JOURNAL OF FUNCTIONAL ANALYSIS 31, 360–376 (1979)

On Normed Jordan Algebras Which Are Banach Dual Spaces

FREDERIC W. SHULTZ

Mathematics Department, Wellesley College, Wellesley, Massachusetts 02181

Communicated by the Editors

Received May 20, 1977

Alfsen, Shultz, and Størmer have defined a class of normed Jordan algebras called JB-algebras, which are closely related to Jordan algebras of self-adjoint operators. We show that the enveloping algebra of a JB-algebra can be identified with its bidual. This is used to show that a JB-algebra is a dual space iff it is monotone complete and admits a separating set of normal states; in this case the predual is unique and consists of all normal linear functionals. Such JB-algebras ("JBW-algebras") admit a unique decomposition into special and purely exceptional summands. The special part is isomorphic to a weakly closed Jordan algebra of self-adjoint operators. The purely exceptional part is isomorphic to $C(X, M_3^8)$ (the continuous functions from X into M_3^8).

INTRODUCTION

In [3] a JB-algebra A is defined as a real Banach space A with product $\circ$ such that $(A, \circ)$ is a Jordan algebra with identity satisfying the norm axioms $\| a^2 \| = \| a \|^2$ and $\| a^2 \| \leqslant \| a^2 + b^2 \|$ for all $a, b \in A$. (In [3] there is a third axiom, $\| a \circ b \| \leqslant \| a \| \| b \|$, which can be shown to be redundant by modifying the proof of the analogous result for C^*-algebras in [4].)

The example which motivates this definition is the class of JC-algebras: norm closed linear spaces of self-adjoint operators on a Hilbert space closed under the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. (In the sequel we will often refer to a JC-algebra with identity as a Jordan operator algebra). Clearly such an algebra is a JB-algebra.

Another example of a JB-algebra is M_3^8: the 3×3 hermitian matrices over the Cayley numbers. It is known that M_3^8 is not isomorphic to any Jordan operator algebra. The main results of [3] show that every JB-algebra can be constructed from these two examples in the manner now to be described. The "JB-factors" are either isomorphic to M_3^8 or to a Jordan operator algebra. For every JB-algebra A there exists a unique Jordan ideal J such that A/J is (isometrically) isomorphic to a Jordan operator algebra and every factor representation of A not annihilating J is onto M_3^8. Thus A will be isomorphic to a Jordan

0022-1236/79/030360–17$02.00/0

operator algebra iff this ideal J is $\{0\}$; this is equivalent to requiring that A satisfy a certain polynomial identity.

One might hope that the ideal J would be a direct summand, so that A could be written as a sum of a Jordan operator algebra and a purely exceptional part. However, this is in general not true, as shown by an example in [3]. By analogy with associative operator algebras (C^*-algebras), this is not surprising: in general a norm-closed algebra will not split apart with respect to a given property. However, such desirable behavior is common if instead one works with weakly closed algebras (von Neumann algebras). This it is natural to investigate the corresponding class of JB-algebras.

Since a JB-algebra cannot in general be represented as an algebra of operators on a Hilbert space, the notion of "weakly closed JB-algebra" makes no sense. However, recall that non-spatial characterizations of von Neumann algebras are known. Kadison [15] showed that an abstract C^*-algebra $\mathscr{A}$ admits a faithful representation as a weakly closed *-algebra of operators iff the self-adjoint part of $\mathscr{A}$ is monotone complete and there exists a separating set of normal states for $\mathscr{A}$. Another characterization is that of Sakai [19] who showed that a C^*-algebra admits such a representation iff it is a Banach dual space.

Both of these characterizations are possible candidates for the notion of a JB-algebra of "weakly closed" character. In §2 below it is shown that fortunately these conditions are equivalent for JB-algebras; a JB-algebra satisfying these equivalent conditions is called a JBW-algebra in the sequel. In the case of a Jordan operator algebra, these properties are equivalent to the existence of a faithful representation as a weakly closed Jordan operator algebra (a so-called "JW-algebra").

The main result of this paper can now be stated. Every JBW-algebra A admits a unique decomposition $A = A_{sp} \oplus A_{ex}$ into special and "purely exceptional" parts. Here A_{sp} will be isomorphic to a JW-algebra, and A_{ex} will be isomorphic to $C(X, M_3^8)$: the continuous functions from X into M_3^8, where X is hyperstonean. We remark that JW-algebras have been studied extensively by Topping [25, 26] and Stormer [23, 24].

We now briefly will summarize the contents of this paper. In Section 1 the relationship of the enveloping algebra and bidual of a JB-algebra is investigated, and it is shown that the enveloping algebra (as defined in [3]) can be identified with the bidual. This result is used in §2 to show that a JB-algebra is a Banach dual space iff it is monotone complete and admits a separating set of normal states; in this case the predual is unique. In §3 the decomposition $A = A_{sp} \oplus A_{ex}$ is established for JBW-algebras. The key result needed to prove this is that if $\varphi: A \to M_3^8$ is a surjective homomorphism of a JBW-algebra A onto M_3^8 then A contains a copy of M_3^8 as a subalgebra. As a corollary along the way, it is shown that matrix units can always be lifted from the quotient of a von Neumann algebra by a norm closed two-sided ideal. (If the quotient is finite dimensional, this follows from [17]).

1. The Enveloping Algebra and Bidual of a JB-Algebra

We will begin by showing that if A is a JB-algebra then A^{**} is also a JB-algebra when equipped with the Arens [5] product. A special case of this is already known: if A is a JC-algebra then A^{**} is isomorphic to a JC-algebra and thus is a JB-algebra [9].

Lemma 1.1. *For each α in an index set I let M_α be a copy of M_3^8. Then $(\sum_{\alpha \in I} M_\alpha)^{**}$ is a JB-algebra for the Arens product.*

Proof. By $\sum M_\alpha$ we mean the l^∞-direct sum of the spaces M_α, i.e., the space of bounded functions from I into M_3^8 with supremum norm and pointwise operations. If X is the Stone-Čech compactification of the discrete space I, then $\sum M_\alpha$ will be isomorphic to the space $C(X, M_3^8)$ of continuous functions from X into M_3^8.

To calculate the bidual of $C(X, M_3^8)$ we first represent this space as a tensor product. (See [21] for background. If E_1 and E_2 are Banach spaces, $E_1 \otimes E_2$ will denote their algebraic tensor product. The least cross norm whose dual norm is a cross norm will be denoted by λ; the greatest cross norm by γ. The completion of $E_1 \otimes E_2$ for a norm α will be written $E_1 \otimes_\alpha E_2$.)

If $g = \sum_1^n f_i \otimes m_i \in C(X) \otimes M_3^8$, define $\bar{g} \in C(X, M_3^8)$ by $\bar{g}(x) = \sum f_i(x)\, m_i$. Then since M_3^8 is finite dimensional, the map $g \to \bar{g}$ is an algebraic and isometric isomorphism from $C(X) \otimes M_3^8 = C(X) \otimes_\lambda M_3^8$ onto $C(X, M_3^8)$ (see [22: p. 355] and [12: p. 90]).

Since M_3^8 is finite dimensional, by a result of Gil de Lamadrid [11: Cor. 5.1], $(C(X) \otimes_\lambda M_3^8)^* \cong C(X)^* \otimes_\gamma (M_3^8)^*$ isometrically. Since the dual norm γ^* of γ is λ, again using finite dimensionality of M_3^8 gives

$$(C(X) \otimes_\lambda M_3^8)^{**} \cong C(X)^{**} \otimes_{\gamma^*} (M_3^8)^{**} = C(X)^{**} \otimes_\lambda M_3^8$$

isometrically, and the last expression coincides with the algebraic tensor product $C(X)^{**} \otimes M_3^8$. An easy calculation shows that the Arens product on $C(X)^{**} \otimes M_3^8$ is the tensor product of the Arens product on each factor, and thus we've established so far that

$$\left(\sum M_\alpha\right)^{**} \cong (C(X, M_3^8))^{**} \cong (C(X) \otimes_\lambda M_3^8)^{**} \cong C(X)^{**} \otimes_\lambda M_3^8.$$

Finally, it is folklore that the bidual of $C(X)$ is algebraically and isometrically isomorphic to $C(Y)$ for some compact Hausdorff space Y. (One way to verify this is to note that $C(X)$ is isomorphic to a JC-algebra, so $C(X)^{**}$ is a JB-algebra by [9]. Furthermore, since $C(X)$ is associative then $C(X)^{**}$ is also associative [5: p. 839], and an associative JB-algebra is isomorphic to some $C(Y)$ [3:

Prop. 2.3]. Alternatively, this could be derived from the relationship of a C^*-algebra and its bidual, e.g., [2, pp. 98-99]). Thus

$$\left(\sum M_\alpha\right)^{**} \cong C(Y) \otimes_\lambda M_3{}^8 \cong C(Y, M_3{}^8)$$

which completes the proof that $(\sum M_\alpha)^{**}$ is a JB-algebra.

THEOREM 1.2. *If A is a JB-algebra then A^{**} is a JB-algebra for the Arens product.*

Proof. We first note that if the theorem holds for certain JB-algebras then it holds for their JB-subalgebras and (finite) direct sums. To verify the former, suppose the theorem holds for a JB-algebra A and B is a JB-subalgebra of A (i.e., a norm closed subalgebra containing an identity). Let $T: B \to A$ be the injection of B in A; then $T^{**}: B^{**} \to A^{**}$ is isometric and preserves the Arens product [7: Thm. 6.1], so B^{**} is isomorphic to a JB-subalgebra of A^{**}. For direct sums, $(A_1 \oplus A_2)^{**} \cong A_1{}^{**} \oplus A_2^{**}$ (algebraically and isometrically) which establishes the claim above.

Now let A be any JB-algebra. By [3: Cor. 5.7, Thm. 8.6], A can be imbedded as a JB-subalgebra of $\sum B_s(H_\alpha) \oplus M_\beta$ where $B_s(H_\alpha)$ is all self-adjoint operators on the Hilbert space H_α and each M_β is a copy of $M_3{}^8$. In turn $\sum B_s(H_\alpha)$ can be imbedded in $B_s(\sum H_\alpha)$, which is a JC-algebra. By [9], the theorem holds for $B_s(\sum H_\alpha)$, and by Lemma 1.1 it holds for $\sum M_\beta$. By the remarks above, this proves the theorem holds for the arbitrary JB-algebra A. ∎

Recall from [3] that the strong topology on A^{**} is the topology determined by the seminorms $a \mapsto \langle a^2, \rho\rangle^{1/2}$ for $0 \leqslant \rho \in A^*$. The next lemma shows that weak* and strongly continuous linear functionals on A^{**} coincide; the argument is that in [20: Thm. 1.8.9] adapted to the present context.

We first make some observations which will be useful in the proof of the lemma. Since the Arens product on A^{**} is commutative (Theorem 1.2) then multiplication on A^{**} is weak*-continuous in each variable separately [5]. Thus for each $a \in A^{**}$ and $\rho \in A^*$ there exists a functional in A^* which we denote $a \circ \rho$ such that $\langle a \circ b, \rho\rangle = \langle b, a \circ \rho\rangle$ for all $b \in A^{**}$. Note in particular that the map $a \mapsto a \circ \rho$ will be continuous for the respective weak topologies $w(A^{**}, A^*)$ and $w(A^*, A^{**})$ and so $\{a \circ \rho \mid a \in A^{**}, \|a\| \leqslant 1\}$ will be a $w(A^*, A^{**})$ compact convex circled subset of A^* for each $\rho \in A^*$.

LEMMA 1.3 *A linear functional ρ on the bidual A^{**} of a JB-algebra A is weak*-continuous iff it is strongly continuous.*

Proof. If ρ is weak*-continuous then ρ can be identified with an element of A^*. Therefore, there will exist positive functionals ρ_1 and ρ_2 in A^* such that $\rho = \rho_1 - \rho_2$ (see, e.g., [1: Prop. II. 1.7]). Now by Schwartz' inequality $\langle a, \rho_i\rangle^2 \leqslant$

$\langle a^2, \rho_i\rangle\langle 1, \rho_i\rangle$ for $i = 1, 2$ so each ρ_i and $\rho = \rho_1 - \rho_2$ are strongly continuous.

Now let ρ be any strongly continuous linear functional on A^{**}. Let τ denote the Mackey topology on A^{**} for the duality of A^{**} and A^*. We claim ρ is τ-continuous on bounded subsets of A^{**}. Suppose $\| a_\alpha \| \leqslant 1$ and $a_\alpha \to 0$ in the τ-topology. Then for each $\omega \in A^*$, $\{a_\alpha \circ \omega\} \subset \{a \circ \omega \mid \| a \| \leqslant 1\}$ and so by definition of the τ-topology,

$$\langle a_\alpha{}^2, \omega\rangle = \langle a_\alpha, a_\alpha \circ \omega\rangle \to 0.$$

Thus $\{a_\alpha\}$ converges strongly to zero, showing that $\langle a_\alpha, \rho\rangle \to 0$ so ρ is τ-continuous on bounded subsets of A^{**}.

In particular, $\rho^{-1}(0)$ will meet the unit ball in a τ-closed (therefore weak*-closed) set, and so by the Krein-Smulian theorem $\rho^{-1}(0)$ is weak*-closed. Thus ρ is weak*-continuous. ∎

THEOREM 1.4. *The enveloping algebra $\tilde{A}$ of a JB-algebra A coincides with the bidual A^{**}.*

Proof. By construction (cf. [3]) $\tilde{A}$ is a subspace of A^{**} containing A, with the inherited Arens product. By the bipolar theorem, the unit ball of A is weak*-dense (for the natural imbedding) in the unit ball of A^{**}, and so the same is true for the unit ball of $\tilde{A} \supseteq A$. Furthermore, by [3, proof of Thm. 3.10] the unit ball of $\tilde{A}$ is strongly complete, and thus in particular is strongly closed in A^{**}. By Lemma 1.3 it will also be weak*-closed, and so by weak*-density will coincide with the unit ball of A^{**}. This shows $\tilde{A} = A^{**}$. ∎

2. JB-ALGEBRA WHICH ARE DUAL SPACES (JBW-ALGEBRAS)

As discussed in the Introduction, our purpose in this section is to establish the equivalence for *JB*-algebras of two properties used by Kadison and Sakai to abstractly characterize von Neumann algebras.

Below L_a denotes the map $b \mapsto a \circ b$ and U_a denotes the "triple product" map $b \mapsto \{aba\} = 2a \circ (a \circ b) - a^2 \circ b$. Recall that in a Jordan algebra an ideal is a subspace invariant under all multiplication maps L_a.

LEMMA 2.1. *If A is a JB-algebra then every weak*-closed ideal J of A^{**} is of the form $U_c(A^{**})$ for a central idempotent $c \in A^{**}$.*

Proof. By [3: Lemma 9.1] J will contain an increasing approximate identity $\{u_\alpha\}$, i.e., $0 \leqslant u_\alpha \leqslant 1$, $\alpha \leqslant \beta$ implies $u_\alpha \leqslant u_\beta$, and $\| u_\alpha \circ a - a \| \to 0$ for all $a \in J$. Since $A^{**} = \tilde{A}$, A^{**} is monotone complete; let c be the least upper bound of $\{u_\alpha\}$ in A^{**}. Then by [3: Thm. 3.10] $u_\alpha \to c$ strongly. It follows that c

is in J and $c^2 = c$ is an identity for J, and thus is also the greatest idempotent in J. Since J is an ideal, then

$$U_c(A^{**}) \subseteq J = U_c(J) \subseteq U_c(A^{**}),$$

which shows $J = U_c(A^{**})$. Furthermore, if $s^2 = 1$, $s \in A^{**}$, then $U_s(c)$ is an idempotent in J and so $U_s(c) \leqslant c$. Since $U_s{}^2 = I$, then by positivity of the map U_s we have $c = U_s{}^2(c) \leqslant U_s(c) \leqslant c$ so $U_s c = c$. Since this holds for every symmetry s, by [3: Lemma 5.3] c is central. ∎

Recall that a JB-algebra A is *monotone-complete* if whenever $\{a_\alpha\} \subseteq A$ is an increasing net bounded above then $a = \text{l.u.b.}\ \{a_\alpha\}$ exists in A. (We then write $a_\alpha \uparrow a$.) If A is a monotone complete JB-algebra we say $\rho \in A^*$ is *normal* if whenever $a_\alpha \uparrow a$ then $\langle a, \rho\rangle = \lim\langle a_\alpha, \rho\rangle$. Note that the subspace of normal linear functionals in A^* is norm closed.

LEMMA 2.2. *Let A be a monotone complete JB-algebra, and N the space of normal linear functionals in A^*. Then for each $a \in A$, $L_a^*(N) \subseteq N$.*

Proof. By virtue of the easily verified identity $L_a = \frac{1}{2}(U_{1+a} - U_a - I)$ it suffices to show $U_a^*(N) \subseteq N$ for each $a \in A$. Suppose $\{b_\alpha\} \subseteq A$ and $b_\alpha \uparrow b$; we must show that for each $\rho \in N$ $\langle b, U_a^*\rho\rangle = \lim\langle b_\alpha, U_a^*\rho\rangle$. Clearly it suffices to show $U_a b_\alpha \uparrow U_a b$.

If a is invertible then by [3: Prop. 2.5, Prop. 2.7] U_a and $U_a^{-1} = U_{a^{-1}}$ are positive so U_a is an order automorphism of A and the result follows. Now for arbitrary $a \in A$ choose $\lambda > 0$ in $\mathbb{R}$ such that both $\lambda 1 + a$ and $\lambda 1 - a$ are invertible; then

$$U_{\lambda 1+a} b_\alpha \uparrow U_{\lambda 1+a} b, \qquad \text{and} \quad U_{\lambda 1-a} b_\alpha \uparrow U_{\lambda 1-a} b. \tag{*}$$

Now for each $c \in A$ from the definition of $U_{\lambda 1+c}$ it easily follows that $U_{\lambda 1+c} = \lambda^2 I + 2\lambda L_c + U_c$. Using this identity and adding the expressions in (*) gives

$$2\lambda^2 b_\alpha + 2U_a b_\alpha \uparrow 2\lambda^2 b + 2U_a b.$$

(Note that in general $c_\alpha \uparrow c$, $d_\alpha \uparrow d$ implies $(c_\alpha + d_\alpha) \uparrow (c + d)$.) Since by assumption $b_\alpha \uparrow b$, then $U_a b_\alpha \uparrow U_a b$ follows, completing the proof. ∎

THEOREM 2.3. *Let A be a JB-algebra. Then A is isometrically isomorphic to a Banach dual space iff A is monotone complete and admits a separating set of normal states. If one of these equivalent conditions holds, then the predual of A is unique and consists of the space N of normal linear functionals in A^* (for the natural pairing of A and N).*

Proof. (1) Assume first that A is a dual space with predual A_*; we will show A is monotone complete. Observe first that the positive cone A^+ is weak*-

closed. (By the Krein-Smulian theorem it suffices to show the order interval $[0, 1] = A^+ \cap A_1$ is weak*-closed; this follows from weak*-compactness of $A_1 = [-1, 1]$ and the fact that $a \mapsto \frac{1}{2}(a + 1)$ is a homomorphism of $[-1, 1]$ onto $[0, 1]$.) Now A is an order unit space with weak*-closed positive cone, so by [10: Thm. 8] A_* is positively generated, i.e., $A_* = (A_*)^+ - (A_*)^+$.

Now suppose $\{a_\alpha\}$ is a bounded increasing net in A. Then for $0 \leqslant \rho \in A_*$, the net $\langle a_\alpha, \rho\rangle$ converges so $\{a_\alpha\}$ is weak*-Cauchy. Assume without loss of generality that $\| a_\alpha \| \leqslant 1$ for all α. Since A_1 is weak*-compact and thus complete, a_α converges weak* to say $a \in A$. Since A^+ is weak*-closed, it follows that the order in A is determined by the functionals in $(A_*)^+$. Thus $a =$ l.u.b. $\{a_\alpha\}$, and each functional in $(A_*)^+$ is normal. This shows A is monotone complete with a separating set of normal states.

(2) Conversely, suppose A is monotone complete with a separating set of normal states; we will show $A = N^*$ for the natural pairing. By Lemma 2.2 $L_a^*(N) \subseteq N$ for each $a \in A$. Let $J = N^0$ (the polar of N in A^{**}); then $L_a(J) \subseteq J$ for each $a \in A \subseteq A^{**}$. Now by separate weak*-continuity of multiplication on A^{**}, weak*-density of A in A^{**}, and the fact that J is weak*-closed, we can conclude that $L_a(J) \subseteq J$ holds for all $a \in A^{**}$, so J is an ideal in A^{**}. By Lemma 2.1 there exists a central idempotent $d \in J$ such that $J = U_d(A^{**})$. Let $c = 1 - d$; we will show that U_c is an isomorphism from A onto $U_c(A^{**})$, and that the latter can be identified with N^*.

The argument in the proof of [3: Prop. 5.6] shows that U_c is a Jordan homomorphism. Since c is central then $U_c + U_{1-c} = I$, so ker $U_c =$ im $U_{1-c} = J = N^\circ$. Since by assumption the normal states on A separate elements of A, then $A \cap N^\circ = \{0\}$, so U_c is one-to-one on A. By [3: Lemma 9.3] U_c is an isometry of A into $U_c(A^{**})$. We will now show that $U_c(A) = U_c(A^{**})$.

We first show that the image $U_c(A)$ is monotone closed in A^{**}, i.e., if $\{b_\alpha\} \subseteq U_c(A)$ and $b_\alpha \uparrow b \in A^{**}$ then $b \in U_c(A)$. Since U_c is an isometry, it suffices to show that if $\{a_\alpha\} \subseteq A$ and $a_\alpha \uparrow a$ (l.u.b. in A) then $U_c a_\alpha \uparrow U_c a$ in A^{**}. Let b be the l.u.b. of $\{U_c a_\alpha\}$ in A^{**}. Let U_c^* denote the dual map of U_c for the pairing of A^{**} and A^*. Then im $U_c^* = (\ker U_c)^\circ = N^{\circ\circ} = N$. Thus for each $\rho \in A^*$, we have $U_c^* \rho \in N$ and so

$$\langle U_c a_\alpha, \rho\rangle = \langle a_\alpha, U_c^* \rho\rangle \to \langle a, U_c^* \rho\rangle = \langle U_c a, \rho\rangle$$

which shows $U_c a_\alpha \to U_c a$ weak*. Now $U_c a_\alpha \uparrow b$ implies $U_c a_\alpha \to b$ weak* [3: Thm. 3.10] which shows $U_c a_\alpha \uparrow U_c a = b$, proving $U_c(A)$ is monotone closed in A^{**}.

It now follows that $\{b \in A^{**} \mid U_c b \in U_c(A)\}$ is monotone closed in A^{**}, and clearly contains A. Now as observed in [3, end of §3] the argument in [18: Lemma 1] can be Jordanized to show that the monotone closure of A in A^{**} is $\tilde{A} (= A^{**}$ by Theorem 1.4). Thus $U_c(A^{**}) = U_c(A)$ as claimed.

Finally, the restriction map is an isometric isomorphism from $U_c(A^{**})$ onto $(U_c^*(A^*))^* = N^*$, so $a \mapsto U_c a \mapsto U_c a|_N$ is an isometry of A onto N^*. Furthermore, for $\rho \in N$, $U_c^* \rho = \rho$ implies $\langle U_c a, \rho \rangle = \langle a, U_c^* \rho \rangle = \langle a, \rho \rangle$, so A is the dual space of N for the natural pairing.

(3) To complete the proof of the theorem, suppose A satisfies one of the equivalent conditions above. Let A_* be any predual of A and let φ be the natural imbedding of A_* in $(A_*)^{**} = A^*$. The argument in (1) shows each functional in A_* is normal so $\varphi(A_*) \subseteq N$. In (2) we showed that $A = N^*$. Since $\varphi(A_*)$ is norm closed in A^*, by the Hahn-Banach theorem $\varphi(A_*) = N$ follows. ∎

Remark. Note that (1) of the proof above shows that if a *JB*-algebra A is a dual space (i.e., A is a *JBW*-algebra) then every normal linear functional in A^* is the difference of normal positive linear functionals.

COROLLARY 2.4. *Let A be a JC-algebra; then the following are equivalent:*

(i) *A is monotone complete and admits a separating set of normal states;*

(ii) *A is a Banach dual space;*

(iii) *A can be faithfully represented as a weakly closed Jordan algebra of self-adjoint operators ("JW-algebra").*

Proof. (i) implies (ii) follows from Theorem 2.3. (ii) implies (iii): By [9] there exists an isometric isomorphism of A^{**} onto a weakly closed *JC*-algebra which is a homeomorphism for the weak* and weak operator topologies, respectively. Now in part (2) of the proof of Theorem 2.3, it was shown that there exists a central idempotent c in A^{**} such that $A \cong U_c(A^{**})$. The latter is weak*-closed (thus weak operator closed in the representation mentioned above), and this shows (ii) implies (iii).

Finally, (iii) implies (i) follows at once from the fact that $B_s(H)$ is monotone complete and admits a separating set of normal states. ∎

3. Decomposition of a JBW-Algebra into Special and Exceptional Summands

We are going to show that every *JBW*-algebra A admits a unique decomposition $A = A_{sp} \oplus A_{ex}$ into special and "purely exceptional" summands. We will also classify purely exceptional *JBW*-algebras. The key result used in proving these results is Proposition 3.8, which shows that every homomorphism from a *JBW*-algebra onto M_3^8 splits. That is, if $\varphi: A \to M_3^8$ is a homomorphism from A onto M_3^8 then there is a subalgebra $A_0 \subseteq A$ such that φ restricted to A_0 is an isomorphism of A_0 onto M_3^8.

We will need to establish that Jordan matrix units can be lifted from any quotient of a *JBW*-algebra. We begin with several results on lifting symmetries.

Recall that an element s of a JB-algebra is a *symmetry* if $s^2 = 1$. We say s is a *partial symmetry* if s^2 is an idempotent; more specifically, if s is a partial symmetry with $s^2 = e$ then we say s is an e-symmetry. Note that in this case s and $s^2 = e$ are compatible (cf. [3: §4]) and so representing the norm closed associative subalgebra generated by s, e, 1 as $C(X)$ [3: Prop. 2.3, Lemma 5.2] $e \circ s = s$ follows, and so s will lie in the subalgebra $\{eAe\}$. Note also that if A is a JBW-algebra and $e^2 = e \in A$ then by [3: Prop. 4.11] $A_e = \{eAe\}$ is a JB-algebra which is monotone complete and admits a separating set of normal states. Thus by Theorem 2.3, A_e will be a JBW-algebra.

We will say elements a, b in a JB-algebra are *orthogonal* when $a \circ b = 0$. Lemmas 3.1, 3.2, 3.3 give some elementary properties of this relation.

LEMMA 3.1. *Let s, t be symmetries in a JB-algebra A and let $x \in A$. Then*

(i) $s \circ x = 0$ *iff* $U_s x = -x$ *iff* $\frac{1}{2}(I - U_s)\, x = x$;

(ii) $s \circ x = 0$ *implies x^2 and s are compatible*;

(iii) $s \circ t = 0$ *implies* $U_s U_t = U_t U_s$.

Proof. For (i) and (ii) recall that the subalgebra generated by 1, s, x is special (by the theorem of Shirshov-Cohn [13: p. 48]). Now (i) follows from the observation that in an associative algebra if $s^2 = 1$ then $sx + xs = 0$ is equivalent to $sxs + x = 0$. In the same associative context $sx + xs = 0$ implies that s and x anti-commute so s and x^2 commute. Define an idempotent e by $e = \frac{1}{2}(s + 1)$; then e and x^2 commute so

$$ex^2e + (1 - e)\, x^2(1 - e) = x^2.$$

By [3: Lemma 2.11] this implies e and x^2 (thus s and x^2) are compatible, which proves (ii).

To prove (iii), let $s \circ t = 0$. Then using (i) and the identity

$$U_a U_b U_a = U_{\{aba\}} \tag{3.1}$$

([13: p. 52]), we get

$$U_s U_t U_s = U_{\{sts\}} = U_{-t} = U_t .$$

Since $U_s{}^2 = I$, $U_s U_t = U_t U_s$ follows. ∎

LEMMA 3.2. *Let e and f be orthogonal idempotents in a JB-algebra and let s be an $(e + f)$-symmetry. Then $(e - f) \circ s = 0$ iff $\{ses\} = f$.*

Proof. If $(e - f) \circ s = 0$ then by Lemma 3.1, $\{s(e - f)\, s\} = -(e - f)$. Since s is an $(e + f)$-symmetry then $\{s(e + f)\, s\} = s^2 = e + f$. Adding gives $\{ses\} = f$.

Conversely, assume $\{ses\} = f$. Then subtracting $\{s(e+f)\,s\} = e + f$ from $2\{ses\} = 2f$ gives $\{s(e-f)\,s\} = f - e$ which by Lemma 3.1 implies $(e-f) \circ s = 0$. ∎

We will say a partial symmetry s *exchanges* idempotents e and f if $\{ses\} = f$ and $\{sfs\} = e$. Note that if s is a g-symmetry with $g \geqslant e$ and $g \geqslant f$, then $\{ses\} = f$ is equivalent to $\{sfs\} = e$ (since $U_s{}^2 = I$ on the subalgebra $\{gAg\}$).

LEMMA 3.3. *Let e and f be orthogonal idempotents in a JB-algebra, and let s be an $(e+f)$-symmetry which exchanges e and f. Let $e_1 \leqslant e$ be an idempotent and define $f_1 = \{se_1s\}$ and $t = \{(e_1 + f_1)\,s(e_1 + f_1)\}$. Then f_1 is an idempotent with $f_1 \leqslant f$ and t is an $(e_1 + f_1)$-symmetry which exchanges e_1 and f_1.*

Proof. Straightforward calculation. ∎

The next lemmas concern lifting symmetries and idempotents from a quotient of a JBW-algebra.

LEMMA 3.4. *Let $\varphi\colon A \to B$ be a homomorphism from a JBW-algebra A onto a JB-algebra B. If $x \in B$ is an idempotent (or a symmetry) then there exists an idempotent (respectively, symmetry) $e \in A$ such that $\varphi(e) = x$.*

Proof. Let $x^2 = x \in B$ and choose $a \in A$ such that $\varphi(a) = x$. Choose bounded continuous functions f and g with $f(0) = g(0) = 0$, $f(1) = g(1) = 1$, and such that

$$f \leqslant \chi_{[1/2,1]} \leqslant g$$

(pointwise on $\mathbb{R}$). Note that $f(x) = g(x) = x$ since the spectrum of x is $\{0, 1\}$. Let $e = \chi_{[1/2,1]}(a)$, so that $e^2 = e$. Then

$$x = f(\varphi(a)) = \varphi(f(a)) \leqslant \varphi(e) \leqslant \varphi(g(a)) = g(\varphi(a)) = x$$

so $\varphi(e) = x$ as required.

The corresponding result for symmetries follows from the fact that $\varphi(1) = 1$ and the one-to-one correspondence of idempotents and symmetries given by the map $e \mapsto 2e - 1$. ∎

The following result is the key technical lemma.

LEMMA 3.5. *Let $\varphi\colon A \to B$ be as above. Let $u_1, u_2, \ldots, u_{n+1}$ $(n \geqslant 1)$ be orthogonal symmetries in B and $s_1, s_2, \ldots, s_n$ orthogonal symmetries in A such that $\varphi(s_i) = u_i$ for $i \leqslant n$. Then there exists an idempotent $e \in A$ and orthogonal e-symmetries $t_1, t_2, \ldots, t_{n+1}$ such that*

(i) *$\varphi(e) = 1$ and $\varphi(t_i) = u_i$ for $i \leqslant n + 1$;*

(ii) *e is compatible with $s_1, \ldots, s_n$ and $t_i = \{es_ie\}$ for $i \leqslant n$.*

Proof. We begin by choosing an element $b \in A$ such that $\varphi(b) = u_{n+1}$. We will modify b to fit our needs, keeping the same image u_{n+1} at each stage.

We first modify b so that it becomes orthogonal to $s_1, \ldots, s_n$. Define

$$c = \tfrac{1}{2}(I - U_{s_1})\, \tfrac{1}{2}(I - U_{s_2}) \cdots \tfrac{1}{2}(I - U_{s_n})b.$$

Observe that each map $\frac{1}{2}(I - U_{s_i})$ is idempotent and commutes with each $\frac{1}{2}(I - U_{s_j})$; it follows that $\frac{1}{2}(I - U_{s_i})\, c = c$ for $i \leqslant n$; by Lemma 3.1 this implies $s_i \circ c = 0$ for $i \leqslant n$.

We next define $e = r(c^2)$. (Recall from [3: Prop. 4.7] that for $0 \leqslant a \in A$, $r(a)$ is the smallest idempotent p such that $a \leqslant \lambda p$ for some $\lambda \in \mathbb{R}$. Also $r(a) = \chi_{(0,\infty)}(a)$ so $r(a)$ is bicompatible with a (that is, compatible with each x compatible with a).)

We now define

$$t_i = \{es_i e\} \quad i \leqslant n; \qquad t_{n+1} = r_e(c^+) - r_e(c^-)$$

where the subscript e denotes that $r(c^{\pm})$ is to be calculated in the JBW-subalgebra A_e. (Note $e = \chi_{\mathbb{R}\setminus\{0\}}(c)$ so $e \circ c = c$; thus $c \in A_e$ so that $r_e(c^{\pm})$ is defined.)

We now verify that $e, t_1, \ldots, t_{n+1}$ satisfy the requirements of the lemma. First observe that $s_i \circ c = 0$ implies that c^2 and s_i are compatible for $i \leqslant n$ by Lemma 3.1. Therefore, $e = r(c^2)$ and s_i are compatible. Calculating in the associative algebra generated by e and a fixed s_i (cf. [3: Lemma 5.2]) we find $t_i^2 = \{es_i e\}^2 = e$ so that each t_i is an e-symmetry for $i \leqslant n$. For $i = n + 1$ we have (calculating in A_e)

$$t_{n+1}^2 = (\chi_{(0,\infty)} - \chi_{(-\infty,0)})^2(c) = \chi_{\mathbb{R}\setminus\{0\}}(c) = \chi_{(0,\infty)}(c^2) = r_e(c^2) = e$$

showing that t_{n+1} is also an e-symmetry.

To verify that $t_1, \ldots, t_n$ are orthogonal, note that by the identity (3.1) for $i, j \leqslant n$

$$U_{t_i} t_j = U_{\{es_i e\}} t_j = U_e U_{s_i} U_e (U_e s_j).$$

Since e and s_i are compatible, then by definition e and s_i operator commute (i.e., left multiplication by e and s_i commute) and so U_e and U_{s_i} commute. Combining this fact with $U_e^2 = U_e$ we have $U_{t_i} t_j = U_e U_{s_i} s_j$. By hypothesis $s_i \circ s_j = 0$, and so by Lemma 3.1 for $i = j$:

$$U_{t_i} t_j = U_e U_{s_i} s_j = -U_e s_j = -t_j.$$

By Lemma 3.1 again we conclude $t_i \circ t_j = 0$ for $i \neq j$, $i, j \leqslant n$.

To show t_{n+1} is orthogonal to $t_1, \ldots, t_n$, note first that $t_i (i \leqslant n)$ and c are orthogonal since (using $c \circ s_i = 0$):

$$U_{t_i} c = U_e U_{s_i} U_e c = U_e U_{s_i} c = -U_e c = -c.$$

Since each t_i is an e-symmetry then each map $U_{t_i}(i \leqslant n)$ is an automorphism of A_e. Thus for $i \leqslant n$

$$U_{t_i}t_{n+1} = U_{t_i}(r_e(c^+) - r_e(c^-)) = r_e(U_{t_i}(c^+)) - r_e(U_{t_i}(c^-))$$
$$= r_e(c^-) - r_e(c^+) = -t_{n+1}.$$

By Lemma 3.1 this proves $t_i \circ t_{n+1} = 0$ for $i \leqslant n$.

There remains only to show $e, t_1, \ldots, t_{n+1}$ have the appropriate images in B. It is easily verified that $\varphi(c) = u_{n+1}$. Since by [3: Lemma 4.6] $c^2 \leqslant \| c^2 \| r(c^2) = \| c^2 \| e$ then

$$\| c^2 \| \varphi(e) \geqslant \varphi(c^2) = u_{n+1}^2 = 1.$$

Since $\varphi(e)$ is an idempotent, this implies $\varphi(e) = 1$. Therefore, for $i \leqslant n$ $\varphi(t_i) = \{\varphi(e)\, \varphi(s_i)\, \varphi(e)\} = u_i$. We will thus be finished if we show that $\varphi(t_{n+1}) = u_{n+1}$.

Observe that $t_{n+1} = r_e(c^+) - r_e(c^-)$ is compatible with c and satisfies $t_{n+1} \circ | c | = c$. Since by [3: Lemma 5.2] $c, | c |, t_{n+1}$ generate an associative subalgebra the same is true for their images in B, and so $\varphi(t_{n+1})$, $\varphi(| c |)$, and $\varphi(c) = u_{n+1}$ are compatible. Also $t_{n+1} = e$ implies $\varphi(t_{n+1})^2 = \varphi(e)^2 = 1$, so $\varphi(t_{n+1})$ is a symmetry in B. Now

$$u_{n+1} = \varphi(c) = \varphi(t_{n+1} \circ | c |) = \varphi(t_{n+1}) \circ \varphi(| c |).$$

Since the elements lie in an associative subalgebra of B and $\varphi(t_{n+1})^2 = 1$ then

$$\varphi(| c |) = u_{n+1} \circ \varphi(t_{n+1}).$$

The right side is the product of compatible symmetries, and so is itself a symmetry. The left side has positive spectrum and so both sides must equal 1. This implies $u_{n+1} = \varphi(t_{n+1})$ which finishes the proof. ∎

The following result is not much more than a reformulation of Lemma 3.5.

LEMMA 3.6. *Let $\varphi: A \to B$ be as above. Let x and y be orthogonal idempotents in B and $x - y$, $u_1, u_2, \ldots, u_n$ orthogonal $(x + y)$-symmetries. Let e_0 and f_0 be orthogonal idempotents in A with $\varphi(e_0) = x$, $\varphi(f_0) = y$. Then there exist idempotents $e \leqslant e_0$, $f \leqslant f_0$ and orthogonal $(e + f)$-symmetries $e - f$, $t_1, \ldots, t_n$ such that $\varphi(e) = x$, $\varphi(f) = y$, $\varphi(t_i) = u_i$ for $i \leqslant n$.*

Proof. We proceed by induction on the number n of symmetries. The result is trivial if $n = 0$. Now assume the lemma holds for a certain value of $n \geqslant 0$; we will show it holds for $n + 1$.

Thus let $x, y, u_1, \ldots, u_{n+1}, e_0, f_0$ be given as above. By the induction hypothesis there will exist idempotents $e' \leqslant e_0$, $f' \leqslant f_0$ and orthogonal $(e' + f')$-symmetries $e' - f'$, $s_1, \ldots, s_n$ such that $\varphi(e') = x$, $\varphi(f') = y$, $\varphi(s_i) = u_i$ for $i \leqslant n$.

We now apply Lemma 3.5 to the JBW-algebra $A_0 = \{(e' + f') A(e' + f')\}$ and the JB-algebra $B_0 = \{(x + y) B(x + y)\}$. Note that $e' - f'$, $s_1, \ldots, s_n$ and $x - y$, $u_1, \ldots, u_{n+1}$ are orthogonal symmetries in A_0 and B_0, respectively. Thus there will exist an idempotent $d \in A_0$ compatible with $x - y$, $s_1, \ldots, s_n$ and orthogonal d-symmetries $t_0, t_1, \ldots, t_{n+1}$ such that $t_0 = \{d(e' - f') d\}$, $t_i = \{ds_i d\}$ for $1 \leqslant i \leqslant n$, $\varphi(t_0) = x - y$, $\varphi(t_i) = u_i$ for $1 \leqslant i \leqslant n + 1$, and $\varphi(d) = x + y$.

Now define $e = \{de'd\}$, $f = \{df'd\}$; note $e + f = d$. Since $d \in A_0 = A_{e'+f'}$ then d is compatible with $e' + f'$ as well as with $e' - f'$, and so is compatible with e' and f'. It follows that e and f are idempotents with $e \leqslant e' \leqslant e_0$ and $f \leqslant f' \leqslant f_0$. Note $t_0 = e - f$, so $e - f$, $t_1, \ldots, t_{n+1}$ are the desired orthogonal $(e + f)$-symmetries. ▮

The following result will not be needed in the sequel but seems of interest in its own right. Recall that a set $\{e_{ij}\}_{i,j=1}^n$ of elements in a C^*-algebra is a set of *matrix units* if $e_{ij}^* = e_{ij}$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all i, j, k, l. (We do not require $\sum e_{ii} = 1$.) The following corollary shows that matrix units can always be lifted from a quotient of a von Neumann algebra by a norm-closed two-sided ideal.

COROLLARY 3.7. *Let $\varphi: \mathscr{A} \to \mathscr{B}$ be a *-homomorphism from a von Neumann algebra $\mathscr{A}$ onto a C^*-algebra $\mathscr{B}$. If $\{x_{ij}\}_{i,j=1}$ is a set of matrix units in $\mathscr{B}$ then there exists a set $\{e_{ij}\}_{i,j=1}$ of matrix units in $\mathscr{A}$ such that $\varphi(e_{ij}) = x_{ij}$ for all i, j.*

Proof. Let A and B be the self-adjoint parts of $\mathscr{A}$ and $\mathscr{B}$, respectively. (Note that A is a JBW-algebra, B is a JB-algebra, and $\varphi: A \to B$ is a Jordan homomorphism of A onto B.) Define $u_j = x_{1j} + x_{j1}$ for $2 \leqslant j \leqslant n$. Observe that $x_{11} - x_{jj}$ and u_j are orthogonal $(x_{11} + x_{jj})$-symmetries for each j.

Our immediate goal to to find orthogonal idempotents $e_{11}, \ldots, e_{nn}$ and $(e_{11} + e_{jj})$-symmetries s_j which exchange e_{11} and e_{jj} and such that $\varphi(e_{jj}) = x_{jj}$, $\varphi(s_j) = u_j$ for all j. We will do this in three steps.

Step 1. Choose orthogonal idempotents $f_1, \ldots, f_n$ mapping onto $x_{11}, \ldots, x_{nn}$ respectively. (To do this, begin by using Lemma 3.4 to choose an idempotent f_1 mapping onto x_{11}. Having chosen $f_1, \ldots, f_k$ choose an idempotent f_{k+1} in $(1 - f_1 - \cdots - f_k) A(1 - f_1 - \cdots - f_k)$ which maps onto $x_{k+1,k+1} \in (1 - x_{11} - \cdots - x_{kk}) A(1 - x_{11} - \cdots - x_{kk})$; then f_{k+1} is orthogonal to $f_1, \ldots, f_k$ and thus inductively we can find $f_1, \ldots, f_n$).

Step 2. For each $j \geqslant 2$ note that since $x_{11} - x_{jj}$ and u_j are $(x_{11} + x_{jj})$-symmetries and are orthogonal, by Lemma 3.6 we can choose idempotents $g_j \leqslant f_1$, $h_j \leqslant f_j$ and a $(g_j + h_j)$-symmetry t_j orthogonal to $g_j - h_j$ with $\varphi(g_j) = \varphi(f_1) = x_{11}$, $\varphi(h_j) = \varphi(f_j) = x_{jj}$, $\varphi(t_j) = u_j$. By making these choices in succession we can also arrange that $g_2 \geqslant g_3 \geqslant \cdots \geqslant g_n$.

Step 3. We now define $e_{11} = g_n$, $e_{jj} = \{t_j e_{11} t_j\}$ for $j \geqslant 2$, and $s_j = \{(e_{11} + e_{jj}) t_j(e_{11} + e_{jj})\}$. By Lemma 3.3, $e_{11}, \ldots, e_{nn}$ and $s_2, \ldots, s_n$ satisfy the conditions described above.

We define in addition $s_1 = e_{11}$. Then for all $j \geqslant 1$, s_j is a partial symmetry which exchanges e_{11} and e_{jj}.

Finally, for $i \neq j$ we define

$$e_{ij} = e_{ii}s_ie_{11}s_je_{jj} \in \mathscr{A}.$$

Note that the same equation holds when $i = j$. It is now easily verified that $\{e_{ij}\}$ is a set of matrix units which φ maps onto $\{x_{ij}\}$. ∎

The following is the main result needed to achieve the decomposition we are after.

PROPOSITION 3.8. *Let A be a JBW-algebra and φ a homomorphism from A onto $M_3{}^8$. Then there exists a subalgebra A' of A such that $\varphi|_{A'}$ is an isomorphism of A' onto $M_3{}^8$.*

Proof. Recall that $M_3{}^8$ is the self-adjoint part of the matrix algebra $M_3(\mathcal{O})$, where $\mathcal{O}$ = Cayley numbers. Let $\{E_{ij}\}_{i,j=1}$ be the usual matrix units of $M_3(\mathcal{O})$.

Now choose orthogonal idempotents f_2 and f_3 in A and an $(f_2 + f_3)$-symmetry t exchanging f_2 and f_3 such that $\varphi(f_2) = E_{22}$, $\varphi(f_3) = E_{33}$ and $\varphi(t) = E_{23} + E_{32}$. (For details see steps 1 and 2 of the proof of Corollary 3.7.) Define $f_1 = 1 - f_2 - f_3$ so that f_1 is an idempotent orthogonal to f_2 and f_3 and satisfies $\varphi(f_1) = E_{11}$.

Now let $1, a_1, \ldots, a_7$ be a basis of $\mathcal{O}$ such that $\bar{a}_i = -a_i$, $\frac{1}{2}(a_ia_j + a_ja_i) = -\delta_{ij}$, for $1 \leqslant i, j \leqslant 7$ (cf. [16]). Define $u_1, \ldots, u_3 \in M_3{}^8$ by $u_1 = E_{12} + E_{21}$, $u_{i+1} = a_iE_{12} - a_iE_{21}$ for $1 \leqslant i \leqslant 7$. Observe that $E_{11} - E_{22}$, $u_1, \ldots, u_3$ are orthogonal $(E_{11} + E_{22})$-symmetries.

Now by Lemma 3.6 we choose idempotents $e_1 \leqslant f_1$, $e_2 \leqslant f_2$ and orthogonal $(e_1 + e_2)$-symmetries $e_1 - e_2$, $s_1, \ldots, s_3$ such that $\varphi(e_1) = E_{11}$, $\varphi(e_2) = E_{22}$, $\varphi(s_i) = u_i$ for $1 \leqslant i \leqslant 8$. Define $e_3 = \{te_2t\}$ and $s_{23} = \{(e_2 + e_3)\, t(e_2 + e_3)\}$. By Lemma 3.3 e_3 is an idempotent $\leqslant f_3$ and s_{23} is an $(e_2 + e_3)$-symmetry which exchanges e_2 and e_3. Note also that $\varphi(e_3) = E_{33}$, $\varphi(s_{23}) = E_{23} + E_{32}$.

Let $e = e_1 + e_2 + e_3$ and $A_e = \{eAe\}$. Then e_1, e_2, e_3 are strongly connected idempotents with sum e. Thus by [13: Thm. 5, p. 133] there exists an alternative algebra $\mathscr{A}$ with involution $\alpha \to \bar{\alpha}$ such that A_e is isomorphic with the self-adjoint part of $M_3(\mathscr{A})$. If $\{F_{ij}\}_{i,j=1}$ are the standard matrix units in $M_3(\mathscr{A})$ then this isomorphism can be chosen so that e_1, e_2, e_3 correspond to F_{11}, F_{22}, F_{33}, respectively, and s_1, s_{23} correspond to $F_{12} + F_{21}$ and $F_{23} + F_{32}$, respectively. We will now identify A_e and $M_3(\mathscr{A})$.

Each $s_i (i = 2, \ldots, 8)$ can then be expressed in the form $s_i = \alpha_iF_{11} + \beta_iF_{12} + \bar{\beta}_iF_{21} + \gamma_iF_{22}$. Using the fact that each s_i is an $(F_{11} + F_{22})$-symmetry, orthogonal to $F_{11} - F_{22}$ and to s_j for $i \neq j$, calculation gives $\alpha_i = \gamma_i = 0$, $\bar{\beta}_i = -\beta_i$, $\frac{1}{2}(\beta_i\beta_j + \beta_j\beta_i) = -\delta_{ij}$ for $i, j = 2, \ldots, 8$.

Now let $\mathscr{A}_0$ be the subalgebra of $\mathscr{A}$ generated by $\beta_2, \ldots, \beta_3$. Since the β_i's anticommute and have squares equal to -1, $\mathscr{A}_0$ will be the linear span of 1 and all products of distinct β_i's; in particular $\mathscr{A}_0$ will be finite dimensional.

Now let A_0 be the self-adjoint part of $M_3(\mathscr{A}_0)$. Then A_0 is a finite dimensional subalgebra of A_e, and contains $s_1, \ldots, s_8, e_1, e_2, e_3, s_{23}$. Thus $\varphi(A_0)$ will contain $E_{11}, E_{22}, E_{33}, E_{12} + E_{21}, E_{23} + E_{32}$, and $u_i = a_i E_{1i} + \bar{a}_i E_{i1}$. It is straightforward to verify that these elements generate all of $M_3{}^8$ and thus $\varphi(A_0) = M_3{}^8$.

Since A_0 is finite dimensional by Lemma 2.1 the ideal $\ker(\varphi|_{A_0})$ will be of the form $A_0 z$ for some central idempotent z of A_0. Defining $A' = A_0(e - z)$ then A' is a subalgebra of A and $\varphi|_{A'}$ is an isomorphism of A' onto $M_3{}^8$, completing the proof. ∎

We will say a JB-algebra A is *purely exceptional* if every factor representation of A is onto $M_3{}^8$. (Recall from [3] that a factor representation of A is a homomorphism $\varphi: A \to M$ where M is a JB-factor and $\varphi(A)$ is strongly dense in M.) Recall also that a compact Hausdorff space X is *hyperstonean* iff $C(X)$ is a dual space [8].

THEOREM 3.9. *Every JBW-algebra A admits a unique decomposition $A = A_{sp} \oplus A_{ex}$ where A_{sp} is special (and therefore isomorphic to a JW-algebra) and A_{ex} is purely exceptional. A_{ex} is isomorphic to $C(X, M_3{}^8)$ where X is hyperstonean, and conversely $C(X, M_3{}^8)$ (for X hyperstonean) is a purely exceptional JBW-algebra.*

Proof. We first prove uniqueness. Suppose z_1 and z_2 are central idempotents such that $A = Az_i \oplus A(1 - z_i)$ is a decomposition as described above for $i = 1, 2$. Then every factor representation of $Az_1 \cap A(1 - z_2) = Az_1(1 - z_2)$ must be into a factor which is both special and exceptional. This implies $z_1(1 - z_2) = 0$ so $z_1 = z_1 z_2$ and similarly $z_2 = z_1 z_2$, which proves uniqueness.

We next establish the existence of such a decomposition. Let $\{z_\alpha\}$ be a maximal orthogonal set of central idempotents such that $Az_\alpha \cong C(X_\alpha, M_3{}^8)$ for some compact Hausdorff space X_α. Define $z_{ex} = \sum z_\alpha$ and $z_{sp} = 1 - z_{ex}$. We will show $A = Az_{sp} \oplus Az_{ex}$ is the desired decomposition.

We first verify that Az_{sp} is special. Suppose not; then by [3: Thm. 9.5] there will exist a factor representation of Az_{sp} onto $M_3{}^8$. By Proposition 3.8, Az_{sp} will contain a subalgebra isomorphic to $M_3{}^8$. By a result of Jacobson [14: Thm. 4] there will exist a decomposition $Az_{sp} = A_0 \oplus A_1$ where $A_1 \cong M_3{}^8 \otimes Z$ where Z is the center of A_1. Thus we can choose a central idempotent $z_0 \neq 0$ such that $A_1 = Az_0$ with $z_0 \leqslant z_{sp}$. Then $Az_0 \cong M_3{}^8 \otimes Z \cong M_3{}^8 \otimes C(Y) \cong C(Y, M_3{}^8)$ where $Z \cong C(Y)$. (The existence of Y follows from [3: Prop. 2.3].) But now z_0 will be orthogonal to $z_{ex} = 1 - z_{sp}$, contrary to the construction of z_{ex}, and this contradiction shows that Az_{sp} is special. (Thus by [3: Lemma 9.4] and Cor. 2.4, Az_{sp} will be isometrically isomorphic to a JW-algebra.)

We next show that Az_{ex} is isomorphic to $C(X, M_3{}^8)$. First observe that Az_{ex} is the l^∞-direct sum of the algebras $Az_\alpha \cong C(X_\alpha, M_3{}^8)$. Now let X_0 be the topological direct sum of the spaces X_α (cf. Bourbaki [6: I.2.4]). Then if

$C_\alpha(X_0, M_3^8)$ denotes the bounded continuous functions from X_0 into M_3^8 then

$$Az_{ex} \cong \sum Az_\alpha \cong \sum C(X_\alpha, M_3^8) \cong C_b(X_0, M_3^8).$$

Finally, let $X = \beta X_0$ be the Stone-Čech compactification of X_0. Since the unit ball of M_3^8 is compact, then $C_b(X_0, M_3^8)$ is isomorphic to $C(X, M_3^8)$ and so we've shown $Az_{ex} \cong C(X, M_3^8)$.

Note that the center of $C(X, M_3^8)$ will be the scalar functions, and so can be identified with $C(X)$. The center of the JBW-algebra Az_{ex} will be monotone complete with a separating set of normal states and so $C(X)$ will be a dual space and X will be hyperstonean.

We next show Az_{ex} is purely exceptional. Note that the constant functions in $C(X, M_3^8)$ are a subalgebra isomorphic to M_3^8. Thus if $\varphi: Az_{ex} \to M$ is any factor representation of M, since M_3^8 is simple and $\varphi(1) \neq 0$ then $\varphi(Az_{ex}) \subseteq M$ will contain a subalgebra isomorphic to M_3^8. Since every JB-factor except M_3^8 is special [3: Thm. 8.6] then $M = M_3^8$ and so $\varphi(Az_{ex}) = M_3^8$, proving that Az_{ex} is purely exceptional.

Finally, let X be hyperstonean and let B be the predual of $C(X)$. Then as shown in the proof of Lemma 1.1,

$$(B \otimes_\gamma M_3^8)^* \cong B^* \otimes_\lambda M_3^8 = C(X) \otimes_\lambda M_3^8 \cong C(X, M_3^8)$$

so $C(X, M_3^8)$ is a dual space and therefore is a JBW-algebra which (as shown above) is purely exceptional. ∎

Acknowledgments

We would like to acknowledge our debt to our previous collaboration [3] with E. Alfsen and E. Størmer. Many of the ideas and techniques in this paper grew out of that joint work.

We are also grateful to the Mathematics Department in Berkeley and especially to Professors William Bade and Marc Rieffel for their hospitality during the time the author spent in Berkeley while this paper was in progress. Finally, the author gratefully acknowledges financial support from Wellesley College which made possible a year spent on leave.

References

1. E. M. Alfsen, Compact convex sets and boundary integrals, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 57, Springer–Verlag, Berlin/Heidelberg/New York, 1971.
2. E. M. Alfsen and F. W. Shultz, Non-commutative spectral theory for affine function spaces on convex sets, *Mem. Amer. Math. Soc.* **172**, Providence, Rhode Island, 1976.
3. E. M. Alfsen, F. W. Shultz, and E. Størmer, A Gelfand–Neumark theorem for Jordan algebras, *Advances in Math.* 28 (1978), 11–56.

4. H. Araki and G. Elliot, On the definition of C^*-algebras, *Publ. Res. Inst. Math. Sci.* (Kyoto Univ.) **9**, No. 1 (1973), 93–112.
5. R. Arens, The adjoint of a bilinear operation, *Proc. Amer. Math. Soc.* **2** (1951), 839–848.
6. N. Bourbaki, General topology, Hermann, Addison–Wesley, Paris/Reading, Mass., 1966.
7. P. Civin and B. Yood, The second conjugate space of a Banach algebra as an algebra, *Pacific J. Math.* **11** (1961), 847–870.
8. J. Dixmier, Sur certains espaces considérés par M. H. Stone, *Summa Brasil Math.* **2**, fasc. 11 (1951), 151–182.
9. E. Effros and E. Størmer, Jordan algebras of self-adjoint operators, *Trans. Amer. Math. Soc.* **127** (1967), 313–316.
10. A. J. Ellis, The duality of partially ordered normed linear spaces, *J. London Math. Soc.* **39** (1964), 730–744.
11. J. Gil de Lamadrid, Measures and tensors, *Trans. Amer. Math. Soc.* **114** (1965), 98–121.
12. A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, *Mem. Amer. Math. Soc.* **16** (1955).
13. N. Jacobson, Structure and representations of Jordan algebras, *Amer. Math. Soc. Colloq. Publ.* **39**, Providence, Rhode Island, 1968.
14. N. Jacobson, A Kronecker factorization theorem for Cayley algebras and the exceptional simple Jordan algebra, *Amer. J. Math.* **76** (1954), 447–452.
15. R. V. Kadison, Operator algebras with a faithful weakly closed representation, *Ann. of Math.* **64** (1956), 175–181.
16. A. G. Kurosh, "Lectures on General Algebra," Chelsea, New York, 1963.
17. K. B. Laursen, A note on lifting of matrix units in C^*-algebras, *Math. Scand.* **33** (1973), 338–342.
18. G. K. Pedersen, Monotone closures in operator algebras, *Amer. J. Math.* **94** (1972), 995–962.
19. S. Sakai, A characterization of W^*-algebras, *Pacific J. Math.* **6** (1956), 763–773.
20. S. Sakai, C^*-algebras and W^*-algebras, Ergebnisse der Mathematik und Ihren Grenzgebiete, Vol. 60, Springer–Verlag, Berlin/Heidelberg/New York, 1971.
21. R. Schatten, A theory of cross-spaces, *in* Annals of Math. Studies, No. 26, Princeton Univ. Press, Princeton, N. J., 1950.
22. Z. Semadeni, Banach spaces of continuous functions, Vol. 1, Monografie Matematyczne, No. 55, Polish Scientific Publishers, Warsaw, 1971.
23. E. Størmer, Jordan algebras of type I, *Acta Math.* **115** (1966), 165–184.
24. E. Størmer, Irreducible Jordan algebras of self-adjoint operators, *Trans. Amer. Math. Soc.* **130** (1968), 153–166.
25. D. Topping, Jordan algebras of self-adjoint operators, *Mem. Amer. Math. Soc.* **53** (1965).
26. D. Topping, An isomorphic invariant for spin factors, *J. Math. Mech.* **15** (1966), 1055–1064.