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The Reconstruction of Maximal Planar Graphs II. Reconstruction

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In the first paper [3], the author, together with Fiorini, has shown that maximal planar graphs are recognizable from their decks of vertex-deleted subgraphs. The aim of this paper is to show that such graphs are reconstructible.

PRELIMINARIES

We assume that the reader is familiar with the work [1] and with the notation and results of the first paper [3]. Further terminology used in this paper will be defined as it appears.

In the first paper the recognition of maximal planar graphs was established. We now prove further that this class of graphs is indeed reconstructible. We first recall that the following result was proved by Fiorini and Manvel [4].

THEOREM. *Every maximal planar graph whose minimum valency is at least 4 is reconstructible.*

In view of this result, there remains to show that maximal planar graphs of minimum valency 3 are also reconstructible. The method used in the above proof depends on a theorem of Chartrand *et al.* [2], which implies that there is a vertex v_0 of G (when the minimum valency of G is at least 4, and G is 3-connected) such that G_{v_0} is 3-connected, and this in turn implies, by a theorem of Whitney [6], that G_{v_0} is uniquely embeddable in the plane. Unfortunately this method fails when G has minimum valency 3. We therefore have to introduce additional concepts.

We first define an *ordinary vertex* to be a vertex whose valency is at least 4. Now, given a maximal planar graph G of minimum valency 3, we can recognize the maximal planarity of G from the deck $\mathscr{D}(G)$ [3]. Therefore for any ordinary vertex v, we need only consider the $\rho(v)$-representations of G_v, any reconstruction of G being obtained from some G_v by adding a vertex

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and joining it to the vertices on the $\rho(v)$-face of a $\rho(v)$-representation of G_v. For this reason, Section 1 of this paper deals with k-representable graphs. Moreover, if for some ordinary vertex w of G, G_w has a unique $\rho(w)$-representation, then G is uniquely reconstructible from G_w. We can therefore assume that for any ordinary vertex w of G, G_w has at least two non-equivalent $\rho(w)$-representations. Such maximal planar graphs are called *degenerate*. These graphs are studied in Section 2.

The proofs of some of the theorems in this paper are long, with many subcases, and therefore only short sketches of such proofs are given. Full details are available from the author upon request.

1. k-Representable Graphs

Terms used in this section which not are defined can be found in the first two chapters of Ore's book [5], henceforth referred to as Ore. In particular we follow Ore's definition of plane equivalence:

Two plane representations R and R' of a planar graph G are said to be *plane equivalent*, or simple referred to as equivalent, if there exists an isomorphism ϕ on G, such that C is a boundary circuit of a face in R if and only if $\phi(C)$ is a boundary circuit of a face in R'.

First we have two lemmas on planar graphs, whose proofs, which are not difficult, are omitted.

Lemma 1.1. *Let R be a plane representation of G, and let C be a circuit bounding a face in R, and $(c_1, c_2,..., c_r)$ a cyclic labelling of C. Let R' be another plane representation of G, such that the vertices of C form a cricuit C' which bounds an r-face in R'. Then the vertices of C appear on C' in the same cyclic order as in R.*

Lemma 1.2. *Let G be a k-representable graph, and R a k-representation of G such that C is the k-circuit bounding the k-face in R. Let R' be another k-representation of G such that C also bounds the k-face in R'. Then R is equivalent to R'.*

Thus we see that if R is a k-representation of G, and C the k-circuit of R, then any other k-representation of G, not equivalent to R, must have a k-circuit different from C bounding the k-face.

Now, let G be a k-representable graph and let R be a k-representation of G. Let C be the circuit bounding the k-face of R, C being labelled in the cyclic order $(c_1, c_2,..., c_k)$. Let $c_i c_{i+1} c_{i+2}$ be a separating triangle for G. Let $T = c_i c_{i+1} c_{i+2}$, and $G - T = (\text{Int } T) \cup (\text{Ext } T)$, where $(\text{Int } T)$ is defined as that component of $G - T$ which has some vertex adjacent to c_{i+1} in G. Then $(\overline{\text{Int } T})$ is a maximal planar graph. Let $c_i y c_{i+2}$ be the face, different from $c_i c_{i+1} c_{i+2}$, which is incident to the edge $c_i c_{i+2}$ in $(\overline{\text{Int } T})$.

We now observe that (Int T) is a bridge for the circuit C. Moreover, if this bridge is transferred to the inside of C (or to the outside of C if the k-face of R is the unbounded face), we obtain another k-representation of G, with the vertex y on the k-face instead of c_{i+1}. We call such a bridge, with three attachments c_i, c_{i+1}, c_{i+2} forming a separating triangle, an *arch*; we call the maximal planar graph $(\overline{\text{Int } T})$, with the three vertices c_i, c_{i+1}, c_{i+2} so labelled, a *span*. Since T is a separating triangle, then the order of $(\overline{\text{Int } T})$ is greater than 3. Therefore each of c_i, c_{i+1}, c_{i+2} has valency at least 3 in $(\overline{\text{Int } T})$. We call these three vertices the *primary vertices* of the span $(\overline{\text{Int } T})$, and c_i, c_{i+2} are called the *pivots* of the span. The vertices c_{i+1} and y are called the *replaced vertex* and the *replacement vertex*, respectively. We emphasize that a span is a maximal planar graph with a labelled face. In general, we adopt the notation $S(abc)$ *or* $S(cba)$ to denote a span with a and c as pivots and b as replaced vertex. We have seen above that if R is a k-representation of a graph G, an arch transfer also gives another k-representation. Now, Ore has shown [5, Theorem 2.5.4] that if R and R' are two plane representations of a connected graph G with no separating vertices, then R can be transformed into a representation equivalent to R' by a sequence of bridge transfers. We show that if R and R' are two k-representations of G, then R can be transformed into a representation equivalent to R' by a sequence of arch transfers.

LEMMA 1.3. *Let G be a k-representable graph, and let R be a k-representation of G. Let C be the circuit bounding the k-face in R and labelled cyclicly as $(c_1, c_2, ..., c_k)$. Let R' be another k-representation of G such that the vertex c_{i+1} is not on the circuit bounding the k-face in R'. Then we have*

(i) $\rho(c_{i+1}) > 2$, *and*

(ii) c_i *is adjacent to* c_{i+2}.

Proof. If $\rho(c_{i+1}) = 2$, then adding a vertex w inside the k-face of R' and joining it to each vertex on the k-face of R', gives a maximal planar graph of order greater than 3, and with a vertex of valency 2, a contradiction. Therefore $\rho(c_{i+1}) > 2$.

Now, since in R' c_{i+1} is not on the k-face, then it must be incident only to triangles. Therefore the subgraph of G induced by $N(c_{i+1})$ must be Hamiltonian.

Now, in R, all the faces which are incident to c_{i+1}, except one, are triangles (Fig. 1.1).

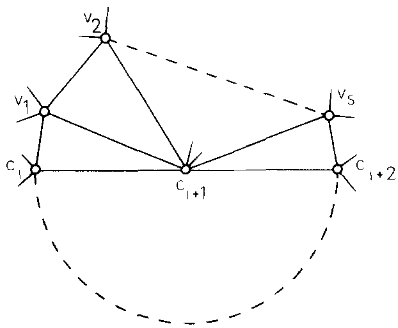

FIGURE 1.1

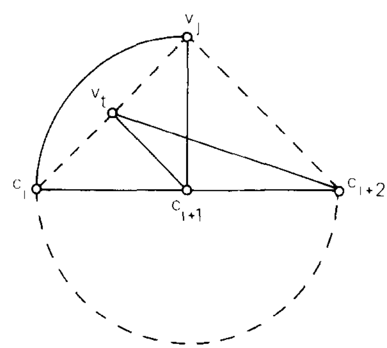

FIGURE 1.2

We can therefore label the neighbours of c_{i+1} in order, proceeding from c_i to c_{i+2} as they appear in R. Therefore $N(c_{i+1}) = (c_i, v_1, v_2, \ldots, v_s, c_{i+2})$.

If $\rho(c_{i+1}) = 3$, then the fact that the subgraph induced by $N(c_{i+1})$ is Hamiltonian implies that c_{i+2} is adjacent to c_i. We may therefore assume that $\rho(c_{i+2}) > 3$.

Now, assume that c_i is not adjacent to c_{i+2}. Since in R' c_{i+1} is incident only to triangles, c_i must be incident to some other vertex from $N(c_{i+1})$, apart from v_1. Let $j = \text{maximum}\{r: v_r \text{ adjacent to } c_i\}$. Thus c_{i+2} cannot be adjacent to v_t for $t < j$, because otherwise G would contain the graph of Fig. 1.2 which is homeomorphic to K_5. Similarly, if $p > j$, v_p cannot be adjacent to any v_t for $t < j$. Therefore v_j is a separating vertex for the subgraph induced by $N(c_{i+1})$, which therefore cannot be Hamiltonian. This contradiction proves that c_i is adjacent to c_{i+2}. ∎

Two corollaries follow from Lemma 1.3.

COROLLARY 1.1. *Let G, R, R' and C be as in Lemma* 1.3, *with c_{i+1} on the k-face in R but not in R'. Then $c_i c_{i+1} c_{i+2}$ is a separating triangle for G.*

Proof. This follows from the fact that c_i is adjacent to c_{i+2} and $\rho(c_{i+1}) > 2$. ∎

COROLLARY 1.2. *Let G, R, R′ and C be as in Lemma* 1.3. *with* c_{i+1} *on the k-face in R but not in R′. Then in any k-representation of G,* c_i *and* c_{i+2} *are on the k-face.*

Proof. From Lemma 1.3(ii), c_i is adjacent to c_{i+2}. Assume that c_i is not on the k-face for some k-representation of G. Then c_{i-1} is adjacent to c_{i+1}. Therefore the edge $c_{i-1}c_{i+1}$ is a transversal for C which is separated by the transversal c_ic_{i+2}. Therefore C can never be a face for G [5, Theorems 2.5.1 and 2.5.3]. But this is a contradiction. Therefore c_i must always be on the k-face. Similarly for c_{i+2}. ∎

We thus see that if in a k-representation R, there exists a vertex c_{i+1} on the k-face and which can be replaced by another vertex in some other k-representation, then $T = c_ic_{i+1}c_{i+2}$ is a separating triangle, and therefore if we define (Int T) as before (Int T) is an arch for C, which when transferred to the inside of C (or to the outside of C, if the k-face of R is the unbounded face) gives another k-representation with c_{i+1} replaced by the replacement vertex, y. The next lemma effectively tells us that y is the only vertex which can replace c_{i+1} on the k-face.

LEMMA 1.4. *Let R be a k-representation of a graph G, with the circuit C bounding the k-face in R, and labelled as usual in a cyclic order, and let* $T = c_ic_{i+1}c_{i+2}$ *be a separating triangle for G. Let y be the replacement vertex of the span with V(T) as primary vertices. Then in any k-representation of G, either y or* c_{i+1} *must appear on the k-face.*

Proof. If we define $(\overline{\text{Int } T})$ to be the span with $V(T)$ as primary vertices, then $(\overline{\text{Int } T})$ is a maximal planar graph in which y is incident to the *face* $c_ic_{i+1}c_{i+2}$ (Fig. 1.3). However, (Ext T) has attachments with c_i and c_{i+2}; therefore there exists no representation of G in which both $c_ic_{i+1}c_{i+2}$ and

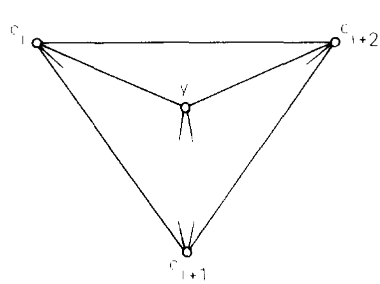

FIGURE 1.3

$c_i y c_{i+2}$ are faces. Therefore there exists no representation in which simultaneously y is incident to the *face* $c_i y c_{i+2}$ and c_{i+1} is incident to the *face* $c_i c_{i+1} c_{i+2}$.

Now, assume that the lemma is false. Therefore there exists a k-representation R' such that both y and c_{i+1} are incident only to triangles. Now, by the above, in R', either y is not incident to the *face* $c_i y c_{i+2}$ or c_{i+1} is not incident to the *face* $c_i c_{i+1} c_{i+2}$. Without loss of generality we can assume that $c_i c_{i+1} c_{i+2}$ is not a face in R'.

Now, we require that H, the subgraph of G induced by $N(c_{i+1})$, be Hamiltonian, since c_{i+1} is bounded only by triangles. But since none of these triangles is $c_i c_{i+1} c_{i+2}$, then $H - c_i c_{i+2}$ must also be Hamiltonian. But this leads to a contradiction as in the proof of Lemma 1.3. Therefore Lemma 1.4 is proved. ▮

We can now prove the main result of this section.

THEOREM 1.1. *Let R and R' be two k-representations of a graph G. Then R' can be changed into a representation equivalent to R by a sequence of arch transfers.*

Proof. Let the circuit C bounding the k-face in R be labelled in a cyclic order as $(c_1, c_2, \ldots, c_k)$. If C also bounds the k-face in R', then by Lemma 1.2, R is equivalent to R'. Therefore we can assume that C does not bound the k-face in R'.

Let c_{i+1} be a vertex on C which does not appear on the k-face in R'. Therefore by Lemma 1.4, the replacement vertex of the span with c_i, c_{i+1}, c_{i+2} as primary vertices and c_{i+1} as replaced vertex must be on the k-face in R'. But then by the transfer of the arch with c_i, c_{i+1}, c_{i+2} as attachments with C, we obtain a k-representation with c_{i+1} on the k-face and $c_i y c_{i+2}$ as a face. We repeat this process for every vertex c_j not on the k-face in R', and after a sequence of arch transfers we obtain a k-representation R'' with the vertices of C on the k-face. But by Lemma 1.1, R'' is equivalent to R. ▮

2. DEGENERATE GRAPHS

In this section, unless otherwise stated, G will be a maximal planar graph having at least one vertex of valency 3. Let v be an ordinary vertex of G. Then G_v is $\rho(v)$-representable. R_v will denote that $\rho(v)$-representation of G_v having the neighbours of v on the $\rho(v)$-face. By Lemma 1.1, the neighbours of v appear on the $\rho(v)$-face of R_v in a unique cyclic order. Thus we can label the set of neighbours of v in a cyclic order as $(v_1, v_2, \ldots, v_{\rho(v)})$, and this labelling is unique up to the choice of initial vertex and orientation. In the sequel, given any ordinary vertex v of G, by a labelling of the set of neighbours of v we mean such a cyclic labelling.

Again, for an ordinary vertex v, consider G_v and R_v. Let S be a span in R_v, having primary vertices v_i, v_{i+1}, v_{i+2}. We then say that the *span* S *is incident to* v. We now distinguish between certain types of spans.

Let S be a maximal planar graph having a face labelled v_i, v_{i+1}, v_{i+2}. Let G be a maximal planar graph having an ordinary vertex v incident to a span $S(v_i v_{i+1} v_{i+2})$ isomorphic to S, and let y be the replacement vertex of this span. R' will denote that $\rho(v)$-representation of G_v obtained from R_v by replacing v_{i+1} by y on the $\rho(v)$-face of R_v. Then if for all such G, R_v is equivalent to R', $S(v_i v_{i+1} v_{i+2})$ will be called a *symmetric span.* A span which is not symmetric will be called *non-symmetric.* Thus, for example, the spans $S(v_1 v_2 v_3)$ in Fig. 2.1 are symmetric.

However the span $S(v_1 v_2 v_3)$ in Fig. 2.2 is non-symmetric, because although in Fig. 2.2(a) the interchange of v_2 and y gives equivalent 6-representations, this is not so in Fig. 2.2(b).

We note that if G is degenerate, then by Theorem 1.1 any ordinary vertex of G must be incident to at least one non-symmetric span. An example of a

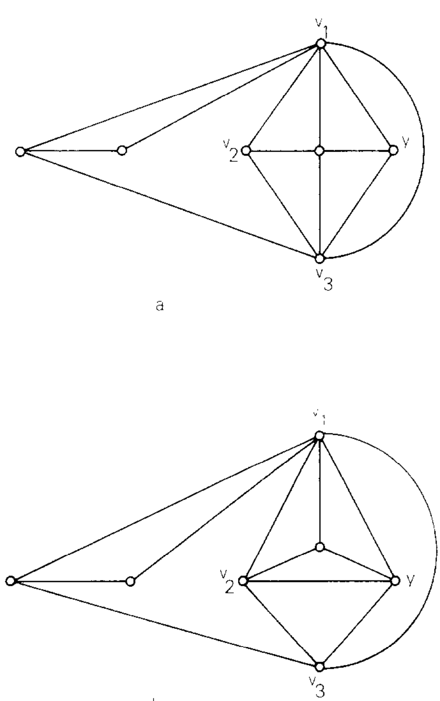

FIGURE 2.1

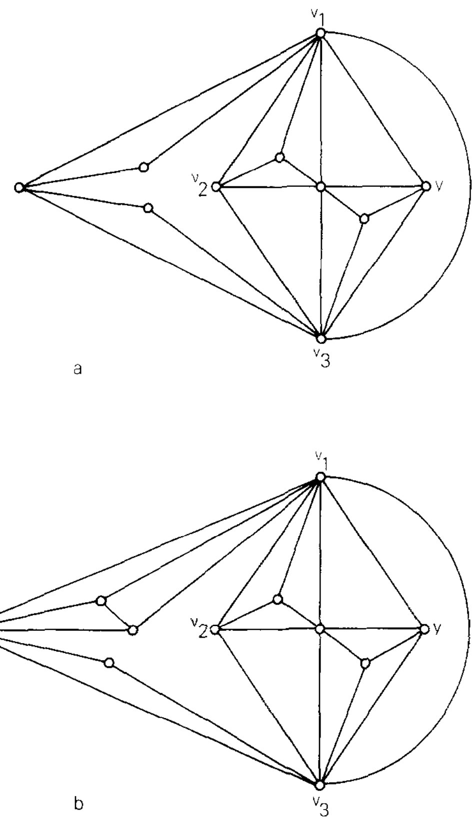

FIGURE 2.2

degenerate graph is given in Fig. 2.3. We also note that if a span is non-symmetric, then it must have at least 6 vertices, because the maximal planar graphs with 4 and 5 vertices are symmetric spans no matter which face is labelled.

We now have four results about spans, the first two following from the definitions.

LEMMA 2.1. *Let G be a maximal planar graph, v an ordinary vertex of G, and* $S(v_i v_{i+1} v_{i+2})$ *a span incident to v. Then* vv_iv_{i+1} *and* $vv_{i+2}v_{i+1}$ *are faces in G.*

The following lemma is a partial converse of the above.

LEMMA 2.2. *Let G be a maximal planar graph, v an ordinary vertex of G, and x, y, z neighbours of v such that vxy and vzy are faces in G. Then if xyz is a separating triangle for G, x, y and z are the primary vertices of a span incident to v, with x and z as pivots.*

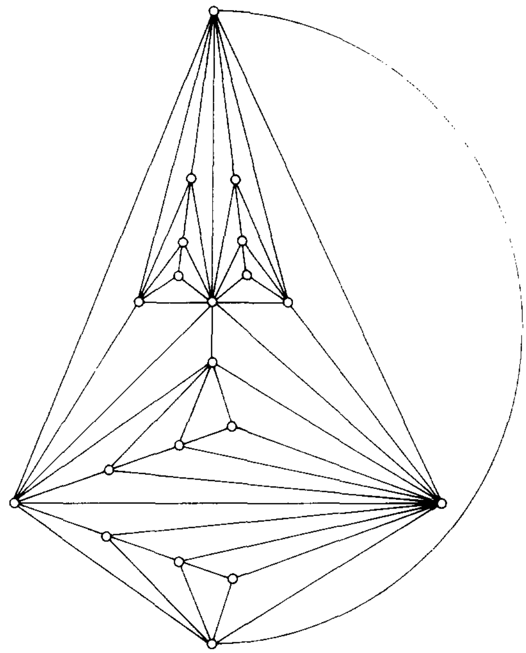

FIGURE 2.3

LEMMA 2.3. *Let G be a maximal planar graph, and v an ordinary vertex of G. Let abc be a triangle of G, and v adjacent to a, b, c. Then if v is incident to a span containing triangle abc, this span must have a, b and c as primary vertices.*

Proof. Let $S = S(xyz)$ be the span incident to v and containing triangle abc. Then x, y, z are the only vertices of S adjacent to v. But a, b, c are three vertices of S adjacent to v; therefore triangle abc is triangle xyz in some order. ▮

LEMMA 2.4. *Let G be a maximal planar graph, and $S = S(w_1 w_2 w_3)$ be a span incident to an ordinary vertex w, with y as replacement vertex. Then if S is symmetric, we have that*

$$\rho(w_2) = 1 + \rho(y).$$

Before proceeding further we require the following definition. Let K be a maximal planar graph, $|V(K)| > 4$, and let abc be a face of K. Let K have the property that for any ordinary vertex $v \in V(K) - \{a, b, c\}$, either

(i) v is incident to a span containing abc, or

(ii) v is adjacent to a, b and c and two of vab, vac, vcb, are faces.

Then we say that K *envelopes triangle abc*.

LEMMA 2.5. *Let abc be a face of a maximal planar graph K, and let K envelope triangle abc. Then there exists an ordinary vertex in K adjacent to a, b and c.*

Proof. We assume that there exists no vertex in K incident to a, b and c. Now, since the order of K is greater than 4, there exists at least one ordinary vertex in $V(K) - \{a, b, c\}$, and every such vertex must be incident to a span containing abc. Among all such spans, let $S = S(w_1 w_2 w_3)$, incident to the ordinary vertex w, be minimal, in the sense that no span containing abc can have less vertices than S. By Lemma 2.1, we have that $ww_1 w_2$ and $ww_3 w_2$ are faces (Figure 2.4).

We define $\mathrm{Int}(ww_1 w_3)$ as that component of $K - (ww_1 w_3)$ which contains w_2, and $\mathrm{Ext}(ww_1 w_3)$ as the other component of $K - (ww_1 w_3)$. Similarly we define $\mathrm{Ext}(w_1 w_2 w_3)$ as that component of $K - (w_1 w_2 w_3)$ containing w, and $\mathrm{Int}(w_1 w_2 w_3)$ as the other component of $K - (w_1 w_2 w_3)$. Thus triangle $abc \subset \overline{\mathrm{Int}(w_1 w_2 w_3)} \subset \overline{\mathrm{Int}(ww_1 w_2)}$.

Now, at least one of w_1, w_2 or w_3 must be different from a, b, c, giving rise to the three cases $w_1 \notin \{a, b, c\}$, $w_2 \notin \{a, b, c\}$, and $w_3 \notin \{a, b, c\}$. We consider only the case $w_2 \notin \{a, b, c\}$, the other cases can be dealt with in a similar manner.

Thus assume that $w_2 \notin \{a, b, c\}$. Since K envelopes triangle abc and since we are assuming that no vertex is adjacent to a, b and c we have that w_2 is incident to a span containing abc, with x, y, z, say, as the primary vertices. The first part of the proof consists in showing that none of x, y, z is equal to w, and hence that xyz is included in $\overline{\mathrm{Int}(w_1 w_2 w_3)}$. Having done this we define $\mathrm{Ext}(xyz)$ to be that component of $K - (xyz)$ containing w_2, and we obtain, by the minimality of S, that the span incident to w_2 with x, y, z as primary vertices is $\overline{\mathrm{Ext}(xyz)}$, which is impossible, since $\overline{\mathrm{Ext}(xyz)}$ contains w_2.

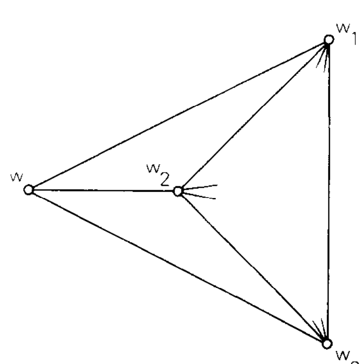

FIGURE 2.4

We thus conclude that there exists a vertex v adjacent to a, b and c in K. However, the fact that the order of K exceeds 4, and since abc is a face, imply that v is ordinary. ▮

We also use the result contained in the following lemma.

LEMMA 2.6. *Let G be a maximal planar graph, and H a maximal planar subgraph of G such that $H \cap (\overline{G-H}) = vab$, and in H there is a vertex c such that vca and vcb are faces in H. Let w be an ordinary vertex in G, $w \in G-H$, and let $S = S(w_1 w_2 w_3)$ be a span incident to w, and containing triangle abc. Then S contains triangle vab.*

Proof. Assuming that the result is false implies that G contains $K(v, c, w_1, w_2, w_3)$. ▮

Before we state our main theorems, we require the definition of a special type of maximal planar graphs, and some results on them.

Let G be a maximal planar graph of order n for which there exists a sequence of nested subgraphs $G_1, G_2, \dots, G_{n-3}$ satisfying the conditions:

(i) $G = G_1$ and $G_{n-3} = K_3$.

(ii) each G_i ($i = 1, 2, \dots, n-5$) has exactly two vertices of valency 3.

(iii) G_{i+1} is obtained from G_i by deleting a vertex of valency 3.

Then we call G a *stitching graph.*

Any maximal planar graph in normal form [5, Sect. 1.3] is a stitching graph. Another example of a stitching graph is given by the graph of Fig. 2.5.

The process (described in the above definition) of reducing a stitching graph G, with two vertices u and v of valency 3, to a triangle, but carried out such that we first delete the vertex u, and then, at each subsequent step, we delete the *new* vertex of valency 3 created by the deletion of the previous

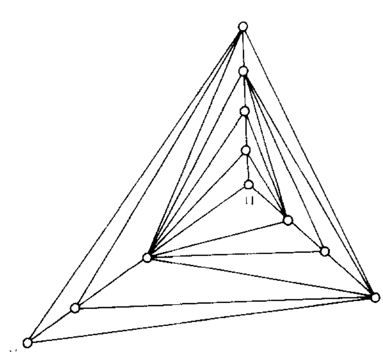

FIGURE 2.5

vertex, will be called the *unstitching of G from u to v*. The unstitching from v to u is similarly defined. Note that in the unstitching of G, from u say, we obtain K_4 immediately before the deletion of the last vertex. At that stage we can delete any vertex different from v to arrive at the final triangle.

Assume that at a stage in the unstitching of G, x is the next vertex to be removed, and y is the new vertex of valency 3 created by the deletion of x. Let a and b be the neighbours of x, apart from y, when x is to be deleted. Then a and b will be called the *extra neighbours* of x. If y in its turn has the same extra neighbours as x we say that *x and y are in the same row*. Since the deletion of the last vertex results in a triangle, and since we have a choice of three vertices to delete at the last step, we shall always consider the unstitching to be carried out in such a way that the last three vertices which are deleted are in the same row. Thus if we define the length of a row to be the number of vertices in the row, we have that the last row in the unstitching of G is always of length greater than 2.

The *stitching sequence of G from u to v* is defined as the sequence of lengths of the rows in the order in which they occur in the unstitching from u. The stitching sequence from v to u is similarly defined. Thus the stitching sequence from u to v for the graph in Fig. 2.5 is (2, 2, 1, 3), while the sequence from v to u is (1, 1, 2, 4).

We now have some results on the stitching sequences of a stitching graph G. The two vertices of valency 3 in G, will be u and v, and the stitching sequences of G from u to v and from v to u will be, respectively,

$$(a_1, a_2, \ldots, a_p) \qquad (b_1, b_2, \ldots, b_q).$$

We only state the following results, without giving the proofs.

1. $\sum_1^p a = \sum_1^q b = n - 3$, where $n = |V(G)|$. Also, $a_p \geqslant 3$ and $b_q \geqslant 3$, except when $n < 6$. If $n < 6$, then G is in normal form and therefore has only one row.

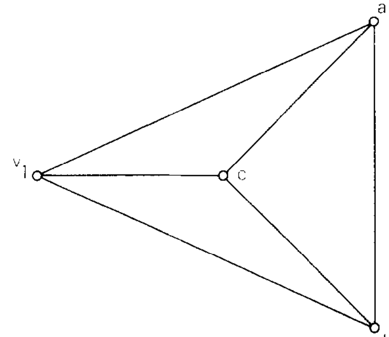

FIGURE 2.6

2. $b_1 = a_p - 2$ and $a_1 = b_q - 2$ if G has more than one row.
3. $p = q$.
4. $a_{i+2} = b_{p-1-i}$ for $i = 0, 1, 2,..., p-3$.

We can now present our main theorems.

THEOREM 2.1. *Let K be a maximal planar graph, and abc a face of K. Let K envelope triangle abc. Then K is a stitching graph, and one of a, b, or c has valency 3 in K.*

Proof. By Lemma 2.5 there an ordinary vertex v_1 in K adjacent to a, b, and c, and since K envelopes abc, then two of abv_1, acv_1, bcv_1 are faces. We can assume with no loss of generality that acv_1 and bcv_1 are faces. Therefore K contains the graph H of Fig. 2.6 as a subgraph, and so vertex c has valency 3, thus proving one part of Theorem 2.1.

Now, if there exist no more ordinary vertices in K, then there is just one more vertex, of valency 3, adjacent to v_1, a and b, and so K would be a stitching graph. We can therefore assume that there exists at least one ordinary vertex in $K - H$. But K envelopes triangle abc; therefore any ordinary vertex in $K - H$ must be incident to a span containing abc. But by Lemma 2.6, any such span must contain triangle abv_1. Therefore the graph $\overline{K - H}$ envelopes triangle abv_1. Thus, by the above, there exists a vertex $z \in \{v_1, a, b\}$, which has valency 3 in $\overline{K - H}$.

We can now apply induction on the number n of vertices of K. We first note that the theorem is certainly true for $n = 5$, since the only maximal planar graph on 5 vertices is a stitching graph. Thus assume that the theorem is true for any graph with less than n vertices. Therefore $\overline{K - H}$ is a stitching graph. But $\rho(z) = 3$ in $\overline{K - H}$. Therefore there exists another vertex w of valency 3 in $\overline{K - H}$, and $w \notin \{v_1, a, b\}$. But K is obtained from $\overline{K - H}$ by adding the vertex c of valency 3, and joining it to v_1, a and b. Therefore K is a stitching graph with c and w as the two vertices of valency 3. ∎

LEMMA 2.7. *Let G be a degenerate graph and let $S = S(w_1 w_2 w_3)$ be a non-symmetric span incident to a vertex w in G. Assume that S has minimal order among all non-symmetric spans in G. Then S must satisfy these three conditions:*

(i) *If u is an ordinary vertex in S, different from w_1, w_2, w_3, and S' is a non-symmetric span incident to u in S, then S' must contain triangle $w_1 w_2 w_3$.*

(ii) *S envelopes triangle $w_1 w_2 w_3$.*

(iii) *If for any $j = 1$, 2 or 3, w_j is ordinary in S, then w_j cannot be incident to a non-symmetric span in S.*

Proof. We only give a sketch of the proof. To prove (i) we first observe that by the minimality of S, u cannot be incident to S' in G. Thus if $S' = S'(pqr)$, this can only arise either if a face of S' is no longer a face in G, or if one of upq, urq, is no longer a face in G. However, the only triangle which is a face in S but not in G is $w_1 w_2 w_3$. But since none of upq or urq is $w_1 w_2 w_3$, the first alternative must be true; hence S contains $w_1 w_2 w_3$.

Now, let t be an ordinary vertex of S, different from w_1, w_2, w_3. In view of (i), we can, in order to prove (ii), assume that t is incident to no non-symmetric spans. But G is degenerate. Therefore t is incident to a non-symmetric span in G. Then if we assume that no span incident to t in S contains triangle w_1, w_2, w_3, we obtain, by arguments similar to those employed in proving (i), that t is adjacent to w_1, w_2, and w_3, and two of $tw_1 w_2$, $tw_1 w_3$, $tw_2 w_3$ are faces in S. Therefore S envelopes triangle $w_1 w_2 w_3$.

Again, using similar arguments as in the first paragraph of this proof, we obtain that if w_1, say, is ordinary in S and is incident to a non-symmetric span in S, then w_2 and w_3 are primary vertices of this span, and hence they are also ordinary in S. But S envelopes triangle $w_1 w_2 w_3$; therefore we have a contradiction to Theorem 2.1. Hence (iii) is proved for w_1. ∎

THEOREM 2.2. *Let G be a degenerate graph. Then there exists an ordinary vertex u in G, such that u is incident to a non-symmetric span S, with a, b, c as primary vertices, S being one of the eight types of graphs shown in Fig.* 2.7, *and σ being a permutation of* $\{a, b, c\}$. (*The labelling of the other vertices will be required in the next theorem.*)

Proof. Since G is degenerate, then every ordinary vertex in G is incident to a non-symmmteric span. Let S be a non-symmetric span incident to an ordinary vertex u in G, such that S has minimal order among all non-symmetric spans of G, and let a, b, c be the primary vertices of S. We now claim that S is one of the eight types of graphs shown in Fig. 2.7.

To prove this we first note that S must satisfy conditions (i), (ii) and (iii) of Lemma 2.7. Therefore by (ii) and by Theorem 2.1, S must be a stitching graph and one of a, b, c has valency 3 in S. We can, with no loss of generality, assume that c has valency 3 in S. Let v be the other vertex of valency 3 in S, and let $(a_1, a_2, \ldots, a_p)$ be the stitching sequence of S from c to v.

We first note that if S has only one row, then it is in normal form. In that case, since c has valency 3, we obtain that S is a graph of type I or II. We can therefore assume that S has more than one row, that is, $p > 1$.

The rest of the proof, of which we only give a sketch, consists in constructing all possible stitching sequences from c to v which S can have. We first show that $a_p = 3$. Assume that $a_p > 3$. Then, since S has more than one row, it is as in Fig. 2.8, where the rest of S is inside triangle xyz, and

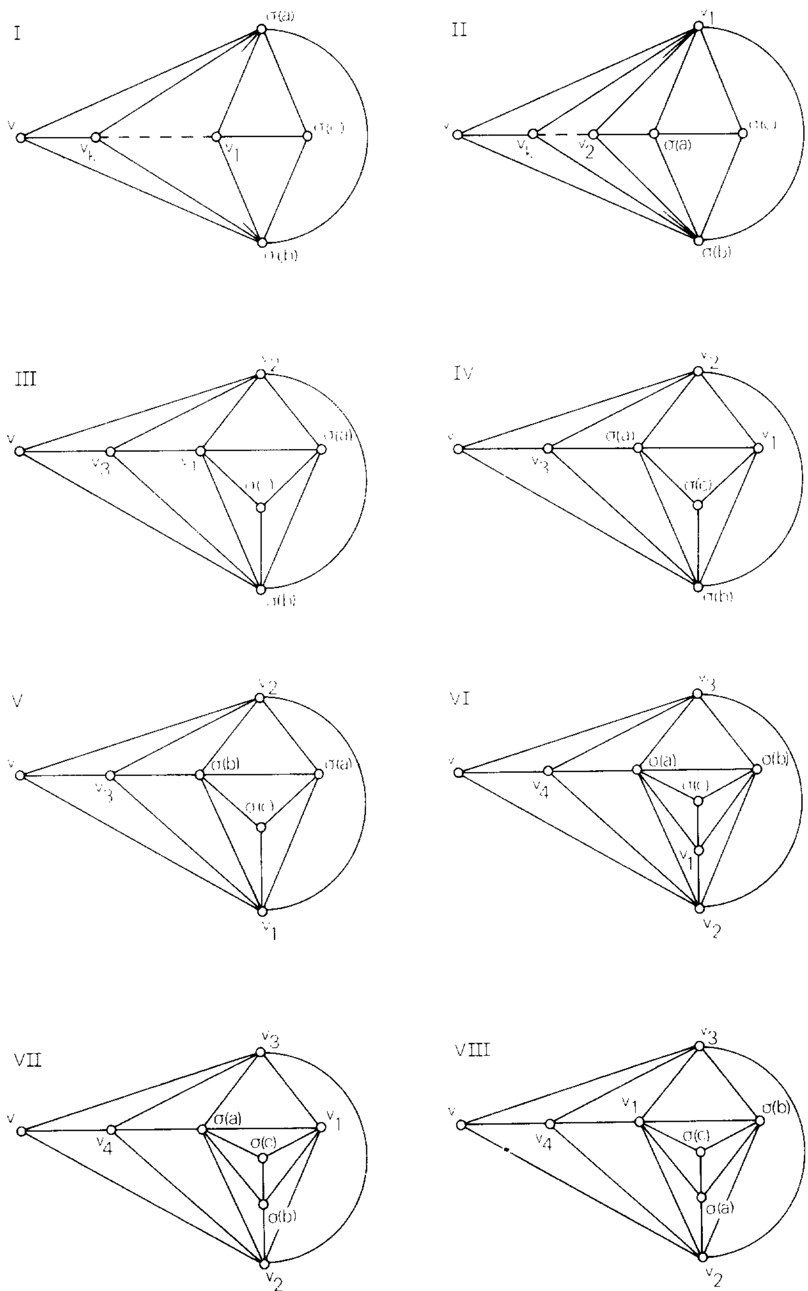
I
σ(a)
v
v1
σ(b)
II
v1
v
v2
σ(a)
σ(b)
III
v
v3
σ(a)
IV
v2
v
σ(a)
v3
v1
σ(c)
σ(b)
V
v2
v
σ(b)
σ(a)
v3
σ(c)
v1
VI
v3
v
σ(a)
σ(b)
v4
v1
v2
VII
v3
v
σ(a)
v1
v4
σ(c)
σ(b)
v2
VIII
v3
v
v1
σ(b)
v4
σ(c)
σ(a)
v2

FIGURE 2.7

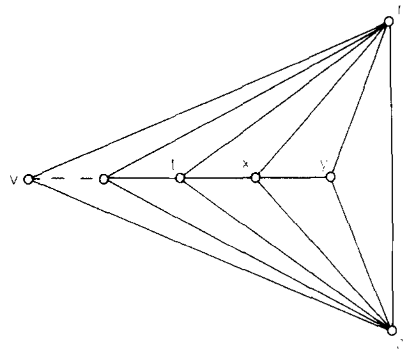

FIGURE 2.8

xyr, zyr are faces. Therefore $\rho(r) \geqslant 6$, and $\rho(t) = 4$. Therefore by Lemma 2.4, the span $S(xrz)$ incident to y is non-symmetric. However this span does not contain triangle abc; therefore we have a contradiction to Lemma 2.7(i). Therefore $a_p = 3$.

We next show that $a_{p-1} = 1$. The proof is again by contradiction; in a manner similar to that in the previous paragraph, we obtain that, if a_{p-1} is greater than 1, S would contain an ordinary vertex incident to a non-symmetric span not containing abc.

Now we note that if S has only two rows, then it is as in Fig. 2.9. But then, since c has valency 3, c must be the vertex w. Therefore one of the triangles ywz, xwz, xwy must be triangle abc. Therefore S is one of types III, IV, V of Fig. 2.7. We can therefore assume that S has more than two rows.

Then, in the same way as we proved that $a_p = 3$, and $a_{p-1} = 1$, we prove that $a_{p-2} = 1$. Then, if S has only three rows, it is as in Figs. 2.10(i)–(ii). But in Fig. 2.10(i), the span $S(yqz)$ incident to x is non-symmetric and does

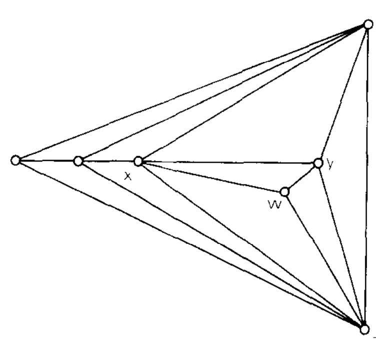

FIGURE 2.9

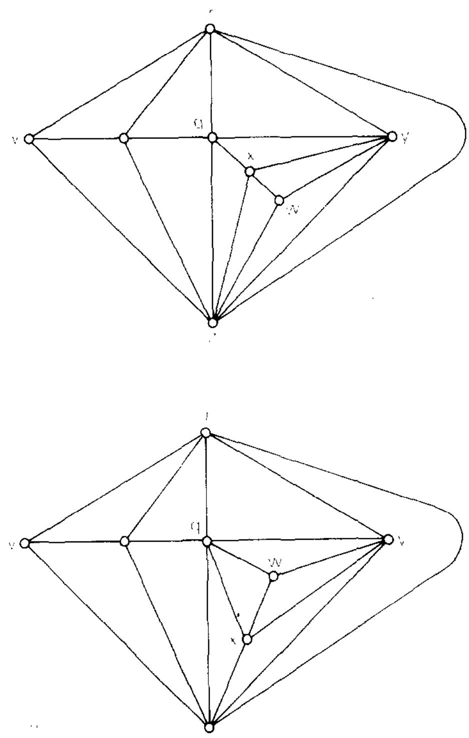

FIGURE 2.10

not contain abc. Therefore S must be as in Fig. 2.10(ii). Hence triangle abc must be one of triangles qwy, qwx, ywx, giving types VI, VII, VIII, respectively.

The proof will now be complete if we can show that S cannot have more than three rows. Again, this is proved in the same way, that is, we assume that S has more than three rows, and we obtain that S contains an ordinary vertex incident to a non-symmetric span not containing abc, a contradiction to Lemma 2.7(i). ∎

Before stating Theorem 2.3, which is the main theorem of this section, we require the following definition.

Let G be a maximal planar graph, and let w be an ordinary vertex of G with the property that w is incident to only one non-symmetric span $S(w_1 w_2 w_3)$ having replacement vertex y. Then if $\rho(w_2) \neq \rho(y) + 1$ we say that w is a *good vertex* of G.

THEOREM 2.3. *Every degenerate graph has a good vertex.*

TABLE I

Type of S	Pivot vertices of S	Good vertex α	Non-symmetric span S_α incident to α	Valency in G of replaced vertex of S_α	Replacement vertex of S_α and its valency in G
I	a and c (or b and c)	v_k	$S_\alpha(bv_{k-1}a)$	$\rho(v_{k-1})=4$	$u; \rho(u)\geqslant 4$
II ($k=2$)	a and b	v_2	$S_\alpha(v_1ab)$	$\rho(a)>5$	$c; \rho(c)=4$
	b and c	v_2	$S_\alpha(v_1ab)$	$\rho(a)=5$	$c; \rho(c)\geqslant 5$
II ($k\geqslant 3$)	a and b	v_k	$S_\alpha(v_1v_{k-1}b)$	$\rho(v_{k-1})=4$	$c; \rho(c)=4$
	b and c	v_k	$S_\alpha(v_1v_{k-1}b)$	$\rho(v_{k-1})=4$	$c; \rho(c)\geqslant 5$
III	Any pair	v_3	$S_\alpha(v_2v_1b)$	$\rho(v_1)=5$	$a; \rho(a)\geqslant 5$
IV	Any pair	v_3	$S_\alpha(v_2ab)$	$\rho(a)>5$	$v_1; \rho(v_1)=4$
V	a and b	v_2	$S_\alpha(v_1ab)$	$\rho(a)\geqslant 6$	$c; \rho(c)=4$
	c and b	v_2	$S_\alpha(v_1ab)$	$\rho(a)=5$	$c; \rho(c)\geqslant 5$
	c and a	v_3	$S_\alpha(v_1bv_2)$	$\rho(b)=6$	$a; \rho(a)>5$
VI	Any pair	v_3	$S_\alpha(v_2ba)$	$\rho(b)>5$	$v_1; \rho(v_1)=4$
VII	Any pair	v_4	$S_\alpha(v_3av_2)$	$\rho(a)>6$	$v_1; \rho(v_1)=5$
VIII	Any pair	v_4	$S_\alpha(v_3v_1v_2)$	$\rho(v_1)=6$	$b; \rho(b)>5$

Proof. Let G be a degenerate graph. Therefore by Theorem 2.2, there exists an ordinary vertex u incident to a non-symmetric span S, S being one of the eight types of graphs in Fig. 2.7.

We have to consider these eight cases. We let a, b, c be the primary vertices of S, and we consider the different cases which arise, depending on which one of a, b, c is the replaced vertex of S considered as a non-symmetric span incident to u. We take σ in Fig. 2.7 to be the identity permutation.

The above table exhibits in each case the good vertex and easily verifiable reasons for the choice.

3. Reconstruction

THEOREM 3.1. *Maximal planar graphs are reconstructible.*

Proof. Since we have seen that every maximal planar graph whose minimum valency is at least 4 is reconstructible we need only consider maximal planar graphs whose minimum valency is 3. Moreover, we have seen under Preliminaries that since the maximal planarity of such graphs is recognizable by [3], we need only consider the reconstruction of degenerate graphs. Thus, let G be a degenerate graph. By Theorem 2.3 we know that G has a good vertex v_0. But then, by Theorem 1.1 and the definition of a good vertex, G_{v_0} has exactly two no-equivalent $\rho(v_0)$-representations R and R', with the property that the valency sequence of the vertices on the $\rho(v_0)$-face of R is different from the valency sequence of the vertices on the $\rho(v_0)$-face of R'. But since we can determine the valency sequence of the neighbours of v_0 in G, we can reconstruct G uniquely from G_{v_0}. ∎

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