



Noncentral elliptical configuration density

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ABSTRACT

The noncentral configuration density, derived under an elliptical model, generalizes and corrects the Gaussian configuration and some Pearson results. Partition theory is then used to obtain explicit configuration densities associated with matrix variate symmetric Kotz type distributions (including the normal distribution), matrix variate Pearson type VII distributions (including t and Cauchy distributions), the matrix variate symmetric Bessel distribution (including the Laplace distribution) and the matrix variate symmetric Jensen-logistic distribution.

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1. Introduction

When the statistical theory of shape was placed in the setting of noncentral multivariate analysis [1], a wide range of standard theories developed over the previous 60 years was available to solve these new distributional problems.

Initial studies in this field assumed, as usual, Gaussian distributions for the landmark components [1,2], and integration over Euclidean and affine transformations provided the required shape and configuration distributions, respectively, in terms of a well-known theory, that of the zonal polynomials of the matrix argument.

The theory of integration over orthogonal and positive definite matrices involving zonal polynomials produced exact distributions, but the problem for large computations remained unresolved for years, and approximations were needed for applications. Recently, with the appearance of efficient algorithms for zonal and hypergeometric functions, the exact distributions can be studied in the corresponding inference problem, and thus the applications can potentially be improved [3].

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However, the classical normal constraint seems inappropriate in most cases, and so new enriched distributions, for example elliptically contoured distributions, could be considered for the landmark components. Nevertheless, the corresponding new integrals under Euclidean or affine transformations require new developments. The Euclidean case has been solved with the usual multivariate analysis and classical integration formulae for zonal and invariant polynomials of several matrix arguments (Goodall and Mardia [1], Díaz-García [2], etc.), but the configuration distribution (based on an affine transformation), for any elliptical model, has not been studied.

There appear to be two reasonable justifications for addressing the configuration problem: firstly, the geometrical meaning for applications, and secondly, the distributional problem. The first of these is clearly the most important for users of shape theory; the transformation refers to problems which are not equally deformed in all directions (as is the case in Euclidean transformation models), but uniformly deformed component by component. This is especially useful in growth theory, mechanical deformations, non-rigid evolution, electrophoretic gel studies, etc., but even the classical applications studied by Euclidean transformations and Gaussian distributions can be re-examined by assuming an elliptical model previously ratified by Schwarz’s dimensional criterion [4], for example, and under an affine transformation. The second problem is a probabilistic one, which fundamentally considers advances in integration over positive definite matrices via zonal polynomials. Noncentral multivariate analysis constitutes the key to studying particular elliptical models one by one, by solving the corresponding multiple integral as in the normal case [1,2] and some integrals given for Pearson models [5], for example. However, this technique provides no solution for any general model and, certainly, some multiple integrals seem prohibitive. Therefore, an interesting problem could be to consider this general direction, by simplifying the integration problem.

Fortunately, some general results for integration on positive definite matrices are available [6] and they can be generalized and used in the setting needed for the configuration distributions. These results mean that our distributions for any elliptical model are easily integrable. However, simplicity in the integration has a price, namely the k -th derivative expressions for elliptical model functions. This is where the second tool for solving the problem is applied, through a partitionial treatment for expressing the derivative in such a way that the single integral can be easily computed [7].

With the distributional problem solved, another important question, that of the computation, remains, but as mentioned before, series of zonal polynomials can now be computed efficiently. Thus, the inference problem based on the exact configuration density is viewed in the present paper as a solvable numerical aspect.

This study is distributed as follows: first, the main integral which supports all the distributional results and the corresponding corollaries are listed in Section 2; subsequently, a subsection is devoted to correcting some published Pearson results which are necessary for the corresponding Pearson configuration density; finally, Section 3 studies the configuration densities corresponding to the classical matrix variate elliptical contoured distributions, including Pearson, Kotz, Bessel and Jensen-logistic distributions (see [7].)

2. The main integral

Noncentral elliptical multivariate distributions involve a number of general integrals to be studied, this number depending on the transformations under consideration, but all of them relying on a single important fact, namely that the elliptically contoured distributions are characterized by a symmetric function, say, $h(\mathbf{U})$, i.e., $h(\mathbf{AB}) = h(\mathbf{BA})$, for any squared matrix \mathbf{A} and \mathbf{B} . The simplest function we can consider is the trace of a positive definite matrix and then the zonal polynomials arise naturally. Thus, Euclidian and affine transformations of the random matrix require integration over the orthogonal group and the positive definite space. In the case of positive definite matrices, Constantine [8] constitutes the source for all subsequent studies, inspiring the following general result for elliptical integration, which can be seen as a combination of Xu and Fang [5] and Teng et al. [6]; see Caro-Lopera [7] for a detailed proof:

Theorem 1. *Let \mathbf{Z} be a complex symmetric $p \times p$ matrix with $\text{Re}(\mathbf{Z}) > \mathbf{0}$ and let \mathbf{Y} be a symmetric $p \times p$ matrix. If k is a nonnegative integer and κ is a partition of k , then, for $\text{Re}(a) > (p - 1)/2$,*

$$\int_{\mathbf{X} > \mathbf{0}} h(\text{tr} \mathbf{XZ}) |\mathbf{X}|^{a-(p+1)/2} C_{\kappa}(\mathbf{XY})(d\mathbf{X}) = \frac{|\mathbf{Z}|^{-a} (a)_{\kappa} \Gamma_p(a) C_{\kappa}(\mathbf{YZ}^{-1})}{\Gamma(pa + k)} S, \tag{1}$$

where

$$S = \int_0^{\infty} h(w) w^{pa+k-1} dw < \infty, \tag{2}$$

and $(a)_{\kappa}$ and $\Gamma_p(a)$ are the generalized hypergeometric coefficient and the multivariate gamma function respectively (see [9, p. 62, 247, 248]).

As a convention, in the present study it is always assumed that the integrals encountered actually exist. A simple consequence is the following.

Corollary 2. Let \mathbf{Z} be a complex symmetric $p \times p$ matrix with $\text{Re}(\mathbf{Z}) > \mathbf{0}$. Then

$$\int_{\mathbf{X} > \mathbf{0}} h(\text{tr} \mathbf{XZ}) |\mathbf{X}|^{a-(p+1)/2} (d\mathbf{X}) = \frac{|\mathbf{Z}|^{-a} \Gamma_p(a)}{\Gamma(pa)} S,$$

where

$$S = \int_0^\infty h(y) y^{pa-1} dy < \infty.$$

Note that, if we take $\mathbf{Z} = \Sigma^{-1}$ and $h(y) = e^{-y/2}$, which implies $S = 2^{pa} \Gamma(pa)$, we have the classical result for a multivariate Gamma function proved in detail by Muirhead [9, pp. 61–63].

2.1. On some published Pearson VII type results

Now, some consequences of [Theorem 1](#) for Pearson VII type aspects can be listed.

Corollary 3. For $m > 0$,

$$\int_{\mathbf{W} > \mathbf{0}} (1 + m^{-1} \text{tr} \mathbf{W})^{-(m+np)/2} |\mathbf{W}|^{\frac{n}{2} - \frac{(p+1)}{2}} C_\kappa(\mathbf{WU}) (d\mathbf{W}) = \frac{m^{\frac{np}{2} + k} \left(\frac{n}{2}\right)_\kappa \Gamma_p\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2} - k\right)}{\Gamma\left(\frac{m}{2} + \frac{np}{2}\right)} C_\kappa(\mathbf{U}).$$

Consider the density of $\mathbf{B} = \mathbf{Y}'\mathbf{Y}$ given in Runze [10, eq. (1.5)]:

$$\frac{\pi^{np/2} c_{n,p}}{\Gamma_p\left(\frac{n}{2}\right)} |\Sigma|^{-n/2} |\mathbf{B}|^{(n-p-1)/2} h(\text{tr} \Sigma^{-1} \mathbf{B}), \quad (3)$$

where $c_{n,p}$ is a normalization constant (see [10]). Then another consequence of [Theorem 1](#) is the expectation of a zonal polynomial with respect to \mathbf{B} with the above-defined density.

Corollary 4. Suppose that \mathbf{B} has density (3) and that \mathbf{A} is an arbitrary symmetric $p \times p$ constant matrix, then

$$E_{\mathbf{B}}[C_\kappa(\mathbf{BA})] = \frac{\pi^{np/2} c_{n,p} \left(\frac{n}{2}\right)_\kappa}{\Gamma\left(\frac{np}{2} + k\right)} C_\kappa(\mathbf{A}\Sigma) J,$$

where

$$J = \int_0^\infty h(y) y^{\frac{np}{2} + k - 1} dy < \infty.$$

Now, if $\kappa = (1, \dots, 1)$ is a partition of k and l_1, \dots, l_p denote the latent roots of \mathbf{A} , then

$$\begin{aligned} C_\kappa(\mathbf{A}) &= d_\kappa l_1 \cdots l_k + \text{terms of lower weight} \\ &= d_\kappa r_k(\mathbf{A}), \end{aligned} \quad (4)$$

where $r_k(\mathbf{A})$ is the k -th elementary symmetric function of l_1, \dots, l_p ; see [9, p. 251]. So we have the following.

Corollary 5. Under the conditions of [Corollary 4](#),

$$E_{\mathbf{B}}[r_j(\mathbf{B})] = \frac{\pi^{np/2} c_{n,p} \left(\frac{n}{2}\right)_\kappa}{\Gamma\left(\frac{np}{2} + k\right)} r_j(\Sigma) J,$$

where $j = 1, \dots, p$.

Corollary 6. Suppose that \mathbf{B} has density (3) and that \mathbf{A} and \mathbf{C} are arbitrary symmetric $p \times p$ constant matrices, then

$$E_{\mathbf{B}}[C_\phi^{\kappa, \lambda}(\mathbf{BC}, \mathbf{BU})] = \frac{\pi^{np/2} c_{n,p} \left(\frac{n}{2}\right)_\phi}{\Gamma\left(\frac{np}{2} + k + l\right)} C_\phi^{\kappa, \lambda}(\mathbf{C}\Sigma, \mathbf{U}\Sigma) J_1,$$

where

$$J_1 = \int_0^\infty h(y) y^{\frac{np}{2} + k + l - 1} dy < \infty,$$

and κ, λ and ϕ are partitions of k, l and $k + l$, respectively; see [11] for the theory of invariant polynomials.

Now, if we take $h(\mathbf{B}) = (1 + m^{-1}\text{tr}\mathbf{B})^{-(m+np)/2}$ in Corollary 4, it follows that $J_1 = m^{k+np/2}B(k + np/2, -k + m/2)$, $C_{n,p} = \Gamma[(m + np)/2]/[\pi^{np/2}m^{np/2}\Gamma(m/2)]$ and

$$E_{\mathbf{B}}[C_{\kappa}(\mathbf{BA})] = \frac{m^k \Gamma[\frac{m}{2} - k] (\frac{n}{2})_{\kappa}}{\Gamma(\frac{m}{2})} C_{\kappa}(\mathbf{A}\Sigma), \tag{5}$$

and Corollary 6 in this case becomes

$$E_{\mathbf{B}}[C_{\phi}^{\kappa,\lambda}(\mathbf{BC}, \mathbf{BU})] = \frac{m^{k+l} \Gamma[\frac{m}{2} - k - l] (\frac{n}{2})_{\phi}}{\Gamma(m/2)} C_{\phi}^{\kappa,\lambda}(\mathbf{C}\Sigma, \mathbf{U}\Sigma). \tag{6}$$

Finally, (5) can again be proved by using the properties of invariant polynomials in (6). In fact, just by taking $\mathbf{C} = \mathbf{0}$, then $C_{\phi}^{\kappa,\lambda}(\mathbf{0}, \mathbf{BU}) = 0$ for $k > 0$ and $C_{\phi}^{\kappa,\lambda}(\mathbf{0}, \mathbf{BU}) = C_{\lambda}(\mathbf{BU})$ for $k = 0$, then

$$E_{\mathbf{B}}[C_{\lambda}(\mathbf{BU})] = E_{\mathbf{B}}[C_{\phi}^{\kappa,\lambda}(\mathbf{0}, \mathbf{BU})] = \frac{m^l \Gamma[\frac{m}{2} - l] (\frac{n}{2})_{\lambda}}{\Gamma(m/2)} C_{\lambda}(\mathbf{U}\Sigma),$$

which corresponds to (5).

Recall that if we take $m = 1$ in the family of Pearson VII distributions, we obtain the multivariate Cauchy distribution; thus by replacing this parameter in (5) it must follow that $\frac{1}{2} \geq k$, which means that the Cauchy distribution has no moments.

Corollary 3–6 were derived by Xu and Fang [5] and Runze [10], but as the reader can confirm, the published results are incorrect. Note that the discrepancy must be clarified because our Pearson configuration density involves the corrected integrals.

3. Configuration density

First, let us recall a definition given by Goodall and Mardia [1].

Definition 7. Two figures $\mathbf{X} : N \times K$ and $\mathbf{X}_1 : N \times K$ have the same configuration, or affine shape, if $\mathbf{X}_1 = \mathbf{X}\mathbf{E} + \mathbf{1}_N e'$, for some translation $e : K \times 1$ and a non-singular $\mathbf{E} : K \times K$.

The configuration coordinates are constructed in the two steps summarized in the expression

$$\mathbf{LX} = \mathbf{Y} = \mathbf{UE}. \tag{7}$$

The matrix $\mathbf{U} : N - 1 \times K$ contains the configuration coordinates of \mathbf{X} . Let $\mathbf{Y}_1 : K \times K$ be non-singular and $\mathbf{Y}_2 : q \times K$, with $q = N - K - 1 \geq 1$, such that $\mathbf{Y} = (\mathbf{Y}'_1 | \mathbf{Y}'_2)'$. Define also $\mathbf{U} = (\mathbf{I} | \mathbf{V})'$; then $\mathbf{V} = \mathbf{Y}_2 \mathbf{Y}_1^{-1}$ and $\mathbf{E} = \mathbf{Y}_1$, where \mathbf{L} is an $N - 1 \times N$ Helmert submatrix.

We shall now establish the Jacobian of the configuration transformation:

Lemma 8. Let $(\mathbf{F}^{1/2})^2 = \mathbf{F} > \mathbf{0}$, \mathbf{H} be a $K \times K$ orthogonal matrix, and $\mathbf{E} = \mathbf{F}^{1/2}\mathbf{H}$, so for $\mathbf{Y} = \mathbf{UF}^{1/2}\mathbf{H}$ then

$$(d\mathbf{Y}) = 2^{-K} |\mathbf{F}|^{(q-1)/2} (d\mathbf{V})(d\mathbf{F})(\mathbf{H}d\mathbf{H}'),$$

where $(\mathbf{H}'d\mathbf{H})$ denotes the Haar measure; see [9, p. 72].

Proof. Let $\mathbf{E} = \mathbf{F}^{1/2}\mathbf{H}$, with \mathbf{E} a $K \times K$ invertible matrix, \mathbf{H} orthogonal and $\mathbf{F}^{1/2} > \mathbf{0}$. Hence $\mathbf{E}'\mathbf{E} = \mathbf{H}'\mathbf{F}\mathbf{H}$, because $\mathbf{E}'\mathbf{E}$ and \mathbf{F} are symmetric and \mathbf{H} is non-singular; then $(d(\mathbf{E}'\mathbf{E})) = |\mathbf{H}|^{K+1} (d\mathbf{F}) = (d\mathbf{F})$. But by Muirhead [9, Theorem 2.1.14], $(d\mathbf{E}) = 2^{-K} |\mathbf{E}'\mathbf{E}|^{-1/2} (d(\mathbf{E}'\mathbf{E}))(\mathbf{H}'d\mathbf{H})$. Thus we obtain $(d\mathbf{E}) = 2^{-K} |\mathbf{H}'\mathbf{F}\mathbf{H}|^{-1/2} (d\mathbf{F})(\mathbf{H}'d\mathbf{H}) = 2^{-K} |\mathbf{F}|^{-1/2} (d\mathbf{F})(\mathbf{H}'d\mathbf{H})$. And by summarizing:

$$\mathbf{E} = \mathbf{F}^{1/2}\mathbf{H} \Rightarrow (d\mathbf{E}) = 2^{-K} |\mathbf{F}|^{-1/2} (d\mathbf{F})(\mathbf{H}'d\mathbf{H}). \tag{8}$$

Now,

$$\mathbf{Y} = \begin{pmatrix} \mathbf{I} \\ \mathbf{V} \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} \mathbf{E} \\ \mathbf{VE} \end{pmatrix}.$$

By differentiating and computing the exterior product, we get $(d\mathbf{Y}) = |\mathbf{E}|^q (d\mathbf{V})(d\mathbf{E})$; but $|\mathbf{E}| = |\mathbf{F}^{1/2}\mathbf{H}| = |\mathbf{F}|^{1/2}$, and so

$$(d\mathbf{Y}) = |\mathbf{F}|^{q/2} (d\mathbf{V})(d\mathbf{E}). \tag{9}$$

By replacing (8) in (9) the desired result is obtained. \square

Now we can state with the help of Theorem 1 the main statistical result of this work, the general case of the configuration density under a non-isotropic noncentral elliptical model.

Theorem 9. If $\mathbf{Y} \sim \mathcal{E}_{N-1 \times K}(\boldsymbol{\mu}_{N-1 \times K}, \boldsymbol{\Sigma}_{N-1 \times N-1} \otimes \mathbf{I}_K, h)$, for $\boldsymbol{\Sigma}$ positive definite ($\boldsymbol{\Sigma} > \mathbf{0}$), $\boldsymbol{\mu} \neq \mathbf{0}_{N-1 \times K}$, then the configuration density is given by

$$\frac{\pi^{K^2/2} \Gamma_K\left(\frac{N-1}{2}\right)}{|\boldsymbol{\Sigma}|^{\frac{K}{2}} |\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}|^{\frac{N-1}{2}} \Gamma_K\left(\frac{K}{2}\right)} \sum_{t=0}^{\infty} \frac{1}{t! \Gamma\left(\frac{K(N-1)}{2} + t\right)} \sum_{r=0}^{\infty} \frac{1}{r!} [\text{tr}(\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu})]^r$$

$$\times \sum_{\tau} \frac{\binom{N-1}{2}_{\tau}}{\binom{K}{2}_{\tau}} C_{\tau}(\mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} (\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U})^{-1}) S, \tag{10}$$

where

$$S = \int_0^{\infty} h^{(2t+r)}(y) y^{\frac{K(N-1)}{2} + t - 1} dy < \infty. \tag{11}$$

Proof. The density of \mathbf{Y} is given by

$$\frac{1}{|\boldsymbol{\Sigma}|^{\frac{K}{2}}} h\{\text{tr}[(\mathbf{Y} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{Y} - \boldsymbol{\mu})]\}.$$

If we factorize \mathbf{Y} according to Lemma 8, then the joint density of \mathbf{U} , \mathbf{F} and \mathbf{H} is

$$\frac{|\mathbf{F}|^{(q-1)/2}}{2^K |\boldsymbol{\Sigma}|^{\frac{K}{2}}} h[\text{tr}(\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} + \mathbf{F} \mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}) + \text{tr}(-2\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} \mathbf{F}^{1/2} \mathbf{H})] (\mathbf{H}' d\mathbf{H})(d\mathbf{F})(d\mathbf{V}).$$

Assuming that h admits a Taylor expansion (see [12] and [13]), the joint density of \mathbf{U} , \mathbf{F} and \mathbf{H} becomes

$$\frac{|\mathbf{F}|^{(q-1)/2}}{2^K |\boldsymbol{\Sigma}|^{\frac{K}{2}}} \sum_{t=0}^{\infty} \frac{1}{t!} \frac{h^{(t)}(\text{tr}(\mathbf{F} \mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}) + \text{tr}(\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}))}{[\text{tr}(-2\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} \mathbf{F}^{1/2} \mathbf{H})]^{-t}} (\mathbf{H}' d\mathbf{H})(d\mathbf{F})(d\mathbf{V}).$$

Now from [14, eq. (22)], then integration with respect to \mathbf{H} gives the joint density of \mathbf{F} and \mathbf{U} as follows:

$$\frac{\pi^{K^2/2} |\mathbf{F}|^{(q-1)/2}}{|\boldsymbol{\Sigma}|^{\frac{K}{2}} \Gamma_K\left(\frac{K}{2}\right)} \sum_{t=0}^{\infty} \frac{1}{t!} h^{(2t)}(\text{tr}(\mathbf{F} \mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}) + \text{tr}(\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu})) \times \sum_{\tau} \frac{1}{\binom{K}{2}_{\tau}} C_{\tau}(\mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} \mathbf{F}) (d\mathbf{F})(d\mathbf{V}).$$

Noting that $h^{2t}(\cdot)$ admits a Taylor expansion, then the joint density of \mathbf{F} and \mathbf{U} finally takes the form

$$\frac{\pi^{K^2/2} |\mathbf{F}|^{(q-1)/2}}{|\boldsymbol{\Sigma}|^{\frac{K}{2}} \Gamma_K\left(\frac{K}{2}\right)} \sum_{t=0}^{\infty} \frac{1}{t!} \sum_{r=0}^{\infty} \frac{1}{r!} h^{(2t+r)}(\text{tr}(\mathbf{F} \mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U})) [\text{tr}(\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu})]^r \times \sum_{\tau} \frac{1}{\binom{K}{2}_{\tau}} C_{\tau}(\mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} \mathbf{F}) (d\mathbf{F})(d\mathbf{V}).$$

Thus, integration over $\mathbf{F} > \mathbf{0}$ gives the configuration density as follows:

$$\frac{\pi^{K^2/2}}{|\boldsymbol{\Sigma}|^{\frac{K}{2}} \Gamma_K\left(\frac{K}{2}\right)} \sum_{t=0}^{\infty} \frac{1}{t!} \sum_{r=0}^{\infty} \frac{1}{r!} [\text{tr}(\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu})]^r \sum_{\tau} \frac{1}{\binom{K}{2}_{\tau}}$$

$$\times \int_{\mathbf{F} > \mathbf{0}} h^{(2t+r)}(\text{tr}(\mathbf{F} \mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U})) |\mathbf{F}|^{(q-1)/2} C_{\tau}(\mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} \mathbf{F}) (d\mathbf{F}). \tag{12}$$

Here, the integral in (12) is reduced by (1) to

$$\int_{\mathbf{F} > \mathbf{0}} h^{(2t+r)}(\text{tr}(\mathbf{F} \mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U})) |\mathbf{F}|^{(q-1)/2} C_{\tau}(\mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} \mathbf{F}) (d\mathbf{F})$$

$$= \frac{|\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}|^{-\frac{N-1}{2}} \binom{N-1}{2}_{\tau} \Gamma_K\left(\frac{N-1}{2}\right) C_{\tau}(\mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} (\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U})^{-1})}{\Gamma\left(\frac{K(N-1)}{2} + t\right)} S,$$

where

$$S = \int_0^{\infty} h^{(2t+r)}(y) y^{\frac{K(N-1)}{2} + t - 1} dy < \infty,$$

and the required result follows. \square

Finally, the central configuration density can be obtained.

Corollary 10. If $\mathbf{Y} \sim \mathcal{E}_{N-1 \times K}(\mathbf{0}_{N-1 \times K}, \boldsymbol{\Sigma}_{N-1 \times N-1} \otimes \mathbf{I}_K, h)$, $\boldsymbol{\Sigma} > \mathbf{0}$, then the central configuration density is invariant under the elliptical contoured distributions and it is given by

$$\frac{\Gamma_K \left(\frac{N-1}{2}\right)}{\pi^{\frac{Kq}{2}} |\boldsymbol{\Sigma}|^{\frac{K}{2}} \Gamma_K \left(\frac{K}{2}\right)} |\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}|^{-\frac{N-1}{2}}.$$

Proof. Take $t = r = 0$ in (10), and noting that $h^{(2t+r)}(y) = h^{(0)}(y) \equiv h(y)$, we have

$$\frac{\pi^{\frac{K^2}{2}} \Gamma_K \left(\frac{N-1}{2}\right) |\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}|^{-\frac{N-1}{2}}}{|\boldsymbol{\Sigma}|^{\frac{K}{2}} \Gamma_K \left(\frac{K}{2}\right) \Gamma \left(\frac{K(N-1)}{2}\right)} S.$$

Now, using [12, p. 59],

$$S = \int_0^\infty h^{(2t+r)}(y) y^{\frac{K(N-1)}{2} + t - 1} dy = \int_0^\infty h(y) y^{\frac{K(N-1)}{2} - 1} dy = \frac{\Gamma \left(\frac{K(N-1)}{2}\right)}{\pi^{K(N-1)/2}},$$

and we obtain the required result. \square

The preceding proposition generalizes Díaz-García et al. [2, Theorem 3.2] which concerned the isotropic case, $\boldsymbol{\Sigma} = \sigma^2 I_{N-1}$ (recall that $|\mathbf{U}' \mathbf{U}| = |\mathbf{I}_K + \mathbf{V}' \mathbf{V}|$). In this case both expressions coincide except for the factor 2^K ; Goodall and Mardia [1] and Díaz-García et al. [2] did not describe the Haar measure ($\mathbf{H}' d\mathbf{H}$) employed in the computation of the Jacobian of $\mathbf{Y} = \mathbf{U} \mathbf{F}^{1/2} \mathbf{H}$. This discrepancy is resolved in Lemma 8 in the present paper.

Most of the applications in statistical theory of shape are based on the isotropic model (see [15]); thus, in the case of the noncentral elliptical configuration density, if we take $\boldsymbol{\Sigma} = \sigma^2 I_{N-1}$ in Theorem 9 we obtain the following.

Corollary 11. If $\mathbf{Y} \sim \mathcal{E}_{N-1 \times K}(\boldsymbol{\mu}_{N-1 \times K}, \sigma^2 I_{N-1} \otimes \mathbf{I}_K, h)$, then the isotropic noncentral configuration density is given by

$$\frac{\pi^{K^2/2} \Gamma_K \left(\frac{N-1}{2}\right)}{|\mathbf{I}_K + \mathbf{V}' \mathbf{V}|^{\frac{N-1}{2}} \Gamma_K \left(\frac{K}{2}\right)} \sum_{t=0}^\infty \frac{1}{t! \Gamma \left(\frac{K(N-1)}{2} + t\right)} \sum_{r=0}^\infty \frac{1}{r!} \left[\text{tr} \left(\frac{1}{\sigma^2} \boldsymbol{\mu}' \boldsymbol{\mu} \right) \right]^r \times \sum_{\tau} \frac{\binom{N-1}{2}_\tau}{\binom{K}{2}_\tau} C_\tau \left(\frac{1}{\sigma^2} \mathbf{U}' \boldsymbol{\mu} \boldsymbol{\mu}' \mathbf{U} (\mathbf{U}' \mathbf{U})^{-1} \right) S,$$

where

$$S = \int_0^\infty h^{(2t+r)}(y) y^{\frac{K(N-1)}{2} + t - 1} dy < \infty.$$

4. Families of elliptical configuration densities

In this section we derive explicit configuration densities for matrix variate symmetric Kotz type distributions (including the normal distribution), matrix variate Pearson type VII distributions (including t and Cauchy distributions), the matrix variate symmetric Bessel distribution (including the Laplace distribution) and the matrix variate symmetric Jensen-logistic distribution (see [16]).

4.1. Pearson type VII configuration density

Some aspects of this distribution were addressed previously in Section 2.1.

We now derive the corresponding configuration.

Corollary 12. If $\mathbf{Y} \sim \mathcal{E}_{N-1 \times K}(\boldsymbol{\mu}_{N-1 \times K}, \boldsymbol{\Sigma}_{N-1 \times N-1} \otimes \mathbf{I}_K, h)$, $\boldsymbol{\Sigma} > \mathbf{0}$, then the non-isotropic noncentral Pearson type VII configuration density is given by

$$\frac{\Gamma_K \left(\frac{N-1}{2}\right)}{\pi^{Kq/2} |\boldsymbol{\Sigma}|^{\frac{K}{2}} |\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}|^{\frac{N-1}{2}} \Gamma_K \left(\frac{K}{2}\right)} {}_1P_1 \left(\left(s - \frac{K(N-1)}{2} \right)_t \left(1 + \frac{\text{tr}(\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu})}{R} \right)^{-s + \frac{K(N-1)}{2} - t} \right);$$

$$\frac{N-1}{2}; \frac{K}{2}; \frac{1}{R} \mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} (\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U})^{-1},$$

with $R > 0, s > K(N-1)/2$, and where

$${}_1P_1(f(t) : a; b; \mathbf{X}) = \sum_{t=0}^\infty \frac{f(t)}{t!} \sum_{\tau} \frac{(a)_\tau}{(b)_\tau} C_\tau(\mathbf{X}).$$

Proof. In this case we take

$$h(y) = \frac{\Gamma(s)}{(\pi R)^{\frac{K(N-1)}{2}} \Gamma\left(s - \frac{K(N-1)}{2}\right)} \left(1 + \frac{y}{R}\right)^{-s},$$

see [17]; then

$$h(y)^{(2t+r)} = \frac{\Gamma(s)(-1)^r(s)_{2t+r}}{(\pi R)^{\frac{K(N-1)}{2}} \Gamma\left(s - \frac{K(N-1)}{2}\right) R^{2t+r}} \left(1 + \frac{y}{R}\right)^{-(s+2t+r)},$$

and after some simplification (11) becomes

$$\delta = \frac{(-1)^r \Gamma\left(\frac{K(N-1)}{2} + t\right) \left(s - \frac{K(N-1)}{2}\right)_{t+r}}{\pi^{\frac{K(N-1)}{2}} R^{t+r}}.$$

Thus (10) takes the form

$$\frac{\Gamma_K\left(\frac{N-1}{2}\right)}{\pi^{Kq/2} |\Sigma|^{\frac{K}{2}} |\mathbf{U}'\Sigma^{-1}\mathbf{U}|^{\frac{N-1}{2}} \Gamma_K\left(\frac{K}{2}\right)} \times \sum_{t=0}^{\infty} \frac{1}{t!} \left\{ \sum_{r=0}^{\infty} \frac{\left(s - \frac{K(N-1)}{2}\right)_{t+r}}{r!} \left[\text{tr}\left(-\frac{1}{R}\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}\right) \right]^r \right\} \\ \times \sum_{\tau} \frac{\left(\frac{N-1}{2}\right)_{\tau}}{\left(\frac{K}{2}\right)_{\tau}} C_{\tau} \left(\frac{1}{R}\mathbf{U}'\Sigma^{-1}\boldsymbol{\mu}\boldsymbol{\mu}'\Sigma^{-1}\mathbf{U}(\mathbf{U}'\Sigma^{-1}\mathbf{U})^{-1}\right).$$

The term in braces preserves the core of the distribution, which in this case is $(a)_t (1+u/R)^{-a-t}$, with $a = s - K(N-1)/2 > 0$, $u = \text{tr}(\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu})$, and then we obtain the non-isotropic noncentral Pearson type VII configuration density. \square

By taking $s = (K(N-1) + R)/2$ in Corollary 12 we obtain the configuration density associated with a matrix variate t -distribution with R degrees of freedom. And by replacing $R = 1$ in the above density, we obtain the respective Cauchy configuration density. Therefore:

Corollary 13. If $\mathbf{Y} \sim \mathcal{E}_{N-1 \times K}(\boldsymbol{\mu}_{N-1 \times K}, \Sigma_{N-1 \times N-1} \otimes \mathbf{I}_K, h)$, with $\Sigma > \mathbf{0}$, then:

(1) the non-isotropic noncentral t configuration density is given by

$$\frac{\Gamma_K\left(\frac{N-1}{2}\right)}{\pi^{Kq/2} |\Sigma|^{\frac{K}{2}} |\mathbf{U}'\Sigma^{-1}\mathbf{U}|^{\frac{N-1}{2}} \Gamma_K\left(\frac{K}{2}\right)} {}_1P_1\left(\left(\frac{R}{2}\right)_t \left(1 + \frac{\text{tr}(\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu})}{R}\right)^{-\frac{R}{2}-t} ; \right. \\ \left. \frac{N-1}{2}; \frac{K}{2}; \frac{1}{R}\mathbf{U}'\Sigma^{-1}\boldsymbol{\mu}\boldsymbol{\mu}'\Sigma^{-1}\mathbf{U}(\mathbf{U}'\Sigma^{-1}\mathbf{U})^{-1}\right),$$

(2) the non-isotropic noncentral Cauchy configuration density is given by

$$\frac{\Gamma_K\left(\frac{N-1}{2}\right)}{\pi^{Kq/2} |\Sigma|^{\frac{K}{2}} |\mathbf{U}'\Sigma^{-1}\mathbf{U}|^{\frac{N-1}{2}} \Gamma_K\left(\frac{K}{2}\right)} {}_1P_1\left(\left(\frac{1}{2}\right)_t \left(1 + \text{tr}(\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu})\right)^{-\frac{1}{2}-t} ; \frac{N-1}{2}; \frac{K}{2}; \mathbf{U}'\Sigma^{-1}\boldsymbol{\mu}\boldsymbol{\mu}'\Sigma^{-1}\mathbf{U}(\mathbf{U}'\Sigma^{-1}\mathbf{U})^{-1}\right).$$

4.2. Kotz type configuration density

In this case, in order to find a closed form of (10), some additional results are needed. In this case the partition theory provides suitable expressions for derivatives. By Caro-Lopera [7] we have the following.

Lemma 14. Let $h(y) = y^{T-1} e^{-Ry^s}$, where $R > 0$, $s > 0$, $2T + K(N-1) > 2$ and $N-1-K \geq 1$. If $w^{(k)}$ denotes $\frac{d^k w}{dy^k}$ and $\sum_{\kappa \in P_k}$ is the summation over all the partitions $\kappa = (k^{\nu_k}, (k-1)^{\nu_{k-1}}, \dots, 3^{\nu_3}, 2^{\nu_2}, 1^{\nu_1})$ of k , with $\sum_{i=1}^k i\nu_i = k$, i.e. κ is a partition of k consisting of ν_1 ones, ν_2 twos, ν_3 threes, etc., see [14, p. 492], then we have

$$h^{(k)} = y^{T-1} e^{-Ry^s} \left\{ \sum_{\kappa \in P_k} \frac{k! (-R)^{\sum_{i=1}^k \nu_i} \prod_{j=0}^{k-1} (s-j)^{\sum_{i=j+1}^k \nu_i}}{\prod_{i=1}^k \nu_i! (i!)^{\nu_i}} y^{\sum_{i=1}^k (s-i)\nu_i} + \sum_{m=1}^k \binom{k}{m} \left[\prod_{i=0}^{m-1} (T-1-i) \right] \right\}$$

$$\times \sum_{\kappa \in P_{k-m}} \left. \frac{(k-m)!(-R)^{\sum_{i=1}^{k-m} v_i} \prod_{j=0}^{k-m-1} (s-j)^{\sum_{i=j+1}^{k-m} v_i}}{\prod_{i=1}^{k-m} v_i!(i!)^{v_i}} y^{\sum_{i=1}^{k-m} (s-i)v_i - m} \right\}. \tag{13}$$

Thus, from [17],

$$h(y) = \frac{sR^{\frac{2T+K(N-1)-2}{2s}} \Gamma\left(\frac{K(N-1)}{2}\right)}{\pi^{K(N-1)/2} \Gamma\left(\frac{2T+K(N-1)-2}{2s}\right)} y^{T-1} e^{-Ry^s},$$

and from (13) we have the required derivative; then integral (11) becomes

$$S = \frac{\Gamma\left(\frac{K(N-1)}{2}\right)}{\pi^{K(N-1)/2} \Gamma\left(\frac{2T+K(N-1)-2}{2s}\right)} \times \left\{ \sum_{\kappa \in P_{2t+r}} \frac{(2t+r)! \prod_{j=0}^{2t+r-1} (s-j)^{\sum_{i=j+1}^{2t+r} v_i} \Gamma\left(\frac{2 \sum_{i=1}^{2t+r} (s-i)v_i + 2T - 2 + K(N-1) + 2t}{2s}\right)}{(-1)^{\sum_{i=1}^{2t+r} v_i} R^{-\frac{\sum_{i=1}^{2t+r} iv_i + t}{s}} \prod_{i=1}^{2t+r} v_i!(i!)^{v_i}} \right. \\ \left. + \sum_{m=1}^{2t+r} \binom{2t+r}{m} \left[\prod_{i=0}^{m-1} (T-1-i) \right] \times \sum_{\kappa \in P_{2t+r-m}} \frac{(2t+r-m)! \prod_{j=0}^{2t+r-m-1} (s-j)^{\sum_{i=j+1}^{2t+r-m} v_i}}{(-1)^{\sum_{i=1}^{2t+r-m} v_i} R^{-\frac{\sum_{i=1}^{2t+r-m} iv_i - m + t}{s}} \prod_{i=1}^{2t+r-m} v_i!(i!)^{v_i}} \right. \\ \left. \times \Gamma\left(\frac{2 \sum_{i=1}^{2t+r} (s-i)v_i + 2T - 2 + K(N-1) + 2t}{2s}\right) \right\}.$$

The Gamma functions exist if the arguments are positive; then, given $t \geq 0, r \geq 0, R > 0, s > 0$ and $N - 1 - K \geq 1$, we select T in such a way that $2T + K(N - 1) + 2t > 2(2t + r + 1)$.

Thus, the corresponding configuration density is obtained.

Corollary 15. If $\mathbf{Y} \sim \mathcal{E}_{N-1 \times K}(\boldsymbol{\mu}_{N-1 \times K}, \boldsymbol{\Sigma}_{N-1 \times N-1} \otimes \mathbf{I}_K, h)$, with $\boldsymbol{\Sigma} > \mathbf{0}$, then the Kotz type III non-isotropic noncentral configuration density is given by

$$\frac{\Gamma_K\left(\frac{N-1}{2}\right)}{\pi^{Kq/2} |\boldsymbol{\Sigma}|^{\frac{K}{2}} |\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}|^{\frac{N-1}{2}} \Gamma_K\left(\frac{K}{2}\right)} \sum_{t=0}^{\infty} \frac{\Gamma\left(\frac{K(N-1)}{2}\right)}{t! \Gamma\left(\frac{K(N-1)}{2} + t\right) \Gamma\left(\frac{2T+K(N-1)-2}{2s}\right)} \\ \times \sum_{r=0}^{\infty} \frac{1}{r!} [\text{tr}(\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu})]^r \sum_{\tau} \frac{\left(\frac{N-1}{2}\right)_{\tau}}{\left(\frac{K}{2}\right)_{\tau}} C_{\tau}(\mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} (\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U})^{-1}) \\ \times \left\{ \sum_{\kappa \in P_{2t+r}} \frac{(2t+r)! \prod_{j=0}^{2t+r-1} (s-j)^{\sum_{i=j+1}^{2t+r} v_i} \Gamma\left(\frac{2 \sum_{i=1}^{2t+r} (s-i)v_i + 2T - 2 + K(N-1) + 2t}{2s}\right)}{(-1)^{\sum_{i=1}^{2t+r} v_i} R^{-\frac{\sum_{i=1}^{2t+r} iv_i + t}{s}} \prod_{i=1}^{2t+r} v_i!(i!)^{v_i}} \right. \\ \left. + \sum_{m=1}^{2t+r} \binom{2t+r}{m} \left[\prod_{i=0}^{m-1} (T-1-i) \right] \times \sum_{\kappa \in P_{2t+r-m}} \frac{(2t+r-m)! \prod_{j=0}^{2t+r-m-1} (s-j)^{\sum_{i=j+1}^{2t+r-m} v_i}}{(-1)^{\sum_{i=1}^{2t+r-m} v_i} R^{-\frac{\sum_{i=1}^{2t+r-m} iv_i - m + t}{s}} \prod_{i=1}^{2t+r-m} v_i!(i!)^{v_i}} \right. \\ \left. \times \Gamma\left(\frac{2 \sum_{i=1}^{2t+r} (s-i)v_i + 2T - 2 + K(N-1) + 2t}{2s}\right) \right\}.$$

$$\times \Gamma \left(\frac{2 \sum_{i=1}^{2t+r-m} (s-i)v_i - 2m + 2T - 2 + K(N-1) + 2t}{2s} \right) \Bigg\}.$$

Some particular cases substantially simplify the above density. If $s = 1$, then we have the following.

Corollary 16. *If $\mathbf{Y} \sim \mathcal{E}_{N-1 \times K}(\boldsymbol{\mu}_{N-1 \times K}, \boldsymbol{\Sigma}_{N-1 \times N-1} \otimes \mathbf{I}_K, h)$, with $\boldsymbol{\Sigma} > \mathbf{0}$, then the Kotz type $s = 1$ non-isotropic noncentral configuration density is given by*

$$\begin{aligned} & \frac{\Gamma_K \left(\frac{N-1}{2} \right)}{\pi^{Kq/2} |\boldsymbol{\Sigma}|^{\frac{K}{2}} |\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}|^{\frac{N-1}{2}} \Gamma_K \left(\frac{K}{2} \right)} \sum_{t=0}^{\infty} \frac{\Gamma \left(\frac{K(N-1)}{2} \right)}{t! \Gamma \left(\frac{K(N-1)}{2} + t \right) \Gamma \left(T - 1 + \frac{K(N-1)}{2} \right)} \\ & \times \sum_{r=0}^{\infty} \frac{1}{r!} [\text{tr}(-R \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu})]^r \sum_{\tau} \frac{\binom{N-1}{2}_{\tau}}{\left(\frac{K}{2} \right)_{\tau}} C_{\tau} (R \mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} (\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U})^{-1}) \\ & \times \left\{ \Gamma \left(T - 1 + \frac{K(N-1)}{2} + t \right) \right. \\ & \left. + \sum_{m=1}^{2t+r} \binom{2t+r}{m} \left[\prod_{i=0}^{m-1} (T-1-i) \right] (-1)^m \Gamma \left(T-1-m + \frac{K(N-1)}{2} + t \right) \right\}. \end{aligned} \tag{14}$$

Now, by assuming $T = 1$ in (14), a confluent hypergeometric class of densities indexed by R is obtained, i.e.

Corollary 17. *If $\mathbf{Y} \sim \mathcal{E}_{N-1 \times K}(\boldsymbol{\mu}_{N-1 \times K}, \boldsymbol{\Sigma}_{N-1 \times N-1} \otimes \mathbf{I}_K, h)$, with $\boldsymbol{\Sigma} > \mathbf{0}$ and $T = 1$, then the Kotz type 1 non-isotropic noncentral configuration density simplifies to*

$$\frac{\Gamma_K \left(\frac{N-1}{2} \right) \text{etr} (R \mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} (\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U})^{-1} - R \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu})}{\pi^{Kq/2} |\boldsymbol{\Sigma}|^{\frac{K}{2}} |\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}|^{\frac{N-1}{2}} \Gamma_K \left(\frac{K}{2} \right)} \times {}_1F_1 \left(-\frac{q}{2}; \frac{K}{2}; -R \mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} (\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U})^{-1} \right).$$

Finally, the normal configuration density can be derived by taking $R = \frac{1}{2}$. In this case we obtain a result from [2] (proposed by Goodall and Mardia [1] with some errors) except for the factor 2^k which comes from their anonymous Jacobian computation.

Another simplification comes from $T = 1$, when the particular configuration density becomes:

Corollary 18. *If $\mathbf{Y} \sim \mathcal{E}_{N-1 \times K}(\boldsymbol{\mu}_{N-1 \times K}, \boldsymbol{\Sigma}_{N-1 \times N-1} \otimes \mathbf{I}_K, h)$, with $\boldsymbol{\Sigma} > \mathbf{0}$, then the Kotz type $T = 1$ non-isotropic noncentral configuration density is given by*

$$\begin{aligned} & \frac{\Gamma_K \left(\frac{N-1}{2} \right)}{\pi^{Kq/2} |\boldsymbol{\Sigma}|^{\frac{K}{2}} |\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}|^{\frac{N-1}{2}} \Gamma_K \left(\frac{K}{2} \right)} \sum_{t=0}^{\infty} \frac{\Gamma \left(\frac{K(N-1)}{2} \right)}{t! \Gamma \left(\frac{K(N-1)}{2} + t \right) \Gamma \left(\frac{K(N-1)}{2s} \right)} \\ & \times \sum_{r=0}^{\infty} \frac{1}{r!} [\text{tr}(\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu})]^r \sum_{\tau} \frac{\binom{N-1}{2}_{\tau}}{\left(\frac{K}{2} \right)_{\tau}} C_{\tau} (\mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} (\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U})^{-1}) \\ & \times \sum_{\kappa \in P_{2t+r}} \frac{(2t+r)! \prod_{j=0}^{2t+r-1} (s-j)^{\sum_{i=j+1}^{2t+r} v_i} \Gamma \left(\frac{2 \sum_{i=1}^{2t+r} (s-i)v_i + K(N-1) + 2t}{2s} \right)}{(-1)^{\sum_{i=1}^{2t+r} v_i} R^{-\frac{\sum_{i=1}^{2t+r} i v_i - t}{s}} \prod_{i=1}^{2t+r} v_i! (i!)^{v_i}}. \end{aligned} \tag{15}$$

Once again, we obtain Corollary 17, and thus the normal case, by taking $s = 1$ in (15) and noting that in this trivial case $v_1 = 2t + r$, $v_i = 0$ for $i = 2, \dots, 2t + r$, and $\prod_{j=0}^{2t+r-1} (s-j)^{\sum_{i=j+1}^{2t+r} v_i} = (1-0)^{\sum_{i=0+1}^{2t+r} v_i} = 1^{2t+r} = 1$. Then the last summation of (15) becomes $(-1)^r R^{t+r} \Gamma \left(\frac{K(N-1)}{2} + t \right)$, and the result is straightforwardly obtained by exponential series, the hypergeometric function definition and Kummer's formula.

Recall that, if the above densities exist, then the arguments in the gamma functions must be positive. This means a careful choice of the Kotz parameters must be made for corrected inferences to be drawn; moreover, given the complexity of the expressions, it is necessary to truncate the series for some t and r in such a way that the parameters T , s and R can be selected [7].

4.3. Bessel configuration density

Another elliptical distribution is known as the Bessel distribution; after some correction in [17], it can be found that the $p \times n$ random matrix has a matrix variate symmetric Bessel distribution with parameters $g, r \in \mathbb{R}, \mu : p \times n, \Sigma : p \times p, \Phi : n \times n$ with $r > 0, g > -\frac{np}{2}, \Sigma > \mathbf{0}$, and $\Phi > \mathbf{0}$ if its probability density function is

$$\frac{[\text{tr}(\mathbf{Y} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \boldsymbol{\mu})\boldsymbol{\Phi}^{-1}]^{\frac{g}{2}} K_g \left(\frac{[\text{tr}(\mathbf{Y} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \boldsymbol{\mu})\boldsymbol{\Phi}^{-1}]^{\frac{1}{2}}}{r} \right)}{2^{g+np-1} \pi^{\frac{np}{2}} r^{np+g} \Gamma \left(g + \frac{np}{2} \right) |\boldsymbol{\Sigma}|^{\frac{n}{2}} |\boldsymbol{\Phi}|^{\frac{p}{2}}},$$

where $K_g(z)$ is the modified Bessel function of the third kind; that is,

$$K_g(z) = \frac{\pi}{2} \frac{I_{-g}(z) - I_g(z)}{\sin(g\pi)}, \quad |\arg(z)| < \pi, \quad g \text{ is integer}$$

and

$$I_g(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k+g+1)} \left(\frac{z}{2} \right)^{g+2k}, \quad |z| < \infty, \quad |\arg(z)| < \pi.$$

In this case the function h takes the form

$$h(y) = \frac{y^{\frac{g}{2}} K_g \left(\frac{1}{r} y^{\frac{1}{2}} \right)}{2^{g+K(N-1)-1} \pi^{\frac{K(N-1)}{2}} r^{K(N-1)+g} \Gamma \left(g + \frac{K(N-1)}{2} \right)}, \tag{16}$$

and the required derivatives for the modified Bessel function are given by

$$K_g^{(k)} = \frac{(-1)^k}{2^k} \sum_{m=0}^k \binom{k}{m} K_{g-k+2m}(z).$$

Then, the k -th derivative of (16) follows after some simplification as

$$h^{(k)} = \frac{1}{2^{g+K(N-1)-1} \pi^{\frac{K(N-1)}{2}} r^{K(N-1)+g} \Gamma \left(g + \frac{K(N-1)}{2} \right)} \times \sum_{m=0}^k \binom{k}{m} \left[\prod_{j=0}^{m-1} \left(\frac{g}{2} - j \right) \right] \sum_{\kappa \in P_{k-m}} \frac{(k-m)! \prod_{j=0}^{k-m-1} \left(\frac{1}{2} - j \right)^{\sum_{i=j+1}^{k-m} v_i}}{2^{\sum_{i=1}^{k-m} v_i} r^{\sum_{i=1}^{k-m} v_i} \prod_{i=1}^{k-m} v_i!(i!)^{v_i}} \times (-1)^{k-m} \sum_{n=0}^{\sum_{i=1}^{k-m} v_i} \binom{\sum_{i=1}^{k-m} v_i}{n} K_{g-\sum_{i=1}^{k-m} v_i+2n} \left(\frac{1}{r} y^{\frac{1}{2}} \right) y^{\sum_{i=1}^{k-m} \left(\frac{1}{2} - i \right) v_i + \frac{g}{2} - m}.$$

Thus the integral S in (11) can now be computed:

$$S = \int_0^{\infty} h^{(2t+r)}(y) y^{\frac{K(N-1)}{2} + t - 1} dy = \frac{1}{\pi^{\frac{K(N-1)}{2}} \Gamma \left(g + \frac{K(N-1)}{2} \right)} \sum_{m=0}^{2t+r} \binom{2t+r}{m} \left[\prod_{j=0}^{m-1} \left(\frac{g}{2} - j \right) \right] \times \sum_{\kappa \in P_{2t+r-m}} \frac{(-1)^{2t+r-m} (2t+r-m)! \prod_{j=0}^{2t+r-m-1} \left(\frac{1}{2} - j \right)^{\sum_{i=j+1}^{2t+r-m} v_i}}{(2r)^{2 \sum_{i=1}^{2t+r-m} i v_i + 2m - 2t} \prod_{i=1}^{2t+r-m} v_i!(i!)^{v_i}}$$

$$\begin{aligned} & \times \sum_{n=0}^{2t+r-m} \sum_{i=1}^{2t+r-m} v_i \binom{2t+r-m}{n} \Gamma \left(\sum_{i=1}^{2t+r-m} (1-i) v_i - m + \frac{K(N-1)}{2} + t - n \right) \\ & \times \Gamma \left(- \sum_{i=1}^{2t+r-m} i v_i + g - m + \frac{K(N-1)}{2} + t + n \right). \end{aligned}$$

The conditions for the existence of S are the same as those indicated for the Bessel distribution plus the conditions imposed by the arguments of the gamma functions, which must be positive.

Thus the configuration density follows from (10).

Corollary 19. If $\mathbf{Y} \sim \mathcal{E}_{N-1 \times K}(\boldsymbol{\mu}_{N-1 \times K}, \boldsymbol{\Sigma}_{N-1 \times N-1} \otimes \mathbf{I}_K, h)$, with $\boldsymbol{\Sigma} > \mathbf{0}$, then the Bessel non-isotropic noncentral configuration density is given by

$$\begin{aligned} & \frac{\Gamma_K \left(\frac{N-1}{2} \right)}{\pi^{\frac{Kg}{2}} |\boldsymbol{\Sigma}|^{\frac{K}{2}} |\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}|^{\frac{N-1}{2}} \Gamma_K \left(\frac{K}{2} \right)} \sum_{t=0}^{\infty} \frac{1}{t! \Gamma \left(\frac{K(N-1)}{2} + t \right) \Gamma \left(g + \frac{K(N-1)}{2} \right)} \\ & \times \sum_{r=0}^{\infty} \frac{1}{r!} [\text{tr}(\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu})]^r \sum_{\tau} \binom{N-1}{\frac{K}{2}}_{\tau} C_{\tau}(\mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} (\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U})^{-1}) \times \sum_{m=0}^{2t+r} \binom{2t+r}{m} \left[\prod_{j=0}^{m-1} \left(\frac{g}{2} - j \right) \right] \\ & \times \sum_{\kappa \in P_{2t+r-m}} \frac{(-1)^{2t+r-m} (2t+r-m)! \prod_{j=0}^{2t+r-m-1} \left(\frac{1}{2} - j \right)^{\sum_{i=j+1}^{2t+r-m} v_i}}{(2r)^{\frac{2t+r-m}{2} \sum_{i=1}^{2t+r-m} i v_i + 2m - 2t} \prod_{i=1}^{2t+r-m} v_i! (i!)^{v_i}} \\ & \times \sum_{n=0}^{2t+r-m} \sum_{i=1}^{2t+r-m} v_i \binom{2t+r-m}{n} \Gamma \left(\sum_{i=1}^{2t+r-m} (1-i) v_i - m + \frac{K(N-1)}{2} + t - n \right) \\ & \times \Gamma \left(- \sum_{i=1}^{2t+r-m} i v_i + g - m + \frac{K(N-1)}{2} + t + n \right). \end{aligned}$$

Finally, if $g = 0$ and $r = \frac{\delta}{\sqrt{2}}$, $\delta > 0$ in Corollary 19, then we have the Laplace non-isotropic noncentral configuration density.

Again it is important to note that, to draw correct inference, the above series must be truncated and the Bessel parameters must be chosen such that the gamma functions exist.

The present study concludes with a distribution which has no closed form for its expression, because the integrals involved cannot be expressed in terms of classical functions.

4.4. Jensen-logistic configuration density

For this case we set h as (see Gupta and Varga [17])

$$h(y) = c \exp(-y) [1 + \exp(-y)]^{-2};$$

then by partition theory the k -th derivative can be computed after some simplification as

$$h^{(k)} = c \sum_{m=0}^k \binom{k}{m} \sum_{\kappa \in P_{k-m}} \frac{(k-m)! \left(\sum_{i=1}^{k-m} v_i + 1 \right)! \exp \left(- \left(1 + \sum_{i=1}^{k-m} v_i \right) y \right)}{(-1)^{m + \sum_{i=1}^{k-m} (1+i) v_i} \prod_{i=1}^{k-m} v_i! (i!)^{v_i} [1 + \exp(-y)]^{2 + \sum_{i=1}^{k-m} v_i}}.$$

Thus, (11) becomes

$$S = \int_0^{\infty} h^{(2t+r)}(y) y^{\frac{K(N-1)}{2} + t - 1} dy$$

$$= c \sum_{m=0}^{2t+r} \binom{2t+r}{m} \sum_{\kappa \in P_{2t+r-m}} \frac{(2t+r-m)! \left(\sum_{i=1}^{2t+r-m} v_i + 1 \right)!}{(-1)^{m+\sum_{i=1}^{2t+r-m} (1+i)v_i} \prod_{i=1}^{2t+r-m} v_i! (i!)^{v_i}} \times \int_0^\infty \frac{e^{-(1+\sum_{i=1}^{2t+r-m} v_i)y}}{(1+e^{-y})^{2+\sum_{i=1}^{2t+r-m} v_i}} y^{\frac{K(N-1)}{2}+t-1} dy.$$

Finally we obtain the configuration from (10).

Corollary 20. *If $\mathbf{Y} \sim \mathcal{E}_{N-1 \times K}(\boldsymbol{\mu}_{N-1 \times K}, \boldsymbol{\Sigma}_{N-1 \times N-1} \otimes \mathbf{I}_K, h)$, with $\boldsymbol{\Sigma} > \mathbf{0}$, then the Jensen-logistic non-isotropic noncentral configuration density is given by*

$$\begin{aligned} & \frac{\pi^{K^2/2} \Gamma_K\left(\frac{N-1}{2}\right)}{|\boldsymbol{\Sigma}|^{\frac{K}{2}} |\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U}|^{\frac{N-1}{2}} \Gamma_K\left(\frac{K}{2}\right)} \sum_{t=0}^\infty \frac{1}{t! \Gamma\left(\frac{K(N-1)}{2} + t\right)} \sum_{r=0}^\infty \frac{1}{r!} [\text{tr}(\boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu})]^r \\ & \times \sum_{\tau} \frac{\left(\frac{N-1}{2}\right)_\tau}{\left(\frac{K}{2}\right)_\tau} C_\tau(\mathbf{U}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \boldsymbol{\mu}' \boldsymbol{\Sigma}^{-1} \mathbf{U} (\mathbf{U}' \boldsymbol{\Sigma}^{-1} \mathbf{U})^{-1}) \times c \sum_{m=0}^{2t+r} \binom{2t+r}{m} \sum_{\kappa \in P_{2t+r-m}} \frac{(2t+r-m)! \left(\sum_{i=1}^{2t+r-m} v_i + 1 \right)!}{(-1)^{m+\sum_{i=1}^{2t+r-m} (1+i)v_i} \prod_{i=1}^{2t+r-m} v_i! (i!)^{v_i}} \\ & \times \int_0^\infty \frac{\exp\left(-\left(1 + \sum_{i=1}^{2t+r-m} v_i\right)y\right)}{[1 + \exp(-y)]^{2+\sum_{i=1}^{2t+r-m} v_i}} y^{\frac{K(N-1)}{2}+t-1} dy. \end{aligned}$$

From the definition of v_i and t , providing that the integral in c exists, we see that the above integral also exists. Nevertheless, for successful inference and for a meaningful configuration sample, the above series must be sufficiently truncated, and then many of the terms in the derivatives vanish.

The inference problem is studied in [7] and [18]. Note, however, that all the preceding densities can be computed by slight modifications of the algorithms described by Koev and Edelman [3]. And this is clearly the most important contribution of the present paper, namely that all the distributions derived here can be computed and thus exact inference can be carried out, without requiring the classical approximations found in the shape theory works.

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