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Covering Orthogonal Polygons with Star Polygons: The Perfect Graph Approach

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This paper studies the combinatorial structure of visibility in orthogonal polygons. We show that the visibility graph for the problem of minimally covering simple orthogonal polygons with star polygons is *perfect*. A star polygon contains a point p, such that for every point q in the star polygon, there is an orthogonally convex polygon containing p and q. This perfectness property implies a polynomial algorithm for the above polygon covering problem. It further provides us with an interesting duality relationship. We first establish that a minimum clique cover of the visibility graph of a simple orthogonal polygon corresponds exactly to a minimum star cover of the polygon. In general, simple orthogonal polygons can have concavities (dents) with four possible orientations. In this case, we show that the visibility graph is weakly triangulated. We thus obtain an $O(n^8)$ algorithm. Since weakly triangulated graphs are perfect, we also obtain an interesting duality relationship. In the case where the polygon has at most three dent orientations, we show that the visibility graph is triangulated or chordal. This gives us an $O(n^3)$ algorithm. © 1990 Academic Press, Inc.

1. INTRODUCTION

One of the most well-studied class of problems in computational geometry concerns the notion of *visibility*. Two points in the plane are said to be visible to each other in the presence of obstacles (which are generally polygonal) if there exists a straight-line path between the two points which does not meet any of the obstacles. Other notions of visibility involve paths which are not straight lines, e.g., rectilinear or staircase paths. There is an intimate connection between visibility problems and polygon covering problems. In his recent book on the Art Gallery Problem,

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O'Rourke [21] states, "It is my belief that some of the fundamental unsolved problems involving visibility in computational geometry will not be solved until the combinatorial structure of visibility is more fully understood." In this paper (and a companion paper [20]) we attempt to study this combinatorial structure. A *visibility graph* has vertices which correspond to geometric components, such as points, lines or regions, and edges which correspond to the visibility of these components to each other. Here, we will be concerned with the visibility graphs for regions inside a simple orthogonal polygon. We show that a certain class of these visibility graphs is perfect. We use this property of the visibility graph to devise polynomial algorithms for a class of polygon covering problems that are NP-hard in general.

For our purposes, a *polygon* is a closed, connected set of points in the plane, bounded by several (circular) sequences of straight line segments. The segments are called edges, their endpoints are called vertices, and their union is called the boundary of the polygon. In our definition, we allow the limiting case where parts of edges of the polygon boundary may coincide. Thus, in the limiting case, we could get "necks" of zero width, and even polygons of zero area.[1] A polygon is said to be *simple* if it has no holes, i.e., the polygon boundary is composed of a single sequence of straight line segments. In this paper, we are only concerned with simple polygons. Note that under our definition, a straight line segment is, in fact, a simple polygon.

An *orthogonal* (or *rectilinear*) polygon (OP), P, is a polygon with all its edges parallel to one of the two coordinate axes. Let n denote the number of edges on the boundary of P. Note that we are only concerned with simple orthogonal polygons. An orthogonal polygon is said to be *horizontally convex* (or *vertically convex*) if its intersection with every horizontal (resp. vertical) line segment is either empty or a single line segment (or a point in the limiting case of a "neck" of zero width). An *orthogonally convex polygon* (OCP) is both horizontally and vertically convex. A collection of polygons, $C = \{P_1, P_2, ..., P_r\}$, where $P_i \subseteq P$, is said to cover a polygon P if the union of all the polygons in C is P. Whenever we speak of a set of covering polygons for an arbitrary P, it will be assumed that each covering polygon is totally contained in P.

The following classification of orthogonal polygons is due to Culberson and Reckhow [7]. Consider the traversal of the boundary of P in the clockwise direction. At each norner (vertex) of P, we either turn 90° right (outside corner) or 90° left (inside corner).[2] A *dent* is an edge of the perimeter of P, both of whose endpoints are inside corners. The direction of traversing a dent gives its *orientation*: for instance, a dent traversed from west to east has a **N** orientation. We will use the

[1] The limiting case can be avoided if polygons are defined as open sets rather than closed sets. The essence of our results is unaffected by the choice of definition, although in the limiting case, the details of our proofs differ. The closed version appears more natural, and makes our proofs more elegant.

[2] Since a 180° vertex is the limit of two 90° vertices as the length of the edge between them goes to zero, we regard a 180° vertex as two outside corners with an edge of length zero between them.

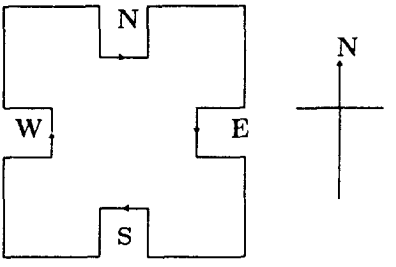

FIG. 1. Orientation of dents.

natural definition of the compass direction, i.e., the positive direction along the y-axis will be referred to as the north direction and so on. Figure 1 illustrates the **N**, **S**, **E**, and **W** dents. For a dent D, $o(D)$ denotes the orientation of D. An OP is classified according the number of orientations of its dents. A class k OP has dents of k different orientations. A class 0 OP does not have dents and is an OCP. A vertically or horizontally convex polygon is a class 2 OP which has only opposing pairs of dents, i.e., either **N** and **S** or **E** and **W**. A class 3 OP without **N** dents is shown in Fig. 2.

The problem of covering general (non-orthogonal) polygons by simpler components has received considerable attention in the literature [5, 6, 15, 16, 28]. Most of these problems are NP-hard, whether or not the polygon to be covered has holes [1, 8, 28]. Several algorithmic results have been obtained for covering orthogonal polygons. For instance, Franzblau and Kleitman have an $O(n^2)$ algorithm for covering a vertically convex orthogonal polygon without holes with a minimum number of rectangles [10]. Keil has provided an $O(n^2)$ algorithm for covering similar polygons with a minimum number of orthogonally convex polygons [17]. Reckhow and Cuberson [23] later provided an $O(n^2)$ algorithm for covering a class 2 orthogonal polygon with a minimum number of orthogonally convex polygons. Recently, the authors [20], and independently, Reckhow, have obtained

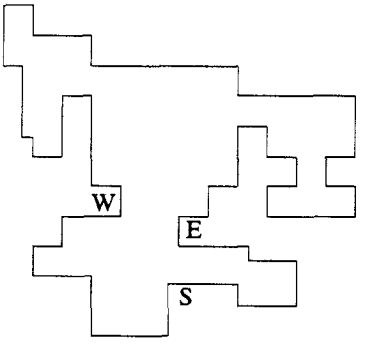

FIG. 2. A polygon with 3 dent orientations.

a polynomial algorithm for minimally covering class 3 polygons with orthogonally convex polygons.

Two kinds of visibility for orthogonal polygons have been studied in the literature [7]. Two points of a polygon P are said to be s-visible if there exists an orthogonally convex polygon in P that contains the two points. Equivalently, we say that two points are s-visible to each other if there exists a staircase path in P that joins them. Two points of the polygon are said to be r-visible if there exists a rectangle that contains the two points. An s-star polygon contains a point p, such that for every point q in the polygon, there is an orthogonally convex polygon containing p and q. An r-star polygon is similarly defined. Thus, an s-star cover is a cover by s-star polygons and an r-star cover is a cover by r-star polygons.

For r-star covers, Keil [17] has provided an $O(n^2)$ algorithm for covering a horizontally convex orthogonal polygon with a minimum number of r-stars, For s-star covers, Culberson and Reckhow [7] provide an $O(n^2)$ algorithm for covering a class 2 orthogonal polygon with a minimum number of s-stars. In general, an orthogonal polygon has dents of four orientations (see Fig. 1). Since this paper does not deal with r-star covers, the word star will be taken to mean an s-star.

The main combinatorial tool of our analysis is the visibility graph. We show that a certain kind of visibility graph, called the *star graph* in this paper, is *perfect*. A perfect graph G [12, 18] has the property that for every induced subgraph H of G, the size of a maximum independent set of H is equal to the size of a minimum clique cover of H. We use this property of the star graph to devise polynomial algorithms for a non-trivial class of the polygon covering problem that is NP-hard in general.

In this paper, we also obtain combinatorial results of the following form concerning coverings of orthogonal polygons. Let an *independent set* of points in a polygon P, with respect to a class of covering polygons C, denote a set of points in P, no two of which can be covered by any covering polygon from the class C. A *duality* theorem for covering problems is of the following form: the size of the minimum cover by polygons from class C is equal to the size of the maximum independent set of points with respect to the class C. Many interesting duality theorems have been obtained for polygonal covering problems. Chvàtal [10] conjectured that a duality theorem holds for the problem of covering orthogonal polygons by rectangles. This conjecture was shown to be false by Szemeredi and Chung (cited in [4]). However, Chaiken *et al.* [4] showed that the duality theorem holds for polygons that are orthogonally convex. Gyóri [13] then showed that the duality relationship holds even if the polygon is only vertically (or horizontally) convex. Later, Saks [26] showed that a graph determined by the boundary squares of the grid induces by the vertices of an OCP is perfect. Other related work includes that of Shearer [27], Boucher [3], and Albertson and O'Keefe [2]. The duality theorem has recently been shown to hold for covering class 3 polygons with orthogonally convex polygons in [20]. In this paper, we show that the duality theorem holds for covering class 4 polygons with star polygons.

We show that for the case where the orthogonal polygon has all four dent orien-

tations, the covering problem can be formulated and solved as the problem of finding a minimum clique cover of the star graph. A minimum clique cover of a graph G is the minimum number of cliques in G such that every vertex of G is in some clique. We further show that the star graph of a class 4 polygon is a weakly triangulated graph [14]. A minimum clique cover for a weakly triangulated graph can be found in $O(v^4)$ time [22], where v is the number of vertices of the graph. This gives us an $O(n^8)$ algorithm, where n is the number of edges of the polygon boundary. Since weakly triangulated graphs are perfect [14], we also obtain the following duality relationship, the minimum number of star polygons needed to cover an orthogonal polygon P is equal to the maximum number of points of P, no two of which can be contained together in a covering star polygon. Finally, we show that the problem of covering orthogonal polygons with three dent orientations with a minimum number of star polygons can be formulated and solved by finding a minimum clique cover for triangulated (chordal) graphs [11, 12]. This gives us an $O(n^3)$ algorithm.

This paper is organized as follows. In Section 2, we discuss the theoretical framework for this problem as discussed in [7, 20, 23]. The star graph is introduced in Section 3. The main results of this paper are presented in Section 4. In Section 5, we provide a theoretical handle on the covering problem by defining the notion of a constriction. The proof of our main theorem is provided in Sections 6 and 7. In Section 8, we address the problem of covering class 3 polygons with star polygons. Our conclusions are presented in Section 9.

2. Preliminaries

In this section we set the theoretical framework and develop some of the basic tools required to analyze the problems of finding minimum OCP and star covers of orthogonal polygons. Some of the fundamental definitions and results stated here are due to Culberson and Reckhow [7, 23]. Throughout this paper, P refers to the simple, orthogonal polygon to be covered.

2.1. *Staircase Paths, OCPs and Star Polygons*

A *staircase path* in P corresponds to a sequence of points $u = x_0, x_1, ..., x_r = v$ contained in P such that (a) each adjacent pair of points, x_i and x_{i+1}, determine a vertical or horizontal line segment which is contained in P and (b) in traversing the staircase path from u to v the edges corresponding to the adjacent pairs of points are traversed in at most two of the four possible compass directions. More informally, a staircase path is a connected sequence of horizontal and vertical edges such that the path alternates between left and right turns. We say $u \vee v$ (read as u *sees* v or u is *visible* to v) if there exists a staircase path joining u and v. Otherwise, we write $u \bar{\vee} v$. Note that staircase paths can share points with the boundary of P, as P is a closed set. Thus, two points u and v that lie on the same edge of P are visible to each other. We say that a staircase path from u to v goes southwest if,

in traversing it from u to v, we go west on all the horizontal segments and south on all vertical segments. Thus, staircase paths between u and v can be of four possible orientations: northeast, northwest, southeast, southwest. The following result of Reckhow and Culberson [23] demonstrates the inherent relationship between staircase paths and OCPs.

LEMMA 1. *For any two points $u, v \in P$, $u \vee v$ if and only if some OCP (contained in P) includes them both.*

Proof. Let an OCP, P', include both u and v. If u and v lie on a horizontal or vertical line in P', then the line segment between them is contained in P', and hence $u \vee v$. Otherwise, without loss of generality, let v lie to the northeast of u. Construct a path in P' from u to v as follows. Starting at u, repeatedly perform the following in order until no further increment to the path is achieved.

(1) Go north until a horizontal edge of P' or the horizontal line through v is reached (whichever comes first).

(2) Go east until a vertical edge of P' or the vertical line through v is reached (whichever comes first).

If we reach v, we have a staricase between u and v, and $u \vee v$. Otherwise, let w, such that $w \neq v$, be the point reached. We consider two cases. First, let w lie on the same horizontal (resp., vertical) line as v. We then have that points neighboring w along the horizontal to its east (resp., along the vertical to its north) are not in P'. Thus the horizontal line segment (resp., vertical line segment), vw, joins two points contained in P', but not fully contained in P'. This contradicts the definition of an OCP. Second, let w lie neither on the horizontal nor on the vertical through v. By construction, w is a point on the boundary of P', such that points neighboring w to its east along the horizontal through w, and points neighboring w to its north along the vertical through w are not in P'. Also by construction, v is to the northeast of w. Now, since P' is connected, some curve c in P' joins v and w. Since v is to the northeast of w, this curve c will intersect either the vertical through w to its north or the horizontal through w to its east. Thus, there exists some line segment, horizontal or vertical, joining two points in P', and not completely contained in P'. This contradicts the fact that P' is an OCP. Thus if an OCP includes u and v, we have that $u \vee v$.

Conversely, if $u \vee v$, then a staircase path in P joins u and v. This staircase path is trivially an OCP that is contained in P, as required. ∎

We now define star polygons. Note that by Lemma 1, this definition is equivalent to the one provided in Section 1.

DEFINITION 1. A *star polygon* (SP) Q is an orthogonal polygon such that there is a point p in Q with the property that p sees every point q in Q.

A *maximal SP* in P is an SP contained in P, but not contained in any other SP contained in P.

LEMMA 2. *Any covering of P by star polygons can be made into a covering of P by the same number of maximal star polygons.*

DEFINITION 2. The *visibility polygon* of $p \in P$, denoted $v(p)$, is the set of all points $q \in P$, such that $p \vee q$.

LEMMA 3. *The boundary between $v(p)$ and any connected component of $P \backslash v(p)$ is a single line segment (horizontal or vertical).*

Proof. The proof is in two parts. We first show that the boundary between $v(p)$ and any connected component Q of $P \backslash v(p)$ is connected. Assume to the contrary that the boundary is disconnected. Then, there is a subset S of P that is bounded by $[v(p) \cup Q]$ and is adjoint from $[v(p) \cup Q]$. Since P is simple, every point of S is contained in P. By definition, no point of S is in $v(p)$. Then, $[S \cup Q]$ is connected and $[S \cup Q] \subset P \backslash v(p)$. Thus, $[S \cup Q]$ is part of a connected component of $P \backslash v(p)$, implying that Q cannot be a connected component of $P \backslash v(p)$ by itself. This gives us a contradiction.

We now show that the boundary between $v(p)$ and Q is a single line segment. Assume to the contrary that the boundary contains adjacent line segments ab, that is horizontal, and bc, that is vertical. Assume without loss of generality, that $v(p)$ lies to the south of ab, and that Q lies to the north of ab. If p lies to the south of ab, then the staircase paths from p to any point on ab goes northeast or northwest. Such a staircase path can be extended vertically upwards to see points in Q, implying that ab is not part of the boundary. Thus, p lies to the north of ab.

If c lies to the north of b then Q is to the west of bc (see Fig. 3). By an argument similar to the one for ab, this implies that p is to the west of bc. Now, the staircase paths from p to a and c, together with ab and bc form an OCP, which must have a nonempty intersection with Q. This implies, by Lemma 1, that p sees points in Q, a contradiction.

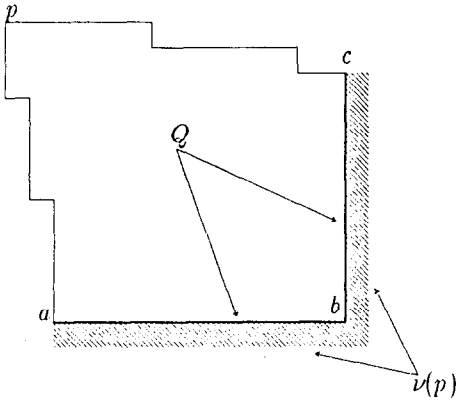

FIG. 3. Proof of Lemma 3; c is to the north of b.

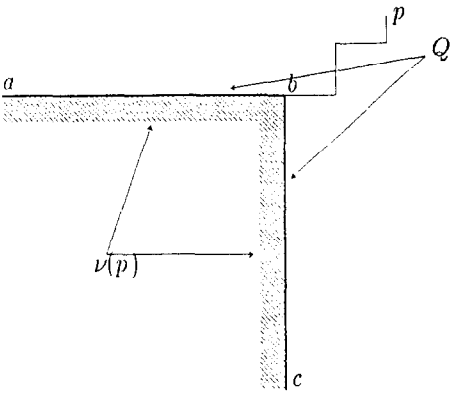

FIG. 4. Proof of Lemma 3; c is to the south of b.

If c lies to the south of b, then Q is to the east of bc (see Fig. 4). As before, p lies to the east of bc. The staircase paths from p to a and b, together with the segment ab form an OCP that includes points in Q. Thus p sees points in Q, a contradiction. ∎

2.2. *Dent Lines and Zones*

For each dent edge D, we construct a *dent line* $\bar{D}$ by extending D in both directions as long as it is contained in P. Notice that under this definition, $\bar{D}$ is in fact the *maximal* line segment that is completely contained in P and that contains the dent edge D. (Although $\bar{D}$ is a *line segment*, it has been called a dent *line* in the literature [20, 23], and we use the same terminology in this paper.) The orientation of $\bar{D}$ is the same as the orientation of D. $\bar{D} \setminus D$ consists of two disjoint line segments, D_l and D_r, one on each side of D. For a dent of **S** orientation, let D_l be the line segment to the left of D and let D_r be the line segment to the right of D (see Fig. 5).

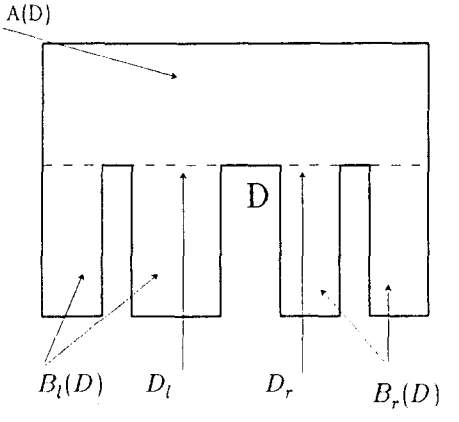

FIG. 5. Dent lines and zones.

D_l and D_r for dents of other orientations are distinguished by rotating the S dent appropriately.

In order to simplify stating the following terms and definitions, let $o(D) = \mathrm{S}$. However, it should not be hard to see that similar statements hold for the other three orientations of dents. Consider $P' = P \backslash \bar{D}$. The set of all connected components of P' meeting $\bar{D}$ to their north is collectively termed the B zone, or $B(D)$. The set of all connected components of P' meeting $\bar{D}$ to their south, together with $\bar{D}$ are collectively termed the A zone, or $A(D)$. Since by definition $A(D)$ includes $\bar{D}$, it is a connected polygon. $B(D)$ can be further subdivided into two zones, $B_l(D)$ and $B_r(D)$, as follows. The connected components of $B(D)$ meeting D_l are together called $B_l(D)$ and the connected components of $B(D)$ meeting D_r are together called $B_r(D)$. Note that under this definition, no point of $\bar{D}$ is in $B(D)$. Also note that under this definition $B_l(D)$ and $B_r(D)$ need not be connected sets; this happens exactly when two dents with the same orientation share the same dent line. Figure 5 shows how zones are delinated. In our definition of zones, $A(D)$, $P \backslash B_l(D)$, and $P \backslash B_r(D)$ are all three connected subsets of P. The following four results about dent lines and zones will be used repeatedly in the rest of this paper.

LEMMA 4. *Let u and v be two points in P. $u \bar{\vee} v$ if and only if there exists a dent D such that $u \in B_l(D)$, $v \in B_r(D)$ or $v \in B_l(D)$, $u \in B_r(D)$.*

Proof. If points u and v are in different B zones of a dent D, then it can easily be shown that $u \bar{\vee} v$. The proof is left to the reader.

Now, assume to the contrary that there exist points u and v in P, such that $u \bar{\vee} v$ and the there exists no dent D in P such that $u \in B_l(D)$, $v \in B_r(D)$ or $v \in B_l(D)$, $u \in B_r(D)$. Construct a connected polygon P' from P as follows: for each dent D, discard from P the appropriate zone, $B_l(D)$ or $B_r(D)$, such that it contains neither u nor v. P' thus defined is guaranteed to contain both u and v. P' is also guaranteed to be connected, as at each stage of the construction, we discard only a B zone of a dent (see the paragraph above). Moreover, P' contains no dents. To see this, observe that each time we perform the construction of discarding the appropriate B zone of each dent D, exactly one new boundary edge is introduced, and this edge is never a dent of P'. Thus, at each stage of the construction, at least one dent is destroyed. Therefore P' is an OCP that includes u and v. By Lemma 1, we conclude that $u \vee v$, a contradiction. ∎

In the case where there is a dent D that satisfies the hypothesis of Lemma 4, we say that D *separates* u and v, and D itself is called a separating dent for u and v. In general, there may be more than one dent separating two points in P. In this case, we will focus our attention on any one separating dent.

LEMMA 5. *Let u, v, and w be three points in P such that a dent D separates u from v. If $u \vee w$ and $v \vee w$, then $w \in A(D)$.*

Proof. Dent D separates u from v, so we can assume without loss of generality

that $u \in B_l(D)$ and $v \in B_r(D)$. Since $u \vee w$, we have that either $w \in B_l(D)$ or $w \in A(D)$. Again, since $v \vee w$, we have that either $w \in B_r(D)$ or $w \in A(D)$. Since $B_l(D) \cap B_r(D) = \varnothing$, we have that $w \in A(D)$, as required. ▮

The following results are stated for particular dent orientations. However, it is not very hard to see that they hold in all of their reflection and rotation symmetric versions.

LEMMA 6. *Let D be a dent and u, w be points such that $u \in B(D)$ and $w \in A(D)$, and let $u \vee w$. Without loss of generality, let $o(D) = \mathbf{N}$, then u lies to the north of $\bar{D}$ and w lies either on $\bar{D}$ or to the south of $\bar{D}$.*

Proof. The boundary between $A(D)$ and $B(D)$ is $\bar{D}$. Any path from a point in $A(D)$ to a point in $B(D)$ must necessarily meet $\bar{D}$. $\bar{D}$ for a dent of **N** orientation is horizontal, such that every point in P neighboring D to its north is in $B(D)$ and every point in P on $\bar{D}$ and neighboring $\bar{D}$ to its sourth is in $A(D)$. $u \vee w$, implying that there is a staircase from u to w in P. Since $u \in B(D)$ and $w \in A(D)$, we conclude that the staircase from u to w travels southwards from u. This implies the hypothesis. ▮

LEMMA 7. *Let u, v and w be points in P such that u lies to the northeast of w and v lies to the northwest of w. If $w \vee u$, $w \vee v$ and $u \bar{\vee} v$ then there is a* **N** *dent separating u from v.*

Proof. If no **N** dent separates u and v, then D, a dent separating them must be a **S**, **E**, or **W** dent. If D is a **S** dent, then by Lemmata 5 and 6, both u and v must be to the south of $\bar{D}$ and w must be either on $\bar{D}$ or to the north of it. Thus, u and v must be to the south of w. But we already have that u and v lie to the northeast and nortwest of w, respectively. If D is a **W** (resp. **E**) dent then, we can show in a similar fashion that u (resp. v) lies to the west (resp. east) of w, which again is a contradiction. So D can only be a **N** dent. ▮

2.3. *Regions*

The set of all dents of P subdivides P into *regions* as follows. For each point p and each dent D of P, we can uniquely specify whether p belongs to $B_l(D)$, $B_r(D)$ or $A(D)$. A *region* u is a maximal subset of P such that for any two points p and q of u, and for any dent D, $p \in Z(D)$ if and only if $q \in Z(D)$, where $Z(D)$ is one of $B_l(D)$, $B_r(D)$, or $A(D)$. In other words, let the dents of P be numbered arbitrarily, and let the *zone vector* of a point in P be the correspondingly numbered list of the (unique) zones of each dent that the point belongs to. Then points p and q belong to the same region if their zone vectors agree.

It is easy to see that a region is a connected subset of P, For, if no two dents of the same orientation share the same dent line, then all zones are connected and a region is the intersection of (a finite number of) zones, and is, therefore connected. If two dents of the same orientation do share the same dent line, then the connected

components of their respective B zones are clearly in different regions, and the same argument applies. It is further clear from the above that the boundary of a region is composed of polygon edges and dent lines. Since there are only $O(n)$ polygon edges and dent lines, and the number of *cells* created in any *arrangement* [9] of $O(n)$ lines in the plane is $O(n^2)$, there are only $O(n^2)$ regions. Note that a region can be a line segment or a point, and thus have zero area under this definition. Figure 6 indicates a polygon with regions that are line segments (marked 9, 10, 11 and 12) and a region that is a point (marked 13). Note also that the zone vector of a *region* is now well-defined. In the following, we relate the notion of visibility by staircase paths with the notion of regions.

DEFINITION 3. Let u and v be regions in P. We say that $u \vee v$ (read as region u *sees* region v) if and only if some OCP (contained in P) includes both u and v.

LEMMA 8 [23]. *Let u and v be regions in P, and let q_u and q_v be arbitrary points in u and v, respectively. Then, there is a staircase path between q_u and q_v if and only if $u \vee v$.*

Proof. If $u \vee v$, then some OCP (contained in P) includes both u and v. Hence, by Lemma 1, there is a staircase path between q_u and q_v.

Conversely, let there be a staircase path between q_u and q_v. It is clear from the preceding definitions that there exists no dent D that separates q_u and q_v. Thus, there exists no dent D such that $q_u \in B_l(D)$ and $q_v \in B_r(D)$, or vice versa. This implies that the connected polygon P' obtained from P by discarding the appropriate B zone of each dent, such that neither q_u nor q_v is in it, will contain q_u and q_v. Since the zone vector of every point in a region is identical, both u and v are completely contained in P'. By our construction above, P' cannot have any dents, and is therefore an OCP, by definition. ∎

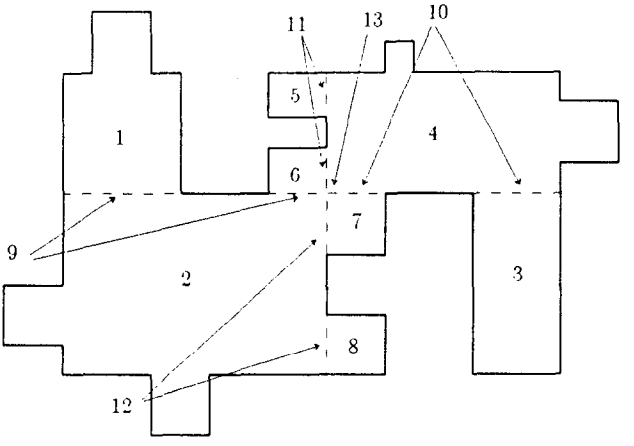

FIG. 6. The regions of an orthogonal polygon.

Note that Lemma 8 indicates that the visibility polygons of two points of a region are identical. Further, every maximal star either includes every point of a region or none. By subdividing P into regions, we have thus discretized the covering problem.

3. The Star Graph

In this section, we define a visibility graph for the problem of covering P with stars, and show a theorem relating a clique cover of this graph and a star cover of P.

The *star graph* $H = (V, E)$ is defined as follows. The vertices of H correspond to the regions of P. Two vertices, u and v are adjacent in H if there is a region w that sees the regions corresponding to u and v.

For two points p, q in P, we write $p \wedge q$ (read as p *indirectly sees* q) if there exists $r \in P$, such that $p \vee r$ and $q \vee r$. Let p and q belong to regions u and v, respectively. We then write that $u \wedge v$. Note that by Lemma 8, if $u \wedge v$ then for every pair of points $p \in u$ and $q \in v$, $p \wedge q$. We refer to the concatenation of the staircase paths $s(p, r)$ and $s(r, q)$ as a 1-*bend* path from p to q, and r is called the *bend point*. If there does not exist any point r that sees both p and q, we write that $p \bar{\wedge} q$ and $u \bar{\wedge} v$.

Notice that the star graph is a visibility graph under indirect visibility. As before, one can physically visualize the star graph H as follows. Arbitrarily choose one point from each region of P. Let this set of points be the vertex set V. An edge of H is a 1-bend path joining two points of V. In the rest of this paper, we work with this physical notion of H.

Having defined the star graph, we now show how this helps in solving the problem.

THEOREM 1. *Let $K = (V(K), E(K))$ be a clique in H. Then, the regions of $V(K)$ can be covered by a single star in P. Thus, a minimum clique cover of H corresponds exactly to a minimum cover of P by star polygons.*

The rest of this section is devoted to proving this theorem.

In a star polygon Q, any two points $p, q \in Q$ see each other indirectly. Thus, every maximal star corresponds to a clique in H. This implies that a cover of P by maximal stars corresponds to a cover of H by cliques. To complete the proof of Theorem 1, we need to show that every clique cover of H corresponds to a star cover of P. It is enough to show that a clique $K = (V(K), E(K))$ in H determines a star in P that covers all the regions corresponding to the vertices of K. We will show that if K is a clique in H, then $\bigcap_{v \in V(K)} v(v)$ is non-empty. Let $p \in \bigcap_{v \in V(K)} v(v)$. Then, $v(p)$ is a star that covers the regions corresponding to the vertices of K. This would complete the proof of Theorem 1.

In the rest of this section, a *cell* denotes a simply connected, compact subset of

the plane. The following Helly-type theorem from topology, due to Molnár [19], will be useful.

THEOREM 2 (Molnár). *Let* $\mathbf{C}$ *be a set of cells in plane. If* $C \cap C'$ *is a cell for every* $C, C' \in \mathbf{C}$ *and* $C \cap C' \cap C'' \neq \emptyset$ *for all* $C, C', C'' \in \mathbf{C}$, *then* $\bigcap \{C \in \mathbf{C}\} \neq \emptyset$.

Observe that if $p \wedge q$, then $v(p) \cap v(q) \neq \emptyset$. The following two results show that if two points indirectly see each other, then the intersection of their visibility polygons is a simply connected region. Further, if three points indirectly see each other, then there is at least one point that sees all three, and, thus, the intersection of their visibility polygons is non-empty. Since every visibility polygon is a cell, Theorem 2 now implies the desired result—viz., a clique C in H corresponds to a maximal star in P.

LEMMA 9. *Let* $p, q \in P$ *such that* $v(p) \cap v(q) \neq \emptyset$. *Then,* $v(p) \cap v(q)$ *is a simply connected polygon.*

Proof. Assume to the contrary that $v(p) \cap v(q)$ is not simply connected. We define $W = v(p) - [v(p) \cap v(q)]$. Since $v(p) \cap v(q)$ is not simply connected. the boundary between $v(q)$ and W is not simply connected. Thus $v(q) \cup W$, which is, of course, $v(p) \cup v(q)$, bounds a region S that is disjoint from $v(q) \cup W$. Every point of S is in P, as P is a simple polygon. Therefore S is a subset of P that is bounded by $v(p) \cup v(q)$. Let R be the connected component of $P - v(p)$ that contains S. By Lemma 3, the boundary of $v(p)$ with R is a single line segment, implying that the boundary of $v(p)$ with S is a single line segment. Similarly, the boundary of $v(q)$ with S is a single line segment. Since at least four orthogonal line segments are required to enclose a region, we obtain a contradiction. ∎

LEMMA 10. *Let* $p, q, r \in P$ *such that* $v(p) \cap v(q) \neq \emptyset$, $v(q) \cap v(r) \neq \emptyset$ *and* $v(r) \cap v(p) \neq \emptyset$. *Then we have that* $v(p) \cap v(q) \cap v(r) \neq \emptyset$.

Proof. Assume to the contrary that $v(p) \cap v(q) \cap v(r) = \emptyset$. Since it is given that $v(p) \cap v(q) \neq \emptyset$, $v(q) \cap v(r) \neq \emptyset$, and $v(r) \cap v(p) \neq \emptyset$, we can easily show that $v(p) \cup v(p) \cup v(r)$ bounds a region, say S, that is disjoint with $v(p) \cup v(q) \cup v(r)$. As in the proof of Lemma 9, we can show that the boundary between $v(p)$ and S is a single line segment, horizontal or vertical. Similar statements can be made about the boundaries between $v(q)$ and S and $v(r)$ and S. This is impossible, because three orthogonal lines cannot bound a region. Therefore, $v(p) \cap v(q) \cap v(r) \neq \emptyset$. ∎

3.1. *Implementation Issues*

In the star graph, the vertex set corresponds to the set of regions of the polygon P. As we mentioned in Section 2.3, the number of regions is $O(n^2)$. Thus, the number of edges in this graph could be $O(n^4)$. A natural question to ask is whether there is a small induced subgraph of the star graph that we can work with and hence reduce computational effort. In this section, we first show that it suffices to

cover a certain subset of the regions of the polygon P, called the sources of P, with maximal stars in order to find a minimum star cover of P. We further show that there are only $O(n)$ sources in a class-3 polygon as against the $O(n^2)$ regions of the polygon.

Let us define a *region* DAG (directed acyclic graph) for P as follows. Note that this definition is slightly different, though equivalent to the region DAG of Reckhow and Culberson [23]. We find our viewpoint more convenient to deal with. For each dent D, let us say that $B_l(D)$ and $B_r(D)$ are both *dominated by* $A(D)$, while $B_l(D)$ and $B_r(D)$ are *incomparable*. For convenience, we also say that $Z(D)$ dominates itself, where $Z(D)$ is one of $B_l(D)$, $B_r(D)$, or $A(D)$. Given the zone vectors of two regions u and v, we say that u *dominates* v if every term of the zone vector of u dominates the corresponding term of the zone vector of v. In this case, we also say that v *is dominated* by u and that u and v are *comparable*. Otherwise, we say that u and v are *incomparable*. The region DAG is the underlying partial order of the comparability graph obtained by placing a vertex for each region and adding a directed edge from vertex v to vertex u if region v is dominated by region u.

Note that if u and v are comparable, then $u \vee v$. For otherwise, there is a dent D that separates u and v, implying that u and v are incomparable. Note also that u and v being incomparable does not imply that $u \not\vee v$. The necessary and sufficient conditions for u and v to see each other is that each of the corresponding terms of their respective zone vectors should be comparable.

A *source* is a region of zero in-degree in the region DAG (see Fig. 7). Similarly, a *sink* is a region of zero out-degree in the region DAG (see Fig. 7). Let us call a dent line $\bar{D}$ of a dent D *pure* if $\bar{D}$ is not shared by some other dent D', such that D and D' have opposite orientations. Let us further say that two regions u and v

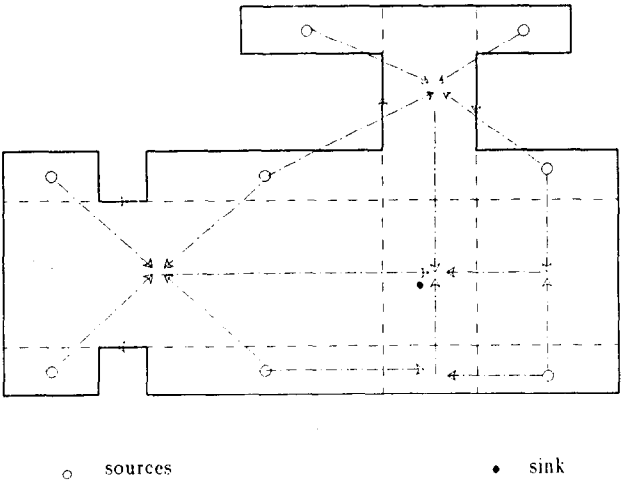

FIG. 7. Sources and sinks in the region DAG.

are *neighbors* if there is a path from a point in u to a point in v that never leaves $u \cap v$. We now claim that no source or sink is bounded by more that one pure dent line of a given orientation. To see that no source is bounded by more than one pure dent line of a given orientation, we reason as follows. Two pure dent lines $\bar{D}_1$ and $\bar{D}_2$ of the same orientation, **S** say, are parallel to each other, such that every point on $\bar{D}_1$ (resp., on $\bar{D}_2$) or neighboring $\bar{D}_1$ (resp., neighboring $\bar{D}_2$) to its north is in $A(D_1)$ (resp., in $A(D_2)$). Without loss of generality, let $\bar{D}_1$ be to the south of $\bar{D}_2$. A region u of P that is bounded by both $\bar{D}_1$ and $\bar{D}_2$ has points that are in $A(D_1)$, and in fact, includes points on $\bar{D}_1$ (note that $\bar{D}_1$ is pure). Let $p \in u$ be a point in u. Let v be a region that neighbors u across the common boundary $\bar{D}_1$. Clearly, every point of v is in $B(D_1)$. Let point q belong to region v. We now claim that u dominates v, and hence cannot be a source. v cannot dominate u, since $u \subseteq A(D_1)$ and $v \subseteq B(D_1)$. If u and v are incomparable, then either some dent separates the two regions, or there exists some dent D such that $u \subseteq B(D)$ and $v \subseteq A(D)$. Both of these are impossible. To see this, note that u and v are neighbors across $\bar{D}_1$, and there is a path joining p and q that only intersects $\bar{D}_1$. Moreover, $\bar{D}_1$ is pure, implying that every dent D that has $\bar{D}_1$ as dent line is of **S** orientation, and hence $u \subseteq A(D)$ and $v \subseteq B(D)$. Therefore, u dominates v, and cannot be a source. By a very similar argument, we can establish that no sink is bounded by more than one pure dent line of a given orientation.

In a class-3 polygon without **N** dents, every **S** dent line is pure. It now follows that a source or a sink in a class 3 polygon can be bounded by at most one horizontal dent line. It is thus clear that every source and sink in class 3 polygons is bounded by some part of the polygon boundary. To count the number of sources of a class 3 polygon, we reason as follows. Let the points where a dent line meets the boundary of P be called *pseudo-vertices.* Since there are at most n dent lines, there are at most $4 \cdot n$ pseudo-vertices. There are two kinds of sources in a class 3 polygon: those that are completely bounded by the polygon boundary, and those that are partly bounded by the polygon boundary and partly by dent lines. The first kind of source always includes a vertex of P. The second kind of source includes a point where a dent line that is part of its boundary meets a polygon edge that is also a part of its boundary. In other words, it includes a pseudo-vertex. This argument shows that class 3 polygons have only $O(n)$ sinks. Note that the polygon in Fig. 7 is a class 4 polygon, which explains why it contains a sink that is completely bounded by dent lines. The following two resultys from [23] show the importance of sources and sinks for the star graph.

LEMMA 11. *If β is a set of maximal star polygons that includes every source of P, then β includes every region of P.*

Proof. Let M be a maximal star that includes a source, u. Let v be a region such that u is dominated by v. From the preceding definitions, we have that every term of the zone vector of v dominates every term of the zone vector of u. Let p be a point M, such that p sees every point in M. In other words, p is a point in the

kernel of M. Thus, there is no dent D such that $p \in B_l(D)$ and $u \subseteq B_r(D)$, or vice versa. From this and from the domination of v over u, we can conclude that if $u \subseteq B_l(D)$ for some dent D, then $v \subseteq A(D) \cup B_l(D)$ and $p \in A(D) \cup B_l(D)$. A symmetric statement can be made about $B_r(D)$. Also, if $u \subseteq A(D)$, then $v \subseteq A(D)$. This shows that there is no dent D that separates p and v. Hence, p sees every point in v. Therefore, v is contained in M, as M is maximal.

To complete the proof of Lemma 11, we note that every region that is not a source dominates some source. We can, therefore, conclude the hypothesis of the lemma. ∎

LEMMA 12. *Let M be a maximal star polygon in P. Then M includes a sink of P that sees every region covered by M.*

Proof. Let p be a point in M that sees every point in M, and let p belong to region u. By Lemma 8, u sees every region of M. Thus, if u is a sink, the lemma is proved. Now, let u not be a sink, and let a sink v dominate u. As noted earlier, this implies that $u \vee v$. Since M is maximal, sink v is included in M.

Let q be an arbitrary point in M. Since $p \vee q$, we conclude that there is no dent D that separates p and q. Thus, for every dent D such that $q \in B_l(D)$, we have that $p \in B_l(D) \cup A(D)$. Since v dominates u, we have that in this case $v \subseteq B_l(D) \cup A(D)$. A similar statement can be made about $B_r(D)$. We can thus conclude that every point in v sees q. Thus, v sees every region covered by M.

We now complete the proof of Lemma 12 by noting that every region that is not a sink is dominated by some sink of P. This proves the lemma. ∎

Lemma 12 tells us that in order to check whether two regions of P indirectly see each other, it suffices to check whether some sink of P sees them both. For a region v and sink u, we can figure out in $O(n)$ time if $u \vee v$ [7]. Thus, in $O(n^5)$ time, one can list the regions seen by the sinks of P. Now, for each pair of regions, we can check in $O(n)$ time if they see each other indirectly, thus constructing H in $O(n^5)$ time.

Let H_s be the subgraph of the star graph H that is induced by the sources of P. We now estimate the complexity of constructing H_s for class 3 polygons. Culberson and Reckhow [7] provide an $O(n^2)$ algorithm to find the sources and sinks of a class 3 polygon. For a source v and sink u, it is easy to figure out in $O(n)$ time if $u \vee v$. Thus, in $O(n^3)$ time. we can list the sources seen by the sinks of P. Now, for each pairs of sources, we can figure out in $O(n)$ time if they see each other indirectly, thus constructing H_s in $O(n^3)$ time. Further, by Theorem 1, we can conclude that a minimum clique cover of H_s corresponds exactly to a minimum cover β of the source of P by stars. By Lemma 11, such a cover can be easily made into a minimum star cover of P by converting the stars in β into maximal stars.

4. Weakly Triangulated Graphs and the Star Graph

In this section, we state our main results about the problem of covering a class 4 polygon with a minimum number of stars. We first assert that the star graph of a class 4 polygon is weakly triangulated. Ryan Hayward [14] proved that weakly triangulated graphs are perfect. By Hayward's theorem (Theorem 4), we then conclude that the star graph is perfect. This gives us the duality theorem for the problem of covering class 4 polygons with stars. Finally , we analyze the computational complexity of finding a minimum star cover of class 4 polygons. We first need the following definition.

Definition 5. A graph G is said to be *weakly triangulated* if neither G nor G^c, the complement of G, contain induced (or chordless) cycles of length greater than four.

We now state our main theorem, the proof of which is contained in the next two sections.

Theorem 3. *The star graph of an orthogonal polygon P is weakly triangulated.*

We now state Hayward's theorem [14]. Theorem 3, together with Hayward's theorem, would then provide us with the duality theorem.

Theorem 4 (Hayward). *Weakly triangulated graphs are perfect.*

Corollary 1 (Duality theorem). *The minimum number of star polygons needed to cover an orthogonal polygon P is equal to the maximum number of points of P, no two of which are contained together in a covering star polygon in P.*

As we mentioned in Section 3.1, the star graph of a class 4 polygon has $O(n^2)$ vertices and $O(n^4)$ edges. Further, it can be found in $O(n^5)$ time. A minimum clique cover a weakly triangulated graph can be found in $O(v^4)$ time [22], where v denotes the number of vertices of the graph. Thus, a minimum clique cover of the star graph can be found in $O(n^8)$ time. It is not very hard to see that a minimum star cover of the polygon P can be easily obtained from this. Thus, there is an $O(n^8)$ algorithm for finding a minimum star cover for P.

5. Constrictions

The main purpose of this section is to provide a theoretical handle on the problem of covering orthogonal polygons with stars. Consider two points, $p, q \in P$, such that they do not have any 1-bend staircase parth between them, i.e., $p \bar{\wedge} q$. Given such a pair of points, we will identify a connected subset of P, Cons(p, q), with the following properties: (a) there is no staircase path from either p or q to

any point in Cons(p, q) and (b) any path in P from p to q must pass through this region. We call this region the *constriction between p* and q. In a sense, the existence of the constriction is the reason why there is no 1-bend path from p to q. We first give a formal definition of a constriction and then obtain certain useful properties.

Let p and q be points of P, and let $p \bar{\wedge} q$. Clearly $v(p) \cap v(q) = \varnothing$. Let $Q_1, ..., Q_k$ denote the connected components of $Q = P \setminus [v(p) \cup v(q)]$. We first claim that there is a unique connected component of Q which shares a boundary with both $v(p)$ and $v(q)$.

LEMMA 13. *There is a unique component* $Q_i \subseteq Q$ *such that* $v(p)$, $v(q)$ *and* Q_i *together form a connected polygon.*

Proof. Clearly, $Q_i \cap Q_j = \varnothing$ for $i \neq j$. Since P is connected. this implies that there exists at least one i such that $P_i = Q_i \cup [v(p) \cup v(q)]$ is connected. Now, let there be i, j, with $i \neq j$, such that P_i and P_j are connected. Then, $[v(p) \cup v(q)] \cup [Q_i \cup Q_j]$ bounds a region S that is disjoint from it. Since P is simple, every point in S is in P. This implies that Q_i and Q_j were not connected components of $P \setminus [v(p) \cup v(q)]$, a contradiction. ∎

DEFINITION 4. The unique component $Q_i \subseteq Q$, which shares a boundary with both $v(p)$ and $v(q)$, is called the *constriction* between p and q. The restriction between p and q is denoted by Cons(p, q) (see Fig. 8).

It is important to note that $P \setminus$Cons(p, q) consists of two connected sets, one containing p and the other containing q. In the rest of this paper, when we say "u is one the same side of Cons(p, q) as p," we mean that point u is in the connected set of $P \setminus$Cons(p, q) that contains p. We now state four important properties concerning Cons(p, q). Properties 1 and 2 deal with the physical characteristics of Cons(p, q).

PROPERTY 1. *The boundary between* Cons(p, q) *and* $v(p)$ *(resp.,* $v(q)$*) is a single line segment (horizontal or vertical), called the p dent line (resp., the q dent line).*

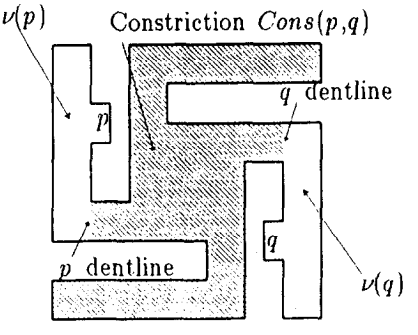

FIG. 8. Cons(p, q), the constriction between p and q.

Proof. Consider the connected components of $P \setminus v(p)$. By Definition 4, $\text{Cons}(p, q)$ and $v(q)$ lie in the same connected component S of $P \setminus v(p)$. The boundary of S with $v(p)$ is the same as the boundary of $\text{Cons}(p, q)$ and $v(p)$, which, by Lemma 3, is a single line segment (horizontal or vertical). Similarly, the boundary of $v(q)$ and $\text{Cons}(p, q)$ is a single line segment (horizontal or vertical). ▮

Property 2 is stated for a particular dent orientation. However, it should not be hard to see that similar statements hold for all four dent orientations.

PROPERTY 2. *Let the p dent line of* $\text{Cons}(p, q)$ *be vertical. Let* r' *be a point on the p dent line, and let r be any point in* $\text{Cons}(p, q)$ *to the east of* r', *such that* $r \vee r'$. *Then, an* **E** *dent separates p from r.*

Proof. Since p sees r', and does not see r to the east of r', every staircase from p to r' has to travel to the northwest or southwest. Thus, r' sees p to its northeast or southeast, and sees r to its east, and $p \bar{\vee} r$. By Lemma 7, there is an **E** dent separating p and r. ▮

Property 2, in fact, gives partial justification for the use of the term "dent line" in referring to the boundary of $\text{Cons}(p, q)$ with $v(p)$ or $v(q)$. This boundary, in fact, corresponds to part of the dent line of the dent postulated to exist by Property 2. It is worth noting that since p sees every point on the p dent line, $\text{Cons}(p, q)$ does not contain any point on the p dent line. A similar situation holds for the q dent line. Properties 3 and 4 are concerned with the nature of paths in P that pass through $\text{Cons}(p, q)$. Property 3 follows immediately from the definition of a constriction.

PROPERTY 3. *Let S be a path in P connecting p and q. Then, there is a connected subpath M of S that joins a point on the p dent line and a point on the q dent line of* $\text{Cons}(p, q)$, *such that every internal point of M is contained in* $\text{Cons}(p, q)$.

PROPERTY 4. *Let* $r \wedge p$ *and* $r \wedge q$.

(1) *If* $r \in \text{Cons}(p, q)$, *there is a* 1*-bend path from a point on the p dent line to a point on the q dent line with r as bend point.*

(2) *If* $r \notin \text{Cons}(p, q)$, *r has a single staircase path that meets both the p dent line and q dent line.*

Proof. (1) Since $r \wedge p$ and $r \wedge q$, we have that r has staircase paths L and M to points in $v(p)$ and in $v(q)$, respectively. Since $r \in \text{Cons}(p, q)$, L and M will constitute staircase paths from r to points on the p and q dent lines. (2) If $r \notin \text{Cons}(p, q)$, assume, without loss of generality, that r is on the same side of $\text{Cons}(p, q)$ in P as p. Clearly, there is a staircase path L from r to some point in $v(q)$, and hence, to the q dent line. L has to intersect the p dent line, thus establishing a staircase from r that meets both the p dent line and the q dent line. ▮

We now define three types of constrictions. The other types of constrictions do not figure in our analysis:

Type I. In a type I constriction, the p dent and the q dent line are parallel and there exist points, r_1 and r_2, on the p dent line, respectively, such that $r_1 \vee r_2$.

Type II. In a type II constriction, the p dent line and the q dent line are orthogonal and there exist points, r_1 and r_2, on the p dent line and the q dent line, respectively, such that $r_1 \vee r_2$.

Type III. In a type III constriction, there exist no two points r_1 and r_2 on the p dent line and the q dent line, respectively, such that $r_1 \vee r_2$. However, there exist points r_1 and r_2 dent line and the q dent line, respectively, such that $r_1 \wedge r_2$. Thus, if $r \wedge p$ and $r \wedge q$, then $r \in \text{Cons}(p, q)$.

The next three results will be used repeatedly in the following three sections. In what follows, $p, q \in P$, such that $p \bar{\wedge} q$. Lemma 14 follows immediately from Property 4 and the definitions of the three types of constrictions.

LEMMA 14. *If* $\exists r \in P$, *such that* $r \wedge p$ *and* $r \wedge q$, *then* $\text{Cons}(p, q)$ *is of type* I, II, *or* III.

Consider a path $S = ps_1 s_2 \cdots s_k q$ in the star graph connecting vertices p and q. S can also be viewed physically as the sequence $(ps_1, s_1 s_2, ..., s_{k-1} s_k, s_k q)$ of 1-bend paths joining points p and q in P. It is easily observed that for any point $x \in S$, some vertex in $\{p, s_1, ..., s_k, q\}$ sees x by a staircase path.

LEMMA 15. *Let* $r \in \text{Cons}(p, q)$, *such that* $r \wedge p$ *and* $r \wedge q$. *Let* $S = (ps_1, s_1 s_2, ..., s_{k-1} s_k, s_k q)$ *be a sequence of* 1*-bend paths, with* $k \geq 1$. *Then,* $\exists i \in \{1, ..., k\}$ *such that* $r \wedge s_i$.

Proof. By Property 4, there is a 1-bend path L from r' on the p dent line to r'' on the q dent line with r as bend point. Without loss of generality, let the p dent line be vertical, and let the staircase path from r to r' be southwest (the other cases are handled similarly). Thus, r sees every point below r' on the p dent line (see Fig. 9).

Assume to the contrary that $r \bar{\wedge} s_i$, $\forall i \in \{1, ..., k\}$. By Property 3, we know that there is a connected subpath M of S, such that M is internally contained in $\text{Cons}(p, q)$ and joins s' on the p dent line and s'' on the q dent line. Every point of M is seen by some s_i, where $i \in \{1, ..., k\}$, since p and q do not see points in $\text{Cons}(p, q)$. Thus, r does not see any points of M, or we are done. By the preceding arguments, let s' be above r' on the p dent line. Clearly, $r \bar{\vee} s'$. Since r' sees s' to its north, and r' sees t to its northeast, Lemma 7 implies that there is a **N** dent D separating r and s'. We have $s' \in B_r(D)$ and $r \in B_l(D)$ (see Fig. 9). $r' \vee s'$ and $r' \vee r$, implying by Lemma 5 that $r' \in A(D)$. Thus, L crosses $\bar{D}_l$ at some point, say l'.

By the definition of a constriction, we have $s'' \notin B_r(D)$. Thus, M crosses $\bar{D}_r$ at m', such that m' is to the west of l'. Hence, $r \vee m'$, implying that $r \wedge s_i$, for some $i \in \{1, ..., k\}$. ∎

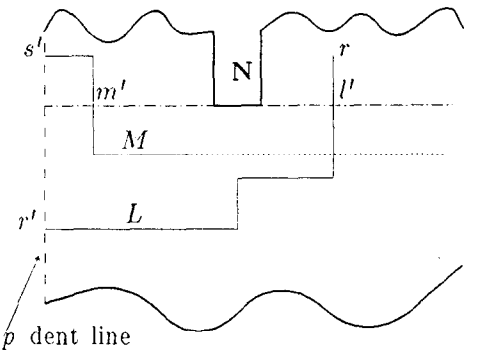

FIG. 9. Proof of Lemma 15.

LEMMA 16. *Let* Cons(p, q) *be of type* II. *Let* $r \notin \text{Cons}(p, q)$, *such that* $r \wedge p$ *and* $r \wedge q$. *Let* $S = (ps_1, s_1s_2, ..., s_{k-1}s_k, s_kq)$ *be a sequence of* 1*-bend paths, with* $k \geq 1$. *Then,* $\exists i \in \{1, ..., k\}$ *such that* $r \wedge s_i$.

Proof. Let r be on the same side of Cons(p, q) as p (the other case is symmetrical). Without loss of generality, assume that the p dent line is vertical and the q dent line is horizontal. By Property 4, there is a staircase path L (southeast, say) from r to r'' on the q dent line that meets the p dent line at r'. Since L goes southeast from r, the point r sees every point of the p dent line below r' and every point of the q dent line to the right of r'' (see Fig. 10).

For r not to see some point of S inside Cons(p, q), S has to meet the p dent line above r', and the q dent line to the left of r''. Thus, S meets L inside Cons(p, q), say at x. Since $x \in \text{Cons}(p, q)$, we have that $p \bar{\vee} x$ and $q \bar{\vee} x$. Thus, for some $i \in \{1, ..., k\}$, $s_i \vee x$. This implies that $r \wedge s_i$. ∎

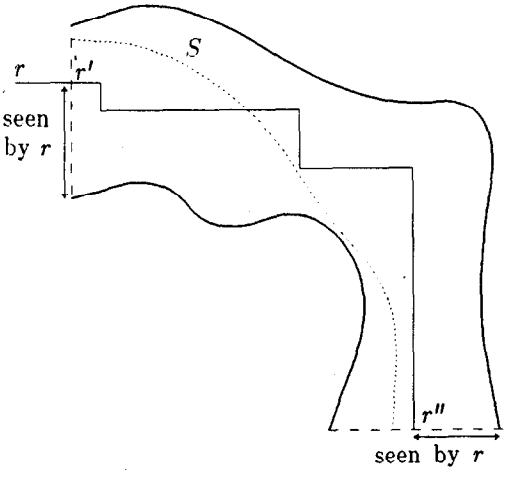

FIG. 10. Proof of Lemma 16.

6. Induced Cycles of the Star Graph

In this section, we establish one part of the proof of Theorem 3, namely that the star graph contains no induced cycles of length greater than four.

Lemma 17. *The star graph H does not contain an induced cycle of length five or more.*

Proof. Assume to the contrary that $C = (V(C), E(C))$ is such an induced cycle, $|V(C)| \geq 5$. For convenience, let $V(C) = \{v_0, v_1, ..., v_{k-1}\}$, $k \geq 5$, and let $\langle v_i, v_{i+1} \rangle \in E'$. Note that all indices here and in the rest of this paper are modulo k. By assumption, edge $\langle v_1, v_3 \rangle \notin E$, implying that $v_1 \bar{\wedge} v_3$. Hence, we have the constriction $\text{Cons}(v_1, v_3)$. Henceforth, $\text{Cons}(v_1, v_3)$ will be referred to as $\text{Cons}(1, 3)$ for notational convenience.

We now assert that point v_2 cannot be in $\text{Cons}(1, 3)$. Assume to the contrary that $v_2 \in \text{Cons}(1, 3)$. Edges $\langle v_3, v_4 \rangle$, $\langle v_4, v_5 \rangle$, ..., $\langle v_{k-1}, v_0 \rangle$, $\langle v_0, v_1 \rangle$ correspond to a sequence S of 1-bend paths connecting points v_1 and v_3. $v_2 \wedge v_1$ and $v_2 \wedge v_3$, and hence by Lemma 15, $v_2 \wedge v_i$, for some $i \in \{4, 5, ..., k-1, 0\}$, thus establishing the chord $\langle v_2, v_i \rangle$. Since point v_2 cannot be in $\text{Cons}(1, 3)$, we have that $\text{Cons}(1, 3)$ is not of type III.

We now show that $\text{Cons}(1, 3)$ is not of type II. Let points v_2 be outside $\text{Cons}(1, 3)$. By Lemma 16, if $\text{Cons}(1, 3)$ is of type II, $v_2 \wedge v_i$, for some $i \in \{4, 5, ..., k-1, 0\}$, again establishing the chord $\langle v_2, v_i \rangle$.

Now, let $\text{Cons}(1, 3)$ be of type I, and let the two dent lines be vertical such that the v_1 dent line is to the west of the v_3 dent line (the other cases are similar). Further, without loss of generality, let v_2 be on the same side of $\text{Cons}(1, 3)$ as v_1. By Property 4, there is a staircase L (southeast, say) from point v_2 to a point in $v(v_3)$, meeting the v_1 dent line and v_3 dent line at v_2' and v_2'', respectively. Therefore, v_2 sees every point below v_2' on the v_1 dent line and every point below v_2'' on the v_3 dent line (see Fig. 11). To prevent chords, S is forced to meet the v_1 dent line and v_3 dent line above v_2' and v_2'', respectively.

Let M be the connected subpath of S that joins a point m' on the v_1 dent line and a point m'' on the v_3 dent line, such that every internal point of M is in $\text{Cons}(1, 3)$. Since P is a simple polygon, every point in the region R that is bounded by M, the staircase L between v_2' and v_2'' and the line segments $v_2'm'$ and $v_2''m''$ is in P. Let x be a southernmost point on M. In other words, no point of M is below the horizontal line through x.

As asserted above, m' and m'' are vertically above v_2' and v_2'', respectively. Further, M does not cross L, else $v_2 \wedge v_i$, for some $i \in \{4, 5, ..., k-1, 0\}$. Thus, the vertical line passing through x meets a point l' of L to the south of x. Further, the line segment xl' is contained in P, since it is completely contained in R. Thus, x sees l' on L to its south. The above argument, together with the fact that L goes southeast from v_2' to v_2'', establishes that x sees a point x'' on the v_3 dent line to its east. For the same reason, x sees a point x', lying either on the v_1 dent line or on L to its west.

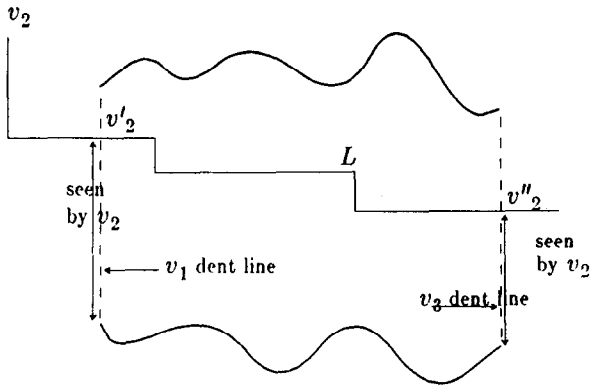

FIG. 11. Proof of Lemma 17.

The point x is either contained in Cons(1, 3) or lies on one of the two dent lines. Let us first analyze the case where x is in Cons(1, 3) (see Fig. 12). If $x \in \{v_4, v_5, ..., v_{k-1}, v_0\}$, we have that $x \vee L'$. Thus, we have shown that $x \wedge v_2$. This gives us the chord $\langle x, v_2 \rangle$. Now let $x \notin \{v_4, v_5, ..., v_{k-1}, v_0\}$. In this case, since M is a connected sequence of 1-bend paths from m' to m'', and x is the southernmost point on M, some point v_i, where $i \in \{4, 5, ..., k-1, 0\}$, sees x by a southeast or southwest path from v_i. This path can clearly be extended into a staircase from v_i to l'. Thus, $v_i \wedge v_2$, giving us the chord $\langle v_2, v_i \rangle$.

Now, let x lie on the v_3 dent line (see Fig. 13). Thus, x is a point in $v(v_3)$, the visibility polygon of v_3. To prevent chords, the only point in $\{v_4, v_5, ..., v_{k-1}, v_0\}$ that can see x is v_4. v_4 cannot see x by an east, west, southeast, or southwest path from v_4, since this path can easily be extended into a staircase from v_4 to l', thus giving us the chord $\langle v_2, v_4 \rangle$. Thus, the staircase from v_4 to x has to travel northeast or northwest. If v_4 were inside Cons(1, 3), then it has to lie on M. To see this,

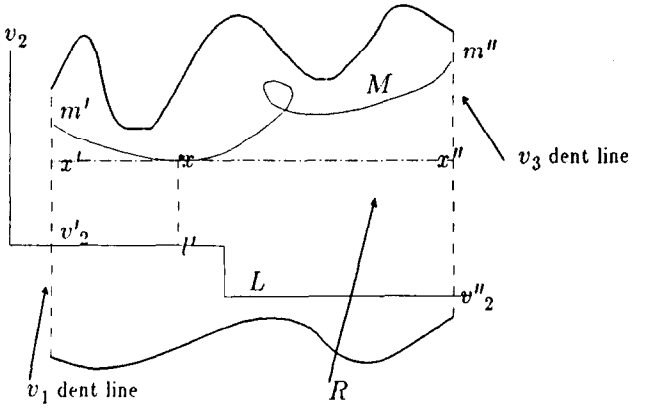

FIG. 12. Proof of Lemma 17; x is in Cons(1, 3).

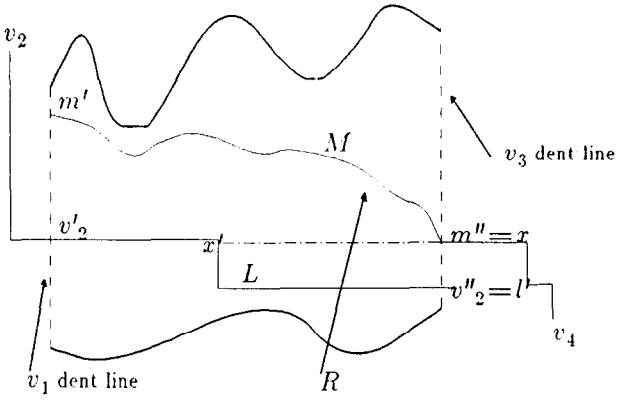

FIG. 13. Proof of Lemma 17; x is on the v_3 dent line.

note that there is no point $v_i \in \{v_5, ..., v_{k-1}, v_0\}$ that has a staircase to a point on the v_3 dent line, else $v_3 \wedge v_i$, a contradiction. Similarly, no point in $\{v_4, ..., v_{k-1}\}$ has a staircase to a point on the v_1 dent line. Thus, there is exactly one connected subpath of S, namely M, that is internally contained in Cons(1, 3). Thus, v_4 lies on M if $v_4 \in$ Cons(1, 3). This tells us that v_4 cannot be in Cons(1, 3), since x was assumed to be a southernmost point of M. Thus, v_4 lies outside Cons(1, 3). v_4 clearly cannot be on the same side of Cons(1, 3) as v_1. This is because $v_4 \wedge v_3$, which implies that there is a staircase from v_4 to some point in $v(v_3)$. This staircase will have to cross the v_1 dent line, thus implying that $v_4 \wedge v_1$, a contradiction. Thus, v_4 lies on the same side of Cons(1, 3) as v_3. It is clear that the bend point of the one bend path between v_4 and v_5 has to lie inside Cons(1, 3), else $v_5 \wedge v_3$. Thus, v_4 sees points inside Cons(1, 3). Since the v_3 dent line is to the east of the v_1 dent line, the staircase from v_4 to x travels northwest. Since x was shown to see either a point x' that is either on the v_1 dent line or on L to its west, we have a northwest staircase from v_4 to x'. This establishes the chord $\langle v_4, v_1 \rangle$ or the chord $\langle v_4, v_2 \rangle$.

By a very similar argument to the one above, if x lies on the v_1 dent line, we can establish that x is seen only by v_0. We can further show that v_0 lies on the same side of Cons(1, 3) as v_1, and that v_0 sees x by a northeast staircase path. Since x sees x'' on the v_3 dent line to its east, we have that $v_0 \vee x''$. Thus, $v_0 \wedge v_3$ giving us the chord $\langle v_3, v_0 \rangle$. ∎

7. INDUCED CYCLES IN THE COMPLEMENT OF THE STAR GRAPH

We establish in this section the other part of the proof of Theorem 3, namely that the complement $H^c = (V, F)$ of the star graph does not have induced cycles of length greater than 4. Note that the presence of edge $\langle u, v \rangle$ in H^c implies that $u \bar{\wedge} v$ in P.

LEMMA 18. *H^c does not contain an induced cycle of length five or greater.*

Proof. Since H does not contain an induced 5-cycle, and the complement of a 5-cycle is also a 5-cycle, H^c cannot contain an induced 5-cycle. Assume to the contrary that there exists an induced cycle $C=(V(C), F(C))$ in H^c, with $|V(C)|>5$. For convenience, let $V(C)=\{v_0, v_1, ..., v_{k-1}\}$, $k>5$, and let $\langle v_i, v_{i+1}\rangle \in F(C)$. Since $v_1 \bar{\wedge} v_2$, we have the constriction Cons(1, 2). Note that since C has no chords, we have that $v_i \wedge v_j$, for $|i-j| \neq 1$.

We now assert that none of $v_4, v_5, ..., v_{k-1}$ can be in Cons(1, 2). Suppose, for instance, that v_4 were in Cons(1, 2). $v_4 \wedge v_1$ and $v_4 \wedge v_2$, and $v_5 \wedge v_1$ and $v_5 \wedge v_2$. From Lemma 15, $v_4 \wedge v_5$, implying that $\langle v_4, v_5\rangle \notin F(C)$, a contradiction. Similar arguments establish that $v_5, ..., v_{k-1}$ cannot be in Cons(1, 2). It is further clear that Cons(1, 2) is not of type III, else each of $v_4, v_5, ..., v_{k-1}$ has to be inside Cons(1, 2) in order to see both v_1 and v_2. Now, let points $v_4, v_5, ..., v_{k-1}$ be outside Cons(1, 2). We now assert that Cons(1, 2) is not of type II. Suppose Cons(1, 2) were indeed of type II. Then, by invoking Lemma 16, we have that $v_4 \wedge v_5$, a contradiction.

We now have that, for all i, Cons($i, i+1$) must be of type I. Let us now further classify type I constrictions as type IA (where the two dent lines are vertical) and type IB (where the two dent lines are horizontal). We now assert that Cons(1, 2) and Cons($i, i+1$) cannot both be of type IA or IB (IA, say), for some $i \in \{4, ..., k-2\}$. To see this, we reason as follows. Since both v_i and v_{i+1} indirectly see both v_1 and v_2, the visibility polygons of both v_i and v_{i+1}, $v(v_i)$ and $v(v_{i+1})$, respectively, share points with $v(v_1)$ and $v(v_2)$. For this reason, neither the v_i dent line nor the v_{i+1} dent line of Cons($i, i+1$) can be inside Cons(1, 2). For the same reason, if the v_i dent line is outside Cons(1, 2), Cons($i, i+1$) and Cons(1, 2) will be on different sides of the v_i dent line. This would imply that $i+1$ cannot indirectly see v_1 and v_2, a contradiction.

A simple combinatorial argument now shows that it is impossible to obtain an induced 7 cycle using only type IA and IB constrictions for cycle edges. Thus, let $k=6$.

It is clear that at least two of Cons(1, 2), Cons(3, 4), and Cons(5, 0) must be of the same type (IA or IB). Assume, without loss of generality, that Cons(1, 2) and Cons(3, 4) are both of type IA. We now assert that this would imply that Cons(2, 3) must be of type IA and that Cons(1, 2) is contained in Cons(2, 3) (see Fig. 14a). This follows easily because $v_1 \wedge v_4$ and $v_2 \wedge v_4$, which implies that the only arrangement possible is as shown in Fig. 14a. Thus, the only way that one could possibly obtain an induced 6 cycle is by using type IA constrictions for three consecutive edges and type IB constrictions for the other three edges. Let edges $\langle v_1, v_2\rangle$, $\langle v_2, v_3\rangle$, and $\langle v_3, v_4\rangle$ correspond to type IA constrictions and let $\langle v_4, v_5\rangle$, $\langle v_5, v_0\rangle$, and $\langle v_0, v_1\rangle$ correspond to type IB constrictions. The arrangement is shown in Figs. 14a and b.

Since we have shown that v_0 is not in Cons(2, 3), and $v_0 \wedge v_2$ and $v_0 \wedge v_3$, Property 4 implies that there is a single staircase path K from v_0 that meets both the v_2 dent line and the v_3 dent line of Cons(2, 3), and passes through Cons(2, 3).

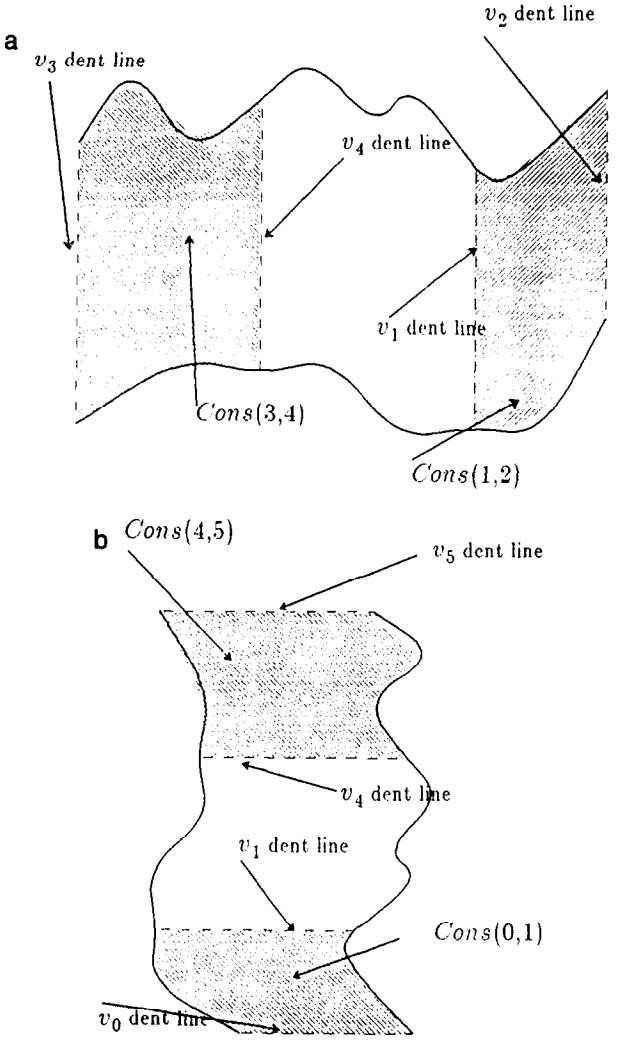

FIG. 14. Proof of Lemma 18.

Since the v_1 dent line of Cons(1, 2) is vertical and is in Cons(2, 3), (see Fig. 14a), K intersects the v_1 dent line of Cons(1, 2), implying that v_0 sees points in $v(v_1)$. This shows that $v_0 \wedge v_1$. But $\langle v_0, v_1 \rangle$ is a cycle edge of H^c, a contradiction. ∎

8. An $O(n^3)$ Algorithm for Class 3 Polygons

In this section, we show that if P is a class 3 polygon, then the star graph H is triangulated [12, 18]. We then show that this gives us an $O(n^3)$ algorithm for finding the star cover for P.

DEFINITION 6 (12, 18]. A graph is said to be *triangulated* (or *chordal*) if it contains no induced cycles of length greater than three.

In general, for class 4 polygons, the star graph H is not triangulated: there exist induced 4-cycles in H (see Fig. 15). For the case where P has only three dent orientations, H is triangulated, as the following theorem shows.

THEOREM 5. *The star graph H of a class 3 polygon is triangulated.*

Proof. By Lemma 17, we have that H does not contain induced cycles of length greater than four. It now suffices to show that H cannot have an induced 4 cycle.

Assume to the contrary, that $\langle v_1, v_2\rangle$, $\langle v_2, v_3\rangle$, $\langle v_3, v_0\rangle$, and $\langle v_0, v_1\rangle$ is an induced 4-cycle in H. Since $\langle v_1, v_3\rangle$ is not present, $v_1 \bar{\wedge} v_3$. Consider Cons(1, 3). Neither v_2 nor v_0 is in Cons(1, 3), else by Lemma 15, $v_2 \wedge v_0$, and $\langle v_2, v_0\rangle$ would be a chord. This also implies that Cons(1, 3) is not of type III.

If Cons(1, 3) is of type II, then by Lemma 16, $v_2 \wedge v_0$, a contradiction.

Thus, Cons(1, 3) is a type I constriction. Without loss of generality, let the two dent lines be vertical and let the v_1 dent line be to the west of the v_3 dent line (see Fig. 16a). Since Cons(1, 3) is a type I constriction, there is a staircase path from a point on the v_1 dent line to a point on the v_3 dent line. This staircase path goes northeast or southeast from the v_1 dent line. Thus, there is a point r' on the v_1 dent line that sees a point r in Cons(1, 3) to its east. By Property 2, we have established the presence of an **E** dent separating v_1 and r. Similarly, one can establish the presence of a **W** dent in P.

Without loss of generality, let v_2 be on the same side of Cons(1, 3) as v_1. By Property 4, there is a staircase L (southeast, say) from v_2 to v_2'' on the v_3 dent line, intersecting the v_1 dent line at v_2'. v_2 sees every point below v_2' on the v_1 dent line and every point below v_2'' on the v_3 dent line.

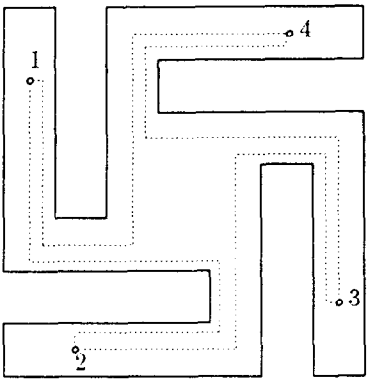

FIG. 15. An induced 4-cycle.

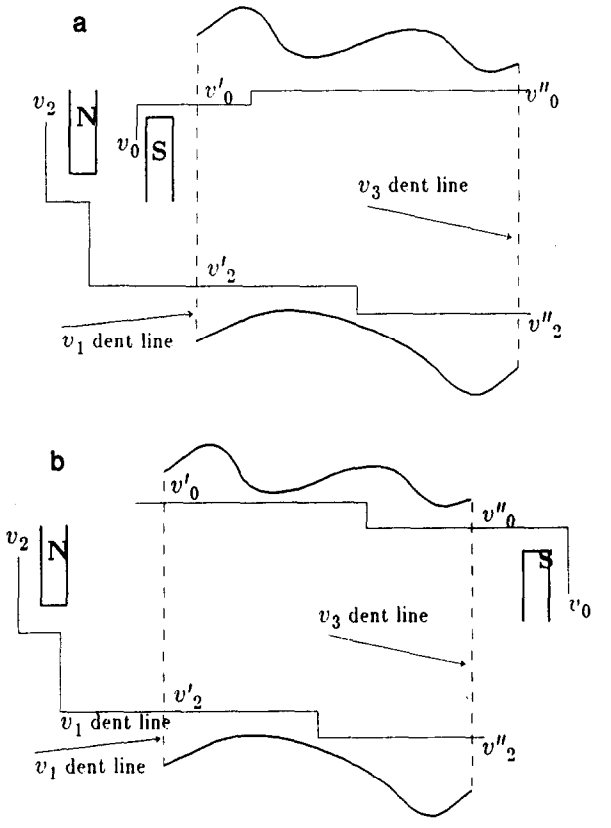

FIG. 16. Proof of Theorem 5.

Case 1. v_0 is on the same side of Cons(1, 3) as v_1. Then, every staircase from v_0 to v''_0 on the v_3 dent line, and hence to v'_0 on the v_1 dent line, has to be northeast (else, $v_0 \wedge v_2$) (see Fig. 16a). Since v_2 does not see v'_0, v'_2 sees v'_0 to its north , and v'_2 sees v_2 to its northwest, Lemma 7 implies that a **N** dent separates v_2 and v'_0. By a similar argument, a **S** dent separates v_0 from v'_2. This establishes that there exist four different dent orientations, a contradiction.

Case 2. v_0 is on the same side of Cons(1, 3) as v_3. Then the staircase from v_0 to v'_0 on the v_1 dent line, and hence to v''_0 on the v_3 dent line, has to be northwest (else, $v_0 \wedge v_2$) (see Fig. 16b). Since v_2 does not see v'_0, v'_2 sees v'_0 to its north, and v'_2 sees v_2 to its northwest, Lemma 7 implies that a **N** dent separates v_2 and v'_0. By a similar argument, a **S** dent separates v_0 from v''_2. This establishes the existence of four different dent orientations, a contradiction. ∎

Let H_s be the subgraph of the star graph H that is induced by the sources of P. Clearly, H_s is chordal for class 3 polygons. As we showed in Section 3.1, all we

require now is a minimum clique cover of H_s. The advantage of this formulation is that instead of dealing wiht $O(n^2)$ regions, we need only consider the $O(n)$ sources. By the algorithm of Section 3.1, H_s can be constructed in $O(n^3)$ time for a class 3 polygon. Gavril's algorithm [11], together with an algorithm due to Rose, Tarjan, and Lueker [25], now gives, in $O(n^2)$ time, the minimum clique cover of H, implying that the star cover of P can be obtained in $O(n^3)$ time.

9. Further Work

The main contribution of this paper has been the demonstration of the intimate connection between star covers and classes of perfect graphs, and deriving the duality relationship of Section 4. The main tool of our analysis has been the visibility graph for regions inside an orthogonal polygon. We have demonstrated certain interesting combinatorial properties of these kinds of graphs. At present, most of the interesting special cases of the problem of covering polygons with simpler polygons that have polynomial time solutions give rise to visibility graphs that are perfect [20, 24, 26, 27]. In the cases of covering simple polygons with convex or star polygons, the visibility graphs are not perfect, and the problems are both NP-hard [1, 8]. Again, in the case of covering orthogonal polygons with a minimum number of rectangles, the visibility graph is not perfect, and this problem is also known to be NP-hard [8]. It is our belief that a careful examination of the combinatrial structure of different kinds of visibility graphs may lead to the solution of other open problems in computation geometry.

(1) For the problem of covering class 4 orthogonal polygons with a minimum number of OCPs, the visibility graph is imperfect, as shown in [20]. We conjecture that this problem is also NP-hard.

(2) Is the visibility graph for the problem of covering orthogonal polygons with a minimum number of *r-stars* perfect? We conjecture that the visibility graph is a quasi-parity graph [18], and hence perfect.

Acknowledgments

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