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Relative invariants, ideal classes and quasi-canonical modules of modular rings of invariants

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A R T I C L E I N F O

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We describe "quasi-canonical modules" for modular invariant rings R of finite group actions on factorial Gorenstein domains. From this we derive a general "quasi-Gorenstein criterion" in terms of certain 1-cocycles. This generalizes a recent result of A. Braun for linear group actions on polynomial rings, which itself generalizes a classical result of Watanabe for non-modular invariant rings.

We use an explicit classification of all reflexive rank one R-modules, which is given in terms of the class group of R, or in terms of R-semi-invariants. This result is implicitly contained in a paper of Nakajima (1982) [15].

1. Introduction

Let k be a field, V a finite dimensional k-vector space of dimension n, $G \subseteq \mathrm{GL}(V)$ a finite group and $A := \mathrm{Sym}(V^*) \cong k[x_1, \ldots, x_n]$, the symmetric algebra over the dual space V^* with its canonical G-action and ring of invariants $R := A^G := \{a \in A \mid ga = a, \ \forall g \in G\}$.

A classical result of K. Watanabe states that if $p = \mathrm{char}(k)$ does not divide $|G|$, then A^G is Gorenstein if $G \subseteq \mathrm{SL}(V)$. If moreover G contains no pseudo-reflection, then the converse holds, i.e. if A^G is Gorenstein, then $G \subseteq \mathrm{SL}(V)$ [18,19]. In the recent paper [4], A. Braun proved an analogue of this result for the modular case, where the characteristic of k is allowed to divide the group order. Consider the following

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Hypothesis ($\mathcal{NR}$). The group $G \subseteq \mathrm{GL}(V)$ contains no pseudo-reflection (neither diagonalizable nor transvection).

Then Braun proved the following result:

Theorem 1.1. *(See [4].) Let k be an arbitrary field and suppose the Hypothesis ($\mathcal{NR}$) holds. Then the following are equivalent*:

(i) $G \subseteq \mathrm{SL}(V)$;
(ii) $A^G \cong \mathrm{Hom}_C(A^G, C)$ *for any polynomial ring $C \subseteq A^G$ with A^G a finitely generated C-module and the homogeneous generators of C of degrees divisible by $|G|$.*

From this he deduces that if G satisfies Hypothesis ($\mathcal{NR}$), then the Cohen–Macaulay and Gorenstein loci of A^G coincide and if A^G is Cohen–Macaulay it is also Gorenstein. He also obtains a modular version of the converse: If G satisfies Hypothesis ($\mathcal{NR}$) and A^G is Gorenstein, then G is contained in $\mathrm{SL}(V)$.

In this paper we generalize Braun's results in two ways: firstly we avoid Hypothesis ($\mathcal{NR}$) altogether. Secondly we neither assume A to be a polynomial ring nor that the parameter algebra C is chosen in any particular way. Instead, our main result applies, whenever A is a (not necessarily graded) k-algebra, which is also a factorial domain with unit group $U(A) = U(k)$. To formulate our main result we need the following definitions and notation:

Let A be a finitely generated (affine) k-algebra, which is also a factorial domain with unit group $U(A) = U(k)$ and let $G \subseteq \mathrm{Aut}(A)$ be a finite group of ring automorphisms of A. We do not assume that G acts trivially on k, so $k' := k^G$ can be a proper subfield of k.

Definition 1. Let $\lambda \in Z^1(G, U(A))$ be a 1-cocycle, i.e. $\lambda : G \to U(A)$ with

$$\lambda(gh) = \lambda(g) \cdot g\big(\lambda(h)\big), \quad \forall g, h \in G.$$

Then we define $A_\lambda := \{a \in A \mid g(a) := \lambda(g)a\}$, the *$R$-module of relative λ-invariants*, or *λ-semi-invariants* in A.

Definition 2. Let $\mathcal{P} \subseteq B$ be an extension of affine k-domains such that the $\mathcal{P}$-module ${}_{\mathcal{P}}B$ is finitely generated and assume that $\mathcal{P}$ is a Gorenstein ring. Then we call the B-module $\mathrm{Hom}_{\mathcal{P}}(B, \mathcal{P})$ a *quasi-canonical module* of B and we call B *quasi-Gorenstein* (w.r.t. $\mathcal{P}$), if $\mathrm{Hom}_{\mathcal{P}}(B, \mathcal{P}) \cong B$ as B-modules (in other words, if B is a pre-symmetric $\mathcal{P}$-algebra).

Remark 1. If in addition B is a graded connected k-algebra and $\mathcal{P}$ a polynomial k-algebra, generated by a homogeneous system of parameters, then B is a Cohen–Macaulay ring, if and only if ${}_{\mathcal{P}}B$ is free. If B is Cohen–Macaulay, then it is well known that $\omega_B := \mathrm{Hom}_{\mathcal{P}}(B, \mathcal{P})$ is a canonical module of B and B is Gorenstein, if and only if $B \cong \omega_B$.

Let $W := W(G) \trianglelefteq G$ be the normal subgroup generated by generalized reflections (see Definition 5) and let $\mathcal{F}$ be any parameter k'-subalgebra $\mathcal{F} \subseteq R := A^G \subseteq S := A^W \subseteq A$, i.e. $\mathcal{F} = k'[f_1, \ldots, f_d]$ is a polynomial ring such that ${}_{\mathcal{F}}R$ and therefore also ${}_{\mathcal{F}}A$ are finitely generated $\mathcal{F}$-modules. Although not explicitly stated in [15], the following facts are implicit in the proofs of that paper:

(i) The class group $\mathcal{C}_R$ of R is isomorphic to the subgroup $\tilde{H}$ of $H^1(G, U(A))$, defined by $\tilde{H} := \{\rho \in H^1(G, U(A)) \mid \mathrm{res}_{I_Q}(\rho) = 1$ in $H^1(I_Q, U(A_Q))$, $\forall Q \in \mathrm{Spec}_1(A)\}$. (See Theorem 3.4).
(ii) There are explicit bijections between the following sets:
  - the divisor class group $\mathcal{C}_R$;
  - the set of iso types of finitely generated reflexive R-modules of rank one;
  - the set of iso types of R-modules of semi-invariants A_χ with $\chi \in Z^1(G, U(A))$.
(iii) If $\chi \in Z^1(G/W, U(A))$, then $A_\chi \cong R \iff [\chi] = 1 \in H^1(G/W, U(A))$.

We can now state the main result of this paper:

Theorem 1.2. *Let A be an affine k-domain which is also a UFD and let $\mathcal{F} \subseteq R \subseteq S \subseteq A$ be as described above. Then the following hold*:

The rings S and A are quasi-Gorenstein $\mathcal{F}$-algebras with natural $\mathcal{F}G$-modules

$$\mathrm{Hom}_{\mathcal{F}}(S, \mathcal{F}) = S \cdot \theta_S \cong S, \qquad \mathrm{Hom}_{\mathcal{F}}(A, \mathcal{F}) = A \cdot \theta_A \cong A \quad \text{and} \quad \mathcal{D}_{A,S}^{-1} \cong \mathrm{Hom}_S(A, S) = A \cdot \theta_{A,S}.$$

Here $\mathcal{D}_{A,R} = \mathcal{D}_{A,S}$ is the Dedekind-different, which is a G-invariant principal ideal in A (see Definition 7).

Let $\chi_S \in Z^1(G/W, U(k))$ and $\chi_A, \chi_{A,S} \in Z^1(G, U(k))$ be the "eigen-characters" of θ_S, θ_A and $\theta_{A,S}$, respectively and view $Z^1(G/W, U(k))$ as a subset of $Z^1(G, U(k))$ in a natural way. Then $\chi_S = \chi_A \cdot \chi_{A,S}^{-1}$ and $\mathrm{Hom}_{\mathcal{F}}(R, \mathcal{F})$ is isomorphic to the R-module of semi-invariants $S_{\chi_S^{-1}} = A_{\chi_S^{-1}}$. In particular the following hold:

(i) *The quasi-canonical R-module $\mathrm{Hom}_{\mathcal{F}}(R, \mathcal{F})$ is isomorphic to a divisorial ideal $I \trianglelefteq R$, with $\mathrm{ch}(\mathrm{cl}(I)) = [\chi_S] = [\chi_A/\chi_{A,S}]$, where $\mathrm{ch}: \mathcal{C}_R \to H^1(G/W, U(k))$ is the isomorphism of Lemma 3.11.*

(ii) *The following are equivalent*:
   (a) *The ring R is quasi-Gorenstein*;
   (b) $[\chi_S] = 1 \in H^1(G/W, U(k))$;
   (c) $[\chi_{A,S}] = [\chi_A] \in H^1(G, U(k))$.

Remark 2.

(i) In [12] Hinic obtained an analogous result to the equivalence of (ii) (a). Furthermore (b) was obtained for $(A, \mathfrak{m})$ a noetherian local Gorenstein ring and under some technical hypotheses on the group action. The most significant one is that the inertia group $H := \{g \in G \mid g_{k(\mathfrak{m})} = \mathrm{id}_{k(\mathfrak{m})}\}$ has order coprime to the characteristic of the residue class field $k(\mathfrak{m})$.

(ii) In the special case, where A is a polynomial ring with k-linear G-action, the equivalence of (ii) (a) and (c) also appears in a paper by A. Broer [5].

Corollary 1.3. *If $[\chi_S] = 1 \in H^1(G/W, U(k))$, then the Cohen–Macaulay and Gorenstein loci of R coincide.*

If $\mathrm{char}(k) = p > 0$, set $\tilde{W} := \langle W, P^g \mid g \in G\rangle$ with P a Sylow p-subgroup of G. In other words, $\tilde{W} \trianglelefteq G$ is the normal subgroup generated by all reflections on A and all elements of order a power of p. We obtain:

Corollary 1.4. *If G acts trivially on k, then $H^1(G/W, U(k)) = \mathrm{Hom}(G/W, U(k)) = \mathrm{Hom}(G/\tilde{W}, U(k))$ and Theorem 1.2 also holds with W and S replaced by $\tilde{W}$ and $\tilde{S} := A^{\tilde{W}}$, respectively. The ring $\tilde{S}$ is a factorial domain and quasi-Gorenstein*; *the subring R is quasi-Gorenstein if and only if $\chi_{\tilde{S}} = 1$.*

Assume for the moment that Hypothesis $(\mathcal{NR})$ holds, then $W = 1$ and $A = S$ with $[\chi_{A,S}] = 1$. Hence in this case R is quasi-Gorenstein, if and only if $[\chi_A] = 1$. If moreover $A = \mathrm{Sym}(V^*)$ with $G \subseteq \mathrm{GL}_k(V)$, then $[\chi_A] = \chi_A = \det^{-1}$ (see Remark 9) and we recover Braun's result (and Watanabe's for $\mathrm{char}(k) \nmid |G|$). More generally:

Corollary 1.5. *Assume that $A = \mathrm{Sym}(V^*)$ and $S := A^W$ is Gorenstein (e.g. a polynomial ring). Assume moreover that $\chi_S = 1$ (note that $\chi_S \in \mathrm{Hom}(G, U(k))$ here). Then $R = A^G$ is Gorenstein, if it is Cohen–Macaulay.*

It is known by a result of Serre [2] that if $\mathrm{Sym}(V^*)^H$ is a polynomial ring for finite $H \subseteq \mathrm{GL}_k(V)$, then $H = W(H)$. Since the converse is false, the hypothesis of the above corollary is not automatic. If however it is satisfied, then the character χ_S can be explicitly described in terms of the G/W-action on the homogeneous generators of A^W (see Section 6).

It is remarkable that Braun's original proof, as well as the one of our generalization, uses techniques from the theory of non-commutative Frobenius and symmetric algebras. A slightly more special version of Theorem 1.2, in which $\mathcal{F}$ is chosen such that the field extension $\mathbb{L} \geqslant \mathrm{Quot}(\mathcal{F})$ is separable (such an $\mathcal{F}$ can always be found by [11, Corollary 16.18, p. 403]), can however be obtained wholly within the "world of commutative algebra", by combining Braun's ideas with methods from algebraic number theory and information hidden in the proofs contained in a classical paper by Nakajima [15]. We will add a sketch of this argument to our proof of Theorem 1.2 in Section 5.

We will use standard notation and will denote with A-*Mod* the category of (left) A-modules and with A-*mod* the full subcategory of *finitely generated* A-modules.

2. The divisor class group and reflexive modules of rank one

In this section we collect some definitions and results from [15], including some information which is implicitly contained via arguments and proofs, but not explicitly stated there. In such a case we include short proofs. Throughout this paper, A will be an affine normal k-domain, which will be further specialized in later sections. We define $\mathbb{L} := \mathrm{Quot}(A)$ and set $\mathrm{Spec}_1(A) := \{\mathrm{P} \in \mathrm{Spec}(A) \mid \mathrm{ht}(\mathrm{P}) = 1\}$. Then for every $\mathrm{P} \in \mathrm{Spec}_1(A)$, the localization A_{P} is a discrete valuation ring with the well-known property that:

(i) $A = \bigcap_{\mathrm{P}\in \mathrm{Spec}_1(A)} A_{\mathrm{P}}$;
(ii) for every $0 \neq \ell \in \mathbb{L}$ the set $\{\mathrm{P} \in \mathrm{Spec}_1(A) \mid \nu_{\mathrm{P}}(\ell) \neq 0\}$ is finite.

Let $\mathcal{D}_A$ denote the divisor group of A, i.e. the free abelian group with basis $\mathrm{Spec}_1(A)$:

$$\mathcal{D}_A := \bigoplus_{\mathrm{P}\in \mathrm{Spec}_1(A)} \mathbb{Z}\,\mathrm{div}(\mathrm{P}).$$

Let $0 \neq J \triangleleft A$ be an ideal with $0 \neq j \in J$. Then $\nu_{\mathrm{P}}(J) \in \mathbb{Z}$ is defined by $JA_{\mathrm{P}} = \mathrm{P}^{\nu_{\mathrm{P}}(J)}A_{\mathrm{P}}$, hence $\nu_{\mathrm{P}}(j) := \nu_{\mathrm{P}}(jA_{\mathrm{P}}) \geqslant \nu_{\mathrm{P}}(J) \geqslant 0$, and it follows that $\nu_{\mathrm{Q}}(J) = 0$ for all but finitely many $\mathrm{Q} \in \mathrm{Spec}_1(A)$. If $I \subseteq \mathbb{L}$ is a fractional ideal, then $\ell I \triangleleft A$ for some $\ell \in A$, hence again $\nu_{\mathrm{Q}}(I) = 0$ for almost all $\mathrm{Q} \in \mathrm{Spec}_1(A)$ and one defines $\mathrm{div}(I) := d(I) := \sum_{\mathrm{P}\in \mathrm{Spec}_1(A)} \nu_{\mathrm{P}}(I)\,\mathrm{div}(\mathrm{P})$. With $\mathcal{H}_A$ we denote the group of principal fractional ideals in A, then the map div embeds $\mathcal{H}_A$ into $\mathcal{D}_A$ as a subgroup with quotient group $\mathcal{C}_A := \mathcal{D}_A/\mathcal{H}_A$, the *divisor class group* of A.

Definition 3. Let $R \subseteq A$ be a subring. For ideals $I \triangleleft R$ or $J \triangleleft A$ we denote with $\bar{I}$ and $\bar{J}$ the corresponding divisorial closures, i.e. $\bar{I} = \bigcap_{q\in \mathrm{Quot}(R),\, I\subseteq Rq} Rq$, and $\bar{J} = \bigcap_{a\in \mathbb{L},\, J\subseteq Aa} Aa$.

Remark 3. If A is a normal noetherian domain and also a UFD, then $\bar{I} = \bigcap_{a\in A,\, I\subseteq aA} aA$.

The following results are standard, so we omit proofs:

Lemma 2.1. *Assume that A is a UFD and $R \subseteq A$ a subring with* $\mathrm{Quot}(R) \cap A = R$. *Let $I \triangleleft R$ and $J \triangleleft A$ be ideals, then*:

1. $\overline{IA} \cap R = \bar{I}$.
2. $\bar{J} = d_J A$ *with* $d_J := \gcd(J)$ *in* A.
3. $\overline{IA} = d_I A$ *with* $d_I := \gcd(I) := \gcd\{r \in I\}$ (taken inside A).
4. *For* $a \in A$: $\overline{aJ} = \bar{J} \cdot a$.
5. *For* $a \in A$ *and divisorial ideal* $J \triangleleft A$, $aJ \triangleleft A$ *is divisorial.*
6. *For any ideals* $J, K \triangleleft A$: $\overline{\bar{J} \cdot K} = \overline{J \cdot K}$.

Let B be an arbitrary commutative ring and $N \in B$-*mod* a finitely generated B-module. Then N is torsion free of rank one $\iff$ there is an ideal $I \trianglelefteq B$ containing a non-zero-divisor, such that $N \cong I \trianglelefteq B$ are isomorphic as B-modules.

For every finitely generated module $M \in A$-*mod* the following hold:

(i) $M^* := \mathrm{Hom}_A(M, A) \cong \bigcap_{\mathrm{p} \in \mathrm{Spec}_1(A)} M_{\mathrm{p}}^* \subseteq \mathbb{L} \otimes_A M^*$.
(ii) If M is torsion free, then the canonical map $c : M \to M^{**}$ induces an isomorphism

$$M^{**} \cong \bigcap_{\mathrm{p} \in \mathrm{Spec}_1(A)} M_{\mathrm{p}}.$$

(iii) The fractional ideal $I \in \mathcal{F}(A)$ is divisorial if and only if I is a reflexive A-module.
(iv) $\ker(c) = \mathrm{Tor}(M)$, the torsion submodule of M, and M^* is reflexive.
(v) For $M, N \in A$-*mod* one has

$$\big(\mathrm{Hom}_A(M, N)\big)^{**} \cong \mathrm{Hom}_A\big(M^{**}, N^{**}\big).$$

Proposition 2.2. *Let A be a normal noetherian domain, then there is a bijection between the divisor class group $\mathcal{C}_A$ and the set of isomorphism classes of finitely generated reflexive A-modules of rank one.*

Proof. If $M, N \in A$-*mod* are f.g. reflexive A-modules of rank one, then $M \cong I$ and $N \cong J$ with divisorial ideals $I, J \lhd A$, so we can assume that $M = I$, $N = J$ are divisorial ideals. Let $\theta : I \to J$ be an isomorphism, then for any $i, i' \in I$, $\theta(ii') = i\theta(i') = i'\theta(i)$, so $\ell := \theta(i)/i \in \mathbb{L}$ with $\ell \cdot I \subseteq J$. By symmetry we have $\ell^{-1} = i/\theta(i) = \theta^{-1}(\theta(i))/\theta(i) = \theta^{-1}(j)/j$ for every $j \in J$, hence $j = \theta^{-1}(j)\ell$ and $J \subseteq \ell I$, so $J = \ell \cdot I$. It follows that the classes $\mathrm{cl}(J) := [\mathrm{div}(J)]$ and $\mathrm{cl}(I) \in \mathcal{C}_A$ coincide.

Now assume $\mathrm{cl}(J) = \mathrm{cl}(I) \in \mathcal{C}_A$, then $\mathrm{div}(I) = \mathrm{div}(J) + \mathrm{div}(\ell A)$ for some $\ell = a/b \in \mathbb{L}$, hence

$$\mathrm{div}(Ib) = \mathrm{div}(I) + \mathrm{div}(bA) = \mathrm{div}(J) + \mathrm{div}(aA) = \mathrm{div}(Ja),$$

and replacing I by $Ib \cong I$ and J by $Ja \cong J$, we can assume that $\mathrm{div}(I) = \mathrm{div}(J)$. Hence $I_{\mathrm{P}} = J_{\mathrm{P}}$ for all $\mathrm{P} \in \mathrm{Spec}_1(A)$, so $I \cong J$, since these are reflexive A-modules. □

3. Relative invariants

Now let $G \subseteq \mathrm{Aut}(A)$ be a finite group of ring automorphisms with corresponding ring of invariants $R := A^G$ and quotient field $\mathbb{K} = \mathbb{L}^G$. The Galois group $G = \mathrm{Gal}(\mathbb{L} : \mathbb{L}^G)$ acts as permutation group on $\mathrm{Spec}_1(A)$ and on the divisor group $\mathcal{D}_A$ and there is an inclusion homomorphism $\rho : \mathcal{D}_{A^G} \to \mathcal{D}_A$ satisfying

$$d(\mathrm{q}) \mapsto e_{\mathrm{q}} \cdot \left(\sum_{\mathrm{Q} \in \mathrm{Spec}_1(A):\ \mathrm{Q} \cap A^G = \mathrm{q}} d(\mathrm{Q}) \right) \in (\mathcal{D}_A)^G,$$

because the ramification index $e_{\mathrm{q},A} := e_{\mathrm{q}} := e(\mathrm{Q} \mid \mathrm{q})$ is constant for all $\mathrm{Q} \in \mathrm{Spec}_1(A)$ over q. The group of invariants $(\mathcal{D}_A)^G$ is a free abelian group with basis consisting of orbit sums

$$d(\mathrm{Q})^+ := \sum_{g \in G/G_{\{\mathrm{Q}\}}} d(g\mathrm{Q}), \quad \mathrm{Q} \in \mathrm{Spec}_1(A).$$

Here $G_{\{\mathrm{Q}\}} := \mathrm{Stab}_G(\mathrm{Q})$ is the stabilizer (i.e. the decomposition group) of Q. Let $\mathfrak{C}$ denote a fixed set of representatives for the G-orbits on $\mathrm{Spec}_1(A)$, i.e.

$$\mathfrak{C} \cong \mathrm{Spec}_1(A)/G \cong \mathrm{Spec}_1\big(A^G\big).$$

Then we have a short exact sequence of abelian groups:

$$0 \to \mathcal{D}_{A^G} \xrightarrow{\rho} (\mathcal{D}_A)^G \to \bigoplus_{\mathrm{Q}\in\mathfrak{C}} \mathbb{Z}/e_{\mathrm{q}}\mathbb{Z} \to 0. \tag{3.1}$$

If $aA \in (\mathcal{H}_A)^G$, then $g(a) = c_g a$ with $c_g \in U(A)$ and $gh(a) = c_{gh}a = g(c_h a) = g(c_h)c_g a$, hence $c_{gh} = c_g \cdot g(c_h)$, so $\lambda := c_{(\cdot)} \in Z^1(G, U(A))$ and $a \in A_\lambda$.

Lemma 3.1. *Let $\chi \in Z^1(G, U(A))$, then $0 \neq A_\chi$ is a reflexive R-module of rank one and is isomorphic to a divisorial ideal of R. The following hold*:

(i) *For every $0 \neq a \in A_{\chi^{-1}}$, $\overline{aA_\chi A} \cap R = \overline{A_\chi a} = aA_\chi \lhd R$ is divisorial.*
(ii) *Let $\lambda \in B(G, U(A))$, i.e. $\lambda(g) = u^{-1}g(u)$ with $u \in U(A)$, and $\mu := \chi \cdot \lambda$. Then $u \cdot A_\chi = A_\mu$ and $A_\mu A = A_\chi A$, which only depends on the class $[\chi] \in H^1(G, U(A))$.*
(iii) *Assume A to be a normal domain. Then for every $\mathrm{Q} \in \mathrm{Spec}_1(A)$, $\nu_{\mathrm{Q}}(\overline{A_\chi A}) < e(\mathrm{Q} \mid \mathrm{q})$.*

Proof. See [15, Lemmas 2.1/2.2]. □

Let $Z^1_A(G, U(A)) := \{\lambda \in Z^1(G, U(A)) \mid A_\lambda \nsubseteq \mathrm{Q},\ \forall \mathrm{Q} \in \mathrm{Spec}_1(A)\}$. If $\lambda, \mu \in Z^1(G, U(A))$ and $\mathrm{Q} \in \mathrm{Spec}_1(A)$, then $A_{\lambda\cdot\mu} \subseteq \mathrm{Q}$ implies $A_\lambda \cdot A_\mu \subseteq A_{\lambda\cdot\mu} \subseteq \mathrm{Q}$, hence $A_\lambda \subseteq \mathrm{Q}$, or $A_\mu \subseteq \mathrm{Q}$. In other words, $Z^1_A(G, U(A))$ is a subgroup of $Z^1(G, U(A))$, containing $B(G, U(A))$ (since $A_\lambda A = A$ for $\lambda \in B(G, U(A))$). Therefore one can define

Definition 4. $H^1_A(G, U(A)) := Z^1_A(G, U(A))/B(G, U(A))$.

Lemma 3.2. *The sequence*

$$0 \to H^1_A\big(G, U(A)\big) \to H^1\big(G, U(A)\big) \xrightarrow{\Psi} \bigoplus_{\mathrm{Q}\in\mathfrak{C}} \mathbb{Z}/e_{\mathrm{q}}\mathbb{Z}$$

with $\Psi : [\chi] \mapsto (\nu_{\mathrm{Q}}(\overline{A_\chi A}))_{\mathrm{Q}\in\mathfrak{C}}$ is an exact sequence of abelian groups.

Proof. See [15, Lemma 2.3]. □

The map

$$\mathcal{C}_{A^G} \to (\mathcal{D}_A)^G/(\mathcal{H}_A)^G \hookrightarrow (\mathcal{D}_A/\mathcal{H}_A)^G = (\mathcal{C}_A)^G$$

is essentially the natural map $\phi : \mathcal{C}_{A^G} \to \mathcal{C}_A$ and we obtain

Corollary 3.3. *The kernel* $\ker(\phi)$ *is naturally isomorphic to $H^1_A(G, U(A)) \cong \ker(\Psi)$. Moreover, ϕ is injective if and only if the A_χ are free R-modules for all $\chi \in Z^1_A(G, U(A))$.*

Proof. See [15, Lemma 2.4]. □

3.1. From now on we assume that A is also a factorial domain

Definition 5.

(i) Let $I_{\mathrm{Q}} := G_{k(\mathrm{Q})} = \{g \in G \mid ga - a \in \mathrm{Q},\ \forall a \in A\}$, the inertia group of $\mathrm{Q} \in \mathrm{Spec}_1(A)$.

(ii) An element $g \in G$ is called a *reflection on* A, if $g \in I_{\mathrm{Q}}$ for some $\mathrm{Q} \in \mathrm{Spec}_1(A)$. The group

$$W := W_A := W_A(G) := \langle I_{\mathrm{Q}} \mid \mathrm{Q} \in \mathrm{Spec}_1(A)\rangle$$

is a normal subgroup (since G acts on $\mathrm{Spec}_1(A)$) and is called the *subgroup of (generalized) reflections on* A.

Theorem 3.4. *Let A and G be as above and assume that A is a factorial domain. Then*

$$\mathcal{C}_{A^G} \cong H^1_A\big(G, U(A)\big) \cong \tilde{H} := \big\{\rho \in H^1\big(G, U(A)\big) \bigm| \mathrm{res}_{I_{\mathrm{Q}}}(\rho) = 1 \text{ in } H^1\big(I_{\mathrm{Q}}, U(A_{\mathrm{Q}})\big),\ \forall \mathrm{Q} \in \mathrm{Spec}_1(A)\big\}.$$

In explicit form: Let I be a divisorial ideal of A^G, then $\overline{IA} = aA$ with semi-invariant $a \in A$. If $\theta_a \in Z^1(A, U(A))$ is the corresponding cocycle, i.e. $g(a) = \theta_a(g)a$ for every $g \in G$, the class $[I] \in \mathcal{C}_{A^G}$ corresponds to the element $[\theta_a] \in \tilde{H}$.

Proof. See [15, Lemma 2.4]. The explicit form can be seen by following up the isomorphism described there. $\square$

Proposition 3.5. *For $\chi \in Z^1(G, U(A))$ the following hold:*

(i) $\overline{A_\chi A} = d_\chi A$, $d_\chi := \mathrm{gcd}(A_\chi) \in A_{\mu_\chi}$ *with* $\mu_\chi \in Z^1(G, U(A))$ *and a uniquely defined element* $[\mu_\chi] \in H^1(G, U(A))$.
(ii) A_χ *defines a unique class* $\mathrm{cl}(A_\chi) \in \mathcal{C}_R$, *which satisfies* $\mathrm{cl}(A_\chi) = [\chi^{-1}\mu_\chi] \in \tilde{H}$ *(see Theorem 3.4).*
(iii) A_χ *is a free R-module if and only if* $[\chi] = [\mu_\chi] \in H^1(G, U(A))$.

Proof. (i): This follows from Lemma 2.1.

(ii): For every $a \in A_{\chi^{-1}}$ the ideal $aA_\chi \lhd R$ is divisorial and we get from Lemma 2.1: $\overline{aA_\chi A} = ad_\chi A$ with $ad_\chi \in A_{\chi^{-1}\mu_\chi}$. Hence $\mathrm{cl}(A_\chi) = [\chi^{-1}\mu_\chi] \in \tilde{H}$ by Theorem 3.4.

(iii): This follows immediately from the above. $\square$

Lemma 3.6. *For $[\chi] \in H^1(G, U(A))$ the following are equivalent:*

(i) $[\chi] \in \tilde{H} = H^1_A(G, U(A))$;
(ii) $d_\chi \in U(A)$;
(iii) $[\chi^{-1}] = \mathrm{cl}(A_\chi) \in \mathcal{C}_R \cong \tilde{H}$;
(iv) $\overline{A_\chi A} = A$.

Proof. "(i) $\Longleftrightarrow$ (ii)": Let $[\chi] \in \tilde{H}$, then there is a divisorial ideal $J \trianglelefteq R$ with $\mathrm{cl}(J) = [\chi^{-1}]$, i.e. $\overline{JA} = fA$ with $f \in A_{\chi^{-1}}$. The divisorial ideal $I := fA_\chi \lhd R$ satisfies

$$\overline{fA_\chi A} = \overline{IA} = f \cdot \overline{A_\chi A} = fd_\chi A.$$

Hence $J = \overline{JA} \cap R = fA \cap R = fA_\chi = I$, so $fd_\chi A = \overline{IA} = \overline{JA} = fA$ and $d_\chi \in U(A)$. On the other hand, if $d_\chi \in U(A)$, then $[\mu_\chi] = 1 \in H^1(G, U(A))$ and $[\chi^{-1}] = [\chi^{-1}][\mu_\chi] \in \tilde{H}$.

"(i) $\Longleftrightarrow$ (iii)" and "(ii) $\Longleftrightarrow$ (iv)" follow from Proposition 3.5. $\square$

Corollary 3.7. *For $\chi \in Z^1(G, U(A))$ we have $A_\chi = d_\chi \cdot A_{\chi\mu_\chi^{-1}}$. Assume $A_\chi = a \cdot S$ with $S \subseteq A$ and $a \in A$. Then $a \mid d_\chi$ and the following hold:*

(i) $a \sim d_\chi \Longleftrightarrow S = A_\lambda$ *with* $[\lambda] = [\chi\mu_\chi^{-1}] \in \tilde{H}$ *(i.e. $A_\lambda \cong A_1 = R$ in R-mod).*
(ii) $1_A \in S \Longleftrightarrow S = R \Longleftrightarrow d_\chi \sim a \in A_\chi$.

Proof. Since $d_\chi = \gcd(A_\chi)$, $A_\chi d_\chi^{-1} \subset A_{\chi\mu_\chi^{-1}}$, hence $d_\chi \cdot A_{\chi\mu_\chi^{-1}} \subseteq A_\chi \subseteq d_\chi \cdot A_{\chi\mu_\chi^{-1}}$, so

$$A_\chi = d_\chi \cdot A_{\chi\mu_\chi^{-1}}.$$

(i): If $A_\chi = aS$ with $S \subseteq A \ni a$, then clearly $a \mid d_\chi$. If $a = ud_\chi$ with $u \in U(A)$, then $d_\chi \cdot A_{\chi\mu_\chi^{-1}} = A_\chi = ud_\chi S$, hence $S = u^{-1} A_{\chi\mu_\chi^{-1}} = A_\lambda$ with $[\lambda] = [\chi\mu_\chi^{-1}]$.

Assume $S = A_\lambda \cong R$, then $d_\chi A = \overline{A_\chi A} = \overline{aA_\lambda A} = a\overline{A_\lambda A} = aA$ by Lemma 3.6; hence $a \sim d_\chi$.

(ii): If $1_A \in S$, then $a \in A_\chi$, therefore $d_\chi \mid a$ and $S = 1/aA_\chi \subseteq R$. Hence $A_\chi \subseteq aR \subseteq A_\chi$ and $R = 1/aA_\chi = S$. If $S = R$, then $A_\chi = aR$, so $\gcd(A_\chi) \ni a \in A_\chi$. If $d_\chi \sim a \in A_\chi$, $aR \subseteq A_\chi$, hence $1_A \in R \subseteq 1/aA_\chi = S$. $\square$

Corollary 3.8. *Let $[\lambda] \in H^1(G, U(A))$ such that $A_\lambda = dR$. Then for every $[\sigma] \in \tilde{H}$ we have $d = \gcd(A_{\lambda\sigma}) \sim d_{\lambda\sigma}$, i.e. d and $d_{\lambda\sigma}$ are associated. In particular $d = d_\lambda \cdot u$ with $u \in U(R)$ and $A_\lambda = Ad_\lambda$.*

Proof. We have

$$\begin{aligned} dA &= d\overline{A_\sigma A} = \overline{dRA_\sigma A} = \overline{A_\lambda A_\sigma A} \subseteq \overline{A_{\lambda\sigma} A} = d_{\lambda\sigma} A = d_{\lambda\sigma} \overline{A_{\sigma^{-1}} A} = \overline{d_{\lambda\sigma} A_{\sigma^{-1}} A} \\ &= \overline{d_{\lambda\sigma} A A_{\sigma^{-1}} A} = \overline{\overline{A_{\lambda\sigma} A} A_{\sigma^{-1}} A} = \overline{A_{\lambda\sigma} A A_{\sigma^{-1}} A} \subseteq \overline{A_\lambda A} = dA. \end{aligned}$$

It follows that $d_\lambda = u \cdot d$ with $u \in U(A) \cap R = U(R)$. $\square$

Theorem 3.9. *Let $\mathcal{P}_{G,A} := \{[\lambda] \in H^1(G, U(A)) \mid A_\lambda = d_\lambda R\}$. Then*

$$\mathcal{P}_{G,A} = \bigl\{[\lambda] \in H^1\bigl(G, U(A)\bigr) \bigm| \operatorname{cl}(A_\lambda) = 1\bigr\},$$

$$\mathcal{P}_{G,A} \cap \tilde{H} = 1 \quad \textit{and} \quad H^1\bigl(G, U(A)\bigr) = \biguplus_{[\lambda] \in \mathcal{P}_{G,A}} \tilde{H} \cdot [\lambda].$$

So $\mathcal{P}_{G,A} \subseteq H^1(G, U(A))$ is a transversal of the cosets of the subgroup $\tilde{H} \subseteq H^1(G, U(A))$.

For every $[\chi] \in H^1(G, U(A))$ let $[\mu_\chi] \in H^1(G, U(A))$ be the character of $d_\chi := \gcd(A_\chi)$, i.e. $d_\chi \in A_{\mu_\chi}$. Then the following hold:

(i) $\operatorname{cl}(A_\chi) = [\chi^{-1}][\mu_\chi]$ *with* $\{[\mu_\chi]\} = \mathcal{P}_{G,A} \cap \tilde{H} \cdot [\chi]$.

(ii) *The map*

$$\mu : H^1\bigl(G, U(A)\bigr) \to \mathcal{P}_{G,A}, \quad [\chi] \mapsto [\mu_\chi]$$

satisfies $\mu \circ \mu = \mu$ and it is a projection operator onto the distinguished transversal $\mathcal{P}_{G,A}$.

Proof. The equation $\mathcal{P}_{G,A} \cap \tilde{H} = 1$ follows from Lemma 3.6 (iv).

Let $[\lambda], [\delta] \in \mathcal{P}_{G,A}$ with $[\sigma] := [\lambda]^{-1}[\delta] \in \tilde{H}$, then $[\delta] = [\lambda][\sigma]$, hence by Corollary 3.8, $d_\delta \sim d_\lambda$ and $[\lambda] = [\delta]$. This shows that every $\tilde{H}$ coset contains at most one element in $\mathcal{P}_{G,A}$.

Let $[\chi] \in H^1(G, U(A))$, then

$$\overline{A_\chi A} = d_\chi A \subseteq A_{\mu_\chi^{(1)}} A \subseteq \overline{A_{\mu_\chi^{(1)}} A} = d_{\mu_\chi^{(1)}} A \subseteq \overline{A_{\mu_\chi^{(2)}} A} = d_{\mu_\chi^{(2)}} A \subseteq \overline{A_{\mu_\chi^{(3)}} A} = \cdots$$

with

$$[\chi] \equiv \bigl[\mu_\chi^{(1)}\bigr] \equiv \bigl[\mu_\chi^{(2)}\bigr] \equiv \cdots \mod \tilde{H}.$$

It is clear that this ascending chain of divisorial ideals must be stationary, hence we will eventually have

$$d_{\mu_\chi^{(i)}}A = d_{\mu_\chi^{(i+1)}}A = d_{\mu_\chi^{(\infty)}}, \quad \text{and therefore} \quad [\mu_\chi^{(i)}] = [\mu_\chi^{(i+1)}] = [\mu_\chi^{(\infty)}] =: [\lambda]$$

with

$$\overline{A_\lambda A} = d_\lambda A \subseteq A_\lambda A \subseteq \overline{A_\lambda A} \quad \text{and} \quad [\chi] \equiv [\mu_\chi] \equiv \cdots \equiv [\mu_\chi^{(\infty)}] = [\lambda] \bmod \tilde{H}.$$

It follows that $d_\lambda = \gcd(A_\lambda) \in A_\lambda$, hence $A_\lambda = d_\lambda R$, so $[\lambda] \in \mathcal{P}_{G,A} \cap \tilde{H} \cdot \chi$.

It now follows from Corollary 3.8 that

$$d_\lambda \sim d_{\mu_\chi^{(i)}} \sim d_{\mu_\chi^{(i-1)}} \sim d_{\mu_\chi^{(i-2)}} \sim \cdots \sim d_{\mu_\chi^{(1)}} \sim d_\chi.$$

So $[\mu_\chi] := [\mu_\chi^{(1)}] \in \mathcal{P}_{G,A} \cap \tilde{H} \cdot [\chi]$. By construction we have $d_{\mu_\chi} \sim d_\chi$, hence $\mu \circ \mu([\chi]) = \mu([\chi])$, which finishes the proof. $\square$

Corollary 3.10. *For every $[\lambda] \in \mathcal{P}_{G,A}$ we have $\mathcal{C}_R = \{\mathrm{cl}(A_\chi) \mid \chi \in \tilde{H} \cdot [\lambda]\}$, i.e. if χ ranges through the full coset $\tilde{H} \cdot [\lambda]$, then the A_χ form a transversal of all isomorphism types of rank one reflexive R-modules.*

Alternatively the set $\{A_{\chi\mu_\chi^{-1}} \mid \chi \in Z^1(G, U(A))\}$ is also a full set of representatives of reflexive rank one R-modules.

Proof. Every rank one reflexive R-module is isomorphic to a divisorial ideal of R, the isomorphism type of which is uniquely determined by its ideal class. From Theorem 3.9 we see that $[\mu_{\sigma\lambda}] =$ eigen character of $(d_{\sigma\lambda}) =$ eigen character of $(d_\lambda) = [\lambda]$, hence we get

$$\mathrm{cl}(A_{\sigma\lambda}) = [\sigma]^{-1}[\lambda]^{-1}[\mu_{\sigma\lambda}] = [\sigma]^{-1}[\lambda]^{-1}[\lambda] = [\sigma]^{-1}.$$

The last statement follows from Corollary 3.7, since $A_\chi = d_\chi \cdot A_{\chi\mu_\chi^{-1}} \cong A_{\chi\mu_\chi^{-1}}$ in R-mod. $\square$

3.2. *A factorial domain, $U(A) = U(k)$*

From now on we assume that the affine k-algebra A is also a factorial domain with $U(A) = U(k)$ with $k \subseteq A$, a field of characteristic $p \geqslant 0$.

Let $\mathrm{P} = a_{\mathrm{P}}A \in \mathrm{Spec}_1(A)$ and $\sigma \in I := I_{\mathrm{P}}$. Then for $u \in k$, $(\sigma - 1)(u) \in k \cap \mathrm{P} = 0$, so $\sigma(u) = u$ and $W \subseteq \mathrm{Aut}_k(A)$. Clearly P is I-stable, so $\sigma(a_{\mathrm{P}}) = \delta_{\mathrm{P}}(g)a_{\mathrm{P}}$ and the map

$$\delta_{\mathrm{P}} : I_{\mathrm{P}} \to U(k), \quad \sigma \mapsto \delta_{\mathrm{P}}(g) = a_{\mathrm{P}}^{-1}\sigma(a_{\mathrm{P}})$$

is an element in $Z^1(I, U(k)) = \mathrm{Hom}(I, U(k))$.

Lemma 3.11. *For $\mathrm{P} \in \mathrm{Spec}_1(A)$, $I := I_{\mathrm{P}}$ and $e := e(\mathrm{P} \mid \mathrm{P} \cap R)$ we have $\mathrm{Hom}(I, U(k)) = \mathrm{Hom}(I, U(A_{\mathrm{P}})) = \langle\delta_{\mathrm{P}}\rangle \cong \mathbb{Z}/e\mathbb{Z}$. There is a short exact sequence*

$$0 \to \mathcal{C}_{A^G} \to H^1(G, U(k)) \to \bigoplus_{\mathrm{Q} \in \mathfrak{C}} \mathrm{Hom}(I_{\mathrm{Q}}, U(k)) \to 0.$$

In particular $\mathcal{C}_{A^G} \cong H^1(G/W, U(k))$.

Proof. See [15, Lemma 2.6]. In addition to this, we only need to show that $\tilde{H} = H^1(G/W, U(k))$. Let $[\chi] \in \tilde{H}$ with $\chi \in Z^1(G, U(k))$, then for $g, h \in W$, $\chi(gh) = \chi(g)g(\chi(h)) = \chi(g)\chi(h)$, since W acts trivially on k. Moreover g and h are products of elements on which χ is 1, hence $\chi_{|W} = 1$. We view $Z^1(G/W, U(k))$ as a subset of $Z^1(G, U(k))$ in a natural way. Then, again since W acts trivially on k we have $B^1(G, U(k)) \subseteq Z^1(G/W, U(k))$, hence $B^1(G, U(k)) = B^1(G/W, U(k))$, so $\mathcal{C}_{A^G} = \tilde{H} \cong Z^1(G/W, U(k))/B^1(G/W, U(k)) = H^1(G/W, U(k))$. $\square$

Remark 4. It follows from Lemma 3.11 that A^W is factorial, a result originally obtained by Dress for polynomial rings [10], inspiring the generalization by Nakajima in [15].

3.3. *A factorial domain, $U(A) = U(k)$ with trivial G-action*

Then $H^1(G, U(A)) = G^* := \mathrm{Hom}(G, U(k))$, the group of linear k-characters of G. If $N \trianglelefteq G$ is a normal subgroup, then the restriction map yields a short exact sequence

$$1 \to (G/N)^* \to G^* \to N^* \to 1.$$

Corollary 3.12. *There is an isomorphism* $\mathrm{ch}: \mathcal{C}_{A^G} \cong \overline{G}^* = \ker(\mathrm{res}_{|W})$, *where* $\overline{G} := G/W$ *and* $\mathrm{res}_{|W}: G^* \to W^*$ *is the restriction map on characters.*

4. Pre-Frobenius algebras

In this section we develop some tools from the theory of non-commutative Frobenius-extensions and symmetric algebras, which will be used to prove the Main Theorem 1.2 in the general case. Although that proof only needs the results in the case of commutative rings, there is little gain in restricting to commutative algebras from the outset, as the methods themselves are “non-commutative” in nature. Let C denote an arbitrary commutative ring with 1, B a possibly non-commutative C-algebra such that ${}_CB$ is a finitely generated module, and D an arbitrary ring. With ${}_B\mathrm{Mod}_D$ we denote the category of B–D-bimodules. For any object $M \in {}_B\mathrm{Mod}_D$ the set $M^* := \mathrm{Hom}_C(M, C)$ is an object in ${}_D\mathrm{Mod}_B$ by the rule: $(d\alpha b)(m) := \alpha(bmd)$ for $\alpha \in M^*$, $d \in D$, $b \in B$. Notice that $d\alpha b$ is in M^* since C is central. Similarly the set $M^\vee := \mathrm{Hom}_B(M, B)$ is an object in ${}_D\mathrm{Mod}_B$ by the rule $dfb(m) := f(md)b$. For any bimodule homomorphism $x \in \mathrm{Hom}_B(N, M)_D$ the map $x^*: M^* \to N^*$, $\alpha \mapsto \alpha \circ x$ is in $\mathrm{Hom}_D(M^*, N^*)_B$ and $x^\vee: M^\vee \to N^\vee$, $\alpha \mapsto \alpha \circ x$ is in $\mathrm{Hom}_D(M^\vee, N^\vee)_B$. Thus we have contravariant functors:

$$()^*: {}_B\mathrm{Mod}_D \to {}_D\mathrm{Mod}_B, \quad M \mapsto M^* \quad \text{and}$$
$$()^\vee: {}_B\mathrm{Mod}_D \to {}_D\mathrm{Mod}_B, \quad M \mapsto M^\vee.$$

Definition 6. The (finite) C-algebra B will be called (left) *pre-Frobenius*, if there is some $\theta \in \mathrm{Hom}_C(B, C)$ such that $\mathrm{Hom}_C(B, C) = B\theta \cong {}_BB$ as a left B-module. If $B\theta = \theta B \cong {}_BB_B$ as B-bimodules, then B is called a *pre-symmetric* algebra over C. A pre-Frobenius (pre-symmetric) algebra ${}_CB$ is called a (left) *Frobenius* (or *symmetric*) C-algebra, if ${}_CB$ is finitely generated projective.

Some of the following results on pre-Frobenius algebras are well known in the context of *artinian* Frobenius-algebras (see e.g. [9, p. 413 ff.], or [14, p. 407 ff]), but we need them in a non-artinian situation. We also do not want to impose the condition that ${}_CB$ is projective from the very beginning, as it is done in the existing literature, including the original source [16]. Therefore we will add proofs, whenever we couldn’t find exact quotes in the literature for the result in question. For the convenience of reading, some of the more technical proofs have been deferred to an extra section (Appendix A).

Let B be a pre-Frobenius algebra; then the left annihilator $L-\mathrm{ann}_B(\theta) := \{b \in B \mid b\theta = 0\}$ is zero. Moreover, the fact that B^* is a B-bimodule implies, that for every $b \in B$ there is a unique $\nu_\theta(b) \in B$ such that $\theta \cdot b = \nu_\theta(b) \cdot \theta$. From

$$\nu_\theta(ab)\cdot\theta=\theta\cdot(ab)=(\theta\cdot a)b=\big(\nu_\theta(a)\cdot\theta\big)b=\nu_\theta(a)\nu_\theta(b)\cdot\theta,$$

we see that ν_θ is multiplicative. If $z\in Z:=Z(B)$, the center of B, then $\theta\cdot z=\theta(z\cdot())=\theta(()\cdot z)=z\cdot\theta$, hence $\nu_\theta(z)=z$ and it follows that $\nu_\theta\in\mathrm{End}_Z(B)$. If $B^*=B\lambda$, then $\lambda=u\cdot\theta$ for some unit $u\in B$, and we get $\nu_\theta(b)=u^{-1}\nu_\lambda(b)u$ for all $b\in B$.

Lemma + Definition 1. Let ${}_CB$ be a (left) pre-Frobenius algebra with $B^*=B\theta$ and center $Z=Z(B)$. Then the following are equivalent:

(i) $\nu_\theta\in\mathrm{Aut}_Z(B)$.
(ii) $B\theta=\theta B$ and $R-\mathrm{ann}_B(\theta):=\{b\in B\mid \theta b=0\}=0$ is zero.
(iii) The map ${}_BB\to{}_BB^*$, $b\mapsto b\cdot\theta=\theta(()b)$ is an isomorphism of left modules and the map $B_B\to B^*_B$, $b\mapsto\theta\cdot b=\theta(b())$ is an isomorphism of right modules.

If any of these equivalent conditions is satisfied, we call ${}_CB$ a *balanced pre-Frobenius algebra*. In the context of artinian Frobenius-algebras, ν_θ is called the *Nakayama-automorphism*.

Proof. (i) $\Rightarrow$ (ii): If $\nu_\theta\in\mathrm{Aut}_C(B)$ and $b\in B$, then $b\theta=\nu_\theta(b')\theta=\theta b'$ for some $b'\in B$, hence $B^*=B\theta\subseteq\theta\cdot B\subseteq B^*$. It follows from the definition of ν_θ, that $R-\mathrm{ann}_B(\theta)\subseteq\ker(\nu_\theta)=0$.

(ii) $\Rightarrow$ (iii): The fact that ${}_CB$ is left pre-Frobenius is equivalent to the statement that the map $b\mapsto b\cdot\theta$ is an isomorphism of left modules. The hypotheses in (ii) imply that the map $b\mapsto\theta\cdot b$ is injective and surjective, hence an isomorphism of right modules.

(iii) $\Rightarrow$ (i): The first isomorphism is just a restatement of the left pre-Frobenius condition and therefore induces the map ν_θ. The second isomorphism clearly implies that ν_θ is injective. It also implies that for every $b\in B$ there is $b'\in B$ with $b\theta=\theta b'=\nu_\theta(b')\theta$, hence ν_θ is surjective. $\square$

Remark 5.

(i) A balanced pre-Frobenius algebra ${}_CB$ with $B^*=B\theta$ is pre-symmetric, if and only if $\nu_\theta=\mathrm{id}_A$. Indeed: $B^*=B\lambda=\lambda B\cong{}_BB_B\iff bb'1_B=b1_Bb'=1_Bbb'\mapsto bb'\lambda=b\lambda b'=\lambda bb'\iff\lambda(()bb')=\lambda(b'()b)=\lambda(bb'())\iff\nu_\lambda=\mathrm{id}\iff\nu_\theta=u^{-1}\nu_\lambda u=\mathrm{id}$ for some unit $u\in B$.
(ii) If ${}_CB$ is a commutative (left) pre-Frobenius algebra, then B is automatically balanced and indeed pre-symmetric.
(iii) If B is an affine commutative graded connected algebra over a field and $C\leqslant B$ is a polynomial ring generated by a homogeneous system of parameters (a parameter subalgebra), then $\mathrm{Hom}_C(B,C)$ is a quasi-canonical module ω_B in the sense of Definition 2. In this case ${}_CB$ is pre-symmetric $\iff\omega_B\cong{}_BB\iff B$ is quasi-Gorenstein.
If moreover B is Cohen–Macaulay, then B is Gorenstein $\iff{}_CB$ is pre-symmetric.

Let C, B, A be rings such that B is a C- and A is a B-bimodule. Then ${}_CA\cong{}_CB\otimes_BA$ and hence by the adjointness of tensor and Hom-functors, we get an isomorphism Ψ of abelian groups:

$$\mathrm{Hom}_C({}_CA,C)=\mathrm{Hom}_C({}_CB\otimes_BA,C)\cong\mathrm{Hom}_B\big({}_BA,\mathrm{Hom}_C({}_CB,C)\big);$$
$$\Psi:\phi\mapsto\big(a\mapsto\big(b\mapsto\phi(b\cdot a)\big)\big);\qquad\Psi^{-1}:\gamma\mapsto\big(a\mapsto\gamma(a)(1)\big).$$

Both abelian groups are also A–C-bimodules and Ψ is an isomorphism of bimodules. Indeed:

$$\begin{aligned}\Psi\big(a'\phi c\big)(a)(b)&=a'\phi c(ba)=\phi\big(baa'\big)\cdot c=\big(\Psi(\phi)\big(aa'\big)(b)\big)\cdot c\\&=\big(\big(\Psi(\phi)\big(aa'\big)\big)c\big)(b)=\big(a'\Psi(\phi)c\big)(a)(b).\end{aligned}$$

Lemma 4.1. *Let A and B be pre-symmetric C-algebras with $A^* = A \cdot \theta_A$ and $B^* = B \cdot \theta_B$. Assume that A is a B-bimodule. Then $\mathrm{Hom}_B(A, B) = A \cdot \lambda$, with $\lambda \in \mathrm{Hom}_B(A, B)$ defined by the equation*

$$\lambda(a) \cdot \theta_B = (a\theta_A)_{|B} = \theta_A(\cdot a)_{|B}.$$

The map λ is a homomorphism of B-bimodules, and the map

$$\phi^{(A,B)} : {}_A A_B \to \mathrm{Hom}_B(A, B), \quad a \mapsto a \cdot \lambda$$

is an isomorphism of A–B-bimodules.

In other words, (A, B) is a (possibly non-commutative) "pre-Frobenius pair".

Proof. See Appendix A. □

We now assume that ${}_C B$ is a balanced pre-Frobenius algebra and we fix $\theta \in B^* = B\theta$, together with $\nu := \nu_\theta \in \mathrm{Aut}_C(B)$. For $M \in {}_B\mathrm{Mod}_D$ we define ${}^\nu M$ to be the same C-module as M but with twisted B-action given by $\nu(a) \cdot {}^{(\nu)}m := {}^{(\nu)}(am)$, or

$$a \cdot {}^{(\nu)}m := \nu^{-1}(a)m.$$

Since $(a \cdot {}^{(\nu)}m)d = (\nu^{-1}(a)m)d = \nu^{-1}(a)(md) = a \cdot {}^{(\nu)}(md)$, it follows that ${}^\nu M \in {}_B\mathrm{Mod}_D$.

Theorem 4.2. *Let B be a balanced pre-Frobenius algebra over the commutative ring C with $B^* = B \cdot \theta$ and corresponding $\nu_\theta \in \mathrm{Aut}_C(B)$. Then for any ring D the function θ induces an isomorphism of functors $()^\vee$, $()^* \circ {}^\nu() : {}_B\mathrm{Mod}_D \to {}_D\mathrm{Mod}_B$:*

$$\theta_* : \mathrm{Hom}_B(M, B) \to \mathrm{Hom}_C\bigl({}^\nu M, C\bigr), \quad \beta \mapsto \theta \circ \beta.$$

Proof. See Appendix A. □

Proposition 4.3. *Let B, θ and ν be as in Theorem 4.2, $M \in {}_B\mathrm{Mod}_D$, such that $M^{**} \cong M$, i.e. M is a finitely generated reflexive C-module. Let $\mathcal{E} := \mathrm{End}_B(M)$, then there is a homomorphism of D-bimodules*

$$\eta : \mathcal{E}^* \to \mathrm{Hom}_B\bigl(M, {}^\nu M\bigr),$$

which is an isomorphism if ${}_B M$ is projective.

Remark 6. In the case when ${}_C B$ is symmetric ($\Longleftrightarrow \nu = \mathrm{id}$), taking $D = \mathcal{E}$ shows that for f.g. B-projective M, $\mathcal{E} = \mathrm{End}_B(M)$ is also a symmetric C-algebra (well known).

Proof of Proposition 4.3. See Appendix A. □

Proposition 4.4. *Let C be a commutative ring, S a finite (not necessarily commutative) C-algebra, i.e. such that the left C-module ${}_C S$ is finitely generated, and let $G \leqslant \mathrm{Aut}_C(S)$ be a finite group. Set $B := S * G$ to be the twisted group algebra with $sgs'g' = sg(s')gg'$ for $s, s' \in S$ and $g, g' \in G$. Then the following are equivalent:*

(i) *$S^* = \mathrm{Hom}_C(S, C) = S \cdot \theta$, i.e. ${}_C S$ is left (balanced) pre-Frobenius.*
(ii) *$B^* = \mathrm{Hom}_C(B, C) = B \cdot \hat{\theta}$ is left (balanced) pre-Frobenius with $\hat{\theta}(Sg) = 0$ for all $g \neq 1$.*

Proof. See Appendix A. □

Theorem 4.5. *Let ${}_C S$ be a finite commutative C-algebra and assume it is pre-Frobenius with $S^* = \mathrm{Hom}_C(S, C) = S \cdot \theta$. Let $G \leqslant \mathrm{Aut}_C(S)$ be a finite group and $B := S * G$ the twisted group ring with center $Z := Z(B)$ and $R := S^G$. Then the following hold*:

(i) *${}_C B$ is balanced pre-Frobenius with $B^* = B \cdot \hat{\theta}$ and corresponding automorphism $\nu := \nu_{\hat{\theta}} \in \mathrm{Aut}_Z(B)$, satisfying $\nu(s) = s$ for all $s \in S$.*
(ii) *The left action of G on S induces a natural right action on S^* given by $(\lambda, g) \mapsto \lambda \circ g$.[1] For $s \in S$, $(s\lambda) \circ g = g^{-1}(s)(\lambda \circ g)$, so G acts by R-module automorphisms.*
(iii) *Corresponding to $\theta \in S^*$ is the cocycle $\chi := \chi_\theta \in Z^1(G, U(S))$ defined by the formula $\theta \circ g^{-1} = g \cdot \theta := \chi_\theta(g) \cdot \theta$ and satisfying $\nu(g) = \chi_\theta^{-1}(g) \cdot g$.*
(iv) *One has for $\theta' \in S^*$:*

$$S \cdot \theta = S \cdot \theta' \quad \Longleftrightarrow \quad \theta' = u\theta \quad \text{with } u \in U(S) \quad \text{and} \quad \chi_{\theta'} = \chi_\theta \cdot \partial u,$$

where $\partial u(g) := g(u)u^{-1} \in B^1(G, U(S))$. In other words, $[\chi_\theta] = [\chi_{\theta'}] \in H^1(G, U(S))$.
(v) *There is an identity of R-modules*

$$S^{\nu^{-1}G} := \{s \in S \mid \nu^{-1}(g)(s) = s,\ \forall g \in G\} = S_{\chi^{-1}},$$

where $S_{\chi^{-1}}$ denotes the module of relative χ^{-1}-invariants.

Proof. See Appendix A. $\square$

We will later use the theory of Galois ring extensions as developed in [7], from where we take the following result:

Theorem 4.6. *Let S be a commutative ring, $G \leqslant \mathrm{Aut}(S)$ a finite group of ring automorphisms and $R := S^G$. Then the following are equivalent*:

(i) *S is a finitely generated projective R-module and $j : B := S * G \to \mathrm{End}_R(S)$ is an isomorphism.*
(ii) *For every $1 \neq \sigma \in G$ and maximal ideal P of S there is $s(\mathrm{P}, \sigma) \in S$ with $s - \sigma(s) \notin \mathrm{P}$.*

If either of these conditions is satisfied, then ${}_B S$ is projective in B-mod and $R \hookrightarrow S$ is called a Galois ring extension *with Galois group G.*

Proof. The equivalence of (i) and (ii) has been shown in [7] and the fact that they imply that ${}_B S$ is finitely generated projective follows for example from [13, Theorem 2.5]. It also follows from the fact that, if $R \hookrightarrow S$ is Galois, then the bimodule ${}_R S_B$ is invertible and induces a Morita-equivalence between *R-mod* and *B-mod*. $\square$

We summarize:

Proposition 4.7. *Let ${}_C S$ be a finite commutative C-algebra, $G \leqslant \mathrm{Aut}_C(S)$ a finite group, with $R := S^G$ and twisted group ring $B = S * G$. Assume ${}_C S$ is a pre-Frobenius algebra with $\mathrm{Hom}_C(B, C) = B\hat{\theta} \cong {}_B B$ and $\nu := \nu_\theta$ as in Theorem 4.5. Then there is a homomorphism*

$$\eta : R^* := \mathrm{Hom}_C(R, C) \to \mathrm{Hom}_B\big(S, {}^\nu S\big) \cong S^{\nu^{-1}G} := \{s \in S \mid \nu^{-1}(g)(s) = s,\ \forall g \in G\}.$$

If moreover $R \leqslant S$ is a Galois-extension, then the following hold:

[1] The corresponding left action is as usual defined by $g\lambda := \lambda \circ g^{-1}$.

(i) *${}_B S$ and ${}_R S$ are f.g. projective generators in B-mod and R-mod respectively.*
(ii) *${}_B S_R$ and ${}_R S_B$ induce Morita-equivalences between R-Mod and B-Mod.*
(iii) *The map η is an isomorphism.*

Proof. Since $S^* \cong {}_C S$, S is C-reflexive with $R = \mathrm{End}_B(S)$ and by Theorem 4.5 ${}_C B$ is balanced C-Frobenius. Hence Proposition 4.3 yields the map η. Moreover we have $\mathrm{Hom}_B(S, {}^\nu S) = (\mathrm{Hom}_S(S, {}^\nu S))^G = ({}^\nu S)^G = S^{\nu^{-1}G}$. The rest follows from the results above. □

To apply this result in the case of noetherian normal domains we need the following technical result (see [3, Proposition 19, p. 537]):

Proposition 4.8. *Let $A \subseteq B$ be a finite extension of commutative noetherian normal domains. Then for $N \in B$-mod, N is reflexive $\Longleftrightarrow$ ${}_A N \in A$-mod is reflexive.*

Corollary 4.9. *If $C \leqslant S$ is a finite extension of (commutative) noetherian normal domains and S is a UFD, then ${}_C S$ is pre-symmetric.*

Proof. The C-module $S^* = \mathrm{Hom}_C(S, C)$ is reflexive with $\mathrm{rank}_C(S^*) = [\mathrm{Quot}(S) : \mathrm{Quot}(C)] = \mathrm{rank}_C(S)$, hence by Proposition 4.8 S^* is S-reflexive with $\mathrm{rank}_S(S^*) = \mathrm{rank}_C(S^*)/\mathrm{rank}_C(S) = 1$. So S^* is isomorphic to a divisorial ideal and since S is a UFD it is free of rank one (see Proposition 2.2). Hence $S^* \cong {}_S S$ and ${}_C S$ is pre-Frobenius. Since S is commutative, ${}_C S$ is pre-symmetric. □

Lemma 4.10. *Let T be a noetherian normal domain, $M, N \in T$-mod finitely generated reflexive modules and $\phi \in \mathrm{Hom}_T(M, N)$ such that $\phi_{\mathfrak p} : M_{\mathfrak p} \to N_{\mathfrak p}$ is an isomorphism for every prime $\mathfrak p \in \mathrm{Spec}(T)$ of height one. Then ϕ is an isomorphism.*

Proof. Let $\mathbb{K} := \mathrm{Quot}(T)$; since $X \in \{M, N\}$ is reflexive, it is torsion free with $X = \bigcap_{\mathfrak p \in \mathrm{Spec}_1(T)} X_{\mathfrak p} \leqslant \mathbb{K} \otimes_T X$. By assumption every $\phi_{\mathfrak p} : M_{\mathfrak p} \to N_{\mathfrak p}$ is an isomorphism, hence so is

$$\phi : M = \bigcap_{\mathfrak p \in \mathrm{Spec}_1(T)} M_{\mathfrak p} \to \bigcap_{\mathfrak p \in \mathrm{Spec}_1(T)} N_{\mathfrak p} = N. \qquad \square$$

Proposition 4.11. *Let ${}_C S$ be a finite extension of noetherian normal domains, $G \leqslant \mathrm{Aut}_C(S)$ a finite group with ring of invariants $R := S^G$ and twisted group ring $B = S * G$. Assume that ${}_C S$ is pre-Frobenius with $\mathrm{Hom}_C(S, C) = S\theta \cong {}_S S$ and let $\chi \in Z^1(G, U(S))$ be defined by the formula $\theta \circ g^{-1} = \chi(g) \cdot \theta$ (see Theorem 4.5). Assume that ${}_B S$ is "projective in height one", i.e. for every $\mathfrak p \in \mathrm{Spec}_1(R)$ (or $\mathfrak p \in \mathrm{Spec}_1(C)$), the localization ${}_{B_{\mathfrak p}} S_{\mathfrak p}$ is projective.*

Then there is an isomorphism of R-modules $\eta : R^* := \mathrm{Hom}_C(R, C) \to S_{\chi^{-1}}$.

Remark 7.

(i) The assumption on ${}_C S$ is satisfied if S is a UFD.
(ii) The assumption on ${}_B S$ is satisfied if the extension $R \hookrightarrow S$ is unramified in height one.

Proof. Define $\hat\theta$ and $\nu := \nu_{\hat\theta}$ as in Theorem 4.5, then $R = \mathrm{End}_B(S)$ and by Proposition 4.7 and Theorem 4.5, $\mathrm{Hom}_B(S, {}^\nu S) = S_\chi$, which is a reflexive R-module by Lemma 3.1. Also R^* is a reflexive C-module, hence by Proposition 4.8, both are reflexive C- and R-modules. Under the hypothesis, it follows from Proposition 4.3 that $\eta_{\mathfrak p} : R^*_{\mathfrak p} \to (\mathrm{Hom}_B(S, {}^\nu S))_{\mathfrak p}$ is an isomorphism for every $\mathfrak p \in \mathrm{Spec}_1(R)$ (or $\mathfrak p \in \mathrm{Spec}_1(C)$[2]), hence, by Lemma 4.10, η is an isomorphism globally.

[2] Using "going up" and "going down".

The first remark follows from Corollary 4.9.

Now assume that $R \hookrightarrow S$ is unramified in height one. Then for every $\mathfrak{p} \in \operatorname{Spec}_1(R)$ the extension $R_\mathfrak{p} \hookrightarrow S_\mathfrak{p}$ is Galois by Theorem 4.6, hence ${}_{B_\mathfrak{p}}S_\mathfrak{p}$ is projective. $\square$

5. Quasi-Gorenstein rings of invariants

Now let k be a field and A a finitely generated normal k-domain with $U(A) = U(k)$, such that the quotient field $\mathbb{L} := \operatorname{Quot}(A)$ is separable over k. Let $G \subseteq \operatorname{Aut}(A)$ be a finite group of ring automorphisms with ring of invariants $R := A^G$. Then k is a separable algebraic extension of $k' := k^G \subseteq \mathbb{K} := \operatorname{Quot}(R)$ and $\mathbb{L}$ as well as $\mathbb{K}$ are separable over k'. By Noether-normalization there is a k'-polynomial ring $\mathcal{F} \subseteq R := A^G$ such that ${}_\mathcal{F}R$ and ${}_\mathcal{F}A$ are finitely generated modules, i.e. $\mathcal{F} = k'[f_1, \dots, f_d]$, with $(f_1, \dots, f_d)$ a system of parameters of R as k'-algebra. It follows from [11, Corollary 16.18], that $\mathcal{F}$ can be chosen such that $\mathbb{L}$ and $\mathbb{K}$ are separable over $\operatorname{Quot}(\mathcal{F})$. If $\mathcal{F}$ is chosen in that way we will mark this by using the notation $\mathcal{F}_{sep}$ instead of $\mathcal{F}$.

Definition 7. For a normal subring $B \subseteq A$ such that $B \hookrightarrow A$ is finite and $\operatorname{Quot}(A)$ is separable over $\operatorname{Quot}(B)$ let $\mathcal{D}_{A,B} \trianglelefteq A$ denote the corresponding Dedekind-different.

It is well known that $\mathcal{D}_{A,B}$ and its inverse $\mathcal{D}_{A,B}^{-1}$ are divisorial (fractional) ideals of A such there is an isomorphism of A-modules

$$\theta : \mathcal{D}_{A,B}^{-1} := \{x \in \operatorname{Quot}(A) \mid \operatorname{tr}_{A,B}(xA) \subset B\} \to \operatorname{Hom}_B(A, B), \quad x \mapsto \operatorname{tr}_{A,B}(x\cdot).$$

More relevant details about the Dedekind-different can be found in [1, Chapter 3].

Lemma 5.1. *Let $G \leqslant \operatorname{Aut}(A)$ be a finite group and $B = A^N$ with $N \trianglelefteq G$ or $B \leqslant A^G$. Then there are natural G-actions on the sets $\mathcal{D}_{A,B}^{-1}$ and $\operatorname{Hom}_B(A, B)$ and the map θ is a G-equivariant isomorphism.*

Proof. Let $B = A^N$, then for any $g \in G$, $\alpha \in \operatorname{Hom}_B(A, B)$, $a \in A$ and $b \in B$ we have $g(b) \in B$, hence

$$\begin{aligned} {}^g\alpha(ba) &:= g\big(\alpha(g^{-1}(ba))\big) = g\big(\alpha\big(g^{-1}(b)g^{-1}(a)\big)\big) = g\big(g^{-1}(b)\alpha\big(g^{-1}(a)\big)\big) \\ &= b \cdot g\alpha\big(g^{-1}(a)\big) = b\,{}^g\alpha(a), \end{aligned}$$

hence G acts on $\operatorname{Hom}_B(A, B)$ by conjugation. For $y \in \operatorname{Quot}(A)$,

$$\operatorname{tr}_{A,B}(y) = \sum_{n \in N} n \cdot y = \sum_{n \in N} n^g \cdot y = g^{-1}\left(\sum_{n \in N} n(g \cdot y)\right) = g^{-1}\operatorname{tr}_{A,B}(gy).$$

For $x \in \mathcal{D}_{A,B}^{-1}$ and $a \in A$ we have $(g\theta(x))(a) = g \cdot \theta(x)(g^{-1}a) = g \operatorname{tr}_{A,B}(x \cdot (g^{-1}a)) = g \operatorname{tr}_{A,B}(g^{-1}(gx \cdot a)) = \operatorname{tr}_{A,B}((gx) \cdot a) = \theta(gx)(a)$, hence G acts on $\mathcal{D}_{A,B}^{-1}$ and θ is G-equivariant.

Now let $B \leqslant A^G$; again G acts on $\operatorname{Hom}_B(A, B)$ by conjugation, with trivial G-action on B. For $g \in G$ and any $y \in \operatorname{Quot}(A)$ we have $\operatorname{tr}_{A,R}(gy) = \sum_{h \in G} hg(y) = \sum_{h \in G} h(y) = \operatorname{tr}_{A,R}(y)$. Now the transitivity of traces yields: $\operatorname{tr}_{A,B}(gy) = \operatorname{tr}_{R,B} \circ \operatorname{tr}_{A,R}(gy) = \operatorname{tr}_{R,B} \circ \operatorname{tr}_{A,R}(y) = \operatorname{tr}_{A,B}(y)$. It follows that for $a \in A$ and $x \in \mathcal{D}_{A,B}^{-1}$: $\theta(gx)(a) = \operatorname{tr}_{A,B}((gx)a) = \operatorname{tr}_{A,B}(g(x.g^{-1}a)) = \operatorname{tr}_{A,B}(x.g^{-1}a) = \theta(x)(g^{-1}a) = g(\theta(x))(a)$, so $\theta(gx) = g\theta(x)$. Again we conclude that G acts on the inverse different and θ is G-equivariant. $\square$

The following is an immediate consequence of Lemma 4.1:

Proposition 5.2. *Let $\mathcal{F} \leqslant S \leqslant A$ with normal domain S. If ${}_\mathcal{F}S$ and ${}_\mathcal{F}A$ are pre-symmetric with $A^* := \operatorname{Hom}_\mathcal{F}(A, \mathcal{F}) = A \cdot \theta_A$ and $S^* = \operatorname{Hom}_\mathcal{F}(S, \mathcal{F}) = S \cdot \theta_S$, then*

$$\mathcal{D}_{A,S}^{-1} \cong \mathrm{Hom}_S(A,S) = A \cdot \theta_{A,S},$$

with $\theta_{A,S}$ defined by the equation $\theta_{A,S}(a) \cdot \theta_S = (a\theta_A)_{|S}$. If $S = A^N$ for $N \trianglelefteq G$, then G acts on $\mathcal{D}_{A,S}^{-1}$ and there are characters χ_A, χ_S and $\chi_{A,S} \in Z^1(G, U(k))$ defined by the equations

$$\theta_A \circ g^{-1} = \chi_A(g) \cdot \theta_A, \qquad \theta_S \circ g^{-1} = \chi_S(g) \cdot \theta_S \quad \text{and} \quad g \circ \theta_{A,S} \circ g^{-1} = \chi_{A,S}(g) \cdot \theta_{A,S}.$$

These characters satisfy

$$\chi_A = \chi_S \cdot \chi_{A,S}.$$

Proof. The first claim follows from Lemma 4.1, which also gives $\theta_A = \theta_S \circ \theta_{A,S}$. For $g \in G$ we have $\chi_A(g)\theta_A = \theta_A \circ g^{-1} = \theta_S \circ \theta_{A,S} \circ g^{-1} = \chi_{A,S}(g) \cdot \theta_S \circ g^{-1} \circ \theta_{A,S} = \chi_{A,S}(g) \cdot \chi_S(g)\theta_S \circ \theta_{A,S} = \chi_{A,S}(g) \cdot \chi_S(g)\theta_A$. □

Now we assume in addition that A is a factorial domain (see Section 3.2).

Let $S := A^W$, then by Lemma 3.11 S is also factorial. The following lemma is well known (at least in the context of Dedekind domains appearing in number theory):

Lemma 5.3. *For any $W \subseteq H \subseteq G$ the following hold:*

(i) $\mathcal{D}_{A,A^G} = \mathcal{D}_{A,A^H}$.
(ii) $\mathcal{D}_{A^H,A^G} = (1) = A^H$.

In particular the extension $A^G \hookrightarrow A^W$ is unramified in height one.

Proof. (i): For $\mathrm{Q} \in \mathrm{Spec}_1(A)$ let $G_{k(\mathrm{Q})} := \{g \in G \mid (g-1)A \subseteq \mathrm{Q}\}$ denote the inertia group of Q and $H_{k(\mathrm{Q})} := H \cap G_{k(\mathrm{Q})}$. It is well known (e.g. from [17, I. No. 7, Proposition 21]) that the ring extensions $A^{G_{k(\mathrm{Q})}} \geqslant A^G$ and $A^{H_{k(\mathrm{Q})}} \geqslant A^H$ are unramified at the prime ideals $\mathrm{Q} \cap A^{G_{k(\mathrm{Q})}}$ and $\mathrm{Q} \cap A^{H_{k(\mathrm{Q})}}$, respectively. A local calculation of the Dedekind-different gives

$$\nu_{\mathrm{Q}}(\mathcal{D}_{A,A^G}) = \nu_{\mathrm{Q}}(\mathcal{D}_{A,A^{G_{k(\mathrm{Q})}}}) = \nu_{\mathrm{Q}}(\mathcal{D}_{A,A^{H_{k(\mathrm{Q})}}}) = \nu_{\mathrm{Q}}(\mathcal{D}_{A,A^H}).$$

Now the claim follows from the fact that the Dedekind-different is a divisorial ideal.

(ii): Let $\mathrm{q} \in \mathrm{Spec}_1(A^H)$; then there exists $\mathrm{Q} \in \mathrm{Spec}_1(A)$ with $\mathrm{q} = \mathrm{Q} \cap A^H$. From the Dedekind-tower theorem and (i) we obtain

$$d(\mathcal{D}_{A,A^G}) = d(\mathcal{D}_{A,A^H}) + d(A \cdot \mathcal{D}_{A^H,A^G}) = d(\mathcal{D}_{A,A^H}),$$

hence $d(A \cdot \mathcal{D}_{A^H,A^G}) = 0$, or equivalently, $A = A \cdot \mathcal{D}_{A^H,A^G}$. It follows that

$$0 = \nu_{\mathrm{Q}}(A \cdot \mathcal{D}_{A^H,A^G}) = e(\mathrm{Q},\mathrm{q}) \cdot \nu_{\mathrm{q}}(\mathcal{D}_{A^H,A^G}),$$

hence $\nu_{\mathrm{q}}(\mathcal{D}_{A^H,A^G}) = 0$. Since $\mathcal{D}_{A^H,A^G} \trianglelefteq A^H$ is divisorial, we conclude that $\mathcal{D}_{A^H,A^G} = A^H$.

The last statement follows from [17, III. No. 5, Theorem 1]. □

Using the fact that ${}_RS$ is unramified in height one we can apply Proposition 4.11 to prove the main result:

Proof of Theorem 1.2. Recall that $\mathcal{F}$ is a parameter subalgebra of $R := A^G \leqslant S := A^W \leqslant A$. Since S and A are UFD's, they are symmetric $\mathcal{F}$-algebras with $S^* = \mathrm{Hom}_{\mathcal{F}}(S,\mathcal{F}) = S \cdot \theta_S$, $A^* = \mathrm{Hom}_{\mathcal{F}}(A,\mathcal{F}) =$

$S \cdot \theta_A$ and the Dedekind-different $\mathcal{D}_{A,R} = \mathcal{D}_{A,S}$ is a principal ideal with $\mathcal{D}_{A,S}^{-1} \cong \mathrm{Hom}_S(A, S) = A \cdot \theta_{A,S}$. The fact that $R^* = \mathrm{Hom}_{\mathcal{F}}(R, \mathcal{F}) \cong S_{\chi_S^{-1}} = A_{\chi_S^{-1}}$ follows from Proposition 4.11 and $\chi_S = \chi_A \cdot \chi_{A,S}^{-1}$ follows from Proposition 5.2. Notice that, using notation from Proposition 3.5, $\chi_S \in \tilde{H}$, hence $[\mu_{\chi_S}] = 1$. It follows that $A_{\chi_S^{-1}}$ represents a divisorial ideal of class given by $[\chi_S]$, from which the remaining statements follow. $\square$

A slightly more special version of Theorem 1.2, with $\mathcal{F}$ replaced by $\mathcal{F}_{sep}$, i.e. which requires to choose the parameter algebra $\mathcal{F}$ in such a way that $\mathbb{L} \geqslant \mathrm{Quot}(\mathcal{F})$ is separable, can be proved in a way which does not depend on the results of Section 4:

Proof of Theorem 1.2, special case with $\mathcal{F}_{sep}$. In this case $R^* \cong \mathcal{D}_{R,\mathcal{F}}^{-1}$ and $\mathcal{D}_{S,\mathcal{F}} = Sd$, a principal ideal, since S is a factorial domain. It follows from Lemma 5.1 that $S^* = \mathrm{Hom}_{\mathcal{F}}(S, \mathcal{F}) \cong \mathcal{D}_{S,\mathcal{F}}^{-1}$ is an isomorphism of $S * G$-modules. Hence $S^* = S\theta_S$, where $\theta_S \in S^*$ can be identified with an element in $\mathrm{Quot}(S)$. Since the fractional ideal $\mathcal{D}_{S,\mathcal{F}}^{-1}$ is G/W-stable θ_S is a relative invariant with character $\chi_S \in \tilde{H}$. By the Dedekind-tower theorem, $\mathcal{D}_{S,\mathcal{F}} = \overline{\mathcal{D}_{S,R}\mathcal{D}_{R,\mathcal{F}}} \trianglelefteq S$, which implies (first locally at height one primes, then globally):

$$S^* = S\theta_S \cong \mathcal{D}_{S,\mathcal{F}}^{-1} = \overline{\mathcal{D}_{S,R}^{-1}\left(S\mathcal{D}_{R,\mathcal{F}}^{-1}\right)} = \overline{S\mathcal{D}_{R,\mathcal{F}}^{-1}} \subseteq \mathrm{Quot}(S).$$

There is a suitable element $r \in R$ with $rS\theta_S \subseteq S$ and therefore $rS\theta_S = r\overline{S\mathcal{D}_{R,\mathcal{F}}^{-1}} \subseteq S$. Hence we get $r\overline{S\mathcal{D}_{R,\mathcal{F}}^{-1}} \cap R = r\mathcal{D}_{R,\mathcal{F}}^{-1} = rS\theta_S \cap R$, so $R^* \cong \mathcal{D}_{R,\mathcal{F}}^{-1} = S\theta_S \cap \mathrm{Quot}(R) \cong S_{\chi_S{}^{-1}} = A_{\chi_S{}^{-1}}$, where the isomorphism is one of R-modules. Since $\chi_S \in \tilde{H}$ we have $[\mu_{\chi_S}] = 1$, so $\mathrm{ch}(\mathrm{cl}(A_{\chi_S{}^{-1}})) = [\chi_S]$. The equation $\chi_S = \chi_A \cdot \chi_{A,S}^{-1}$ follows immediately from

$$\overline{\mathcal{D}_{A,R}\mathcal{D}_{R,\mathcal{F}}} = \overline{\mathcal{D}_{A,S}\mathcal{D}_{S,\mathcal{F}}}$$

and $\mathcal{D}_{A,S} = \mathcal{D}_{A,R}$. The remaining statements follow immediately. $\square$

Proof of Corollary 1.4. Since $\tilde{W}/W$ is generated by p-elements, it follows that $\mathcal{C}_{\tilde{S}} = \mathrm{Hom}(\tilde{W}/W, U(k)) = 1$, hence $\mathrm{Hom}(G/W, U(k)) = \mathrm{Hom}(G/\tilde{W}, U(k))$ and $\tilde{S}$ is a factorial domain, hence quasi-Gorenstein. Using Lemma 5.3 the remaining arguments are exactly as above with W replaced by $\tilde{W}$ and S by $\tilde{S}$. $\square$

For the proof of Corollary 1.3 we need the following proposition:

Proposition 5.4. *Let $\mathcal{P} \subseteq B$ be as in Definition 2 and assume that ${}_{\mathcal{P}}B$ is quasi-Gorenstein. Then for every* $\mathrm{Q} \in \mathrm{Spec}(B)$, *the localization B_{Q} is Cohen–Macaulay if and only if B_{Q} is Gorenstein. In other words, the Cohen–Macaulay and Gorenstein loci of B coincide.*

Proof. Let $\mathrm{Q} \in \mathrm{Spec}(B)$ be such that B_{Q} is Cohen–Macaulay. Set $\mathrm{q} = \mathrm{Q} \cap \mathcal{P} \in \mathrm{Spec}(\mathcal{P})$ and let $\mathrm{Q} := \mathrm{Q}_1, \ldots, \mathrm{Q}_k$ be the primes of B lying over q. Since $\mathrm{Hom}_{\mathcal{P}}(B, \mathcal{P}) \cong B$ and $\widehat{B_{\mathrm{q}}} \cong \bigtimes_{i=1}^k \hat{B}_{\mathrm{Q}_i}$, we get $\widehat{B_{\mathrm{q}}} \cong (\widehat{\mathrm{Hom}_{\mathcal{P}}(B, \mathcal{P})})_{\mathrm{q}} \cong \mathrm{Hom}_{\mathcal{P}}(B, \mathcal{P}) \otimes_{\mathcal{P}} \hat{\mathcal{P}}_{\mathrm{q}} \cong \mathrm{Hom}_{\hat{\mathcal{P}}_{\mathrm{q}}}(\widehat{B_{\mathrm{q}}}, \hat{\mathcal{P}}_{\mathrm{q}}) \cong \bigtimes_{i=1}^k \mathrm{Hom}_{\hat{\mathcal{P}}_{\mathrm{q}}}(\hat{B}_{\mathrm{Q}_i}, \hat{\mathcal{P}}_{\mathrm{q}}) \cong \bigtimes_{i=1}^k \hat{B}_{\mathrm{Q}_i}$. Let $1_{\widehat{B_{\mathrm{q}}}} = \sum_{i=1}^k e_i$ with $e_i e_j = \delta_{ij}$, then $\hat{B}_{\mathrm{Q}_i} = e_i \widehat{B_{\mathrm{q}}} \cong e_i \mathrm{Hom}_{\hat{\mathcal{P}}_{\mathrm{q}}}(\widehat{B_{\mathrm{q}}}, \hat{\mathcal{P}}_{\mathrm{q}}) \cong \mathrm{Hom}_{\hat{\mathcal{P}}_{\mathrm{q}}}(e_i \widehat{B_{\mathrm{q}}}, \hat{\mathcal{P}}_{\mathrm{q}}) \cong \mathrm{Hom}_{\hat{\mathcal{P}}_{\mathrm{q}}}(\hat{B}_{\mathrm{Q}_i}, \hat{\mathcal{P}}_{\mathrm{q}})$. Since B_{Q_1} is Cohen–Macaulay, so is $\hat{B}_{\mathrm{Q}_1}$ and it is finite over $\hat{\mathcal{P}}_{\mathrm{q}}$. It follows that $\mathrm{Hom}_{\hat{\mathcal{P}}_{\mathrm{q}}}(\hat{B}_{\mathrm{Q}_i}, \hat{\mathcal{P}}_{\mathrm{q}})$ is the unique canonical module $\omega_{\hat{B}_{\mathrm{Q}_i}}$ (up to isomorphism) of $\hat{B}_{\mathrm{Q}_i}$. Therefore

$\omega_{\hat{B}_{\mathrm{Q}_i}} \cong \hat{B}_{\mathrm{Q}_i}$. It is generally true, that for a finitely generated B_{Q}-module M, the completion $M \otimes_{B_{\mathrm{Q}}} \hat{B}_{\mathrm{Q}}$ is canonical for $\hat{B}_{\mathrm{Q}}$, if and only if M is canonical for B_{Q}, so we conclude that $\omega_{B_{\mathrm{Q}}} \cong B_{\mathrm{Q}}$ and B_{Q} is Gorenstein. $\square$

Proof of Corollary 1.3. This follows immediately from Theorem 1.2 and Proposition 5.4. $\square$

6. The graded connected case

The application of Theorem 1.2 depends on the determination of $[\chi_S]$ or, equivalently $[\chi_A]$ and $[\chi_{A,S}]$. If G acts trivially on k, then these are linear characters in $\mathrm{Hom}(G/W, U(k))$ or $\mathrm{Hom}(G, U(k))$, respectively. In this section we investigate these characters in the case where A is a graded connected Cohen–Macaulay ring.

In the case where k has positive characteristic we will make use of the concept of *Brauer-characters*, as defined in modular representation theory of finite groups. In this case, k will be part of a p-modular system (K, R, k), where R is a discrete valuation ring with quotient field K of characteristic 0 and $k = R/\mathrm{Rad}(R)$ of characteristic $p > 0$. In this case one can define for every linear representation of G on the finite dimensional k-vector space V a K-valued class function χ_V in such a way that for every $g \in G$, $\chi_V(g) \in R$ and $\overline{\chi_V(g)} \in k$ coincides with the trace $\mathrm{trace}(g_{|V})$. The class function χ_V is called the *Brauer-character* of V. For more details we refer to [8, p. 402 ff]. Throughout this section $A = \sum_{i \geqslant 0} A_i$ is an $\mathbb{N}_0$ graded connected noetherian normal k-algebra, i.e. $A_0 = k$ with $U(A) = U(k)$ and $G \subseteq \mathrm{Aut}_k(A)$ a finite group of graded k-algebra automorphisms. We will also assume that A is a Cohen–Macaulay domain, i.e. A is a free module over some (and then every) parameter algebra $\mathcal{F} \subseteq A$. We keep the previous notation, so $R = A^G \hookrightarrow A$ is a finite extension of noetherian normal domains. Let $y_1, y_2, \ldots, y_d \in R$ be a homogeneous system of parameters (hsop) with $d_i := \deg(y_i)$, $d = \mathrm{Dim}(R) = \mathrm{Dim}(A)$, and set $\mathcal{F} := k[y_1, \ldots, y_d]$.

Definition 8. Let $V := \bigoplus_{n \geqslant 0} V_n$ be an $\mathbb{N}_0$ graded k-vector space and G a finite group acting on V by graded k-linear automorphisms. We define the (*Brauer-*) *character series*

$$H_{V,g}^{(Br)}(t) := \sum_{n=0}^{\infty} \chi_{V_n}(g) t^n,$$

where χ_{V_n} is the (Brauer-) character afforded by the action of G on V_n. Note that $H_{V,g}(t) \in k[[t]]$, whereas $H_{V,g}^{Br}(t) \in \mathbb{Q}(\epsilon)[[t]]$, where ϵ is a primitive order(g)-th root of unity in $\mathbb{C}$.

Note that

$$H_A(t) := H_{A,\mathrm{id}}^{Br}(t) = \sum_{i \geqslant 0} \dim_k(A_i) t^i \in \mathbb{Q}(t)$$

is the ordinary Hilbert-series of A. Let $U := \overline{A} := A/\mathcal{F}^+ A$, where $\mathcal{F}^+ := (y_1, \ldots, y_d) \trianglelefteq \mathcal{F}$ is the unique maximal homogeneous ideal of $\mathcal{F}$. Then $\mathcal{F} \otimes_k U$ is the projective cover of ${}_{\mathcal{F}}A$ in $\mathcal{F}$-*mod*, hence, as ${}_{\mathcal{F}}A$ is free, we have $\mathcal{F} \otimes_k U \cong A$ as $\mathcal{F}$-modules. Moreover $U = \bigoplus_{i=0}^{\beta} U_i = \bigoplus_{i=1}^{\ell} k\xi_i$, where we choose a homogeneous k-basis $\{\xi_i \mid i = 1, \ldots, \ell\}$ with $\deg(\xi_i) =: \beta_i \leqslant \beta_{i+1}$, $\beta := \beta_\ell$ and $\ell := \dim_k(U)$. We also will choose an $\mathcal{F}$-basis $\mathcal{B} := \{s_i \mid i = 1, \ldots, \ell\}$ of A, such that $s_i + \mathcal{F}^+ A = \overline{s_i} = \xi_i$ for $i = 1, \ldots, \ell$.

Note that G acts on A and U and if $g(\xi_i) = \sum_{j=1}^{\ell} g_{ji} \xi_j$ with $(g_{ji}) \in k^{\ell \times \ell}$, then

$$g(s_i) = \sum_{j=1}^{\ell} g_{ji} s_j + \mathcal{X}$$

with $\mathcal{X} \in \mathcal{F}^+ A$. For each j let $\tilde{A}_j := \langle \mathcal{B} \rangle_k \cap A_j$, then $A_i = \bigoplus_{m+n=i} \mathcal{F}_m \otimes_k \tilde{A}_n$ and it is easily seen that

$$\chi_{A_i}(g) = \sum_{m+n=i} \dim_k(\mathcal{F}_m) \cdot \chi_{U_n}(g) = \mathrm{coeff}_i\big(H^{Br}_{\mathcal{F}}(t) \cdot H^{Br}_{U,g}(t)\big).$$

Hence $H^{Br}_{A,g}(t) = H_{\mathcal{F}}(t) \cdot H^{Br}_{U,g}(t)$. Since $H_{\mathcal{F}}(t) = \frac{1}{\prod_{i=1}^{\ell}(1-t^{d_i})}$ and $H^{Br}_{U,g}(t) \in \mathbb{Q}(\epsilon)[t]$, we get

Lemma 6.1. *The Brauer-character series of A are rational, i.e. $H^{Br}_{A,g}(t) \in \mathbb{Q}(\epsilon)(t)$.*

Now we assume in addition that A is Gorenstein. It is then well known that

$$H_A(t) = (-1)^d t^{a(A)} H_A(1/t),$$

where $a(A) = \deg(H_A(t))$ is the degree of $H_A(t)$. This symmetry is induced by the duality of the corresponding artinian Gorenstein algebra $U = \overline{A} := A/\mathcal{F}^+ A$, where $\mathcal{F}^+ := (y_1, \ldots, y_d) \trianglelefteq \mathcal{F}$ is the unique maximal homogeneous ideal of $\mathcal{F}$. For later use we recall the details:

There is a graded embedding $U/U^+[-\beta] \hookrightarrow k[-\beta] \subseteq {}_U U_\beta$, $k = ((U/U^+)[-\beta])_\beta \ni \lambda \mapsto \lambda \xi_\ell$. It follows from [6] that ${}_U U$ is injective with $\mathrm{Soc}(U) \cong k$ (up to shift), hence

$$k[-\beta] \cong U/U^+[-\beta] \cong \mathrm{Soc}({}_U U).$$

It is well known that ${}^*E(k) \cong U^* := \mathrm{Hom}_k(U, k)$, where ${}^*E(k)$ denotes the graded *injective hull of $k = U_0$ (see [6] for the definition of *injectivity). Note that $U = \bigoplus_{i=0}^{\beta} U_i$; choosing a homogeneous dual k-basis $\{\xi_i^* \mid i = 1, \ldots, \ell\}$ (such that $\xi_i^*(\xi_j) = \delta_{i,j}$ and $\deg(\xi_i^*) = -\deg(\xi_i)$), we see that $U^* = \bigoplus_{i=0}^{\beta} (U^*)_{-i}$ with $k \cong \mathrm{Soc}(U^*) \cong U_0^*$ and $\dim_k(U^*)_{-i} = \dim_k U_i$. Since ${}_U U$ is injective and indecomposable we conclude

$${}_U U \cong {}^*E\big(\mathrm{Soc}({}_U U)\big) \cong {}^*E\big(k[-\beta]\big) \cong {}^*E(k)[-\beta] \cong U^*[-\beta].$$

It follows that $\dim_k(U_i) = \dim_k(U^*[-\beta]_i) = \dim_k((U^*)_{i-\beta}) = \dim_k(U_{\beta-i})$, hence $H_U(t) = t^\beta H_U(1/t) = H_U^*(t)$. Since $\mathrm{Rad}(U^*) = \mathrm{Soc}(U)^\perp = \langle \xi_1^*, \ldots, \xi_{\ell-1}^* \rangle$ we have ${}_U U^* = U \cdot \xi_\ell^*$ as well as a non-degenerate associative bilinear form

$$\kappa(\,,\,) : U \times U \to k, \quad \kappa\big(\xi, \xi'\big) = \xi_\ell^*\big(\xi \cdot \xi'\big).$$

It follows from $\mathrm{Soc}(U) = k \cdot \xi_\ell$, that for $g \in G$, $g(\xi_\ell) = \lambda(g)\xi_\ell$, with some linear character $\lambda \in \mathrm{Hom}(G, U(k))$. Since the G-action preserves degrees, we have $g(\xi_j) \in \sum_{n<\beta} U_n$, hence $g^{-1}\xi_\ell^*(\xi_j) = \xi_\ell^*(g(\xi_j)) = 0$ for every $j < \ell$ and $g^{-1}\xi_\ell^*(\xi_\ell) = \xi_\ell^*(g(\xi_\ell)) = \lambda(g) \cdot 1$; hence $g\xi_\ell^* = \lambda(g)^{-1}\xi_\ell^*$ for every $g \in G$. It follows that $\kappa(g(\xi_i), g(\xi_j)) = \lambda(g) \cdot \kappa(\xi_i, \xi_j)$.

Proposition 6.2. *Let A be a graded connected Gorenstein algebra, then the Brauer-character series of A and U satisfy the following identities*:

(i) $H^{Br}_{U,g}(t) = \hat{\lambda}(g) \cdot t^\beta H^{Br}_{U,g^{-1}}(1/t)$;

(ii) $\frac{H^{Br}_{A,g}(t)}{H^{Br}_{A,g^{-1}}(1/t)} = (-1)^d t^{a(A)} \hat{\lambda}(g)$ *with* $a(A) = \beta - \sum_i d_i = \deg(H^{Br}_{A,1}(t))$.

In particular

$$\hat{\lambda}(g) = (-1)^d \cdot \lim_{t \to 1} \frac{H^{Br}_{A,g}(t)}{H^{Br}_{A,g^{-1}}(1/t)}.$$

Remark 8. It follows from (i) that the character λ only depends on A and not on the choice of $\mathcal{F}$. Therefore we denote it by λ_A and we will denote the corresponding Brauer-character by $\hat{\lambda}_A$.

Proof of Proposition 6.2. (i): Let $\mathfrak{A} := \{a_1, \ldots, a_m\}$ and $\mathfrak{B} := \{b_1, \ldots, b_m\}$ be k-bases of U_i and $U_{\beta-i}$, respectively, then $\kappa(g(a_i), g(b_j)) = \lambda(g) \cdot \kappa(a_i, b_j)$. On the other hand, this is equal to $M_{\mathfrak{A}}(g)^{tr} \circ Q \circ M_{\mathfrak{B}}(g)$, where $Q = (\kappa(a_i, b_j)) \in k^{m\times m}$. For every $0 \leqslant \nu \leqslant \beta$ with $\nu \neq \beta - i$ we have

$$U_\nu \subseteq U_i^{\perp} := \bigl\{a \in U \bigm| \kappa(a, U_i) = 0\bigr\}.$$

Hence the map $U_i \times U_{\beta-i} \to k$, $(a, b) \mapsto \kappa(a, b)$ is a perfect pairing, in particular Q is a non-singular matrix. Therefore $M_{\mathfrak{A}}(g)^{tr} = \lambda(g) \cdot Q \circ M_{\mathfrak{B}}(g)^{-1} \circ Q^{-1}$ and $\operatorname{trace}(g_{|U_i}) = \lambda(g) \operatorname{trace}(g^{-1}_{|U_{\beta-i}})$, from which (i) follows immediately.

(ii): Using (i), the LHS is equal to

$$\frac{H^{Br}_{U,g}(t)}{H^{Br}_{U,g^{-1}}(1/t)} \cdot \frac{\prod_i (1 - t^{d_i})^{-1}}{\prod_i (1 - t^{-d_i})^{-1}} = \hat{\lambda}(g) \cdot t^{\beta - \sum_i d_i} (-1)^d. \qquad \square$$

Remark 9. Let $g \in \mathrm{GL}(V)$ semisimple, $A := \mathrm{Sym}(V^*) \cong k[x_1, \ldots, x_n]$ with $x_1, \ldots, x_n$ a basis of V^*. We can assume that $g(x_i) = \lambda_i x_i$ with eigenvalues $\lambda_i \in U(k)$, so with slight abuse of notation we obtain

$$H^{Br}_{A,g}(t) = \widehat{\operatorname{trace}(g_{|A})} = \prod_{i=1}^{n} \bigl(1 + \widehat{\lambda_i} t + \widehat{\lambda_i}^2 t^2 + \cdots\bigr) = \prod_{i=1}^{n} \frac{1}{1 - \widehat{\lambda_i} t} = \frac{1}{\widehat{\det(1 - tg)}}.$$

It follows that

$$\begin{aligned} H^{Br}_{A,g^{-1}}(1/t) &= \frac{1}{\widehat{\det(1 - g^{-1} 1/t)}} = \frac{t^n}{\widehat{\det(t - g^{-1})}} = \frac{t^n \widehat{\det(g)}}{\widehat{\det(gt - 1)}} \\ &= (-1)^n t^n \widehat{\det}(g) \cdot \frac{1}{\widehat{\det(1 - gt)}} = (-1)^n t^n \widehat{\det}(g) \cdot H^{Br}_{A,g}(t). \end{aligned}$$

Hence $\widehat{\lambda_A}(g) = \widehat{\det}(g)^{-1}$.

Proposition 6.3. *Let A be a graded connected Gorenstein domain and also a factorial domain. Then* $\mathrm{Hom}_{\mathcal{F}}(A, \mathcal{F}) \cong A\theta_A$ *with* $\chi_A^{-1} = \lambda_A$ *as defined in Remark* 8. *Moreover* $\mathrm{Hom}_{\mathcal{F}}(R, \mathcal{F}) \cong A_\lambda$, *where*

(i) $\lambda := \lambda_A$ *if* $\mathrm{char}(k)$ *does not divide* $|G|$;
(ii) $\lambda := \lambda_S \in \mathrm{Hom}(G/W, U(k))$, *if* $S = A^W$ *is Cohen–Macaulay* (*and therefore Gorenstein*).

In each of those cases $R = A^G$ is quasi-Gorenstein if and only if $\lambda = 1$.

Proof. It follows from Theorem 1.2 that there exists some function $\theta := \theta_A$ with $\mathrm{Hom}_{\mathcal{F}}(A, \mathcal{F}) \cong A \cdot \theta$. From [6, Proposition 3.3.3 (a)] we get

$$\overline{A} \cdot \bar{\theta} = \mathrm{Hom}_{\mathcal{F}}(A, \mathcal{F}) \otimes \mathcal{F}/\mathcal{F}^+ \cong \mathrm{Hom}_U(U, k) = U \cdot \xi_\ell^*,$$

hence $\bar{\theta} = c \cdot \xi_\ell^*$ with some non-zero scalar $c \in k$. Setting $\lambda := \lambda_A$, it follows that $\overline{g\theta} = g(\bar{\theta}) = \lambda(g)^{-1}\bar{\theta}$, so $g(\theta) - \lambda(g)^{-1} \cdot \theta \in \mathcal{F}^+ \mathrm{Hom}_{\mathcal{F}}(A, \mathcal{F})$. On the other hand G maps θ onto another module generator and therefore $g(\theta) = s_g \cdot \theta$ with a unit $s_g \in k = A_0$. It follows that $g(\theta) - \lambda(g)^{-1} \cdot \theta \in k \cdot \theta \cap \mathcal{F}^+ A\theta = 0$ and we conclude $g(\theta) = \lambda(g)^{-1} \cdot \theta$. This shows $\chi_A = \lambda_A^{-1}$.

Since S is a factorial domain, it is Gorenstein if Cohen–Macaulay, so the same argument as above gives $\chi_S = \lambda_S^{-1}$. The statement about $\mathrm{Hom}_{\mathcal{F}}(R, \mathcal{F})$ follows from Theorem 1.2.

For the rest of the proof we can assume that $\mathrm{char}(k)$ does not divide $|G|$. We consider the restriction map $\mathrm{res}: \mathrm{Hom}_{\mathcal{F}}(A, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{F}}(R, \mathcal{F})$, $\Psi \mapsto \Psi_{|R}$. Since $t: A \to R$, $s \mapsto |G|^{-1} \sum_{g \in G} g(s)$ is an epimorphism of $\mathcal{F}$-modules and ${}_{\mathcal{F}}R$ is free, we have ${}_{\mathcal{F}}A = {}_{\mathcal{F}}R \oplus {}_{\mathcal{F}}X$ for some complement ${}_{\mathcal{F}}X \subseteq A$, hence res is surjective. Let $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, \mathcal{F})$, then there is $s \in A$ with $\phi = s \cdot \theta(|_R) = \theta(s \cdot ())$. For any $r \in R$ we get

$$\phi(r) = \frac{1}{|G|} \sum_{g \in G} \phi(gr) = \frac{1}{|G|} \sum_{g \in G} \theta(sgr) = \frac{1}{|G|} \sum_{g \in G} \theta\big(g\big(g^{-1}(s)r\big)\big)$$
$$= \frac{1}{|G|} \sum_{g \in G} \lambda(g)\theta\big(g^{-1}(s)r\big) = \theta\big(t_\lambda(s)r\big),$$

where $t_\lambda := \frac{1}{|G|} \sum_{g \in G} \lambda(g) g^{-1} : A \to A_\lambda$ is the projection operator in $\mathrm{Hom}_{\mathcal{F}}(A, A_\lambda)$. Thus we have $\mathrm{Hom}_{\mathcal{F}}(R, \mathcal{F}) \subseteq \mathrm{res}(A_\lambda \cdot \theta)$. Again it follows from ${}_{\mathcal{F}}A = {}_{\mathcal{F}}R \oplus {}_{\mathcal{F}}X$, that $\mathrm{Hom}_{\mathcal{F}}(X, \mathcal{F})^G = 0$, hence

$$\mathrm{Hom}_{\mathcal{F}}(A, \mathcal{F})^G \cong \mathrm{res}_R\big(\mathrm{Hom}_{\mathcal{F}}(A, \mathcal{F})^G\big) = \mathrm{Hom}_{\mathcal{F}}\big(A^G, \mathcal{F}\big).$$

Clearly $A_\lambda \cdot \theta \in \mathrm{Hom}_{\mathcal{F}}(A, \mathcal{F})^G$, so $\mathrm{Hom}_{\mathcal{F}}(R, \mathcal{F}) = \mathrm{res}(A_\lambda \cdot \theta) \cong {}_R A_\lambda$. If $\lambda = 1$, then $\mathrm{Hom}_{\mathcal{F}}(R, \mathcal{F}) = R \cdot \mathrm{res}(\theta)$ is a cyclic R-module, so $\omega_R \cong \mathrm{Hom}_{\mathcal{F}}(R, \mathcal{F}) \cong {}_R R$ and R is Gorenstein. $\square$

Remark 10. In the special case where $A = \mathrm{Sym}(V^*)$ with linear G-action the result above for the non-modular case also appears in [15, Corollary 3.2]. The proof indicated there depends on the results of [18,19]. In contrast to this our proof above is elementary and independent of Watanabe's results as well as of our Theorem 1.2.

One can apply the results above for example in the situation where $A := \mathrm{Sym}(V^*)$ for finite dimensional kG-module V, and $S = A^W$ or $S = A^{\tilde{W}}$, with $\overline{G} := G/W$ or $G/\tilde{W}$ acting on S. However, even if $S = k[x_1, \ldots, x_n]$ is a polynomial ring (with $\deg(x_i) =: d_i \geqslant 1$), then action of $\overline{G}$ will in general be non-linear and the k-space $\langle x_1, \ldots, x_n \rangle_k$ will be not $\overline{G}$-stable. Nevertheless we can use Remark 9 to determine $\lambda_S = \chi_S^{-1}$.

Let M be a finite dimensional kG-module with kG-submodule $N \subseteq M$. As a vector space we have $M = N \oplus U$, with $U \cong M/N$ as a kG-module. Even though M and $N \oplus U$ are in general not isomorphic as kG-modules, one has $\chi_M = \chi_N + \chi_{M/N}$. It follows that $\mathrm{Sym}(M) \cong \mathrm{Sym}(N) \otimes_k \mathrm{Sym}(U)$ as a k-algebra, but in general not as kG-module. Nevertheless we have $H^{Br}_{\mathrm{Sym}(M),g}(t) = H^{Br}_{\mathrm{Sym}(N),g}(t) \cdot H^{Br}_{\mathrm{Sym}(M/N),g}(t)$. Even more generally, the following lemma includes the case of a graded, but non-linear G-action on the algebra generators:

Lemma 6.4. *Let G act on A by graded algebra automorphisms and $B \leqslant A$ a G-stable graded subalgebra. Assume that $A \cong B \otimes_k A/B_+A$ as a k-algebra (not necessarily as kG-module). Then*

$$H^{Br}_{A,g}(t) = H^{Br}_{B,g}(t) \cdot H^{Br}_{A/B_+A,g}(t).$$

Proof. Let $A/B_+A =: C$ and identify the k-algebras $B \otimes_k C \cong A$, via $b \otimes c = bc$. Let $x_1, \ldots, x_\mu$ be a k-basis of B_m and $y_1, \ldots, y_\nu$ a k-basis of C_n. Then $g(y_j) = \sum_t g_{C;tj} y_t + \mathfrak{BC}$ with $\mathfrak{BC} \in \sum_{r=1}^n B_r C_{n-r}$ and the matrix $(g_{C;tj})$ describing the representation of g on the kG-module $C_j \cong (A/B_+A)_j$. Hence

$$g(x_i y_j) = g(x_i) g(y_j) = \sum_s (g_{B;si} x_s) \sum_t (g_{C;tj} y_t + \mathfrak{BC})$$
$$= g_{B;ii} \cdot g_{C;jj} \cdot x_i y_j + \sum_{(s,t) \neq (i,j)} g_{B;si} g_{C;tj} x_s y_t + \mathcal{X},$$

with $\mathcal{X} \in \sum_{r=1}^{n} B_{m+r}C_{n-r} \subseteq B_+A$. It follows that $\chi_{A_{m+n},g} = \chi_{B_m,g} \cdot \chi_{C_n,g}$ and therefore $H^{Br}_{A,g}(t) = H^{Br}_{B,g}(t) \cdot H^{Br}_{C,g}(t)$. $\square$

Proposition 6.5. *Let $A = k[x_{11},\dots,x_{1j_1},x_{21},\dots,x_{2j_2},\dots,x_{\ell 1},\dots,x_{\ell j_\ell}]$ be a polynomial ring with generators of degrees $1 \leqslant d_1 < d_2 < \cdots < d_\ell$. For $i := 1,\dots,\ell$ let U_i denote the kG-module $A_{d_i}/A^+A^+ \cap A_{d_i} \in kG$-mod and $\det_i : G \to k$, $g \mapsto \det(g_{|U_i})$. Then for every $g \in G$:*

$$H^{Br}_{A,g}(t) = \prod_{i=1}^{\ell} H^{Br}_{\mathrm{Sym}(U_i),g}(t) = \prod_{i=1}^{\ell} \frac{1}{\widehat{\det(1 - t^{d_i} g)}} \quad \text{and} \quad \widehat{\lambda_A}(g) = \prod_{i=1}^{\ell} \widehat{\det}_i(g)^{-1}.$$

Proof. The subalgebra $B := k[x_{11},\dots,x_{1j_1}] = \mathrm{Sym}(U_1) \subseteq A$ is G-stable and we have $A = B \otimes_k A/B^+A$ with polynomial ring $A/B^+A \cong k[\bar{x}_{21},\dots,\bar{x}_{2j_2},\dots,\bar{x}_{\ell 1},\dots,\bar{x}_{\ell j_\ell}]$. Now the first equality follows from Lemma 6.4 and an obvious induction. The rest follows in a way similar to Remark 9. $\square$

If $\mathrm{char}(k) = p > 0$, then by definition p does not divide $[G : \tilde{W}]$, hence if $A^{\tilde{W}}$ is Cohen–Macaulay, so is A^G. With regard to the Gorenstein property we obtain the following:

Corollary 6.6. *Let $A := \mathrm{Sym}(V^*)$ with finite dimensional kG-module V and assume that $A^{\tilde{W}} \cong k[x_{11},\dots,x_{1j_1},x_{21},\dots,x_{2j_2},\dots,x_{\ell 1},\dots,x_{\ell j_\ell}]$ is a polynomial ring with generators of degrees $1 \leqslant d_1 < d_2 < \cdots < d_\ell$. Then A^G is Cohen–Macaulay and A^G is Gorenstein if and only if $\prod_{i=1}^{\ell} \widehat{\det}_i(g)^{-1} = 1$ for all $g \in G$.*

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Appendix A. Proofs of results of Section 4

Proof of Lemma 4.1. It is clear that $\mathrm{Hom}_B(A,B)$ is a natural A–B-bimodule. For each $a \in A$, the map $a \cdot \theta_A = \theta_A((\cdot) \cdot a)|_B$ is in B^*. Hence for each a there is a unique element $\lambda(a) \in B$, such that $\lambda(a) \cdot \theta_B = (a\theta_A)|_B$. Denote $\phi^{(A,C)} : A \to A^*$, $a \mapsto a \cdot \theta_A$ and $\phi^{(B,C)} : B \to B^*$, $b \mapsto b \cdot \theta_B$, then we have the bijection:

$$\phi^{(A,B)} := \left(\left(\phi^{(B,C)}\right)^{-1}\right)_* \circ \Psi \circ \phi^{(A,C)} : A_B \to \mathrm{Hom}_B({}_BA, B),$$

which maps $a \in A$ to the function

$$\left(\left(\phi^{(B,C)}\right)^{-1}\right)_* \left(a' \mapsto \left(b \mapsto \left(a \cdot \theta_A\left(ba'\right)\right)\right)\right) = a' \mapsto \left(\phi^{(B,C)}\right)^{-1} \left(\theta_A\left((\cdot)a'a\right)\big|_B\right)$$

hence $a \mapsto a \cdot \lambda$. The fact, that $\lambda \in \mathrm{Hom}_B({}_BA, B)$ can also be seen directly:

indeed, $(b\lambda(a))\theta_B(b') = \lambda(a) \cdot \theta_B(b'b)$ which is, by definition $\theta_A(b'ba)$, but this again is by definition $\lambda(ba)\theta_B(b')$.

Obviously $\phi^{(A,B)}$ is an isomorphism of left A-modules. The fact that $\phi^{(A,B)}$ is a bimodule homomorphism follows, if λ is a B-bimodule homomorphism, so we verify this:

$$\lambda(ab) = \lambda(a) \cdot b.$$

By the assumptions we have

$$\lambda(a)\cdot b\cdot\theta_B\big(b'\big)=\theta_B\big(b'\lambda(a)b\big)=\theta_B\big(bb'\lambda(a)\big)=\theta_A\big(bb'a\big)=\theta_A\big(b'ab\big)=\lambda(ab)\theta_B\big(b'\big).$$

This finishes the proof. $\square$

Proof of Theorem 4.2. For $\beta\in\mathrm{Hom}_B(M,B)$, $d\in D$ and $b\in B$ we have

$$\theta_*(d\beta b)(m)=\theta\big(\beta(md)b\big)=\theta\big(\nu^{-1}(b)\beta(md)\big)=\theta\big(\beta\big(\nu^{-1}(b)md\big)\big).$$

On the other hand,

$$\big(d\theta_*(\beta)b\big)(m)=\theta_*(\beta)(b\cdot md)=\theta\big(\beta(b\cdot md)\big)=\theta\big(\beta\big(\nu^{-1}(b)md\big)\big),$$

hence θ_* is a morphism of D–B-bimodules.

For $x\in\mathrm{Hom}_B(M,N)_D$ we get $x^*(\theta_*(\beta))=\theta_*(\beta)\circ x=\theta\circ\beta\circ x=\theta_*(x^\vee(\beta))$. Hence θ_* is a morphism of functors.

We have to exhibit an inverse of θ_*: For any $m\in M$ and $\alpha\in M^*$ the map $\widehat{\alpha,m}:b\mapsto\alpha(bm)\in C$ is an element in $\mathrm{Hom}_C(B,C)$, hence $\widehat{\alpha,m}=b_{\alpha,m}\cdot\theta$ for a unique $b_{\alpha,m}\in B$. Consider the map

$$\chi(M):\big({}^\nu M\big)^*\to M^\vee,\quad \alpha\mapsto(m\mapsto b_{\alpha,m}\in B).$$

Since $\theta(b''b_{\alpha,b'm})=\alpha(b''b'm)=\theta(b''b'b_{\alpha,m})$, we have $b_{\alpha,b'm}=b'b_{\alpha,m}$. Hence $\chi(M)(\alpha)\in\mathrm{Hom}_B(M,B)$. For $\beta\in\mathrm{Hom}_B(M,B)$ and $m\in M$ we get

$$\chi\circ\theta_*(\beta)(m)=b_{\theta\circ\beta,m}$$

with $\theta(bb_{\theta\circ\beta,m})=\theta(\beta(bm))=\theta(b\beta(m))$ hence $b_{\theta\circ\beta,m}=\beta(m)$. For $\alpha\in({}^\nu M)^*$: $\theta_*\circ\chi(\alpha)(m)=\theta(\chi(\alpha)(m))=\theta(b_{\alpha,m})=\alpha(m)$. So, indeed, χ is the two-sided inverse of θ_*. Naturality, as well as the fact that χ consists of bimodule isomorphisms follow automatically from this. $\square$

Proof of Proposition 4.3. Note that for any rings X, Y, Z and $U\in{}_X\mathrm{Mod}_Y$, $V\in{}_X\mathrm{Mod}_Z$ the set $\mathrm{Hom}_X(U,V)\in{}_Z\mathrm{Mod}_Y$ by the rule $z\alpha y(u):=\alpha(uy)\cdot z$. Hence the right D-action on M makes $\mathcal{E}=\mathrm{Hom}_B(M,M)$, $\mathrm{Hom}_B(M,{}^\nu M)$ and $\mathcal{E}^*$ into D-bimodules. Similarly $M^\vee=\mathrm{Hom}_B(M,B)\in{}_D\mathrm{Mod}_B$, hence $M^\vee\otimes_B M$ is a D-bimodule. Let t denote the canonical homomorphism

$$t:M^\vee\otimes_B M\to\mathrm{Hom}_B(M,M),\quad \alpha\otimes m\mapsto\big(m'\mapsto\alpha\big(m'\big)m\big).$$

Then $t(d\alpha\otimes md')(m')=d\alpha(m')md'=\alpha(m'd)md'=(\alpha(m'd)m)d'=(\alpha\otimes m)(m'd)d'=d(t(\alpha\otimes m))d'(m')$, hence t is a homomorphism of D-bimodules. Consider the homomorphism of C-modules:

$$\begin{aligned}\mathcal{E}^*&=\mathrm{Hom}_C\big(\mathrm{Hom}_B(M,M),C\big)\to^{t^*}\mathrm{Hom}_C\big(M^\vee\otimes_B M,C\big)\\&\cong\mathrm{Hom}_B\big(M,\mathrm{Hom}_C\big(M^\vee,C\big)\big)\to^{(\chi^*)_*}\mathrm{Hom}_B\big(M,\mathrm{Hom}_C\big(\big({}^\nu M\big)^*,C\big)\big)\\&=\mathrm{Hom}_B\big(M,\big({}^\nu M\big)^{**}\big)=\mathrm{Hom}_B\big(M,{}^\nu M\big).\end{aligned}$$

The middle map $\Psi:\mathrm{Hom}_C(M^\vee\otimes_B M,C)\to\mathrm{Hom}_B(M,\mathrm{Hom}_C(M^\vee,C))$ satisfies $\Psi(\gamma)(m)(\beta)=\gamma(\beta\otimes m)$. Both sides are D-bimodules and we get $\Psi(d\gamma d')(m)(\beta)=d\gamma d'(\beta\otimes m)=\gamma(d'\beta\otimes md)=\Psi(\gamma)(md)(d'\beta)=[\Psi(\gamma)(md)d'](\beta)=[d\Psi(\gamma)d'](m)(\beta)$, hence Ψ is an isomorphism of D-bimodules. The canonical map $M\to M^{**}$ with $m^{**}(\theta)=\theta(m)$ satisfies $(md)^{**}(\theta)=\theta(md)=d\theta(m)=m^{**}(d\theta)=m^{**}d(\theta)$, so it is a D-module homomorphism. It follows that the intermediate maps from $\mathcal{E}^*$ to $\mathrm{Hom}_B(M,{}^\nu M)$ are morphisms of D-bimodules. All of them are also isomorphisms, except possibly t^*.

It is well known that if ${}_BM$ is f.g. projective, then t and therefore t^* are isomorphisms as well. (Since ${}_CM$ is finitely generated, so is ${}_BM$.) $\square$

Proof of Proposition 4.4. For $\mu \in B^*$ and $b \in B$ we have $b \cdot \mu = \mu(()b) = (() \cdot b)^*\mu$, hence $g \cdot \mu = (()g)^*(\mu)$. Therefore the commutative diagram of C-modules with obvious projections and inclusions

$$\begin{tikzcd} S \arrow[r, "i_1"] \arrow[d, "()g"'] & B \arrow[r, "\pi_1"] \arrow[d, "()g"] & S \arrow[d, "()g"] \\ Sg \arrow[r, "i_g"'] & B \arrow[r, "\pi_g"'] & Sg \end{tikzcd}$$

induces the dual diagram

$$\begin{tikzcd} S^* \arrow[r, "\pi_1^*"] & B^* \arrow[r, "i_1^*"] & S^* \\ (Sg)^* \arrow[r, "\pi_g^*"'] \arrow[u, "g\cdot"] & B^* \arrow[r, "i_g^*"'] \arrow[u, "g\cdot"'] & (Sg)^*. \arrow[u, "g\cdot"'] \end{tikzcd}$$

For $s \in S$: $\pi_1 \circ ()s = ()s \circ \pi_1$, i.e. for $b = \sum_{h\in G} s_h h \in B$ we have $\pi_1(bs) = \pi_1(\sum_{h\in G} s_h h(s)h) = s_1 s = \pi_1(b) \cdot s$. Hence for $\mu \in S^*$, $\pi_1^*(s\mu)(b) = (s\mu)(\pi_1(b)) = \mu(\pi_1(b)s) = \mu(\pi_1(bs)) = \pi_1^*(\mu)(bs) = (s\pi_1^*(\mu))(b)$, so $\pi_1^* \in \mathrm{Hom}_S(S^*, B^*)$. For $\nu \in B^*$ and $s, s' \in S$ we have $i_1^*(s\nu)(s') = (s\nu)(s') = \nu(s's) = (s \cdot i_1^*(\nu))(s')$, so $i_1^* \in \mathrm{Hom}_S(B^*, S^*)$.

(i) $\Rightarrow$ (ii): Assume that ${}_CS$ is left pre-Frobenius. For every $\lambda_g \in Sg^* = \mathrm{Hom}_C(Sg, C)$, there is $\ell(g) \in S$ with $g\lambda_g = \ell(g)\theta$. Hence $g \cdot \pi_g^*(\lambda_g) = \pi_1^*(\ell(g)\theta) = \ell(g) \cdot \pi_1^*(\theta)$. So $\pi_g^*(\lambda_g) = g^{-1}\ell(g) \cdot \pi_1^*(\theta) \in B \cdot \pi_1^*(\theta)$. It follows that

$$B^* = \bigoplus_{g\in G} \pi_g^*\big(Sg^*\big) \subseteq B \cdot \pi_1^*(\theta) \subseteq B^*.$$

Let $\hat{\theta} := \pi_1^*(\theta)$ and $b = \sum_g s_g g \in B$ with $\hat{\theta}(bB) = 0$, then for every $h \in G$, $0 = \sum_g \hat{\theta}(s_g g S h^{-1}) = \theta(s_h S)$, so $s_h = 0$ and $b = 0$. It follows that ${}_CB$ is left pre-Frobenius.

Now assume that ${}_CS$ is balanced pre-Frobenius. Then $\hat{\theta}(Bb) = 0$ implies $0 = \hat{\theta}(h^{-1}S \sum_g s_g g) = \sum_g \hat{\theta}(Sh^{-1}(s_g)h^{-1}g) = \theta(Sh^{-1}(s_h))$, so $h^{-1}(s_h) = 0$ and $s_h = 0$ for every $h \in G$. In a similar way as before we see that $B^* = \hat{\theta} \cdot B$, so ${}_CB$ is balanced pre-Frobenius in the way as described.

(ii) $\Rightarrow$ (i): Assume first that ${}_CB$ is left pre-Frobenius as described. Since $i_1^*(g\hat{\theta})(s) = \hat{\theta}(sg) = 0$ for all $g \neq 1$, we have $S^* = i_1^*(B^*) = i_1^*(B \cdot \hat{\theta}) = i_1^*(S\hat{\theta}) = Si_1^*(\hat{\theta})$. If $s \in S$ with $si_1^*(\hat{\theta}) = 0$, then $\hat{\theta}(Ss) = 0$, hence $\hat{\theta}(Bs) \subseteq \sum_g \hat{\theta}(Sgs) \subseteq \sum_g \hat{\theta}(Sg(s)g) = \hat{\theta}(Ss) = 0$, so $s = 0$. It follows that S is left pre-Frobenius. If ${}_CB$ is balanced pre-Frobenius, it follows in a similar way that ${}_CS$ is balanced with $\theta := i_1^*(\hat{\theta})$. $\square$

Proof of Theorem 4.5. We use the notation of Proposition 4.4.

(i): Since ${}_CS$ is (balanced) pre-Frobenius it follows that ${}_CB$ is balanced pre-Frobenius with $B^* = B\hat{\theta}$ and $\hat{\theta} = \pi_1^*(\theta)$. Since $\theta \cdot s(s') = \theta(ss') = \theta(s's) = s\theta(s')$ we get $\hat{\theta}s(s'g) = \hat{\theta}(ss'g) = \hat{\theta}(s'sg) = \hat{\theta}(s'g(s)g) = \theta(s'gs) = s\hat{\theta}(s'g)$. It follows that $\nu(s) = s$.

(ii)–(iv): For $s \in S$,

$$(s\lambda) \circ g\big(s'\big) = (s\lambda)\big(gs'\big) = \lambda\big(g\big(s'\big)s\big) = \lambda\big(g\big(s'g^{-1}(s)\big)\big) = \lambda \circ g\big(s'g^{-1}(s)\big) = g^{-1}(s)(\lambda \circ g)\big(s'\big),$$

so $(s\lambda)\circ g = g^{-1}(s)(\lambda \circ g)$. Let $\lambda \in S^*$, then $\lambda \circ g^{-1} = t \cdot \theta$ for some $t \in S$. Set $s := g^{-1}(t)$, then $\lambda \circ g^{-1} = g(s)\theta$, so $\lambda = (g(s)\cdot\theta)\circ g = s\cdot(\theta\circ g) \in S(\theta \circ g)$. It follows $S^* = S\cdot\theta = S\cdot(\theta\circ g)$, hence $\theta \circ g = s_g\theta$ with $s_g \in U(S)$.

For $g \in G$ let $\nu(g) = \sum_{h' \in G} \nu_{h'}(g)h'$ with $\nu_{h'}(g) \in S$; then for $s' \in S$ and $h \in G$ we have $\hat{\theta}\cdot g(s'h) = \hat{\theta}(gs'h) = \theta(g(s'))\delta_{gh,1} = \nu(g)\cdot\hat{\theta}(s'h) = \hat{\theta}(s'h\nu(g)) = \sum_{h'\in G}\hat{\theta}(s'h(\nu_{h'}(g))hh') = \theta(s'h(\nu_{h^{-1}}(g)))$. For $h \neq g^{-1}$ we get $\theta(s'h(\nu_{h^{-1}}(g))) = 0$ for all s', hence $\nu_{h^{-1}}(g) = 0$ and for $h = g^{-1}$ we get $\theta(s'g^{-1}(\nu_g(g))) = \theta \circ g(s')$, so $\theta \circ g = g^{-1}(\nu_g(g))\cdot\theta$. Define $\chi(g)$ by $\chi(g)\theta = g\cdot\theta = \theta\circ g^{-1}$, then $\chi(g) = g(\nu_{g^{-1}}(g^{-1}))$ and $\nu(g) = g(\chi(g^{-1}))g$. It is straightforward to see that $\chi \in Z^1(G, U(S))$, hence $1 = \chi(gg^{-1}) = \chi(g)g(\chi(g^{-1}))$ and $g(\chi(g^{-1})) = \chi(g)^{-1} = \chi^{-1}(g)$.

Clearly $S\theta = S\theta' \iff \theta' = u\theta$ with unit $u \in U(S)$. Set $\chi' := \chi_{\theta'}$, then it follows $\theta'\circ g^{-1} = \chi'(g)\theta' = \chi'(g)u\theta = (u\theta)\circ g^{-1} = g(u)(\theta\circ g^{-1}) = g(u)\chi(g)\theta$. Hence $\chi'(g) = \chi(g)g(u)u^{-1}$.

(v): Let $\nu^{-1}(g) = \sum_h s_h h$ with $s_h \in S$, then $g = \sum_h \nu(s_h)\nu(h) = \sum_h s_h\chi^{-1}(h)h$, so $s_h = 0$ for all $h \neq g$ and $s_g = \chi^{-1}(g)^{-1} \in S$. It follows that $s \in S^{\nu^{-1}G} \iff \nu^{-1}(g)(s) = s \iff \chi(g)g(s) = s \iff s \in S_{\chi^{-1}}$. $\square$

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