Theoretical Computer Science 112 (1993) 399–412
Elsevier

Note

Irrational speeds of configurations growth in generalized Pascal triangles

Ivan Korec*

Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

Communicated by A. Salomaa
Received May 1991
Revised February 1992

Abstract

Korec, I., Irrational speeds of configurations growth in generalized Pascal triangles, Theoretical Computer Science 112 (1993) 399–412.

A method is presented for obtaining irrational speeds of growth of configurations in generalized Pascal triangles and one-dimensional cellular automata. This method does not use diagonalization or simulation of Turing machines but uses a number-theoretic idea.

1. Introduction

Generalized Pascal triangles (they will be defined below) can be considered as computations of one-dimensional cellular automata from finite initial configurations. (As usual, a configuration is called finite if its support, i.e., the set of all cells which are not in the quiescent state, is finite.) They most immediately correspond to the cases of two-element neighborhood, but they can be used to arbitrary fixed neighborhoods as well.

Correspondence to: I. Korec, Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia.

* Supported by Grant 364/1991 of Slovak Academy of Sciences. Submitted during author's stay at University of Turku supported by the Academy of Finland.

0304-3975/93/$06.00

Cellular automata are often used to simulate some physical processes; of course, two- and three-dimensional ones are more suitable for this purpose but in some cases also one-dimensional cellular automata are useful. In the present paper only one-dimensional case is considered. Imagine, for example, waves spreading on a water surface (in a narrow and very long reservoir for one-dimensional case). If we throw several stones into water the waves will spread at the water surface. The size and the form of the waves will depend on the number of thrown stones, their weights etc., but the speed of wave head will be usually independent on these circumstances. Hence, it is natural to ask that a cellular automaton used to simulate this waves has the following property:

In every computation with finite nonempty initial configuration, the length of the support is (asymptotically) linear function of time, and the speed of its growth does not depend on the initial configuration.

Of course, this property can be easily obtained if the speed is the maximal one allowed by the considered type of neighborhood (sometimes called "speed of light"). Also some lower rational speeds can be easily obtained. On the other hand, by the simulation of a suitable Turing machine also irrational speeds can be obtained; however, in this case the speed usually depends on the initial configuration (and not necessarily exists for all initial configurations).

The aim of the present paper is to present another method to obtain irrational speeds. It is based on the following observation. Let the configurations of a cellular automaton A be considered as b-adic numbers and one step of its computation corresponds to the multiplication by q. Then (under some assumptions) the lengths of configurations in a computation of A grows with the speed $\log_b q$. We shall use this idea to construct generalized Pascal triangles (and, hence, implicitly also one-dimensional cellular automata) with this speed of configuration growth also for some noninteger b and q. For this purpose we shall generalize the notion of b-adic number system in a suitable way and we shall investigate a special set K of reals.

2. Generalized Pascal triangles

The set of nonnegative integers will be denoted by $\mathbf{N}$; the operators DIV, MOD will denote the quotient and the remainder by the integer division (like in the programming language Pascal). Brackets [ ] will denote the integer part of a real. The set of all (rational) integers will be denoted by $\mathbf{Z}$. Further we denote

$$D=\{(x, y)\in\mathbf{Z}\times\mathbf{Z};\ x+y\geqslant 0\}.$$

If A is an alphabet (i.e., a finite nonempty set) then A^+ will denote the set of all nonempty words in the alphabet A. The length of a word w will be denoted $|w|$; it must be distinguished from the absolute value of a real by the context. The ith symbol of w will be denoted by $w(i)$ [the starting symbol is $w(0)$].

By an algebra we shall always understand an algebra $\mathscr{A}=(A;*,\mathrm{o})$ of signature (2, 0) and satisfying the identity $\mathrm{o}*\mathrm{o}=\mathrm{o}$. We shall usually consider finite algebras; the exceptions will be explicitly mentioned.

Definition 2.1. To every algebra $\mathscr{A}=(A;*,\mathrm{o})$ and every $w\in A^+$ we associate the function $G=\mathrm{GPT}(\mathscr{A},w)$ with the domain D by the formulae

$$G(x,y)=\begin{cases} w\left(x+\left[\frac{|w|}{2}\right]\right) & \text{if } x+y=0 \text{ and } 0\leqslant x+\left[\frac{|w|}{2}\right]<|w| \\ \mathrm{o} & \text{if } x+y=0 \text{ and not } \left(0\leqslant x+\left[\frac{|w|}{2}\right]<|w|\right) \\ G(x-1,y)*G(x,y-1) & \text{if } x+y>0. \end{cases}$$

The functions of the form $\mathrm{GPT}(\mathscr{A},w)$ for a finite algebra $\mathscr{A}$ and a word $w\in A^+$ will be called generalized Pascal triangles (*abbreviation* GPT).

Example 2.2. Let us imagine the classical Pascal triangle written in the usual way in a plane. Let its rows be completed by infinitely many zeros at both sides. Then it is natural to consider it as the function P with the domain D and satisfying

$$P(x,y)=\binom{x+y}{x} \quad \text{if } (x,y)\in\mathbf{N}\times\mathbf{N}, \qquad P(x,y)=0 \quad \text{otherwise}.$$

We shall always assume the corresponding coordinate system in the plane (the axis x is directed right-down and the axis y is directed left-down). The function P can be expressed as $\mathrm{GPT}(\mathscr{N},1)$, where $\mathscr{N}=(\mathbf{N};+,0)$ and $+$ is the usual addition on $\mathbf{N}$. This analogy explains the term "generalized Pascal triangle". However, P is not a GPT because the algebra $\mathscr{N}$ is not finite. Nice (and very often studied) examples of GPT are Pascal triangles modulo n, particularly if n is a prime or a prime power. They are obtained if the values of P are reduced modulo a positive integer n. We can express them in the form $\mathrm{GPT}(\mathscr{N}_n,1)$, where $\mathscr{N}_n=(\{0,1,\ldots,n-1\};+,0)$ and $+$ denotes the addition modulo n.

Definition 2.3. Let $\mathscr{A}=(A;*,\mathrm{o})$ be an algebra, $w\in A^+$ and $t\in\mathbf{N}$.

(a) The tth row of $G=\mathrm{GPT}(\mathscr{A},w)$ will be the function $h:\mathbf{Z}\to A$ defined by $h(x)=G(x,t-x)$ for all $x\in\mathbf{Z}$.

(b) The substantial part $\mathrm{SP}(\mathscr{A},w,t)$ of the tth row of $G=\mathrm{GPT}(\mathscr{A},w)$ will be the empty word if $G(x,t-x)=\mathrm{o}$ for all $x\in\mathbf{Z}$; otherwise, $\mathrm{SP}(\mathscr{A},w,t)$ will be the word consisting of

$$G(u,t-u),\ G(u+1,t-u-1),\ldots,G(v-1,t-v+1),\ G(v,t-v),$$

where u is the least and v is the greatest integer such that $G(u,t-u)\neq\mathrm{o}$ and $G(v,t-v)\neq\mathrm{o}$.

In other words, $\mathrm{SP}(\mathscr{A}, w, t)$ is the least (connected) subword of the tth row of $\mathrm{GPT}(\mathscr{A}, w)$ which contains all its symbols distinct from o.

Remark 2.4. Originally (for example, in [2, 3]) generalized Pascal triangles were associated to the algebras of signature (0, 1, 1, 2) (but the approach introduced above was also mentioned). Further, the domains of GPT were some proper subsets of D (for example $\mathbf{N} \times \mathbf{N}$). A GPT in the old sense can be transformed into a GPT in the new sense if we extend it to the domain D by the new value o. The algebra corresponding now to the old algebra $\mathscr{B}=(B; K, \mathrm{l}, \mathrm{r}, .)$ can be $\mathscr{A}=(A; *, \mathrm{o})$, where $A=B \cup \{\mathrm{o}\}$, $\mathrm{o} \notin B$ and

$$x * y = x . y, \qquad x * \mathrm{o} = \mathrm{r}(x), \qquad \mathrm{o} * y = \mathrm{l}(y), \qquad \mathrm{o} * \mathrm{o} = \mathrm{o}$$

for all $x, y \in B$. The constant K [which was used to define $\mathrm{GPT}(\mathscr{B}) = \mathrm{GPT}(\mathscr{B}, K)$] has no analogy there; therefore, $\mathrm{GPT}(\mathscr{A})$ is not defined now. Note also that sometimes it is more suitable to choose $\mathrm{o} \in B$; e.g., if we consider the Pascal triangle modulo a positive integer n, the choice $\mathrm{o}=0$ is suitable. Another difference is that now the coordinates of the initial symbol of w are not fixed as (0, 0) and, hence, the set of all GPT is closed under horizontal shifts. (This property had no meaning by the original definition of GPT.) All mentioned differences (which were marginally considered also already in [2]) are technically advantageous but not very substantial.

GPT correspond very naturally to the computations of (one-dimensional) one-sided cellular automata. However, they can be used also for one-dimensional cellular automata with arbitrary neighborhoods. For example, if the neighborhood $\{-1, 0, 1\}$ is considered, then an element of the GPT will code the states of two neighboring cells. (Two partitions into such pairs are possible; one will be used in the odd moments and the other in the even moments of the discrete time.)

3. Speeds of configuration growths

Definition 3.1. Let $\mathscr{A}=(A; *, \mathrm{o})$ be a finite algebra and α be a real.

(i) We shall say that α is the speed of configuration growth of $\mathscr{A}$, and write $\mathrm{SpGr}(\mathscr{A})=\alpha$, if there is a constant c such that for every $w \in A^+ - \{\mathrm{o}\}^+$ and every $t \in \mathbf{N}$

$$\big|\, |\mathrm{SP}(\mathscr{A}, w; t)| - (|\mathrm{SP}(\mathscr{A}, w; 0)| + \alpha t) \,\big| \leqslant c \tag{3.1}$$

(ii) We shall write $\mathrm{SpGr}_1(\mathscr{A})=\alpha$ if for every $w \in A^+ - \{\mathrm{o}\}^+$ there is a constant c such that (3.1) holds.

If $\mathrm{SpGr}(\mathscr{A})$ is defined then $\mathrm{SpGr}_1(\mathscr{A})$ is also defined and it holds that $\mathrm{SpGr}_1(\mathscr{A}) = \mathrm{SpGr}(\mathscr{A})$. The converse is not true: it may happen that $\mathrm{SpGr}_1(\mathscr{A})$ is defined and $\mathrm{SpGr}(\mathscr{A})$ does not exist. Analogously, it may happen that

$$\lim_{t \to \infty} \frac{|\mathrm{SP}(\mathscr{A}, w; t)|}{t}$$

exists for every $w \in A^+ - \{o\}^+$ and does not depend on w, and, despite this, $\mathrm{SpGr}_1(\mathscr{A})$ does not exist. Hence, Definition 3.1 (i) is rather strong. It asks that the configuration growth is "as linear as possible" and simultaneously "as uniform as possible". This makes the existence results about SpGr also rather strong, e.g. stronger than that about SpGr_1.

Since always $|\mathrm{SP}(\mathscr{A}, w; t)| \geqslant 0$ and $|\mathrm{SP}(\mathscr{A}, w; t+1)| \leqslant |\mathrm{SP}(\mathscr{A}, w; t)| + 1$ we have the following lemma.

Lemma 3.2. *For every algebra $\mathscr{A}$, if $\mathrm{SpGr}_1(\mathscr{A})$ is defined then*

$$0 \leqslant \mathrm{SpGr}_1(\mathscr{A}) \leqslant 1.$$

Of course, the same holds also for SpGr. For every rational α, $0 \leqslant \alpha \leqslant 1$, a finite algebra $\mathscr{A}$ such that $\mathrm{SpGr}(\mathscr{A}) = \alpha$ can be easily constructed. We do not give this construction here. Instead, we give an example of irrational $\mathrm{SpGr}_1(\mathscr{A})$. Its idea will be used also in the proof of the main theorem of the present paper.

Example 3.3. Let $\mathscr{A} = (A; *, o)$, where $\mathrm{A} = \{0, 1, 2, \ldots, 9\}$, $o = 0$ and for all $x, y \in A$

$$x * y = (2x) \text{ MOD } 10 + (2y) \text{ DIV } 10.$$

(Hence, e.g., $6 * 7 = 12$ MOD $10 + 14$ DIV $10 = 2 + 1 = 3$.)

Consider $\mathrm{GPT}(\mathscr{A}, w)$ for some $w \in A^+ - \{o\}^+$; for $w = 57$ it is displayed in Fig. 1. If w is the decadic representation of the integer m then tth row of $\mathrm{GPT}(\mathscr{A}, w)$ contains the decadic representation of $m \cdot 2^t$; this representation consists of the substantial part of the tth row and (may be) several additional zeros from the right. Since the length of the decadic representation of an integer is approximately its decadic logarithm we could prove $\mathrm{SpGr}_1(\mathscr{A}) = \log_{10} 2$. However, $\mathrm{SpGr}(\mathscr{A})$ is not defined because, e.g., if w represents 5^n then $|\mathrm{SP}(\mathscr{A}, w; t)|$ for $t \leqslant n$ decreases to 1, and only then increases with the speed $\log_{10} 2$.

4. The set K

This section has an auxiliary character. A special set K of reals (more precisely, of nonnegative algebraic integers, as we shall see later) will be introduced and studied

```
0 0 0 5 7 0 0 0 0
 0 0 1 1 4 0 0 0 0
0 0 0 2 2 8 0 0 0
 0 0 0 4 5 6 0 0 0
0 0 0 0 9 1 2 0 0
 0 0 0 1 8 2 4 0 0
```

Fig. 1.

there. It is important to know that K is sufficiently rich because the parameters in our main theorem will be chosen from K. The other properties of K are not so important now.

Definition 4.1. (a) A finite sequence

$$(\beta_1, \beta_2, \ldots, \beta_n) \tag{4.1}$$

of positive reals will be called a K-system if there are nonnegative integers a_{ijk}, $i, j, k \in \{1, 2, \ldots, n\}$, such that for all $i, k \in \{1, 2, \ldots, n\}$ it holds that

$$\beta_i \beta_k = \sum_{j=1}^{n} a_{ijk} \beta_j. \tag{4.2}$$

(b) A K-system (4.1) will be called a K-system for a real α if there are nonnegative integers $b_1, b_2, \ldots, b_n$ such that

$$\alpha = b_1 \beta_1 + b_2 \beta_2 + \cdots + b_n \beta_n. \tag{4.3}$$

(c) K is the set of all reals for which a K-system exists.

(d) The product of two finite sequence of reals

$$(\alpha_1, \alpha_2, \ldots, \alpha_m), \quad (\beta_1, \beta_2, \ldots, \beta_n)$$

will be the sequence

$$(\alpha_1 \beta_1, \ldots, \alpha_m \beta_1, \alpha_1 \beta_2, \ldots, \alpha_m \beta_2, \ldots, \alpha_1 \beta_n, \ldots, \alpha_m \beta_n).$$

Instead of "linear combination with the coefficient in **N**" we shall say shortly "additive combination". Instead of "linear combination with the coefficient in **Z**" we shall say shortly "integer linear combination".

Lemma 4.2. (a) *If $(\beta_1, \beta_2, \ldots, \beta_n)$ is a K-system then the sequence $(1, \beta_1, \beta_2, \ldots, \beta_n)$ is also a K-system.*

(b) *If $(\beta_1, \beta_2, \ldots, \beta_n)$ is a K-system for $\alpha > 0$ then the sequence $(\beta_1, \beta_2, \ldots, \beta_n, \alpha)$ is a K-system.*

(c) *The product of two K-systems is also a K-system.*

The proof is straightforward and will be omitted.

Lemma 4.3. *For every real α the following conditions are equivalent:*

(i) $\alpha \in K$.

(ii) *There are positive reals $\gamma_1, \gamma_2, \ldots, \gamma_k$ and nonnegative integers d_{ij}, $i, j \in \{1, 2, \ldots, k\}$ such that α is an additive combination of $\gamma_1, \gamma_2, \ldots, \gamma_k$ and*

$$\begin{pmatrix} \gamma_1^2 \\ \gamma_2^2 \\ \vdots \\ \gamma_k^2 \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1k} \\ d_{21} & d_{22} & \cdots & d_{2k} \\ \vdots & \vdots & & \vdots \\ d_{k1} & d_{k2} & \cdots & d_{kk} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_k \end{pmatrix}. \tag{4.4}$$

(iii) *There are positive reals $\delta_1, \delta_2, \ldots, \delta_s$, positive integers $n_0, n_1, n_2, \ldots, n_s$ and polynomials with nonnegative integer coefficients*

$$g(x_1, x_2, \ldots, x_s), f_1(x_1, x_2, \ldots, x_s), \ldots, f_s(x_1, x_2, \ldots, x_s) \tag{4.5}$$

of degrees less than $n_0, n_1, \ldots, n_s$, respectively, such that

$$\alpha = g(\delta_1, \delta_2, \ldots, \delta_s) \tag{4.6}$$

and for all $i \in \{1, 2, \ldots, s\}$

$$\delta_i^{n_i} = f_i(\delta_1, \delta_2, \ldots, \delta_s). \tag{4.7}$$

Proof. The implications (i) $\to$ (ii) and (ii) $\to$ (iii) are almost obvious; hence, it suffices to prove (iii) $\to$ (i). Let us assume that (iii) holds. Then let $n = n_1 n_2 \ldots n_s$ and let $(\beta_1, \beta_2, \ldots, \beta_n)$ be the product of the finite sequences

$$(1, \delta_i, \ldots, \delta_i^{n_i - 1}), \quad i \in \{1, 2, \ldots, s\}.$$

Then $(\beta_1, \beta_2, \ldots, \beta_n)$ is a K-system for α. To show that, compute formally the value $g(\delta_1, \delta_2, \ldots, \delta_s)$ and the values $\beta_i \beta_j$ for all $i, j \in \{1, 2, \ldots, n\}$. If a product $\delta_1^{j_1} \delta_2^{j_2} \ldots \delta_s^{j_s}$ with $j_i \geqslant n_i$ for some $i \in \{1, \ldots, s\}$ occurs there then by (4.7) it can be replaced by a sum of similar product of less degree. At the end we obtain an additive combination of the reals $\beta_1, \ldots, \beta_n$.

Theorem 4.4. *Let $n_1, n_2, \ldots, n_s$ be positive integers and*

$$f_1(x_1, x_2, \ldots, x_s), f_2(x_1, x_2, \ldots, x_s), \ldots, f_s(x_1, x_2, \ldots, x_s)$$

be nonzero polynomials with nonnegative real coefficients and of degrees less than $n_1, n_2, \ldots, n_s$, respectively. Then there is exactly one s-tuple $(\delta_1, \delta_2, \ldots, \delta_s)$ of positive reals such that (4.7) *holds.*

Proof. Consider the metric space of all s-tuples of positive reals with the metric d defined by

$$d((\alpha_1, \ldots, \alpha_s), (\beta_1, \ldots, \beta_s)) = \max\left\{ \left| \ln \frac{\alpha_i}{\beta_i} \right| : i \in \{1, \ldots, s\} \right\}.$$

This metric space is complete. Further, consider the operator F which maps every s-tuple $(\alpha_1, \ldots, \alpha_s)$ of positive reals to the s-tuple $(\beta_1, \ldots, \beta_s)$ defined by

$$\beta_i^{n_i} = f_i(\alpha_1, \ldots, \alpha_s) \quad \text{for all } i \in \{1, \ldots, s\}.$$

It can be shown that the operator F is contractive: for all s-tuples X, Y of positive reals

$$d(F(X), F(Y)) \leqslant \frac{n-1}{n} d(X, Y),$$

where $n = \max(n_1, \ldots, n_s)$. Therefore, by the Banach fixed point theorem F has the unique fixed point, and it is the solution of (4.7). $\square$

Theorem 4.4 shows that the equations of the form (4.6), (4.7) [and analogously their special cases (4.3) with β_i replaced by γ_i, (4.5)] can be used to determine uniquely concrete elements of K. The equations (4.2) and (4.3) are not so suitable for this purpose because they are solvable only if the integers a_{ijk} fulfil some rather complicated conditions. However, they will be most suitable in a proof below.

Theorem 4.5. (a) *The set K contains all nonnegative integers.*

(b) *The set K is closed under addition, multiplication and kth roots* ($k = 2, 3, 4, \ldots$).

(c) *If $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in K$ and γ is a positive real such that*

$$\gamma^n = \alpha_{n-1}\gamma^{n-1} + \cdots + \alpha_1\gamma + \alpha_0 \tag{4.8}$$

then $\gamma \in K$.

Proof. (a) is almost obvious; the trivial K-system (1) suffices for the proof. For the sum and product in (b), consider a common K-system X for both α and β. [To construct it, add 1 to K-systems for α, β, and then form their product. See Lemma 4.2(a) and (c).] Then X is also a K-system for $\alpha + \beta$ and $\alpha\beta$ and, therefore, these reals belong to K. The statement for the roots will follow from (c).

To prove (c), consider the product Y of the finite sequences

$$(\beta_1, \beta_2, \ldots, \beta_k), \; (1, \gamma, \gamma^2, \ldots, \gamma^{n-1}),$$

where the first one is a common K-system for all $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ [it can be constructed from separate K-systems for α_i by Lemma 4.2(a) and (c)]. Hence, Y consists of all $\beta_i\gamma^j$, $1 \leqslant i \leqslant k$, $0 \leqslant j < n$. We may assume $\beta_1 = 1$, and then γ belongs to Y. It remains to show that Y is a K-system. To do that, consider the (formal) product of arbitrary two elements of Y. The nth and the higher powers of γ can be eliminated by (4.8), all α_j can be replaced by additive combination of β_i and, finally, (4.2) can be used to replace the products of (several) β_i by the additive combinations of β_i. $\square$

Theorem 4.5 shows that the set K is sufficiently rich. Theorem 4.7 will give a necessary condition for the elements of K. To prove it, the following lemma will be useful.

Lemma 4.6. *For every positive integer k, every sequence of k-dimensional vectors with integer components contains a member which can be expressed as an integer linear combination of the previous members.*

The proof can be done by induction with respect to k and will be omitted. (Note that the mentioned vectors form a noetherian module over the ring of integers; see, e.g., [5].)

Theorem 4.7. (a) *The set K contains no negative real and no real x, $0<x<1$.*

(b) *All elements of K are algebraic integers.*

Proof. For (a) consider arbitrary K-system and its least element. We can easily see that this element is $\geqslant 1$. Hence, all elements of K-systems are $\geqslant 1$, and it suffices to use (4.3).

For (b), take $\alpha \in K-\{0\}$ and a K-system for α. Let us express the powers of α as integer linear combinations of the K-system and consider the sequence of vectors of coefficients. Now we can apply Lemma 4.6 to this sequence. $\square$

Remark 4.8. For many elements $\alpha \in K$ there is a K-system (for α) of the form

$$(1, \alpha, \alpha^2, \ldots, \alpha^{m-1}). \tag{4.9}$$

This obviously happens if in the irreducible polynomial

$$f(x)=x^n-a_{n-1}x^{n-1}-\cdots-a_1x-a_0 \in \mathbf{Z}[x] \tag{4.10}$$

with the root α all α_i, $0 \leqslant i<n$, are nonnegative. [To write $f(x)$ is very often the most natural way how to determine α.] In this case the choice $m=n$ is possible. The above situation takes place, e.g., for all $\alpha \in \mathbf{N}$ and all $a+\sqrt{b}$, $a, b \in \mathbf{N}$, $b>a^2$.

However, in some cases $m>n$ must be chosen. For example, the positive real α satisfying $\alpha^3=6\alpha^2-2\alpha+6$ belongs to K because

$$\begin{aligned}\alpha^4 &= 6\alpha^3-2\alpha^2+6\alpha=5\alpha^3+\alpha^3-2\alpha^2+6\alpha \\ &= 5\alpha^3+(6\alpha^2-2\alpha+6)-2\alpha^2+6\alpha \\ &= 5\alpha^3+4\alpha^2+4\alpha+6.\end{aligned}$$

Hence, (4.9) is a K-system (for α) if $m=4$, but it is not if $m=3$.

The real $\alpha=2+\sqrt{2}$ also belongs to K, because, e.g., $(1, \sqrt{2})$ is a K-system for it, but in this case for no $m \in \mathbf{N}$ (4.9) is a K-basis. Otherwise, $f(2+\sqrt{2})=0$ for a polynomial (4.10) with nonnegative a_i, $0 \leqslant i<n$. Then also $f(2-\sqrt{2})=0$ and, hence, $f(x)$ has (at least) two positive roots, which contradicts Theorem 4.4.

There are also $\beta \in K$ such that for no $\alpha \in K$ and no $m \in N$ (4.9) is a K-basis for β. We show that $\beta=\sqrt{2+\sqrt{2}}$ has this property. Assume, conversely, that for some α, m (4.9) is a K-system, for β. Then since $1 \leqslant \beta<2$ we have $\beta=\alpha^k$ and, hence, $\alpha=\sqrt[2k]{2+\sqrt{2}}$ for some k. Since (4.9) is a K-system α is a root of a polynomial (4.10) with $n=m$ and all $a_i \in \mathbf{N}$. Then $\sqrt[2k]{2-\sqrt{2}}$ is also a root of this polynomial, which contradicts Theorem 4.4.

Finally, consider $\gamma=\sqrt{5}+\sqrt{[(1+\sqrt{5})/2]}$. It holds $\gamma\in K$ and, analogously as above, (4.9) with no m is a K-system for γ. A K-system for γ is

$$(1, \delta, \delta^2, \delta^3, \sqrt{5}, \sqrt{5}\delta), \tag{4.11}$$

where $\delta=\sqrt{[(1+\sqrt{5})/2]}$. (Note that $\sqrt{5}\delta^2$, $\sqrt{5}\delta^3$ need not be included because $\sqrt{5}\delta^2=2+\delta^2$.) This K-system is a minimal one for γ, even if its members are linearly dependent (because $1-2\delta^2+\sqrt{5}=0$). To show the minimality, consider the (approximate) numerical values of (4.11):

$$(1.000, 1.272, 1.618, 2.058, 2.236, 2.844)$$

and denote by M (the set of components of) a minimal K-system for γ which is contained in (4.11). Since there is only one way how to represent $\gamma=\sqrt{5}+\delta\doteq 3.508$ as an additive combination of (4.11), we have δ, $\sqrt{5}\in M$. Then also δ^2, δ^3, $\sqrt{5}\delta\in M$ because they must be additive combinations of M, and they have no nontrivial representation as additive combinations of (4.11). Finally, $1\in M$ because it is necessary to represent $\delta^4\doteq 2.618$.

5. Irrational speeds of configuration growths

We start with a generalization of positional (i.e., usual) number system. The generalization will be made for the unique purpose, to help the proof of Theorem 5.3. The representation of numbers will not be unique in general. Neither computational aspects nor further possible generalizations are considered here.

Definition 5.1. (i) A positional number system is an ordered triple

$$(\mathbf{A}, \mathrm{val}, b),$$

where A is an alphabet (of digits), val is a mapping of A into the set of nonnegative reals and the real $b>1$ is the basis.

(ii) The function val is extended to the set A^+ by the formula

$$\mathrm{val}(u_n u_{n-1}\dots u_1 u_0)=\sum_{i=0}^{n} \mathrm{val}(u_i)b^i$$

for every $n\in\mathbf{N}$ and all $u_n, u_{n-1}, \dots, u_1, u_0\in A$.

Lemma 5.2. *For every positional number system* (A, val, b) *there is a constant* c *such that for all* $u\in A^+$ *with* $\mathrm{val}(u(0))\neq 0$ *it holds*

$$\big|\,|u|-\log_b \mathrm{val}(u)\,\big|\leqslant c. \tag{5.1}$$

Proof. Denote by m, M the minimal positive and the maximal member of the set $\{\mathrm{val}(x): x\in A\}$; the values M, m exist because A is finite and contains a positive member $u(0)$. Since

$$m \leqslant \mathrm{val}(u(0)) \leqslant M \quad \text{and} \quad 0 \leqslant \mathrm{val}(u(i)) \leqslant M \quad \text{for all } i,\ 1 \leqslant i < |u|,$$

we have

$$mb^{|u|-1} \leqslant \mathrm{val}(u) < \frac{M \,.\, b^{|u|}}{b-1}$$

and, hence,

$$\frac{m}{b} \leqslant \frac{\mathrm{val}(u)}{b^{|u|}} < \frac{M}{b-1}.$$

The bounds on both sides are positive and independent on u. Hence, the b-adic logarithm of the central member, i.e. $\log_b \mathrm{val}(u) - |u|$ is bounded by their b-adic logarithms. This allows us to find a c suitable for (5.1).

Theorem 5.3. *If α, $\beta \in K$, $b \in \mathbf{N}$ and $0 < \alpha\beta < b$ then there is a finite algebra $\mathscr{A} = (A; *, \mathrm{o})$ such that*

$$\mathrm{SpGr}(\mathscr{A}) = \frac{\log \alpha}{\log b - \log \beta}. \tag{5.2}$$

Proof. Before the technical details, we explain the idea: Substantial parts of rows of $\mathrm{GPT}(\mathscr{A}, w)$ will be understood as numbers written in a positional number system with the base $\delta = b/\beta$ and forming of a new row will correspond to the multiplication by α. To avoid the difficulty mentioned in Example 3.3 we arrange that the number of finishing zeros will not increase.

Let $(\gamma_1, \gamma_2, \ldots, \gamma_n)$, $\gamma_1 = 1$, be a K-system for both α, β and let B be the set of all additive combinations of $\gamma_1, \gamma_2, \ldots, \gamma_n$. Let us define two unary operations div_b, mod_b on the set B as follows.

If $x = 0$ then let $\mathrm{div}_b(x) = \mathrm{mod}_b(x) = 0$.

For $x > 0$ let $\mathrm{div}_b(x)$ be the greatest element of $\{u \in B\colon x - bu \in B - \{0\}\}$ (this set is finite and nonempty) and let $\mathrm{mod}_b(x)$ be such that

$$x = b\,\mathrm{div}_b(x) + \mathrm{mod}_b(x).$$

The latest formula obviously holds also for $x = 0$. Further, $x \neq 0$ implies $\mathrm{mod}_b(x) \neq 0$. We can also see that the values $\mathrm{mod}_b(x)$, $x \in B$ do not exceed

$$M = \max\{(b-1)(\gamma_1 + \gamma_2 + \cdots + \gamma_n),\ b \max\{\gamma_1, \gamma_2, \ldots, \gamma_n\}\}.$$

Now we define for all x, $y \in B$

$$x * y = \mathrm{mod}_b(\alpha x) + \beta\,\mathrm{div}_b(\alpha y)$$

and choose a suitable real m so that the set $A = \{x \in B\colon x \leqslant m\}$ contains at least two members and is closed under $*$.

We shall show that the choice $m=bM/(b-\alpha\beta)$ suffices. Since $m>1$ we have $0, 1\in A$; it remains to show $x*y\in A$ for arbitrary $x, y\in A$. Since

$$\mathrm{div}_b(\alpha y)\leqslant\frac{\alpha y}{b}\leqslant\frac{\alpha m}{b},$$

we have

$$x*y\leqslant M+\beta\,\frac{\alpha m}{b}=\frac{(b-\alpha\beta)m}{b}+\frac{\alpha\beta m}{b}=m$$

and, hence, $x*y\in A$.

Now we put $\mathrm{o}=0$ and $\mathscr{A}=(A;*,\mathrm{o})$. It remains to show that (5.2) holds. To do that, consider the positional number system $(A, \mathrm{val}, b/\beta)$, where $\mathrm{val}(x)=x$ for every $x\in A$. For every $w\in A^+-\{\mathrm{o}\}^+$ and $t\in\mathbf{N}$

$$\mathrm{val}(\mathrm{SP}(\mathscr{A}, w; t+1))=\alpha\ \mathrm{val}(\mathrm{SP}(\mathscr{A}, w; t)). \tag{5.3}$$

We shall prove it formally later; now we only note that any element u which occurs in SP $(\mathscr{A}, w; t)$ at the rth position (from the right) contributes by

$$u_1=\beta\,\mathrm{div}_b(\alpha u), \qquad u_0=\mathrm{mod}_b(\alpha u)$$

into the $(r+1)$th and the rth position of $\mathrm{SP}(\mathscr{A}, w; t+1)$, respectively. Their common contribution to $\mathrm{val}(\mathrm{SP}(\mathscr{A}, w; t+1))$ is

$$u_1\delta^{r+1}+u_0\delta^r=\alpha u\delta^r,$$

i.e., α-times greater than the contribution of u into $\mathrm{val}(\mathrm{SP}(\mathscr{A}, w; t))$.

The equality (5.3) immediately implies that

$$\mathrm{val}(\mathrm{SP}(\mathscr{A}, w; t))=\alpha^t\ \mathrm{val}(\mathrm{SP}(\mathscr{A}, w; 0))$$

and, hence,

$$\log_\delta \mathrm{val}(\mathrm{SP}(\mathscr{A}, w; t))=t\ \log_\delta\alpha+\log_\delta\ \mathrm{val}(\mathrm{SP}(\mathscr{A}, w; 0)),$$

where $\delta=b/\beta$. Now let c be the constant from Lemma 5.2 for our positional system. Applying Lemma 5.2 for $u=\mathrm{SP}(\mathscr{A}, w; t)$ and for $u=\mathrm{SP}(\mathscr{A}, w; 0)$ we can obtain

$$\big|\,|\mathrm{SP}(\mathscr{A}, w; t)|-(t\ \log_\delta\alpha+|\mathrm{SP}(\mathscr{A}, w; 0)|)\,\big|\leqslant 2c.$$

Hence,

$$\mathrm{SpGr}(\mathscr{A})=\log_\delta\alpha=\frac{\log\alpha}{\log\delta}=\frac{\log\alpha}{\log b-\log\beta},$$

what we wanted to prove.

It remains to prove (5.3). Let $w\in A^+-\{0\}^+$, $G=\mathrm{GPT}(\mathscr{A}, w)$ and $t\in\mathbf{N}$. Let s be the minimal integer such that $G(t-s, s)\neq 0$, i.e., $G(t-s, s)$ is the rightmost symbol of

$\mathrm{SP}(\mathscr{A}, w; t)$. Then we have

$$\mathrm{val}(\mathrm{SP}(\mathscr{A}, w; t)) = \sum_{y=s}^{\infty} G(t-y, y)\,\delta^{y-s}.$$

(The sum is only formally infinite; almost all summands are 0.) Since $x * y \neq 0$ for $x \neq 0$ we have

$$G(t+1-s, s) = G(t-s, s) * G(t+1-s, s-1) \neq 0.$$

Further, obviously, $G(t+1-y, y) = 0$ for all $y < s$ and, therefore,

$$\begin{aligned}
\mathrm{val}(\mathrm{SP}(\mathscr{A}, w; t+1)) &= \sum_{y=s}^{\infty} G(t+1-y, y)\,\delta^{y-s} \\
&= \sum_{y=s}^{\infty} (G(t-y, y) * G(t+1-y, y-1))\delta^{y-s} \\
&= \sum_{y=s}^{\infty} (\mathrm{mod}_b(\alpha G(t-y, y)) + \beta\,\mathrm{div}_b(\alpha G(t+1-y, y-1)))\delta^{y-s} \\
&= \sum_{y=s}^{\infty} \mathrm{mod}_b(\alpha G(t-y, y))\,\delta^{y-s} \\
&\quad + \sum_{y=s}^{\infty} \beta\,\mathrm{div}_b(\alpha G(t+1-y, y-1))\,\delta^{y-s} \\
&= \sum_{y=s}^{\infty} \mathrm{mod}_b(\alpha G(t-y, y))\delta^{y-s} \\
&\quad + \sum_{y=s-1}^{\infty} \beta\,\mathrm{div}_b(\alpha G(t-y, y))\delta^{y+1-s} \\
&= \sum_{y=s}^{\infty} \mathrm{mod}_b(\alpha G(t-y, y)\delta^{y-s} \\
&\quad + \sum_{y=s}^{\infty} \beta\,\mathrm{div}_b(\alpha G(t-y, y))\delta^{y+1-s} \\
&= \sum_{y=s}^{\infty} (\mathrm{mod}_b(\alpha G(t-y, y)) + \beta\delta\,\mathrm{div}_b(\alpha G(t-y, y)))\delta^{y-s} \\
&= \sum_{y=s}^{\infty} \alpha G(t-y, y)\delta^{y-s} = \alpha\,\mathrm{val}(\mathrm{SP}(\mathscr{A}, w; t)),
\end{aligned}$$

and the proof of (5.3) is finished. □

Theorem 5.3 can be applied with many rather complicated parameters α, β; we do not present such examples. One of the simplest possibilities to obtain an irrational speed of growth by Theorem 5.3 is the following.

Example 5.4. Let $\alpha=2$, $\beta=1$ and $b=3$. In this case the trivial K-system (1) and $m=6$ are sufficient (even if the formula from the proof gives a higher value); then $A=\{0, 1, 2, 3, 4, 5, 6\}$ and $\mathrm{SpGr}(\mathscr{A})=\ln 2/\ln 3=\log_3 2$.

Example 5.5. The same speed of growth as above can be also obtained by an algebra of cardinality 6. Let $A=\{o, 0, 1, 2, 3, 4\}$, $b=3$, $\mathrm{val}(o)=0$ and $\mathrm{val}(i)=i$ for the other $i\in A$; hence, (A, val, b) is a positional systems "with two zeros". Let $\mathscr{A}=(A; *, o)$, where $o*o=o*1=o$, $o*0=1$ (the exception!) and

$$x*y=(2\,\mathrm{val}(x))\ \mathrm{MOD}\ 3+(2\,\mathrm{val}(y))\ \mathrm{DIV}\ 3$$

in all other cases.

The trailing zeros are not a problem because they are represented by 0's, and not by the quiescent state o. The problem with leading zeros is solved by the exceptional rule $o*0=1$ which causes them not to exist after the initial step. If the exceptional formula $o*0=1$ is not used in the computation of $\mathrm{SP}(\mathscr{A}, w; t+1)$ then (5.3) (with $\alpha=2$) holds. This is true for all $t>0$ because then $\mathrm{SP}(\mathscr{A}, w; t)$ does not contain any subword o0. The change of $|\mathrm{SP}(\mathscr{A}, w; t)|$ in the initial step can be estimated directly (i.e., without using val), and then $\mathrm{SpGr}(\mathscr{A})=\log_3 2$ can be proved.

References

[1] K. Culik II, L.P. Hurd and S. Yu, Computation theoretic aspects of cellular automata, *Physica D* **45** (1990) 357–378.
[2] I. Korec, Generalized Pascal triangles (in Slovak), Dr.Sc. Thesis, UK Bratislava, 1984
[3] I. Korec, Generalized Pascal triangles. Decidability results, *Acta Math. Univ. Comenian.* **XLVI–XLVII** (1985) 93–130.
[4] I. Korec, Generalized Pascal triangles, in: K. Halkowska and S. Stawski, eds, *Proc. 5th Universal Algebra Symp.* Turawa, Poland, 1988 (World Scientific, Singapore, 1989) 198–218.
[5] S. Lang, *Algebra* (Addison-Wesley, Reading, MA, 1965).