_0(x) \subset G_1(x)$ is coconnected in $F(x)$, then for any submapping $F_0 \subset G_0$ we define a G_1-*coconnectification* of F_0 as a multivalued mapping taking a point $x \in X$ to the $G_1(x)$-coconnectification of $F_0(x)$.

Lemma 2.14. *Suppose that equi-LC^0 multivalued mapping $F : X \to Y$ contains proper submappings $G_0 \subset G_1$ such that G_1 is compact and for any $x \in X$ the pair $G_0(x) \subset G_1(x)$ is coconnected in $F(x)$. Then for any compact submapping $F_0 \subset G_0$ its G_1-coconnectification F_1 is a compact submapping of G_1.*

Proof. Since the $G_1(x)$-coconnectification of the set $F_0(x)$ is closed in $F(x)$ by Lemma 2.13 and is contained in $G_1(x)$, the set $F_1(x)$ is compact.

Let us prove that F_1 is upper semicontinuous. Suppose to the contrary that for some point $x \in X$ and for some $\varepsilon > 0$ there is a sequence of points $\{x_i\}_{i=1}^{\infty}$ in X converging to x such that $F_1(x_i) \not\subset O(F_1(x), \varepsilon)$ for all i. That means we can fix points $y_i \in F_1(x_i) \setminus O(F_1(x), \varepsilon)$ for all i. Since G_1 is a compact map and $y_i \in G_1(x_i)$ for all i, there is a limit point $y \in G_1(x)$ for the sequence $\{y_i\}_{i=1}^{\infty}$. Without loss of generality we assume that the sequence $\{y_i\}_{i=1}^{\infty}$ converges to y. Since $y_i \notin O(F_1(x), \varepsilon)$, we have $y_i \notin F_1(x)$. Fix a point $z \in F(x) \setminus G_1(x)$. The points y and z belong to connected set $F(x) \setminus F_1(x)$ which is open in $F(x)$ and therefore is locally path connected (since $F(x) \in LC^0$). Hence, there exists a path $s : [0, 1] \to F(x) \setminus F_1(x)$ such that $s(0) = y$ and $s(1) = z$. Since F is lower semicontinuous and G_1 is upper semicontinuous, there is a sequence of points $\{z_i \in F(x_i) \setminus G_1(x_i)\}_{i=M}^{\infty}$, converging to z.

Using equi-LC^0 property of the mapping F we can choose a sequence of maps $\{s_i : [0, 1] \to F(x_i)\}_{i=M'}^{\infty}$ such that $s_i(0) = y_i$, $s_i(1) = z_i$ and the paths s_i converge to the path s uniformly (this follows from general results on continuous selections [8], although an elementary proof exists which is straightforward but too technical). Since the path s does not intersect $F_0(x)$ and F_0 is upper semicontinuous, for all but finitely many i the path s_i does not intersect $F_0(x_i)$. It means that the points y_i and z_i belong to the same connected component of the set $F(x_i) \setminus F_0(x_i)$, which contradicts to the choices of y_i and z_i. $\square$

Definition 2.15. The mapping $f : X \to Y$ is said to be *topologically regular* provided that if $\varepsilon > 0$ and $y \in Y$, then there is a positive number δ such that $\mathrm{dist}(y, y') < \delta$, $y' \in Y$, implies that there is a homeomorphism of $f^{-1}(y)$ onto $f^{-1}(y')$ which moves no point as much as ε (i.e. an ε-homeomorphism).

Note that since the Poincaré conjecture is true, any Serre fibration of LC^2-compacta with all fibers homeomorphic to some fixed compact three-dimensional manifold is topologically regular [6].

Lemma 2.16. *If $p : E \to B$ is a topologically regular mapping of compacta with all fibers homeomorphic to some fixed compact three-dimensional manifold, then the multivalued mapping $p^{-1} : B \to E$ is*

- *equi locally hereditarily coconnectedly aspheric,*
- *equi locally coconnected,*
- *equi locally polyhedrally* 2*-connected.*

Proof. Fix a point $q \in E$ and $\varepsilon > 0$. We will find $\delta > 0$ such that for any point $x \in p(O(q,\delta))$ there exist subsets D^3 and O^3 of the fiber $p^{-1}(x)$ such that

$$O(q,\delta) \cap p^{-1}(x) \subset D^3 \subset O^3 \subset O(q,\varepsilon) \cap p^{-1}(x)$$

where D^3 is homeomorphic to closed 3-ball and O^3 is homeomorphic to $\mathbb{R}^3$. Then the first property of p^{-1} follows from the fact that O^3 is hereditarily coconnectedly aspheric (see Remark 2.3). The last two properties follow from coconnectedness of the pair $D^3 \subset O^3$ and contractibility of D^3 respectively.

Take a neighborhood O^3_q of the point q in the fiber $p^{-1}(p(q))$ such that O^3_q is homeomorphic to $\mathbb{R}^3$ and is contained in $O(q,\varepsilon/2)$. Note that if h is $\varepsilon/2$-homeomorphism of O^3_q, then $h(O^3_q)$ is contained in $O(q,\varepsilon)$. Let D^3_q be a neighborhood of q in $p^{-1}(p(q))$ homeomorphic to closed 3-ball. Take a positive number $\sigma < \varepsilon$ such that $O(q,\sigma) \cap p^{-1}(p(q))$ is contained in D^3_q. Choose a positive number $\delta < \sigma/2$ such that for any point $x \in O(p(q),\delta)$ there exists $\sigma/2$-homeomorphism of the fiber $p^{-1}(p(q))$ onto $p^{-1}(x)$. Now take a point $x \in p(O(q,\delta))$ and fix $\sigma/2$-homeomorphism h of the fiber $p^{-1}(p(q))$ onto $p^{-1}(x)$. By the choice of σ, the set $h(D^3_q)$ contains $O(q,\sigma/2) \cap p^{-1}(x)$. Therefore, we have

$$O(q,\delta) \cap p^{-1}(x) \subset h\bigl(D^3_q\bigr) \subset h\bigl(O^3_q\bigr) \subset O(q,\varepsilon) \cap p^{-1}(x). \qquad \square$$

3. Fibrations with 3-manifold fibers

Lemma 3.1. *Let $F : X \to Y$ be equi locally hereditarily coconnectedly aspheric, equi locally coconnected, equi-LC^1 compact-valued mapping of a space X to a Banach space Y. Assume that F is a submapping of an equi-LC^0 compact map $\Phi : X \to Y$ such that $F(x) \neq \Phi(x)$ only if $F(x) = pt$ in which case $\Phi(x) \setminus F(x)$ is connected. Suppose that a compact submapping $\Psi : A \to Y$ of F is defined on a compactum $A \subset X$ and admits continuous approximations. Then for any $\varepsilon > 0$ there exists a neighborhood OA of A and a compact submapping $\Psi' : OA \to Y$ of $F|_{OA}$ such that $\Gamma_{\Psi'} \subset O(\Gamma_\Psi, \varepsilon)$, Ψ' admits a compact approximately connected filtration of infinite length, and* $\operatorname{cal}\Psi' < \varepsilon$.

Proof. Fix a positive number ε. Apply Lemma 2.9 to the maps F and Ψ with α being equi local hereditary coconnected asphericity to get a positive number $\varepsilon_3 < \varepsilon$. Apply Lemma 2.9 again with α being equi local coconnectedness to get a positive number $\varepsilon_2 < \varepsilon_3/2$. By Lemma 2.7 the mapping F is equi locally polyhedrally 2-connected. Subsequently applying Lemma 2.9 with α being equi local polyhedral n-connectedness for $n = 2, 1, 0$, we find positive numbers ε_1, ε_0, and δ such that $\delta < \varepsilon_0 < \varepsilon_1 < \varepsilon_2$ and for every point $(x,y) \in O(\Gamma_\Psi,\delta)$ the pair $(O(y,\varepsilon_1) \cap F(x), O(y,\varepsilon_2) \cap F(x))$ is polyhedrally 2-connected, the pair $(O(y,\varepsilon_0) \cap F(x), O(y,\varepsilon_1) \cap F(x))$ is polyhedrally 1-connected, and the intersection $O(y,\varepsilon_0) \cap F(x)$ is not empty. Notice that for every point $(x,y) \in O(\Gamma_\Psi,\delta)$ we also have the pair $(O(y,\varepsilon_2) \cap F(x), O(y,\varepsilon_3/2) \cap F(x))$ being coconnected and the set $O(y,\varepsilon_3) \cap F(x)$ being hereditarily coconnectedly aspheric.

Let $f : A \to Y$ be a continuous single-valued mapping whose graph is contained in $O(\Gamma_\Psi,\delta)$. Let $f' : \mathcal{O}A \to Y$ be a continuous extension of the mapping f over some neighborhood $\mathcal{O}A$ such that the graph of f' is contained in $O(\Gamma_\Psi,\delta)$. Now we can define a polyhedrally connected filtration $G_0 \subset G_1 \subset G_2 : \mathcal{O}A \to Y$ of the mapping $F|_{\mathcal{O}A}$ by the equality

$$G_i(x) = O\bigl(f'(x), \varepsilon_i\bigr) \cap F(x).$$

Since the set $\bigcup_{x \in \mathcal{O}A}\{\{x\} \times O(f'(x),\varepsilon_i)\}$ is open in the product $\mathcal{O}A \times Y$ and the mapping F is complete, then G_i is also complete. Clearly, the set $G_2^\Gamma(x)$ is contained in $O(\Gamma_\Psi, 2\varepsilon_2)$. Now, applying Lemma 2.10 to the filtration $G_0 \subset G_1 \subset G_2$, we obtain a compact approximately connected subfiltration $F_0 \subset F_1 \subset F_2 : \mathcal{O}A \to Y$. Define a map $G_3 : \mathcal{O}A \to Y$ by the equality

$$G_3(x) = \overline{O\bigl(f'(x), \varepsilon_3/2\bigr)} \cap \Phi(x).$$

Compactness of G_3 follows from compactness of Φ. By the choice of ε_2 the pair $G_2(x) \subset G_3(x)$ is coconnected in $\Phi(x)$ when $\Phi(x) = F(x)$, otherwise $G_2(x) = F(x) = pt$ and the pair $G_2(x) \subset G_3(x)$ is coconnected in $\Phi(x)$ because $\Phi(x) \setminus F(x)$ is connected. By Lemma 2.14 we find a coconnectification F_3 of F_2 (with respect to Φ) inside G_3. Notice that F_3 is also a coconnectification of F_2 with respect to F. By the choice of ε_3, F_3 is a compact submapping of F having approximately aspheric point-images. Therefore, the infinite filtration $F_0 \subset F_1 \subset F_3 \subset F_3 \subset F_3 \subset \cdots$ is approximately connected and we can put $\Psi' = F_3$. $\square$

Theorem 3.2. *Let $F : X \to Y$ be equi locally hereditarily coconnectedly aspheric, equi locally coconnected, equi-LC^1 compact-valued mapping of locally compact ANR-space X to Banach space Y. Assume that F is a submapping of an equi-LC^0 compact map $\Phi : X \to Y$*

such that $F(x) \neq \Phi(x)$ only if $F(x) = pt$ in which case $\Phi(x) \setminus F(x)$ is connected. Suppose that a compact submapping $\Psi : A \to Y$ of $F|_A$ is defined on a compactum $A \subset X$ and admits continuous approximations. Then for any $\varepsilon > 0$ there exist a neighborhood OA of A and a single-valued continuous selection $s : OA \to Y$ of $F|_{OA}$ such that $\Gamma_s \subset O(\Gamma_\Psi, \varepsilon)$.

Proof. Fix $\varepsilon > 0$. By Lemma 3.1 there are a neighborhood U_1 of A in X and a compact submapping $\Psi_1 : U_1 \to Y$ of $F|_{U_1}$ such that $\Gamma_{\Psi_1} \subset O(\Gamma_\Psi, \varepsilon)$, Ψ_1 admits a compact approximately connected filtration of infinite length, and $\operatorname{cal}\Psi_1 < \varepsilon$. Since X is locally compact and A is compact, there exists a compact neighborhood OA of A such that $OA \subset U_1$. By Theorem 2.12 the mapping $\Psi_1|_{OA}$ admits continuous approximations. Take $\varepsilon_1 < \varepsilon$ such that the neighborhood $\mathcal{U}_1 = O(\Gamma_{\Psi_1}(OA), \varepsilon_1)$ lies in $O(\Gamma_\Psi, \varepsilon)$.

Now by induction with the use of Lemma 3.1, we construct a sequence of neighborhoods $U_1 \supset U_2 \supset U_3 \supset \cdots$ of the compactum OA, a sequence of compact submappings $\{\Psi_k : U_k \to Y\}_{k=1}^\infty$ of the mapping F, and a sequence of neighborhoods $\mathcal{U}_k = O(\Gamma_{\Psi_k}(OA), \varepsilon_k)$ such that for every $k \geqslant 2$ we have $\operatorname{cal}\Psi_k < \varepsilon_{k-1}/2 < \varepsilon/2^k$, and $\mathcal{U}_k$ is contained in $\mathcal{U}_{k-1}$. It is not difficult to choose the neighborhood $\mathcal{U}_k$ of the graph Γ_{Ψ_k} in such a way that for every point $x \in U_k$ the set $\mathcal{U}_k(x)$ has diameter less than $3/2^k$.

Then for any $m \geqslant k \geqslant 1$ and for any point $x \in OA$ we have $\Psi_m(x) \subset O(\Psi_k(x), 3/2^k)$; this implies the fact that the sequence of compacta $\{\Psi_k(x)\}_{k=1}^\infty$ is Cauchy (in Hausdorff metric). Since $\Psi_k(x) \subset F(x)$ for all k and every $x \in OA$, there exists a limit $s(x) \in F(x)$ of this sequence (recall that $F(x)$ is compact). The mapping $s : OA \to Y$ is single-valued by the condition $\operatorname{cal}\Psi_k < 1/2^k$ and is upper semicontinuous (and, therefore, is continuous) by the upper semicontinuity of all the mappings Ψ_k. Thus s is a selection of the mapping F. $\square$

Theorem 3.3. *Let $p : E \to B$ be a Serre fibration of LC^2-compacta with all fibers homeomorphic to some fixed compact three-dimensional manifold. If $B \in ANR$, then any section of p over closed subset $A \subset B$ can be extended to a section of p over some neighborhood of A.*

Proof. Note that the Serre fibration p is topologically regular [6].

Let $s : A \to E$ be a section of p over A. Embed E into Hilbert space l_2 and consider a compact-valued mapping $F : B \to l_2$ defined as follows:

$$F(b) = \begin{cases} s(b), & \text{if } b \in A, \\ p^{-1}(b), & \text{if } x \in B \setminus A. \end{cases}$$

We may consider F as a submapping of equi-LC^0 compact map $p^{-1} : B \to l_2$. It follows from Lemma 2.16 that the mapping F is equi locally hereditarily coconnectedly aspheric, equi locally coconnected, and equi-LC^1. We can apply Theorem 3.2 to the mapping F and its submapping s to find a single-valued continuous selection $\tilde{s} : OA \to l_2$ of $F|_{OA}$. Since the restriction $F|_A$ is single-valued and equal to s, we have $\tilde{s}|_A = s$. Clearly, $\tilde{s}$ defines a section of the fibration p over OA extending s. $\square$

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