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INTERPOLATION OF TRACK DATA WITH RADIAL BASIS METHODS

R. E. CARLSON
Lawrence Livermore National Laboratory
P.O. Box 808, Livermore, CA 94550, U.S.A.
T. A. FOLEY
Computer Science Department, Arizona State University
Tempe, AZ 85287-5406, U.S.A.

Abstract—The multiquadric and thin plate spline radial basis methods, together with the triangle-based minimum norm network algorithm and the modified quadratic Shepards' method, are applied to various sets of data that are sampled densely along *tracks* in the plane. The effectiveness of these methods on track data has been questioned in the past. We observe that both radial basis methods and the minimum norm network method performed well on smoothly varying track data sets, while the multiquadric method with a small value for the parameter R^2 was the only method that was effective on rapidly varying track data.

1. INTRODUCTION

Suppose we are given a set $S = \{(x_i, y_i) : i = 1, 2, \ldots, n\}$ of n distinct points in the plane. Also, suppose we are given the value of a function $f(x, y)$ at the points in S, i.e., $V = \{f(x_i, y_i) : i = 1, 2, \ldots, n\}$. The points in S are said to be *scattered data points* since there is no assumption that they form (or lie on) a rectangular grid. A procedure for constructing a function $F(x, y)$ satisfying

$$F(x_i, y_i) = f(x_i, y_i), \qquad \text{for all } (x_i, y_i) \in S, \tag{1}$$

is called a *scattered data interpolation algorithm.*

Scattered data interpolation algorithms usually are quite different from algorithms used to solve gridded data interpolation problems. Gridded data algorithms typically use tensor product methods, while scattered data methods tend to be more ad hoc, more diverse, and generally more complicated.

For most scattered data problems, it is usually assumed that there is a randomness about the points in S. However, in this paper, we are concerned with data sets S in which the points lie along *tracks* or paths in the plane, hence the name **track data**. No assumption is made that the tracks are along straight lines. One of the distinguishing features of track data is that two points which are adjacent to each other along a given track are usually orders of magnitude closer together than points on different tracks. Track data may arise because methods for sampling data have been severly restrictive. For example, in the Big Sur data set described in Section 3, data was generated by taking water temperature measurements from a boat. Also, track data may be obtained by digitizing contour plots. This problem is considered in Section 4.

Four scattered data interpolation methods that performed well in the critical comparison of Franke [1,2] are the multiquadric (MQ) method of Hardy [3,4], thin plate spline (TPS) of Duchon [5], minimum norm network (MNN) of Nielson [6], and the modified quadratic Shepards' (QS) method of Franke and Nielson [7]. These methods are discussed in the next section

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX and subsequently applied to several track data samples. With respect to accuracy and visual smoothness, MQ and TPS were the most effective of the many methods tested in Franke [1,2], while MNN was the most effective of the triangle-based methods. TPS, MNN, and QS performed poorly on the Big Sur data, and MQ did well when a small value for the user-defined parameter is used. We constructed several other track data test cases in order to determine if the problems are caused by the fact that the 2-D points in S fall on tracks, or because the function values, V, vary rapidly.

2. INTERPOLATION METHODS

An important class of scattered data interpolation methods is known as *radial basis function* methods. These methods are of the form

$$F(x,y) = \sum_{i=1}^{n} \lambda_i \Phi(\|(x,y) - (x_i, y_i)\|). \tag{2}$$

For our purposes here, we shall assume $\|\cdot\|$ means the Euclidean norm.

Two of the most effective radial basis function methods are multiquadric and thin plate splines. The MQ interpolant is given by (2) with

$$\Phi(\|(x,y) - (x_i,y_i)\|) = \sqrt{\|(x,y) - (x_i,y_i)\|^2 + R^2}. \tag{3}$$

Coefficients $\{\lambda_i\}$ are computed by solving the symmetric system of linear equations obtained from (1). An important unsolved problem is how to compute the optimal value for R^2. The authors' earlier results [8] demonstrate that the optimal R^2 depends primarily on the set V, and it is less dependent on the set S, other than the scale of the domain. Prior to this result, most formulas for R^2 depended only on S. It is shown in [8] that the MQ interpolant is invariant under a uniform scaling of the x and y coordinates if R is scaled in the same way. That is, if $F(x,y)$ is the MQ interpolant to (x_i, y_i, f_i) using R^2 and $G(x,y)$ is the MQ interpolant to (cx_i, cy_i, f_i) using the parameter $(cR)^2$, then $F(x,y) = G(cx, cy)$. The MQ method is easily seen to be invariant under translations and rotations of the data in the xy plane using the same value of R^2. The MQ interpolant is also a linear operator with respect to the function values V, assuming that S and R^2 are fixed. Additional information on multiquadric interpolation can be found in [4, 9–12].

The TPS interpolant is defined by

$$F(x,y) = \sum_{i=1}^{n} \lambda_i \Phi(\|(x,y) - (x_i,y_i)\|) + \sum_{i=1}^{3} \mu_i \, p_i(x,y), \tag{4}$$

where

$$\Phi(\|(x,y) - (x_i,y_i)\|) = \|(x,y) - (x_i,y_i)\|^2 * \ln(\|(x,y) - (x_i,y_i)\|) \tag{5}$$

and $\{p_i(x,y)\}$ are the monomials 1, x, and y. Coefficients $\{\lambda_i\}$ and $\{\mu_i\}$ are computed by solving the system of linear equations of order $n+3$ obtained from (1) and

$$\sum_{i=1}^{n} \lambda_i p_k(x_i, y_i) = 0, \qquad k = 1,2,3. \tag{6}$$

REMARK. Using the polynomial terms $\{p_i(x,y)\}$ gives polynomial precision. These terms could also be added to the MQ interpolant. As noted in [8], our experience with MQ is that polynomial precision is generally not an improvement, and may degrade the accuracy of the interpolant on some data sets. It is generally advisable to add the constant term for stability, and to add the linear or quadratic terms if the data can be closely approximated by a low degree polynomial.

A different class of scattered data interpolation methods involves constructing a network of triangles in the plane using the points in S as vertices. Such a triangulation is not unique, and one of the most commonly used triangulation algorithms maximizes the minimum angle of the triangles. This algorithm is called the Delunay triangulation [13,14].

After constructing the Delunay triangulation, the MNN method computes first order partial derivatives at the vertices (x_i, y_i), which define cubic polynomials over each edge of the triangulation. The first order partial derivatives are computed by solving a sparse linear system of equations so that the network of piecewise cubic polynomials minimizes the following expression which is integrated over all edges in the triangulation:

$$\int \left[\frac{d^2}{ds^2} F(x(s), y(s))\right]^2 ds. \tag{7}$$

Because of this minimization, the piecewise cubic network can be considered to be a bivariate analog of the univariate cubic spline. In the triangle T_{ijk}, the MNN interpolant has the form

$$F(x,y) = W_i(x,y)Q_i(x,y) + W_j(x,y)Q_j(x,y) + W_k(x,y)Q_k(x,y), \tag{8}$$

where $W_i(x,y)$ is a rational weight function, and $Q_i(x,y)$ is a minimum norm interpolant to the piecewise cubic network on T_{ijk} and cross boundary derivatives on the edge opposite the vertex (x_i, y_i). The MNN method is a C^1 interpolant defined on the convex hull of S, and it is perhaps the most effective of the triangle-based methods tested in [1,2].

Because triangle-based methods are defined only for the convex hull of S, it is necessary to extrapolate outside the convex hull in order to evaluate the interpolant in the unit square. This may be accomplished by a C^0 extension (see [6] for details).

The fourth algorithm used in our comparison is the modified quadratic Shepards' method described in Franke and Nielson [7] and Renka [15]. The QS method has the form

$$F(x,y) = \frac{\sum_{i=1}^{n} W_i(x,y)L_i(x,y)}{\sum_{i=1}^{n} W_i(x,y)}, \tag{9}$$

where $L_i(x,y)$ is a weighted least squares quadratic fit to the neighboring points of (x_i, y_i) and

$$W_i(x,y) = \left[\frac{(T-d_i)_+}{Td_i}\right]^2,$$

where $d_i = \|(x,y) - (x_i, y_i)\|$ and T is a positive constant dependent on the number of data points and the diameter of the point set S. We used the same weights as those used in Franke [1] for our comparisons. We also used the parameters suggested in Renka [15] and found there was no improvement.

3. TRACK DATA AND THE BIG SUR PROBLEM

There is a limited amount of information available on scattered data interpolation methods applied to solving track data problems. However, the effectiveness of the MQ method on track data has been questioned by Hardy [4], Foley [16], and Franke [private communication]. Their concern apparently stems from results on a data set which has become known as the *Big Sur* data set [16]. It consists of 64 data points lying along five distinct tracks as shown in Figure 1 scaled to the unit square.

The Big Sur data consists of water temperature measurements taken in a boat traveling nearly perpendicular to the shore off the coast of Big Sur, California. A plot of the surface constructed from the MQ interpolant using $R^2 = 0.001$ on the Big Sur data is shown in Figure 2.

Note the rather large excursions of the surface between the tracks. In light of the excellent success MQ has enjoyed on more randomly scattered data, it was generally assumed that the above excursions were produced because MQ was ineffective for solving track data problems. The authors examined the Big Sur data set more closely and noticed excursions along tracks 2 and 3 in Figure 1. These excursions are clearly indicated in Figure 3 in which the data long two of the tracks was treated as though the track was a straight line. Therefore, it is not surprising that the surfaces exhibited some excursions because of the steep gradients in the data along these tracks.

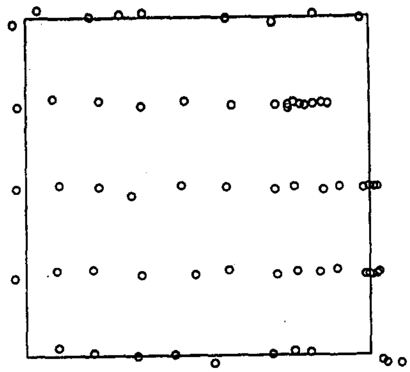

Figure 1. Big Sur data locations scaled to the unit square.

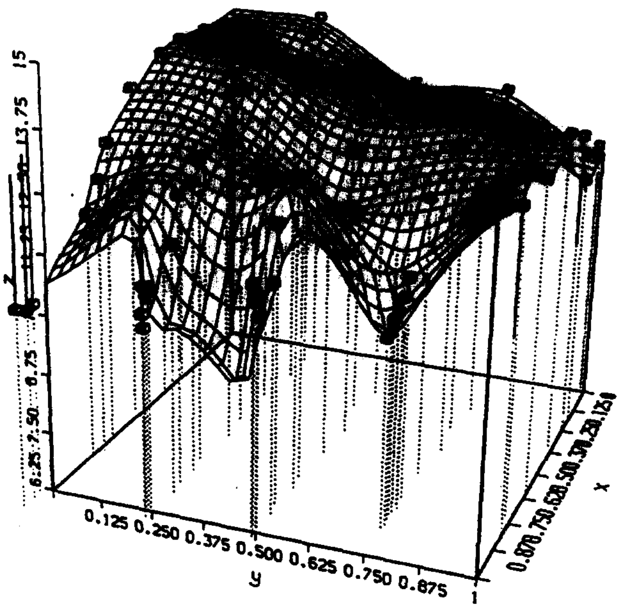

Figure 2. MQ interpolant of the Big Sur data, $R^2 = 0.001$.

The authors were able to obtain a MQ interpolant free from excursions between the tracks by choosing R^2 to be much smaller than the value used for the plots in Figure 2. (It is well known that decreasing the value of R^2 "tightens" the surface somewhat because the basis functions approach $\|(x, y) - (x_i, y_i)\|$.) The new surface is shown in Figure 4 using $R^2 = 10^{-5}$.

The other three methods were not effective on the Big Sur data. The TPS method was reasonable for the most part. However, undesirable oscillations occurred in a manner similar to the MQ method using $R^2 = 0.001$ shown in Figure 2. The MNN and QS method exhibited severe oscillations in several locations.

REMARK. In [16], Foley produced a visually pleasing interpolant to the Big Sur data using a multi-stage method. Because this method seems to work best when the tracks are roughly parallel, multi-stage methods are not discussed in this paper.

4. GENERATING A SET OF TEST PROBLEMS

To study the performance of scattered data algorithms on track data, we mapped the 64 data points from the Big Sur data set into the unit square. To obtain a fair comparison the RMS errors for the MNN method, we modified two points in Figure 1 so that the unit square was contained in the convex hull of S. The point in the lower left of Figure 1 was changed to $(0,0)$ and the point in the upper right was changed to $(1,1)$. Next, we generated function values at these points by evaluating each of Franke's six test functions [1,2]. Interpolants were constructed

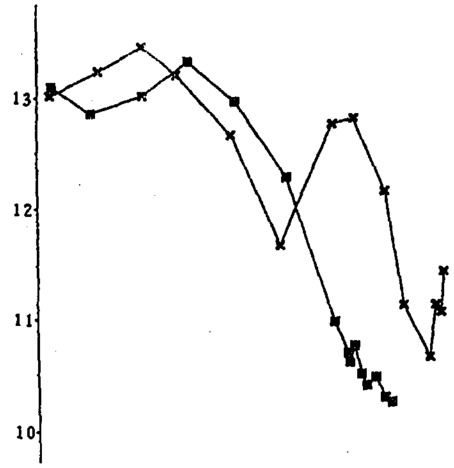

Figure 3. Piecewise linear interpolant of Big Sur data along tracks 2 and 3.

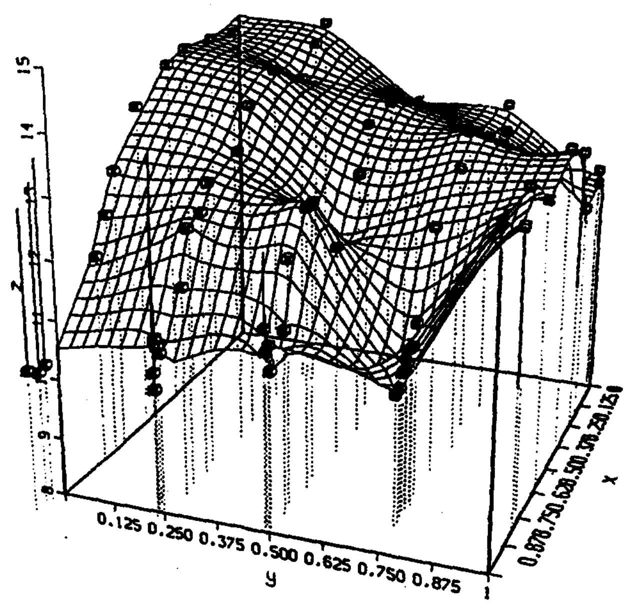

Figure 4. MQ interpolant of the Big Sur data, $R^2 = 10^{-5}$.

using the MQ, TPS, MNN, and QS methods. Numerous MQ interpolants were constructed in order to determine the effect of the parameter R^2 on the accuracy of the interpolant.

To determine the impact of track data on these four methods, we computed the RMS error for each interpolant. This error was computed by evaluating the interpolant and the underlying function on a uniform 33×33 grid defined on the unit square. Since the accuracy of the MQ method depends on R^2, plots of the RMS error as a function of R^2 for each of the six test functions and all data sets are shown in Figure 5. The RMS errors from the more randomly scattered data sets supplied by Franke (consisting of 25, 33, and 100 points) are shown for comparison. The small circles in Figure 5 denote the RMS error for the R^2 computed by the algorithm in [8]. All problems were solved using double precision on a Cray X-MP.

Tables 1-6 below compare MQ, TPS, MNN, and QS, where the MQ errors are computed using the formula proposed by the authors in [8] which depends on the set V, and for the optimum R^2 as determined from Figure 5.

A second type of track data was generated by discretizing contour plots of the first test function. Many points were computed on the contour curves of the exact test function on the right side of Figure 6, and then two out of every three points were discarded at random. (Our experience has been that users tend to be overzealous in digitizing contour plots, resulting in far more points than are needed to get an accurate reproduction of the contours.) This procedure yielded a set

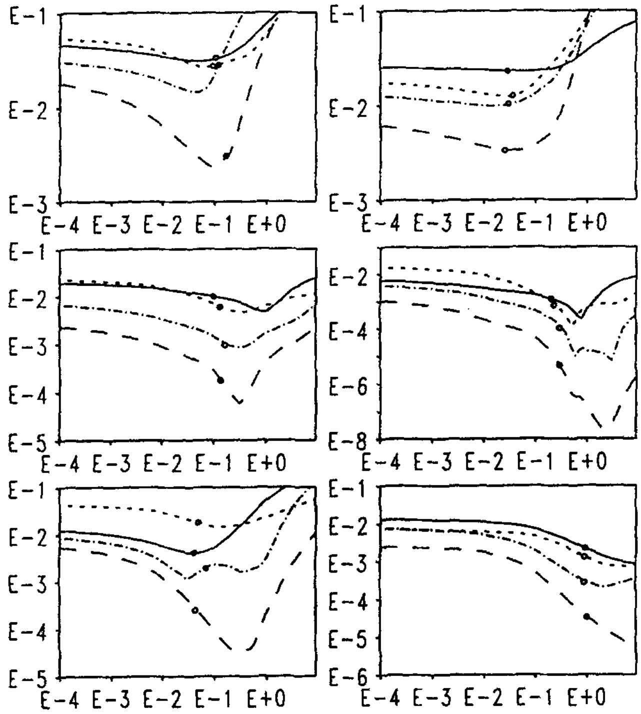

Figure 5. RMS Error as a Function of R^2.

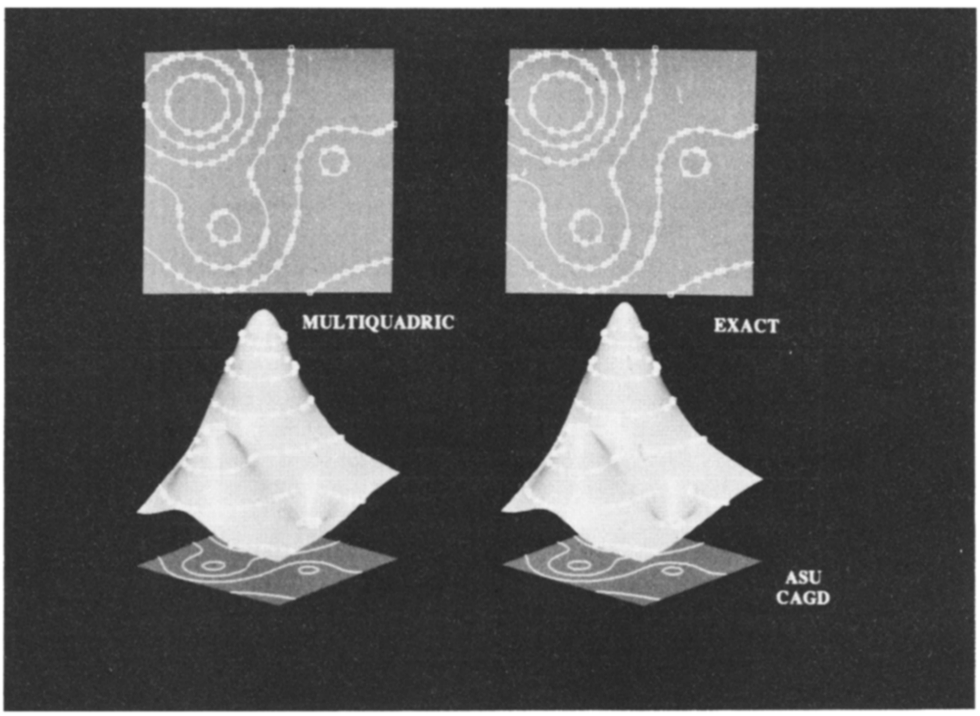

Figure 6. The MQ interpolant to digitized data from contour plots.

of 122 data points as shown in Figure 6. The images on the left side of Figure 6 are of the MQ interpolant using $R^2 = 0.01$. The RMS errors for the MQ method is 0.0073. The TPS method also yielded a visually pleasing surface whose RMS error is 0.0102.

Table 1. RMS errors for Function 1.

	25 pts	33 pts	64 pts	100 pts
MQ (R^2 from [2])	0.0330	0.0270	0.0315	0.0031
MQ (opt R^2)	0.0305	0.0264	0.0154	0.0026
TPS	0.0348	0.0421	0.0226	0.0095
MMN	0.0328	0.0437	0.0237	0.0094
QS	0.0486	0.0478	0.0298	0.0128

Table 2. RMS error for Function 2.

	25 pts	33 pts	64 pts	100 pts
MQ (R^2 from [2])	0.0230	0.0125	0.0090	0.0033
MQ (opt R^2)	0.0228	0.0125	0.0090	0.0033
TPS	0.0235	0.0134	0.0092	0.0044
MMN	0.0242	0.0140	0.0090	0.0043
QS	0.0314	0.0206	0.0187	0.0055

Table 3. RMS errors for Function 3.

	25 pts	33 pts	64 pts	100 pts
MQ (R^2 from [2])	0.0098	0.0059	0.0010	0.00011
MQ (opt R^2)	0.0046	0.0046	0.0010	0.00005
TPS	0.0137	0.0140	0.0022	0.00092
MMN	0.0172	0.0159	0.0040	0.00200
QS	0.0183	0.0139	0.0074	0.00194

Table 4. RMS errors for Function 4.

	25 pts	33 pts	64 pts	100 pts
MQ (R^2 from [2])	0.0012	0.0007	0.00007	0.000004
MQ (opt R^2)	0.0002	0.0002	0.00005	0.00000002
TPS	0.0035	0.0071	0.00173	0.00030
MMN	0.0043	0.0056	0.00295	0.00069
QS	0.0067	0.0068	0.00458	0.00089

Table 5. RMS errors for Function 5.

	25 pts	33 pts	64 pts	100 pts
MQ (R^2 from [2])	0.0041	0.0181	0.0018	0.00021
MQ (opt R^2)	0.0040	0.0146	0.0013	0.00003
TPS	0.0065	0.0296	0.0022	0.00217
MMN	0.0069	0.0228	0.0030	0.00229
QS	0.0126	0.0220	0.0074	0.00361

Table 6. RMS errors for Function 6.

	25 pts	33 pts	64 pts	100 pts
MQ (R^2 from [2])	0.0021	0.0012	0.0003	0.00002
MQ (opt R^2)	0.0007	0.0007	0.0003	0.000006
TPS	0.0093	0.0055	0.0044	0.00150
MMN	0.0077	0.0046	0.0041	0.00165
QS	0.0034	0.0136	0.0034	0.00050

5. CONCLUDING REMARKS

The MQ and TPS radial basis methods can yield effective interpolants to track data. If the function values $F(x_i, y_i)$ come from a smooth function, such as Franke's test functions, then both of these methods generated visually smooth surfaces with relatively small RMS errors. The MQ interpolant generally produced smaller RMS errors than all of the methods. The MNN method also worked well on the smooth track data sets in Section 4. This was a little surprising because the triangulation consisted of several long, thin triangles connecting adjacent tracks. The probable reason for the effectiveness of the MNN interpolant is that the first order partial derivatives were computed to solve an appropriate global minimization problem. The QS method was not very effective on most of the smooth track data sets in certain regions using default parameters. The RMS errors and the visual smoothness of the MQ, TPS, and MNN interpolants on the smooth track data sets were consistent with their behavior on the more uniform data sets of 25, 33, and 100 points in Franke [1].

For the rapidly varying Big Sur track data in Section 3, the only method that yielded effective results was the MQ interpolant using a small value for R^2. The TPS interpolant worked well in most regions, but it had undesirable oscillations in some areas. The MNN and QS methods had major problems with the Big Sur data set. An obvious observation is that track data is not a problem for the MQ, TPS, and MNN methods. The undesirable behavior of these interpolants on the Big Sur Data set is caused by the rapidly varying function values. With a small value for R^2, the MQ interpolant can overcome this problem. The generalized MNN method in Nielson and Franke [17] which uses exponential tension splines for the network curves may be able to effectively handle the Big Sur data if appropriate tension values are used.

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