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Dyson–Schwinger equation and quantum phase transitions in massless QCD

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Abstract

We study the stability of the highest symmetric solution (Wigner-solution) of Dyson–Schwinger equations in chiral limit and at zero temperature. Our results confirm that if the chemical potential is not very large, the QCD vacuum is in the chiral symmetry breaking phase and the quantum phase-transition of the chiral symmetry restoration is in first order. Meanwhile, it seems that there is not competition between chiral breaking phase and color superconductivity phase since the color superconductivity phase appears only if the chemical potential is very large. Moreover, we propose that chiral symmetry breaking arises from the positive feedback with respect to the mass perturbation.

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It has been known that the results of perturbative renormalization-group in QCD will encounter divergence at low energy region. This behavior indicates an important fact that the vacuum (Wigner-vacuum) defined within Feynman's original path-integral framework which possesses the highest symmetries is unstable and incorrect [1]. In addition, it should be emphasized that the effective action (viz. minus effective potential) based on Wigner-vacuum should have the same global and modified BRST symmetry [2] as the original action [3]. Thus, the gaps corresponding to the running coupling coefficients of the variational one-particle-irreducible (1PI) vertexes, which break the global symmetry and BRST gauge symmetry, should definitely be zero! Nevertheless, the statement, which is correct in high energy region and is the key to prove gauge theory is perturbatively renormalizable [2], often deviates from observations to real QCD in low energy region. Therefore, the real QCD vacuum in low energy channel is different from the Wigner-vacuum. Anyway, we should note that energy or free energy (if chemical potential $\mu \neq 0$) of Wigner-vacuum is definitely the lowest in Euclidean formalism of QCD at zero temperature (temperature $T = 0$ means a unbounded imaginary time). Thus, we cannot justify a phase-transition by comparing the energy between the symmetrical vacuum and the symmetry-broken one. Indeed, there has been a powerful general argument that, while ground states of a macroscopic system are degenerate, the true vacuum is the asymmetric one rather than the Wigner-vacuum because the Wigner-vacuum is extremely unstable against a perturbation (viz. symmetric system will undergo a dramatic transforms against an even arbitrarily small perturbation) [4]. However, such an idea has not yet been proved solidly. It is then interesting to study under what conditions the vacuums of QCD tend to be degenerate, and how to understand the phase-transitions in a reasonable and systematic way. It has been known that Dyson–Schwinger (D–S) equations approach provides a nonperturbative framework to study the vacuum properties of strong interaction and hadron properties in free space and to simulate the chiral symmetry restoration and deconfinement in the system with finite temperature and/or finite chemical potential [5–8]. In this Letter, we intend to shed some light on these questions by analyzing the stability of Wigner-vacuum in the framework of Dyson–Schwinger equation approach.

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Within Feynman's framework, one can easily derive a series of dynamical integral-equations (Dyson–Schwinger equations) [5] and a series of identities which come from the symmetries. If we combine these two series of results, we will get a theory based on Wigner-vacuum which gives Wigner-solution of the equations for any correlations and gaps. On the other hand, if we relax the restrictions of the symmetries, the Dyson–Schwinger equation will be correct not only for the symmetrical vacuum but also for the asymmetrical vacuum (if existing!) and, besides the Winger-solution, we can get a new class of solutions (Nambu-solution) of the D–S equation for the correlations and gaps. We must note here that the Nambu-solution is important if and only if the Wigner-vacuum is unstable under perturbations, viz. vacuums are degenerate, otherwise, Nambu-solution is not the real ground state but a dynamically stable excited state. For example, as we will see at below, the fact that chiral susceptibility of Wigner mass function in the chiral limit is negative indicates the chiral-symmetry breaking of the QCD vacuum in a natural sense, and the restoration of chiral symmetry is closely related to the presence of positive chiral susceptibility beyond some critical chemical potential.

It is well known that D–S equation is a series unclosed equations where the equation for n-point Green functions depend on $(n+1)$-point Green functions. One should then truncate it with indispensable approximations before taking it to evaluate any physics quantity practically. Furthermore, in order to obtain a full and reliable understanding of QCD phase-diagram with respect to the medium density (or the chemical potential), we should investigate the stability of the Wigner-vacuum against all possible perturbations simultaneously which are allowed by the truncation to D–S equation. In this work, for simplicity, we take the rainbow approximation and investigate the stability of the Wigner-vacuum against the allowed chiral-perturbation and diquark-perturbation.

We begin with single-flavor QCD in chiral limit. The action in Euclidean space is usually given as

$$S=\int d^4x\left[\bar{\Psi} i\gamma\cdot D\Psi+\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}-i\mu\bar{\Psi}\gamma_0\Psi\right], \tag{1}$$

where the γ matrices are chosen to satisfy $[\gamma_\mu,\gamma_\nu]_+=-2\delta_{\mu\nu}$ and $D_\mu=\partial_\mu+ig_\Lambda\frac{\lambda_a}{2}A_{a\mu}$.

Representing the 1PI vertexes of $\bar{\psi}\psi$, $\psi\psi$, $\bar{\psi}\bar{\psi}$ as Σ_1, Σ_2, Σ_3, respectively, we can write the dressed quark propagator without diquark component as

$$\tilde{G}=\frac{1}{p\cdot\gamma-i\mu\gamma_0+\Sigma_1}. \tag{2}$$

Meanwhile, the relations between the 1PI vertexes and the full propagators can be written as

$$G_1=\frac{\tilde{G}}{1-\Sigma_3\tilde{G}\Sigma_2\tilde{G}},\qquad G_{2,3}=\tilde{G}(-\Sigma_{2,3})G_1. \tag{3}$$

With these relations we can easily obtain the corresponding D–S equation

$$\Sigma_{1,2,3}(p)=-g_\Lambda^2\int\frac{d^4q}{(2\pi)^4}\gamma_\mu\frac{\lambda_\eta}{2}G_{1,2,3}(q)\gamma_\nu\frac{\lambda_\eta}{2}D_{\mu\nu}(k), \tag{4}$$

where $k\equiv p-q$ stands for the momentum transferred. Furthermore, if we are only interested in the condensates in chiral and color $\bar{3}$ diquark channels which preserves the parity and in which single gluon exchange interaction is attractive [1], the 1PI vertexes in strong interaction matter with chemical potential μ can be expressed as

$$\Sigma_1(p)=\left[A(p)-1\right]\vec{p}\cdot\vec{\gamma}+\left[C(p)-1\right](p_0-i\mu)\gamma_0+\Delta_1(p), \tag{5}$$

$$\Sigma_2(p)=\Delta_2(p)M\gamma_5\hat{C}, \tag{6}$$

$$\Sigma_3(p)=\Delta_3(p)\hat{C}M\gamma_5, \tag{7}$$

where M is a matrix in color space corresponding to the color $\bar{3}$ diquark channel, $\hat{C}$ is the charge conjugation operator, which can be given explicitly as

$$M_{\alpha\beta}=\epsilon_{1\alpha\beta}, \tag{8}$$

$$\hat{C}\Psi=\gamma_2\gamma_0\bar{\Psi}^{\mathrm{T}},\qquad \hat{C}\bar{\Psi}=\Psi^{\mathrm{T}}\gamma_2\gamma_0. \tag{9}$$

It is evident that the Wigner-solution is characterized by $\Delta_{1,2,3}\equiv 0$.

Instead of exploiting Nambu-solution of D–S equation, we prefer to investigate whether the Wigner-vacuum is stable against perturbations of the condensates $\Delta_1^b\bar{\Psi}\Psi$, $\Delta_2^b\bar{\Psi}M\gamma_5\hat{C}\Psi$ and $\Delta_3^b(\hat{C}\bar{\Psi})M\gamma_5\Psi$. For our purpose, here we have defined the perturbed Wigner-solutions for the gaps as

$$\Delta_1^{\mathrm{Wigner}}(p)=F_1(p)\Delta_1^b, \tag{10}$$

$$\Delta_2^{\mathrm{Wigner}}(p)=F_2(p)\Delta_2^b, \tag{11}$$

$$\Delta_3^{\mathrm{Wigner}}(p)=F_3(p)\Delta_3^b. \tag{12}$$

The F_1, $F_{2,3}$ stands for the susceptibility with respect to the chiral channel, diquark channels, respectively.

From the D–S equation with infinitesimal but explicit mass term, we find that, when $\Delta_1^b \neq 0$, $\Delta_{2,3}^b = 0$, F_1 satisfies equation

$$F_1(p) = 1 - g_\Lambda^2 \int \frac{d^4q}{(2\pi)^4} \gamma_\mu \frac{\lambda_\eta}{2} \frac{F_1(q)}{A_W^2(q)\vec{q}^2 + C_W^2(q)(q_0 - i\mu)^2} \gamma_\nu \frac{\lambda_\eta}{2} D_{\mu\nu}(k). \tag{13}$$

From the D–S equation with infinitesimal but explicit *diquark* term, we obtain that, when $\Delta_2^b \neq 0$, $\Delta_{1,3}^b = 0$, F_2 satisfies equation

$$\begin{aligned} &F_2(p) M \gamma_5 \hat{C} \\ &\quad = M\gamma_5 \hat{C} + g_\Lambda^2 \int \frac{d^4q}{(2\pi)^4} \gamma_\mu \frac{\lambda_\eta}{2} \frac{F_2(q)}{A_W(q)\vec{q}\cdot\vec{\gamma} + C_W(q)(q_0 - i\mu)\gamma_0} M\gamma_5\hat{C} \frac{1}{A_W(q)\vec{q}\cdot\vec{\gamma} + C_W(q)(q_0 - i\mu)\gamma_0} \gamma_\nu \frac{\lambda_\eta}{2} D_{\mu\nu}(k). \end{aligned} \tag{14}$$

Here A_W, C_W are Wigner-solutions for the A, C which satisfy the D–S equation with $\Delta_{1,2,3} \equiv 0$. After some derivation, the equations can be explicitly written as

$$A_W(p) = 1 - \frac{4g_\Lambda^2}{3\vec{p}^2} \int \frac{d^4q}{(2\pi)^4} \frac{A_W(q)[2p_i q_j D_{ij}(k) - \vec{p}\cdot\vec{q} D_{\mu\mu}(k)] + 2C_W(q)[(q_0 - i\mu)p_i D_{i0}(k)]}{A_W^2(q)\vec{q}^2 + C_W^2(q)(q_0 - i\mu)^2}, \tag{15}$$

$$C_W(p) = 1 - \frac{4g_\Lambda^2}{3(p_0 - i\mu)^2} \int \frac{d^4q}{(2\pi)^4} \frac{2A_W(q)(p_0 - i\mu)q_i D_{0i}(k) + C_W(q)(p_0 - i\mu)(q_0 - i\mu)D_{\mu\mu}^M(k)}{A_W^2(q)\vec{q}^2 + C_W^2(q)(q_0 - i\mu)^2}, \tag{16}$$

where $D_{\mu\mu}^M \equiv D_{00} - D_{ii}$.

It is apparent that to solve the equation practically, one needs the effective gluon propagator as an input. We then take the effective gluon propagator as the Tübingen model [9]

$$D_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) D(k), \tag{17}$$

with

$$g_\Lambda^2 D(k) = 4\pi^2 d \frac{k^2}{\omega^2} e^{-k^2/\omega^2}.$$

By using relation

$$\frac{\lambda_\eta}{2} M \frac{\lambda_\eta^{\mathrm{T}}}{2} = -\frac{2}{3} M, \tag{18}$$

we could conclude that the stability of Wigner-vacuum against chiral and diquark perturbation is characterized by linear integral equations

$$F_1(p) = 1 + 4g_\Lambda^2 \int \frac{d^4q}{(2\pi)^4} \frac{F_1(q) D(k)}{A_W^2(q)\vec{q}^2 + C_W^2(q)(q_0 - i\mu)^2}, \tag{19}$$

$$F_{2,3}(p) = 1 - 2g_\Lambda^2 \int \frac{d^4q}{(2\pi)^4} \frac{F_{2,3}(q) D(k)[A_W^2(q)\vec{q}^2 + C_W^2(q)(q_0^2 + \mu^2)]}{[A_W^2\vec{q}^2 + C_W^2(q_0^2 - \mu^2)]^2 + [2C_W^2 \mu q_0]^2}. \tag{20}$$

Up to now, what we have discussed is only one flavor of quark. In fact, we can deal with the case of two or three flavors of quarks without any technical difficulty in chiral limit. Since in chiral limit, flavor has nothing to do with dynamics, the gapped flavor channel should be chosen to leave the maximal unbroken symmetries [1], such as $\bar{\Psi}_{i\alpha}\epsilon^{1\alpha\beta}\epsilon^{ij}\gamma_5\hat{C}\Psi_{j\beta}$ in case of two flavors and $\bar{\Psi}_{i\alpha}\epsilon_A{}^{\alpha\beta}\epsilon^{Aij}\gamma_5\hat{C}\Psi_{j\beta}$ in case of three flavors (α, β are color indexes and i, j are flavor indexes), which are the well-known two-flavor–color superconductivity (2CS) and color–flavor locking (CFL) channels, respectively. However, all these considerations, which do not change the equations of the susceptibilities in chiral limit, are nontrivial in case of real mass spectrum of $\{u, d, s\}$ quarks because it has strongly indicated a transition from the 2CS phase to the CFL phase at some relatively high chemical potential. In the present work, as we work in chiral limit, one-flavor should be sufficient.

To obtain the quantitative result, we solve the D–S equation with the Tübingen model of the effective gluon propagator and parameters $\omega = 0.4$ GeV, $d = 45.0\ \mathrm{GeV}^{-2}$, with which the pion properties and some other low energy chiral observables have been described well [9]. The obtained results of the chemical potential dependence of the Wigner-solutions $A_W[\mu]$, $C_W[\mu]$ are illustrated in Fig. 1. The obtained chemical potential dependence of the Nambu dynamical mass function as the ratio Δ_{1N}/A_N of the Nambu-solutions is illustrated in Fig. 2. And the chemical potential dependence of the susceptibilities corresponding to the chiral quark gap and diquark gap of Wigner-solution ($F_1[\mu]$ and $F_2[\mu]$) are displayed in Fig. 3 and Fig. 4, respectively. Fig. 1 shows obviously that the Wigner-solutions $A(p, \mu)$ and $C(p, \mu)$ are equal only if $\mu = 0$ (another point shown in Fig. 1 is not universal

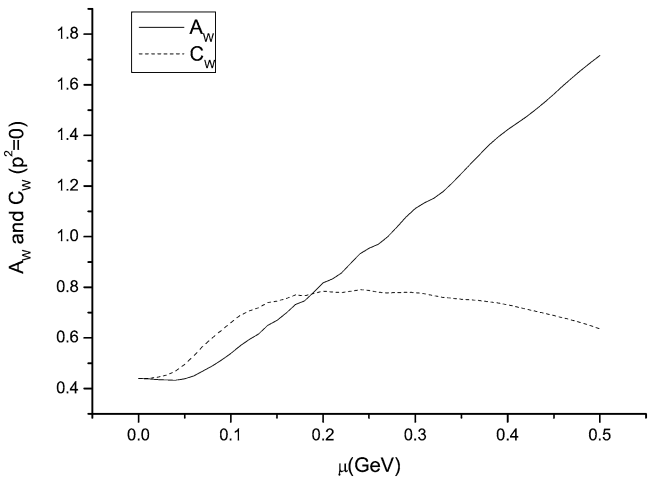

Fig. 1. Chemical potential μ dependence of Winger-solutions in extremely low energy channel ($p^2 = 0$).

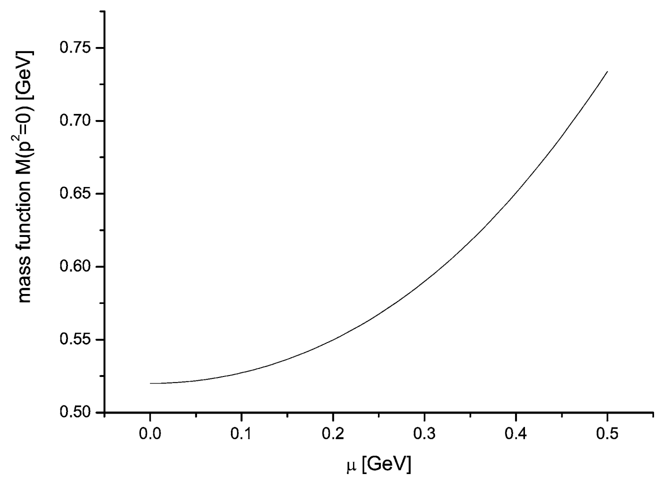

Fig. 2. Chemical potential μ dependence of the dynamical quark mass function in extremely low energy channel ($p^2 = 0$).

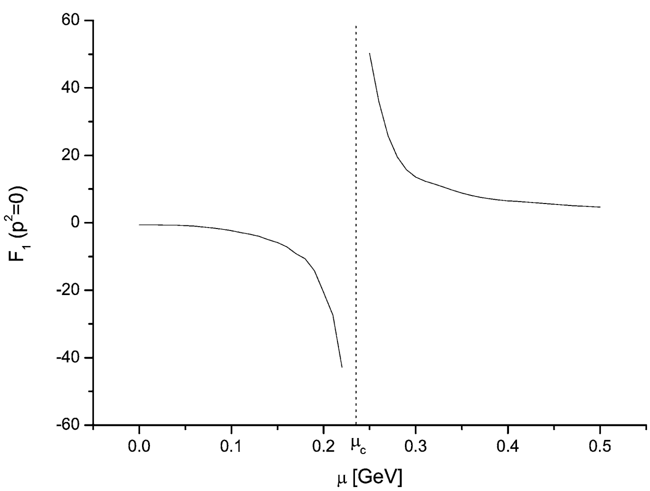

Fig. 3. Chemical potential μ dependence of the susceptibility F_1 in extremely low energy channel ($p^2 = 0$).

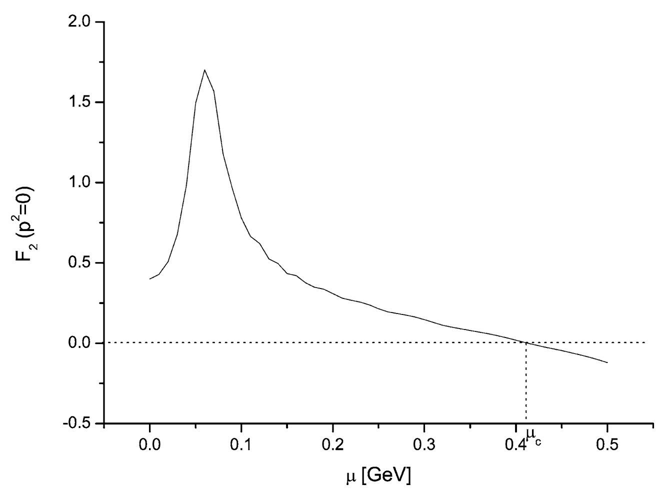

Fig. 4. Chemical potential μ dependence of the susceptibility F_2 in extremely low energy channel ($p^2 = 0$).

for other momentum modes), but remarkably separated from each other once $\mu \neq 0$. Since the A_W and C_W contain the crucial information of the QCD-vacuum in our current framework, we have calculated them as precisely as we can, rather than taking any approximation such as $A = C$ [10]. In fact, because of the obvious oscillating behavior of $A_W(\mu)$ and $C_W(\mu)$ caused by chemical potential, which will be restrained by chiral gap in Nambu-solutions, the numerical calculation for Wigner-solutions is much more difficult than for Nambu-solutions.

Because the susceptibility is the response rate of the dynamical mass with respect to the perturbation of current quark mass, with the above obtained result we discuss the quantum phase-transition in QCD. At first we take, for instance, a free fermion system in chiral limit with $F_1 \equiv 1$ as a starting point. Corresponding to the presence of the repulsive potential in chiral channel induced by mass vertex of the system, the influence of small and positive mass term on the system is nothing more than a gapped dispersion relation $E = (\vec{p}^2 + m^2)^{1/2}$. Changing the sign of the perturbative mass term just trivially results in a redefinition of fermion and anti-fermion, and of course the same physics. Therefore, the influence of small mass term on free system is really perturbative, and we can infer safely that the free vacuum should preserve chiral symmetry. Now, we turn to the case of QCD. Because QCD is an asymptotically free theory, the chiral susceptibility in high energy mode is positive to maintain the system stable in high energy region. If the F_1 in low energy mode is also *positive*, the QCD should be similar to the free case by means of that the vacuum is stable against mass perturbation. In fact, our calculation indicates that, in low chemical potential region, the F_1 is *negative* in low energy channel. In this case, a small and positive mass term $m\bar{\Psi}\Psi$ results in an attractive (not repulsive as the free case!) potential in the chiral channel whose strength is characterized by $|mF_1|$. This attractive interaction, even if quite weak, should induce a small $\langle\bar{\Psi}\Psi\rangle$ condensate, correspondingly. Since the induced small condensate can be treated as a positive mass-like vertex, which will be dressed by QCD interaction and results in a deeper attractive potential (if chiral susceptibility is still negative) in chiral channel

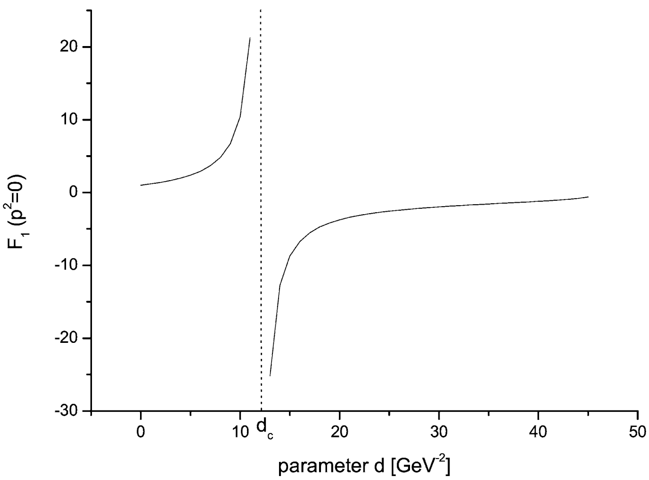

Fig. 5. Coupling constant d dependence of the susceptibility F_1 at zero chemical potential and zero momentum.

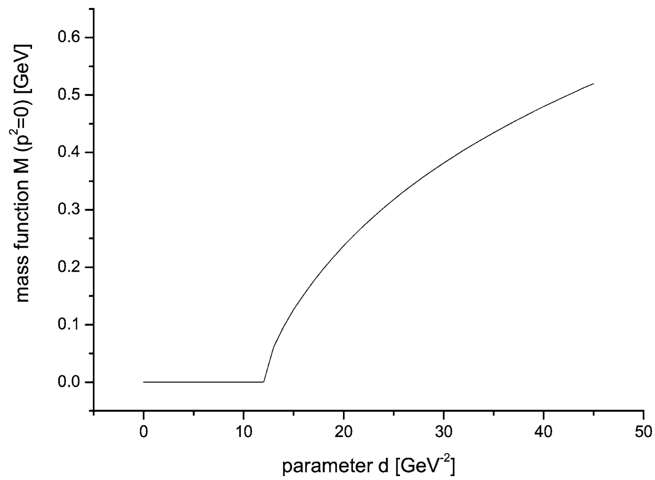

Fig. 6. Coupling constant d dependence of the dynamical mass function at zero chemical potential and zero momentum.

and a "larger" condensate! Such and such, the chiral condensate will become "larger" and "larger" as far as the chiral susceptibility become positive. In the limit $m \to 0^+$, this process tends to be quasistatic, without change of free energy or entropy or energy, it means that a spontaneous quantum phase transition from Wigner-vacuum to Nambu-vacuum takes place! If the perturbative mass is negative, the situation is similar, but leads the Wigner-vacuum to another physically equivalent Nambu-vacuum which can be affirmed in the framework of D–S equation (in the chiral limit, we can always find two Nambu-solutions for mass function with the same quantity and opposite sign, if the interaction in infrared region is strong enough [5,11,12]). From Fig. 3, one can easily recognize that the susceptibility F_1 is definitely negative if the chemical potential $\mu < 0.24$ GeV, it becomes positive if the chemical potential $\mu > 0.24$ GeV, and more significantly, the F_1 is divergent and disconnected at $\mu = 0.24$ GeV. Meanwhile, Fig. 4 indicates that the susceptibility F_2 is positive if the chemical potential $\mu < 0.42$ GeV and it changes to negative continuously as the chemical potential gets larger than 0.42 GeV. These behaviors manifest that the chiral symmetry breaking phase transits to the chiral symmetry preserving phase at chemical potential $\mu = 0.24$ GeV, and the phase transition is of first order, and that the diquark channel of Wigner-vacuum is stable while $\mu < 0.42$ GeV, and the phase transition is of second order. Therefore we can reach a conclusion for the massless-QCD phase transition in the present truncated D–S equation approach as: there is a first order phase-transition of chiral restoration at a quite large chemical potential and color superconductivity phase emerges as a second order phase transition at a very large chemical potential where chiral symmetry has already been restored.

To make our criterion more robust, we also investigate the coupling constant dependence of the chiral susceptibility F_1 of Wigner-solution at $\mu = 0$ and that of the Nambu dynamical mass function $M = \Delta_{1N}/A_N$ at $\mu = 0$. The obtained results are displayed in Fig. 5 and Fig. 6, respectively. It is evident that if the interaction strength parameter $d < 12.2\ \mathrm{GeV}^{-2}$, the chiral symmetry preserving phase (with dynamical mass $M = 0$) is stable. Only if the interaction in the infrared region is strong enough (in the present numerical case with $\omega = 0.4$ GeV, the d should be larger than $12.2\ \mathrm{GeV}^{-2}$), can the chiral symmetry breaking take place. Meanwhile, it should be emphasized that while the critical coupling constant shown in Fig. 5 can be easily found by straightforward solving the D–S equation for dynamical mass function (see Fig. 6), the critical chemical potential shown in Fig. 3 cannot be obtained by solving the D–S equation for dynamical mass function (see Fig. 2).

In summary, by solving the Dyson–Schwinger equations, we have studied the chemical potential dependence of the solutions of the D–S equation and of the Wigner-vacuum susceptibility in the chiral and diquark channels as well as the effect of the interaction strength. It shows that if the chemical potential of the system is not very large and the interaction in the infrared region is strong enough, the QCD vacuum is in the chiral symmetry breaking phase. If the chemical potential gets larger and reaches a critical value, the chiral symmetry can be restored by means of a first order phase transition. If the chemical potential is much larger so as to arrive at another critical value, the color superconductivity phase emerges. Meanwhile, the process of the chiral symmetry breaking is proposed to be a positive feedback with respect to the perturbation. Considering the criteria to characterize the chiral phase transition, we would like to mention that it is not advisable to introduce any approximate expression for free energy to judge which solution (Wigner-solution or Nambu-solution) is free energy favorable. First, no body knows how to construct an exact expression for free energy. It is more serious that an approximate expression can hardly satisfy the fact that Wigner-vacuum should always belong to the ground state Hilbert-subspace no matter the true vacuum is degenerate or not. Second, more importantly, it is not necessary to construct a free energy expression for understanding phase transition because the D–S equations (not solutions of D–S equations) contain naturally the full phase information of the dynamical system. For instance, as we have shown in this Letter, the susceptibilities of the Wigner-vacuum can be widely used to understand dynamical symmetry breaking of the physical vacuum.

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