Stability of $\theta$-Methods for Advanced Differential Equations with Piecewise Continuous Arguments

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(Received August 2003; revised and accepted February 2005)

Abstract—This paper deals with the stability analysis of numerical methods for the solution of advanced differential equations with piecewise continuous arguments. We focus on the behaviour of the one-leg $\theta$-method and the linear $\theta$-method in the solution of the equation $x'(t) = ax(t) + a_0x([t]) + a_1x([t+1])$, with real $a$, $a_0$, $a_1$ and $[\cdot]$ designates the greatest-integer function. The stability regions of two $\theta$-methods are determined. The conditions under which the analytic stability region is contained in the numerical stability region are obtained and some numerical experiments are given. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Delay differential equations, Piecewise continuous arguments, Asymptotic stability.

1. INTRODUCTION

This paper is concerned with the numerical solution of the delay differential equations with piecewise continuous arguments (EPCA),

$$x'(t) = f(t, x(t), x(\alpha_1(t)), x(\alpha_2(t))),$$

where the arguments $\alpha_i(t)$, $i = 1, 2$, have intervals of constancy.

In recent years, considerable work has been done on differential equations with piecewise continuous arguments. The first contribution is due to [1,2]. Also, there exists an extensive literature dealing with EPCA in [3–9]. In their papers, it has been shown that, in general, the properties of...
solutions to EPCA and the corresponding differential equations without delay are strikingly different. The task of investigating EPCA is also of considerable applied interest since they include, as particular cases, impulsive and loaded equations of control theory and are similar to those found in some biomedical models. A typical EPCA contains arguments that are constant on certain intervals. A solution is defined as a continuous, sectionally smooth function that satisfies the equation within these intervals. Continuity of a solution at a point joining any two consecutive intervals leads to recursion relations for the solution at such points. Hence, the solutions are determined by a finite set of initial data, rather than by an initial function, as in the case of general functional differential equation. It seems to us that the strong interest in differential equation with piecewise continuous arguments is motivated by the fact that describe hybrid dynamical systems. They combine the properties of differential equations and difference equations. As is well known, the stability of numerical solutions is one of the most attracting topics.

In this paper, we consider the advanced equation with piecewise continuous argument,

$$\dot{x}(t) = ax(t) + a_0 x([t]) + a_1 x([t + 1]), \quad t \geq 0,$$

$$x(0) = x_0,$$

(1.2)

where \(a, a_0, a_1, x_0\), are real constants and \([.]\) denotes the greatest integer function. In [10], some properties of the solution of equation (1.2) are investigated.

**Definition 1.** A solution of equation (1.2) on \([0, \infty)\) is a function \(x(t)\) that satisfies the following conditions.

1. \(x(t)\) is continuous on \([0, \infty)\).
2. The derivative \(x'(t)\) exists at each point \(t \in [0, \infty)\), with the possible exception of the points \([t] \in [0, \infty)\) where one-sided derivatives exist.
3. Equation (1.2) is satisfied on each interval \([n, n + 1) \subset [0, \infty)\) with integral end-points.

**Theorem 1.** If \(a_1 < a/(e^a - 1)\), then equation (1.2) has on \([0, \infty)\) a unique solution,

$$x(t) = (m_0 ([t]) + \lambda m_1 ([t])) \lambda^{[t]} x_0,$$

where \([t]\) is the fractional part of \(t\) and

$$m_0(t) = e^{at} + (e^{at} - 1) a^{-1} a_0, \quad m_1(t) = (e^{at} - 1) a^{-1} a_1.$$

Equation (1.2) is asymptotically stable (i.e., \(x(t) \to 0\) as \(t \to 0\) for all \(x_0\)), if and only if

$$a + a_0 + a_1 \left( a_1 - a_0 - \frac{a(e^a + 1)}{e^a - 1} \right) > 0.$$

(1.3)

**2. THE \( \theta \)-METHODS**

Let \(h = 1/m\) be a given stepsize with integer \(m \geq 1\) and the gridpoints \(t_n = nh\) \((n = 1, 2, \ldots)\), we consider the linear \( \theta \)-methods to (1.1),

$$x_{n+1} = x_n + h \{ \theta f((n + 1) h, x_{n+1}, x^h(\alpha_1 ((n + 1) h)), x^h(\alpha_2 ((n + 1) h)))$$

$$+ (1 - \theta) f(nh, x_n, x^h(\alpha_1 (nh)), x^h(\alpha_2 (nh))) \} \quad (n = 1, 2, \ldots),$$

(2.1)

and the one-leg \( \theta \)-methods applied to (1.1),

$$x_{n+1} = x_n + h f((n + \theta) h, x^h(\alpha_1 ((n + \theta) h)), x^h(\alpha_2 ((n + \theta) h))) \quad (n = 1, 2, \ldots).$$

(2.2)
Here, \( \theta \) is a parameter, with \( 0 \leq \theta \leq 1 \), specifying the method, \( x^h(\alpha_i(t)) \) (\( i = 1, 2 \)) denotes an approximation to \( x(\alpha_i(t)) \), \( i = 1, 2 \), and \( x^h(t) \) is an approximation to \( x(t) \) defined by

\[
x^h(t) = \frac{t - nh}{h} x_{n+1} + \frac{(n+1)h - t}{h} x_n, \quad \text{for } nh < t \leq (n+1)h, \quad n = 0, 1, \ldots.
\]

Applying (2.1) and (2.2) to (1.2), we arrive at the following recurrence relations, respectively,

\[
x_{n+1} = x_n + \theta \left( a x_{n+1} + a_0 x^h \left( \left( n + 1 \right) h \right) \right) + (1 - \theta) \left( a x_n + a_0 x^h \left( \left( n h \right) \right) + a_1 x^h \left( \left( n + 1 \right) h \right) \right),
\]

(2.3)

\[
x_{n+1} = x_n + \theta \left( a x_{n+1} + (1 - \theta) x_n \right) + a_0 x^h \left( \left( n + \theta \right) h \right) + a_1 x^h \left( \left( n + \theta \right) h + 1 \right),
\]

(2.4)

Let \( n = km + l \) (\( l = 0, 1, 2, \ldots, m-1 \)). Then, we define \( x^h([t_n + \delta h]) \), \( 0 \leq \delta \leq 1 \), as \( x_{km} \) according to Definition 1. As a result, (2.3) and (2.4) reduce to the same recurrence relation,

\[
x_{km+l+1} = a x_{km+l} + \beta x_{km} + \gamma x_{(k+1)m}, \quad l = 0, 1, 2, \ldots, m-1,
\]

(2.5)

where

\[
\alpha = 1 + \frac{ha}{1 - \theta ha}, \quad \beta = \frac{ha_0}{1 - \theta ha}, \quad \gamma = \frac{ha_1}{1 - \theta ha}.
\]

In fact, in each interval \([n, n+1)\), equation (1.2) can be seen as an ordinary differential equation, hence, the \( \theta \)-methods for equation (1.2) are convergent of order 1 if \( \theta \neq 1/2 \) and order 2 if \( \theta = 1/2 \).

### 3. STABILITY ANALYSIS

It is easily seen that (2.5) is equivalent to

\[
x_{(k+1)m} = \frac{\alpha^m + (a_0/a)(\alpha^m - 1)}{1 - (a_1/a)(\alpha^m - 1)} x_{km},
\]

\[
x_{km+l+1} = \left( a^{l+1} + \frac{a_0}{a} (a^{l+1} - 1) \right) x_{km} + \frac{a_1}{a} (a^{l+1} - 1) x_{(k+1)m}, \quad 0 \leq l \leq m - 2.
\]

**DEFINITION 2.** The process (2.3) or (2.4) for equation (1.2) is called asymptotically stable at \((a, a_0, a_1)\) if and only if there exists a constant \( M \), such that for any given \( x_0 \), relation (3.1) defines \( x_n \) that satisfies \( x_n \to 0 \) for \( n \to \infty \) whenever \( m > M \).

**DEFINITION 3.** The set of all triples \((a, a_0, a_1)\) at which the process (2.3) or (2.4) for equation (1.2) is asymptotically stable is called the asymptotic stability region.

In the following, we take \( M = |a| \) and denote the asymptotical stability region by \( S_\theta \) and the set consisting of all triples \((a, a_0, a_1)\) which satisfy condition (3.1) by \( H \), i.e.,

\[
H = \left\{ (a, a_0, a_1) : \left( a + a_0 + a_1 \left( a_1 - a_0 - \frac{a \left( e^a + 1 \right)}{e^a - 1} \right) \right) > 0 \right\}.
\]

(3.2)

We will investigate which conditions lead to \( H \subseteq S_\theta \). For convenience, we divide the region \( H \) into three parts,

\[
H_0 = \{(a, a_0, a_1) \in H : a = 0 \},
\]

\[
H_1 = \{(a, a_0, a_1) \in H \setminus H_0 : a + a_0 + a_1 < 0 \}.
\]

\[
H_2 = \{(a, a_0, a_1) \in H \setminus H_0 : a + a_0 + a_1 > 0 \}.
\]

In view of (3.1),

\[
|x_{km+l+1}| \leq N |x_{km}| \quad (0 \leq l \leq m - 1),
\]

(3.3)

where

\[
N = \left\{ \begin{array}{ll}
\max_{0 \leq l \leq m-1} \left\{ a^{l+1} + \frac{a_0}{a} (a^{l+1} - 1) \right\} + \frac{a_1 (a^{l+1} - 1) (a^m + (a_0/a)(a^m - 1))}{a - a_1 (a^m - 1)} & a \neq 0, \\
\max_{0 \leq l \leq m-1} \left\{ \frac{l + 1 + 1}{a_0} + \frac{a_1 (l+1) (l+1 + a_0)}{m (1 - a_0)} \right\} & a = 0.
\end{array} \right.
\]

Hence, we have the following lemma.
Lemma 1. \( x_n \to 0 \) as \( n \to \infty \) if and only if \( x_{km} \to 0 \) as \( k \to \infty \).

It is well known by (3.1) that \( x_{km} \to 0 \) as \( k \to \infty \) if and only if
\[
|\lambda| < 1, \tag{3.4}
\]
where
\[
\lambda = \frac{\alpha^m + (a_0/a)(\alpha^m - 1)}{1 - (a_1/a)(\alpha^m - 1)}.
\]
Equation (3.4) is equivalent to
\[
(a + a_0 + a_1) \left( a_1 - a_0 - a - \frac{2a}{\alpha^m - 1} \right) > 0. \tag{3.5}
\]

In the similar way, we denote
\[
S_0 = \{(a, a_0, a_1) \in S : a = 0\},
\]
\[
S_1 = \{(a, a_0, a_1) \in S \setminus S_0 : a + a_0 + a_1 < 0\},
\]
\[
S_2 = \{(a, a_0, a_1) \in S \setminus S_0 : a + a_0 + a_1 > 0\}.
\]

Therefore, in order to investigate which conditions lead to \( H \subseteq S_0 \), we only need to prove \( H_i \subseteq S_i \), \( i = 0, 1, 2 \), i.e.,
\[
(a + a_0 + a_1) \left( \frac{2a}{e^a - 1} - \frac{2a}{\alpha^m - 1} \right) \geq 0. \tag{3.6}
\]

It is a simple matter to verify the following lemma which will be used in the proof of the main results in the paper.

Lemma 2. Let \( \varphi(x) = 1/x - 1/(e^x - 1) \), then \( \varphi(x) \) is a decreasing function and \( \varphi(-\infty) = 1 \), \( \varphi(0) = 1/2 \), and \( \varphi(+\infty) = 0 \).

Lemma 3. For all \( m \geq M \),
\begin{itemize}
  \item[(1)] \( (1 + a/(m - \theta a))^m \geq e^a \) if and only if \( 1/2 \leq \theta \leq 1 \) for \( a > 0 \); \( \varphi(-1) \leq \theta \leq 1 \), for \( a < 0 \);
  \item[(2)] \( (1 + a/(m - \theta a))^m \leq e^a \) if and only if \( 0 \leq \theta \leq 1/2 \), for \( a < 0 \); \( 0 \leq \theta \leq \varphi(1) \), for \( a > 0 \).
\end{itemize}

Proof. It is easily seen that for \( m > M \), \( m - \theta a > 0 \) and \( 1 + a/(m - \theta a) > 0 \).

(1) Suppose that for all \( m \geq M \),
\[
\left( 1 + \frac{a}{m - \theta a} \right)^m \geq e^a. \tag{3.7}
\]
Then, for all \( m \geq M \),
\[
\theta \geq \frac{m}{a} - \frac{1}{e^{a/m} - 1}. \tag{3.8}
\]
According to Lemma 2, (3.8) is equivalent to
\[
\frac{1}{2} \leq \theta \leq 1, \quad \text{for } a > 0,
\]
\[
\varphi(-1) \leq \theta \leq 1, \quad \text{for } a < 0.
\]

(2) Suppose that for all \( m \geq M \),
\[
\left( 1 + \frac{a}{m - \theta a} \right)^m \leq e^a. \tag{3.9}
\]
Then, for all \( m \geq M \),
\[
\theta \leq \frac{m}{a} - \frac{1}{e^{a/m} - 1}. \tag{3.10}
\]
In view of Lemma 2, (3.10) is equivalent to
\[
0 \leq \theta \leq \frac{1}{2}, \quad \text{for } a < 0,
\]
\[
0 \leq \theta \leq \varphi(1), \quad \text{for } a > 0.
\]
We note for $a \in \mathbb{R}$,
\[ \frac{a}{e^a - 1} > 0. \]  \hfill (3.11)

For all $m > M$, we have
\[ 1 + \frac{a}{m - \theta a} > 1, \quad \text{for } a > 0, \]
\[ 0 < 1 + \frac{a}{m - \theta a} < 1, \quad \text{for } a < 0, \]
which implies
\[ \frac{a}{a^m - 1} > 0. \]

**Theorem 2.** $H_1 \subseteq S_1$ if and only if
\[ 0 < \theta \leq \varphi (1), \quad \text{for all } a > 0, \]
\[ \varphi (-1) \leq \theta \leq 1, \quad \text{for all } a < 0. \]

**Proof.** From $(a, a_0, a_1) \in H_1$, we have $a \neq 0$, $a + a_0 + a_1 < 0$, and (3.6) reduces to
\[ \frac{a}{e^a - 1} < \frac{a}{(1 + \frac{a}{(m - \theta a)})^m - 1}, \]  \hfill (3.13)

which from (3.11) and (3.12) is equivalent to
\[ \left(1 + \frac{a}{m - \theta a}\right)^m \leq e^a, \quad \text{for } a > 0, \]
\[ \left(1 + \frac{a}{m - \theta a}\right)^m \geq e^a, \quad \text{for } a < 0. \]

In view of Lemma 3, the theorem is proved.

**Theorem 3.** $H_2 \subseteq S_2$ if and only if
\[ \frac{1}{2} \leq \theta \leq 1, \quad \text{for all } a > 0, \]
\[ 0 \leq \theta \leq \frac{1}{2}, \quad \text{for all } a < 0. \]

**Proof.** Assume $(a, a_0, a_1) \in H_2$, we have $a \neq 0$, $a + a_0 + a_1 > 0$, and (3.6) reduces to
\[ \frac{a}{e^a - 1} \geq \frac{a}{(1 + \frac{a}{(m - \theta a)})^m - 1}, \]  \hfill (3.14)

which from (3.11) and (3.12) is equivalent to
\[ \left(1 + \frac{a}{m - \theta a}\right)^m \geq e^a, \quad \text{for } a > 0, \]
\[ \left(1 + \frac{a}{m - \theta a}\right)^m \leq e^a, \quad \text{for } a < 0. \]

In view of Lemma 3, the proof is complete.
**THEOREM 4.** For $\theta$-methods with $0 \leq \theta \leq 1$, we have $H_0 = S_0$.

**Proof.** Assume $a = 0$. Then, $\lambda = (1 + a_0)/(1 - a_1)$ and (3.1) reduces to

$$
(a_0 + a_1) (a_1 - a_0 - 2) > 0,
$$

(3.15)

which is the same as (1.3) with $a = 0$.

**Remark 1.** From the above discussion, we have $1 - \theta a > 0$ for $m > M$. In view of (3.12), it is easy to see $1 - (a_1/a)(\alpha^m - 1) \neq 0$ in $H_1$ and $H_2$. Therefore, processes (2.3) and (2.4) can be going on for $m > M$.

**Remark 2.** Assuming $(a, a_0, a_1) \in H_1$ and $M = 1$, then from the proofs of Lemma 3 and Theorem 2, it is easy to see that the stability conditions are

$$
0 \leq \theta \leq \varphi(a), \quad \text{for } a > 0,
$$

$$
\varphi(a) \leq \theta \leq 1, \quad \text{for } a < 0.
$$

(3.16)

From Lemma 2, $\varphi(a) \leq \varphi(1)$ for $a > 1$ and $\varphi(a) \geq \varphi(-1)$ for $a < -1$. Hence, the range of $\theta$ is smaller than that when $M = |a|$.

**4. NUMERICAL EXPERIMENTS**

In order to give a numerical illustration of the conclusions in the paper, we consider the following four problems,

$$
\begin{align*}
    x'_1(t) &= 5x_1(t) - 7x_1([t]) - 3x_1([t + 1]), & x_1(0) &= 1, \\
    x'_2(t) &= -10x_2(t) - 9x_2([t]) + x_2([t + 1]), & x_2(0) &= 1, \\
    x'_3(t) &= 5x_3(t) - 7x_3([t]) + 2.1x_3([t + 1]), & x_3(0) &= 1, \\
    x'_4(t) &= -0.5x_4(t) - x_4([t]) + 2x_4([t + 1]), & x_4(0) &= 1.
\end{align*}
$$

It can be seen from (1.3) that $(5, -7, -3), (-10, -9, 1) \in H_1$, $(5, -7, 2.1), (-0.5, -1, 2) \in H_2$.

We shall use the $\theta$-method with the stepsize $h = 1/m$ to get the numerical solution at $t = 10$, where the true solutions are $x_1(10) \approx 1.30610137261282E - 2$, $x_2(10) \approx 9.989915755667499E - 1$, $x_3(10) \approx 6.08892607159937E - 1$, and $x_4(10) \approx 9.42689510390442E - 6$ from Theorem 1 and the range of $\theta$ is about $[0, 0.4180]$ for (4.1), $[0.5820, 1]$ for (4.2), $[1/2, 1]$ for (4.3) and $[0, 1/2]$ for (4.4).

In Tables 1 and 2, we have listed the absolute errors $AE$ and the relative errors $RE$ at $t = 10$ of the $\theta$-methods, and the ratio of the errors of the case $m = 50$ over that of $m = 100$. We can see from these tables that the methods preserve their order of convergence. All numerical experiments are in agreement with the conclusions in this paper.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$AE$ with $\theta = 0$</th>
<th>$RE$ with $\theta = 0$</th>
<th>$AE$ with $\theta = 0.5$</th>
<th>$RE$ with $\theta = 0.5$</th>
<th>$AE$ with $\theta = 1$</th>
<th>$RE$ with $\theta = 1$</th>
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<td>-</td>
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<td>4.4045E-3</td>
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Table 2. Problem in the case of $a < 0$.

<table>
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<tr>
<th>m</th>
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<th>(4.3) with $\theta = 0.5$</th>
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<td>8.6817E - 2</td>
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REFERENCES