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F-Sets in Graphs

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A subset S of the vertex set of a graph G is called an F-set if every $\alpha \in \Gamma(G)$, the automorphism group of G, is completely specified by specifying the images under α of all the points of S, and S has a minimum number of points. The number of points, $k(G)$, in an F-set is an invariant of G, whose properties are studied in this paper. For a finite group Γ we define $k(\Gamma) = \max\{k(G) \mid \Gamma(G) = \Gamma\}$. Graphs with a given Abelian group and given k-value ($k \leqslant k(\Gamma)$) have been constructed. Graphs with a given group and k-value 1 are constructed which give simple proofs to the theorems of Frucht and Bouwer on the existence of graphs with given abstract/permutation groups.

1. Introduction

In this paper we consider finite ordinary graphs. Generally, we follow the notations and terminology in [3]. Let G be a graph whose automorphism group $\Gamma(G)$ is not the identity group. A subset S of the vertex set $V(G)$ is called an f-set if every $\sigma \in \Gamma(G)$ is completely specified by giving the images of the points of S alone. An f-set with a minimum number of points is called an F-set. The cardinality of an F-set S of G is denoted by $k(G)$. If $\Gamma(G) = \{e\}$, let $k(G) = 1$.

The aim of this paper is to study the properties of $k(G)$ (this section) and the existence of graphs with a given value for $k(G)$ (Section 2). One particularly interesting class of graphs with $k(G) = 1$ provides alternative proofs for the theorems of Frucht [2] and Bouwer [1].

THEOREM 1. *If there exists a $\sigma \in \Gamma(G)$ such that σ is completely specified by giving the images of $S(\subseteq V(G))$, then S is an f-set.*

Proof. If not, there exist $\sigma_1, \sigma_2 \in \Gamma(G)$ such that $\sigma_1(s) = \sigma_2(s)\ \forall s \in S$ and $\sigma_1 \neq \sigma_2$. But then, $\sigma_2^{-1}\sigma_1(s) = s\ \forall s \in S$ and $\sigma_2^{-1}\sigma_1 \neq e$, the identity automorphism. This implies $\sigma\sigma_2^{-1}\sigma_1(s) = \sigma(s)\ \forall s \in S$ and $\sigma\sigma_2^{-1}\sigma_1 \neq \sigma$, a contradiction establishing the theorem.

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COROLLARY 1.1. *To check whether a set $S \subseteq V(G)$ is an f-set, it is enough to check whether identity is the only automorphism of G which fixes S pointwise.*

THEOREM 2. *If S is an f-set, then $\sigma(S)$ is also an f-set for any $\sigma \in \Gamma(G)$.*

The simple proof is omitted.

Note 1. A minimal f-set of G need not be an F-set. For example, consider the graph in Fig. 1. Here $\{1, 2\}$ is a minimal f-set but not an F-set, since $k(G) = 1$ and $\{g_1\}$ is an F-set.

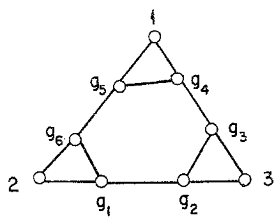

FIGURE 1

LEMMA 1. *If $k(G) = 1$ and if $\{v\}$ is an F-set of G, then $|\Gamma(G)| =$ the number of points which are similar to v.*

Proof. If u is similar to v, there exists a unique $\alpha \in \Gamma(G)$ such that $\alpha(v) = u$.

DEFINITION. A set of elements of a group Γ is said to be *independent* if no element of the set can be generated by the remaining elements of the set.

THEOREM 3. *Let Γ be a finite group. If G is a graph with $\Gamma(G) = \Gamma$, then $k(G) \leqslant \max\{|X| \mid X$ is an independent set of $\Gamma\}$.*

Proof. Let $S = \{1, 2,..., k\}$ be an F-set of G. Define subgroups H_i of Γ as $H_i = \{\sigma \in \Gamma(G) \mid \sigma(j) = j, j \neq i\}$. Since S is an F-set, each $H_i \neq \{e\}$ and $H_i \cap H_j = \{e\}$ if $i \neq j$. Since any element of the subgroup generated by $\{H_j \mid j \neq i\}$ keeps i fixed, no element ($\neq e$) of H_i can be generated by $\{H_j \mid j \neq i\}$. So a set containing one element ($\neq e$) from each H_i forms an independent set of Γ. Hence $k(G) = k \leqslant \max\{|X| \mid X$ is an independent set of $\Gamma\}$.

COROLLARY 3.1. *If Γ is a finite cyclic group and not a direct product of nontrivial subgroups then $k(G) = 1$ for any graph with $\Gamma(G) = \Gamma$.*

Proof. Any maximal independent set of Γ contains only one element.

DEFINITION. Let $k(\Gamma) = \max\{k(G) \mid G$ such that $\Gamma(G) = \Gamma\}$.

The following are easy to see: $k(K_n) = n - 1$ and $k(K_{m,n}) = m + n - 2$ if $m + n > 2$.

THEOREM 4. *Let G be a block-graph with blocks $\{B_i\}$. Let r_i be the number of non-cut-points of B_i. Then, $\sum_{r_i>1}(r_i - 1) \leqslant k(G) \leqslant \sum_{r_i>1}(r_i - 1) + M$, where*

$$M = \sum_{r_i=1} r_i - 1 \quad \text{if} \quad \sum_{r_i=1} r_i \neq 0$$
$$= 0 \quad \text{otherwise.}$$

Proof. In the block B_i, if $r_i \geqslant 2$, then all the non-cut-points of B_i except one have to be in any F-set. Hence the first inequality.

Let $S = \{\cup A_i\} \cup B$ where(1) A_i is any one subset of $r_i - 1$ non-cut-points of B_i, if $r_i > 1$ and (2) if C is the set of all non-cut-points in the B_i's with $r_i = 1$, then B is any one subset of C with $|C| - 1$ points if $C \neq \varnothing$ and $B = \varnothing$ if $C = \varnothing$. It can be easily seen that if every point of S is fixed then all the non-cut-points of G are fixed and hence all the cut-points are also fixed. Hence S is an f-set of G. This gives the second inequality.

COROLLARY 4.1. *If T is a tree then $k(T) \leqslant$ number of pendent vertices* of T.

Note 2. The above inequalities may be strict inequalities or equalities. Figure 2 gives four examples in which all the combinations are realized.

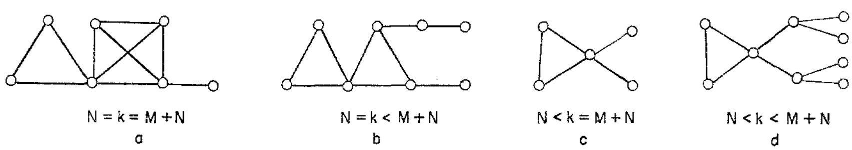

FIGURE 2

2. GRAPHS WITH GIVEN $k(G)$

The graph products considered here are Cartesian products.

THEOREM 5. *Let $\{G_i\}$ be a finite number of connected prime graphs. A set of necessary and sufficient conditions for a subset S of $V(G)$ to be an f-set of $G = \prod G_i$ is*

(1) *$p_i(S)$ is an f-set of G_i, $\forall i$, where p_i is the projection mapping of G to the i th coordinate space G_i.*

(2) *the map $\alpha: p_i(S) \to p_j(S)$ given by $\alpha(p_i(S)) = p_j(S)\ \forall s \in S$ is not a restriction of an isomorphism of G_i to G_j, $\forall i \neq j$.*

Proof. Picture the points of G_i as plotted on the ith axis of an n-dimensional space, where $i \in \{1, 2,..., n.\}$ The points of G are then among the lattice points in the nonnegative orthant.

Since graph multiplication is commutative, we can write $G = G_i \times (\prod_{i \neq j} G_j)$. If condition (1) is not satisfied then by Corollary 1.1, there exists a nontrivial automorphism of $\Gamma(G_i)$, fixing $p_i(S)$. This naturally extends to an automorphism of G which fixes all the points whose ith coordinates are in the set $p_i(S)$. Hence condition (1) is necessary. If (2) were not satisfied, then there exists an isomorphism $\alpha: G_i \to G_j$ such that $\alpha(p_i(s)) = p_j(s)$. Arrange the points of G_j such that $(g_i, \alpha(g_i))$ forms a diagonal in the (i, j)th plane, as shown in Fig. 3. Now, there is an automorphism of $G_i \times G_j$

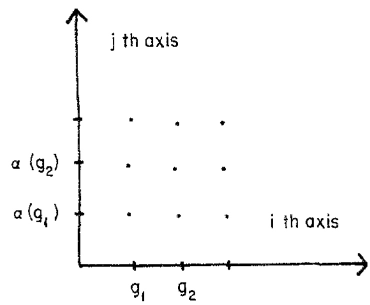

FIGURE 3

fixing the diagonal points $(g_i, \alpha(g_i))$, and hence a nontrivial automorphism of G fixing the points of S, which implies that S is not an f-set of S, a contradiction.

Since the automorphisms of G are only of the above two types, it is clear that conditions (1) and (2) are sufficient to ensure that S is an f-set. This completes the proof.

THEOREM 6. *Let G and H be two connected graphs which are prime to each other. Then $k(G \times H) = \text{Max}(k(G), k(H))$.*

Proof. Let $\{1, 2,..., m\}$ be an F-set of G and $\{1', 2',..., n'\}$ be an F-set of H. Let $m \geqslant n'$. By Theorem 5, it is clear that $k(G \times H) \geqslant m$. Since G and H are prime to each other the second condition of Theorem 5 is always satisfied for any subset S of $V(G \times H)$. Consider $S = \{(1, 1'), (2, 2'),...,(n, n'),$

$(n+1, n'),\ldots, (m, n')\}$. It is easy to see that the first condition of Theorem 5 is also satisfied by S and hence, having the minimum number of points, S is an F-set of $G \times H$, proving the theorem.

COROLLARY 6.1. *Let* $G = \prod_{i=1}^{m} G_i^{r_i}$, *where* G_i *are prime to each other. Then* $k(G) = \max_i k(G_i^{r_i})$.

Now, let us consider F-sets of G^n, where G is a prime graph. Let $\{s_i = (v_{i1}, v_{i2},\ldots, v_{in}) \mid i = 1, 2,\ldots, k\}$ be an F-set of G^n. By Theorem 5, $\{v_{ij} \mid i = 1, 2,\ldots, k\}$ is an f-set of G for $j = 1, 2,\ldots, n$, and there do not exist automorphisms of G such that $v_{il} \to v_{im}$, $i = 1,\ldots, k$ for any $l, m \in \{1, 2,\ldots, n\}$. It is not necessary that the elements v_{il} should be different for a given l. Let us say that two ordered f-sets $(v_1,\ldots, v_k)$ and $(v_1',\ldots, v_k')$ with k (not necessarily distinct) points of G are *distinct* if there does not exist an automorphism of G taking $v_i \to v_i'$. So, by the above discussion, if there are m_k distinct ordered f-sets of G with k points, then writing these sets as columns of a matrix and considering the rows as points of G^{m_k}, we get an f-set of G^{m_k} with k points. Hence $k(G^{m_k}) \leqslant k$. First we prove that m_k is a strictly increasing function of k. List all the m_{k-1} ordered f-sets with $k-1$ points. Let $u \neq v$ be points of G. To each of these ordered f-sets add the point u at the end. Obtain another f-set by adding v to one of the original m_{k-1} f-sets. It is easily seen that these $m_{k-1}+1$ ordered f-sets are distinct f-sets with k points. Hence $m_{k-1} < m_k$. Suppose $r = k(G^{m_k}) < k$. Then consider an F-set S of G^{m_k}. Consider the m_k projections of S into the coordinate spaces. They give m_k ordered f-sets for G, with r elements each. Since $m_r < m_k$, at least two of these are not distinct, and by Theorem 5, S is not an f-set of G^{m_k}, a contradiction. Hence $k(G^{m_k}) \geqslant k$. Thus $k(G^{m_k}) = k$ and we have proved

THEOREM 7. $k(G^n) = k$ *if* $m_{k-1} < n \leqslant m_k$.

Note 3. It seems very difficult to find m_k for a given graph, even for small values of k. (It is obvious that $k \geqslant k(G)$.)

THEOREM 8. *Let* $G = K_2$. *Then*

$$m_k = 2^{k-1} \qquad \textit{if } k \textit{ is odd},$$
$$= 2^{k-1} + \tfrac{1}{2}\binom{k}{k/2} \qquad \textit{otherwise}.$$

Proof. $m_1 = 2^0 = 1$ is clear. Let the points of K_2 be $\{0, 1\}$. The only nontrivial automorphism of K_2 interchanges 0 and 1. If $a \in \{0, 1\}$, let $\bar{a}$ denote the other element. If $\{b_1, b_2,\ldots, b_k\}$ is an ordered f-set then $\{\bar{b}_1, \bar{b}_2,\ldots, \bar{b}_k\}$ is not distinct from $\{b_1,\ldots, b_k\}$. So the number of distinct f-sets with k points (not necessarily distinct) are given by different placings of 0's and 1's in the k places. In other words, if there are r zeros and $(k - r)$ ones, then it is just choosing the r places for the zeros. This is done in $\binom{k}{r}$ ways. Since $r = 0, 1,..., k/2$ if k is even and $r = 0, 1,..., (k - 1)/2$ if k is odd, (the other choices for 0 giving no new distinct ordered f-set, as noted before), we get the number of distinct ordered f-sets with k points as $\binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{k/2}$ if k is even and $\binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{(k-1)/2}$ if k is odd. That is $2^k/2$ if k is odd and $(2^k/2) + \frac{1}{2}\binom{k}{k/2}$ if k is even.

Note 4. Since Q_n, the n-dimensional cube is just $(K_2)^n$, we have calculated $k(Q_n)$.

Let us now turn our attention to graphs with prescribed k-values.

THEOREM 9. *Let Γ be a finite Abelian group and $1 \leqslant k \leqslant$ the maximum number of elements in any independent set of Γ. Then there exists a graph G (indeed infinitely many) such that $\Gamma(G) = \Gamma$ and $k(G) = k$.*

Proof. Since Γ is Abelian, it is a direct product of cyclic groups. Let $\Gamma = \prod_1^n \Gamma_i$ where each Γ_i is cyclic and is not a direct product of nontrivial groups. This n is nothing but the maximum number of elements in an independent set of Γ. Let $\{H_i\}$ be graphs such that $\Gamma(H_i) = \Gamma_i$ and H_i are mutually prime. For example, if $| \Gamma_i | = 3$ we can take any of the graphs in Fig. 4, as H_i.

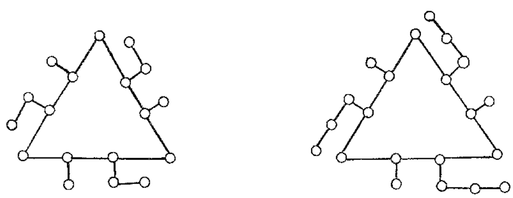

FIGURE 4

Consider $H = \prod_1^{n-k+1} H_i$. By Corollary 3.1, $k(H_i) = 1$, $\forall i = 1, 2,..., n$, and by Theorem 6, $k(H) = 1$. Now construct G as follows. Let

$$V(G) = V(H) \cup \{V(H_i) \mid i = n - k + 2,..., n\} \cup \{u\},$$

and

$$E(G) = E(H) \cup \{E(H_i) \mid i = n - k + 2,..., n\}$$
$$\cup \{(u, h)/h \in V(H) \text{ or } V(H_i), i = n - k + 2,..., n\}.$$

It is easily seen that $\Gamma(G) = \prod_1^n \Gamma_i = \Gamma$ and that any f-set of G must contain one point from H and each of $\{H_i/i = n - k + 2,..., n\}$. Since such a set actually forms an F-set, $k(G) = k$.

We leave the similar results for non-Abelian groups as the following two conjectures.

Conjecture 1. If Γ is a finite non-Abelian group, $k(\Gamma) =$ maximum number of elements in an independent set of Γ.

Conjecture 2. If Γ is a finite non-Abelian group and $1 \leqslant k \leqslant k(\Gamma)$, then there exists a graph G with $\Gamma(G) = \Gamma$ and $k(G) = k$.

3. Graphs with Given Group

In this section we construct graphs with $k(G) = 1$ which are simpler than the Frucht graphs [2] and Bouwer graphs [1]. We start with the construction of graphs G_n which are basic in our construction of graphs with a given group. G_2 is defined to be K_2 and G_3 is as in Fig. 1.

The points 1, 2, 3 are called the S-points (special points) of G_3 and the other points are called the G-points (group points) of G_3. For $n \geqslant 3$, G_n is constructed inductively from K_n as follows. Start with a K_n. The points of K_n (in G_n) are the S-points of G_n. On each line of K_n introduce two new points. In the resulting homeomorph of K_n, each S-point has a neighborhood containing $n - 1$ points. Identify the S-points of a copy of G_{n-1} with these $n - 1$ points. Thus, for each point of K_n, we have introduced a copy of G_{n-1}. The resulting graph is G_n. The collection of the G-points of all the copies of G_{n-1} present in G_n constitute the set of G-points of G_n. It is clear that there are $n!$ G-points in G_n. All these have degree three and are similar to each other. Further, if any one of these points, say g, is fixed, then the whole graph is fixed. (For g belongs to a unique G_2 which is included in a unique G_3, etc., and fixing g fixes the points of this G_2 and in turn this G_3 and so on.) Thus each G-point is an F-set of G_n. Name the S-points of G_n as 1, 2,..., n. Name any of the G-points as $g_1 = e \in S_n$. Since g_1 is an F-set, if $\alpha(g_1)$ (for $\alpha \in \Gamma(G_n)$) is given then α is specified completely. This α restricted to the S-points of G_n gives an element, say g_2, of S_n. Name $\alpha(g_1)$ as g_2. Similarly all the G-points of G_n can be named (uniquely and unambiguously) by the elements of S_n. Now by Lemma 1, $\Gamma(G_n) = S_n$.

Theorem 10. *Any automorphism α of G_n when restricted to the G-points, is a left multiplication by $\alpha(e)$.*

Proof. Let $\alpha(e) = x$ and $\alpha(y) = z$. Let us denote the automorphism taking e to x as α_x. So $\alpha_x(y) = z$ and $\alpha_y(e) = y$. Therefore $\alpha_x\alpha_y(e) = z$, i.e., $\alpha_{xy}(e) = z$, which implies $xy = z$. Hence $\alpha_x(y) = xy = \alpha_x(e)\,y$.

Theorem 11 (Frucht [2]). *Given a finite group Γ, there exists a graph G such that $\Gamma(G) = \Gamma$.*

Proof. As any finite group can be viewed as a subgroup of some S_n, let Γ be a subgroup of S_n, for some n. In the G_n corresponding to this n, for each G-point (named with the elements of Γ) take a copy of K_2 and identify one of its points with this G-point. The new graph, say H_n, is the required graph. For, $e = g_1$ is an F-set in H_n also, and only the G-points named with elements of Γ are similar to g_1, since, under an automorphism α_g, where $g \in \Gamma$ (by Theorem 10) the points with the names of elements of Γ go among themselves. By Lemma 1, $\Gamma(H_n) \simeq \Gamma$.

COROLLARY 11.1 (Bouwer [1]). *Given a permutation group P and a graph Y such that $P \subseteq \Gamma(Y)$ (as permutation groups), there exists a graph G such that*

(1) *Y is an induced subgraph of G.*

(2) *When $\Gamma(G)$ is restricted to the vertex set of Y, it is isomorphic to P as a permutation group.*

(3) *$\Gamma(G) \simeq P$ (as abstract groups).*

Proof. Construct H_n as before with the automorphism group P. Identify the points of Y with the S-points of H_n (i.e., the points having the same label are identified). It is clear that this new graph is the required G with the S-points forming the subgraph Y.

Note 5. The proof for Bouwer's theorem is simple here, as we start with a new type of graph with a given group.

Note 6. By replacing each line joining two G-points of G_n with a graph whose group is the cyclic group of order 2, where two points which are similar are identified with the G-points, we get an infinite number of graphs with a given group.

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