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On the Size of Edge Chromatic Critical Graphs

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In this paper, by applying the discharging method, we prove that if $G=(V,E)$ is a Δ-critical graph, then $|E|\geqslant\frac{1}{4}|V|(\Delta+\sqrt{2\Delta-1})$. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

All graphs $G=(V,E)$ are finite and simple. The *chromatic index* $\chi'(G)$ of a graph G is the minimum number of colors required to color the edges of G so that two adjacent edges receive different colors. A graph G of maximum degree Δ is *class one* if $\chi'(G)=\Delta$. Otherwise, Vizing's theorem [8] guarantees $\chi'(G)=\Delta+1$ and G is said to be *class two.* A *Δ-critical graph* G is a connected graph of maximum degree Δ such that G is class two and $G-e$ is class one for each edge e of G. The following is a well-known conjecture of Vizing proposed in 1968.

Conjecture (Vizing [10]). If G is a Δ-critical graph, then $|E|\geqslant\frac{1}{2}\{|V|(\Delta-1)+3\}$.

Vizing's conjecture has been proved for the case $\Delta\leqslant 5$ [5, 7]. It was proved in 1975 by Fiorini [2] that if G is a Δ-critical graph, then $|E|\geqslant\frac{1}{4}|V|(\Delta+1)$.

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More than 20 years later, Fiorini's result was improved by two recent papers which are summarized by the following theorem.

THEOREM 1 (See Clark and Haile [1]; Haile [4]). *If G is a Δ-critical graph, then* $|E| \geqslant f(\Delta)|V|$, *where*

$$f(\Delta) = \begin{cases} \dfrac{\Delta+1}{3} & \text{if } 6 \leqslant \Delta \leqslant 8, \\ \dfrac{3(\Delta+2)}{10} & \text{if } \Delta = 9, 11, 13, \\ \dfrac{15+\sqrt{29}}{4} & \text{if } \Delta = 15, \\ \dfrac{\Delta}{4} + \dfrac{6\Delta}{4(\Delta+4)} & \text{if } \Delta \geqslant 10,\ \Delta \text{ is even}, \\ \dfrac{\Delta}{4} + \dfrac{7\Delta+3}{4(\Delta+5)} & \text{if } \Delta \geqslant 7,\ \Delta \text{ is odd}. \end{cases}$$

For more results about the above conjecture, see [3, 6, 11]. Let $k = \frac{1}{2}(\Delta + \sqrt{2\Delta - 1})$. In this paper, by applying the discharging method which was used to solve the 4-color problem, we give a simple and short proof of the following theorem which is a stronger result than Theorem 1 for $\Delta \geqslant 10$ and $\Delta \neq 11$.

THEOREM 2. *Let G be a Δ-critical graph, then* $|E| \geqslant \dfrac{k|V|}{2}$.

Comparing Theorem 2 with Theorem 1, one can easily check that $\frac{k}{2} - f(\Delta) \geqslant 0$ for $\Delta \in \{10, 12, 13, \ldots\}$ and $\lim_{\Delta \to \infty}(\frac{k}{2} - f(\Delta)) = \infty$. Furthermore, the method in this paper is totally different from one used in the proofs of previous results and this is the first time that the discharging method is applied to a graph theory problem in which Euler's formula is not used and embeddings of graphs in surfaces are not mentioned.

Before proceeding, we introduce some notation. For $x \in V$, the degree of x is denoted by $d(x)$. An i-vertex, $\geqslant i$-vertex or $\leqslant i$-vertex is a vertex of degree i, at least i or at most i. We define $d_i(x)$ to be the number of i-vertices adjacent to x.

2. PROOF OF THEOREM 2

Before we prove Theorem 2, we need the following lemmas.

LEMMA 1 (Vizing's Adjacency Lemma [9]). *If x is a vertex of a Δ-critical graph and $d_i(x) \geqslant 1$, then $d_\Delta(x) \geqslant \max\{2, \Delta - i + 1\}$.*

LEMMA 2. *Let* $h(x) = \frac{x-k}{x+(n-\Delta-1)}$. *Then* $h(x)$ *is decreasing if* $\Delta - k \geqslant n-1$, *and* $h(x)$ *is increasing if* $\Delta - k \leqslant n-1$.

The proof of Lemma 2 is easy. Hence we omit it.

CLAIM 1. *If* $n \geqslant 2$, *then the following two inequalities are true*:

$$k \leqslant \frac{n(n+\Delta-1)}{2n-1}, \tag{1}$$

$$k - n - \frac{n(\Delta-k)}{n-1} \leqslant 0 \tag{2}$$

Proof. Since $k = \frac{1}{2}(\Delta + \sqrt{2\Delta-1})$, we have

$$\begin{aligned}\frac{n(n+\Delta-1)}{2n-1} - k &= \frac{n(n+\Delta-1) - 2nk + k}{2n-1}\\ &= \frac{n^2 - (\sqrt{2\Delta-1}+1)n + \frac{\Delta}{2} + \frac{1}{2}\sqrt{2\Delta-1}}{2n-1}\\ &= \frac{(n - \frac{1}{2}(\sqrt{2\Delta-1}+1))^2}{2n-1} \geqslant 0.\end{aligned}$$

Thus, (1) is true. Similarly, we can show (2). ∎

CLAIM 2. *Let* $n \geqslant 2$. *If* $\Delta - k \leqslant n-1$, *then*

$$g(n) = k - n - \frac{(\Delta-k+1)(\Delta-k)}{n-1} \leqslant 0.$$

Proof. If $n = \Delta - k + 1$, then one can check that

$$\begin{aligned}g(n) &= k - (\Delta-k+1) - (\Delta-k+1)\\ &= 3k - 2\Delta - 2 = \frac{3}{2}\sqrt{2\Delta-1} - \frac{\Delta}{2} - 2\\ &= -\frac{1}{2}\left(\sqrt{\Delta - \frac{1}{2}} - \frac{3}{2}\sqrt{2}\right)^2 \leqslant 0.\end{aligned}$$

Assume that $g(n) \leqslant 0$ for some $n \geqslant \Delta - k + 1$. Since $\Delta - k \leqslant n-1$ and $\Delta - k + 1 \leqslant n$, it follows that

$$(\Delta-k+1)(\Delta-k) \leqslant n(n-1) \quad \text{or } (\Delta-k+1)(\Delta-k)\left(\frac{1}{n-1} - \frac{1}{n}\right) - 1 \leqslant 0.$$

Adding the above inequality to $g(n)$ gives $g(n+1) \leqslant 0$. ∎

Proof of Theorem 2. Suppose that $G = (V, E)$ is a Δ-critical graph with $|E| < \frac{1}{2}k|V|$. Denote $k - d(x)$ by $M(x)$, for each vertex x. Then

$$\sum_{x \in V(G)} M(x) = \sum_{x \in V(G)} (k - d(x)) = k|V| - 2|E| > 0.$$

We call the number $M(x)$ the *initial charge* of x for $x \in V$. We will assign a new charge denoted by $M'(x)$ to each $x \in V$ according to the *discharging rule* R below:

R. Let x be a $<k$-vertex. Then x sends $\frac{d(y)-k}{d(x)+d(y)-\Delta-1}$ to each $>k$-vertex y adjacent to it.

Note that the rule only moves charge around, and does not affect the sum. Therefore,

$$\sum_{x \in V(G)} M(x) = \sum_{x \in V(G)} M'(x).$$

Now we check $M'(x)$ and show $M'(x) \leqslant 0$ for each $x \in V$. Let x be a k-vertex. Then by R, $M(x) = M'(x) = 0$. Let x be an n-vertex with $n > k$. Then $M(x) = k - n$. Let m be the lowest degree of a neighbor of x. By Lemma 1, x is adjacent to at least $\Delta - m + 1$ Δ-vertices. By R, each of the at most $n - (\Delta - m + 1)$ ($<k$)-vertices adjacent to x sends at most $\frac{n-k}{m+n-\Delta-1}$ to x and thus $M'(x) \leqslant k - n + [n - (\Delta - m + 1)]\frac{n-k}{m+n-\Delta-1} = 0$.

Let x be an n-vertex with $n < k$. Then $M(x) = k - n$. Since G is critical with $\Delta > 1$, we have $n \geqslant 2$. We consider two cases according to $\Delta - k \geqslant n - 1$ or $\Delta - k \leqslant n - 1$.

Case 1. $\Delta - k \geqslant n - 1$. By Lemma 1, if y is adjacent to x, then $d(y) \geqslant \Delta + 2 - d(x) = (\Delta - n + 1) + 1 \geqslant k + 1$. By R and Lemma 2, x sends at least $\frac{\Delta-k}{n-1}$ to each neighbor, and the total charge that x sends out is at least $\frac{n(\Delta-k)}{n-1}$. Thus, by Claim 1, we have $M'(x) \leqslant k - n - \frac{n(\Delta-k)}{n-1} \leqslant 0$.

Case 2. $\Delta - k \leqslant n - 1$. Let m be the number of Δ-neighbors of x. First we assume that $m \geqslant \Delta - k + 1$. Since x sends out $\frac{\Delta-k}{n-1}$ to each Δ-vertex adjacent to it, our claim 2 implies that $M'(x) \leqslant k - n - (\Delta - k + 1)\frac{\Delta-k}{n-1} \leqslant 0$. Now we consider the case when $m < \Delta - k + 1$. Let y be a vertex adjacent to x. By Lemma 1, x has at least $\Delta - d(y) + 1$ Δ-neighbors, that is, $m \geqslant \Delta - d(y) + 1$. Hence we have

$$d(y) \geqslant \Delta - m + 1 > \Delta + 1 - (\Delta - k + 1) = k.$$

Since $\Delta - k \leqslant n - 1$, by Lemma 2, x sends at least $\frac{\Delta-m+1-k}{n-m}$ to each $<\Delta$-vertex adjacent to it. Thus x sends out at least

$$m\frac{\Delta - k}{n - 1} + (n - m)\frac{\Delta - m + 1 - k}{n - m} = m\frac{\Delta - k}{n - 1} + \Delta - m + 1 - k$$

to its neighbors. Since $\frac{\Delta-k}{n-1} \leqslant 1$, we have

$$m\frac{\Delta - k}{n-1} + \Delta - m + 1 - k \geqslant (\Delta - k + 1)\frac{\Delta - k}{n-1},$$

which leads to

$$M'(x) \leqslant k - n - \left(m\frac{\Delta - k}{n-1} + \Delta - m + 1 - k\right) \leqslant k - n - (\Delta - k + 1)\frac{\Delta - k}{n-1} \leqslant 0.$$

Hence, we have

$$0 < \sum_{x \in V(G)} M(x) = \sum_{x \in V(G)} M'(x) \leqslant 0,$$

a contradiction. ∎

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