Generalized Plasticity Theory for Phase Transformations

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Abstract

In this work we derive a new version of generalized plasticity theory, suitable to describe phase transformations. In particular, we present a general multi–surface formulation of non–isothermal generalized plasticity capable of describing the multiple and interacting loading mechanisms, which occur during phase transformations. Our formulation has a geometric basis and takes place within the context of tensor analysis in Euclidean spaces. The new theory, besides its theoretical interest, is also important for application purposes such as the description and prediction of the response of shape memory alloy materials.

Keywords: Generalized Plasticity; phase transformations; shape memory alloys

1. Development of a generalized plasticity theory for phase transformations

Generalized plasticity [2], [3], [4] is a local internal variable theory of rate–independent behavior which is based primarily on the loading–unloading irreversibility. As in all internal–variable type of theories it is assumed that the local thermomechanical state in a body is determined uniquely by the couple \((G, Q)\) where \(G\) stands for the vector of the controllable state variables and \(Q\) stands for the vector of the internal variables, which are related to phase transformations. In this work we follow a material (referential) approach within a strain–space formulation. Accordingly, \(G\) may be identified by \((E, T)\) where \(E\) is the referential (Green–Saint Venant) strain tensor and \(T\) is the temperature. Depending on the

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nature of the (material) internal variable vector \( Q \), the theory may in principle be formulated equivalently
with respect to the \textit{macro –, meso –, or micro – scale structure} of the material.

The central concept of generalized plasticity is that of the elastic range (e.g. see Lubliner, [4]), which
is defined at any material state as the region in the strain – temperature space comprising the strains
which can be attained elastically (i.e. with no change in the internal variables) from the current strain –
temperature point. It is assumed that the elastic range is a regular set in the sense that it is the closure
of an open set. The boundary of this set is defined as a loading surface [4] (at \( Q \)). In turn, a material state
may be defined as elastic if it is an interior point of its elastic range and inelastic if it is a boundary point
of its elastic range; in the latter case the material state lies on a loading surface. It should be added that the
notion of process is introduced implicitly here. By assuming that the loading surface is smooth at the
current strain - temperature point and by invoking some basic axioms and results from set theory and
topology Lubliner, [4], (see also [8]) showed that the rate equations for the evolution of the internal
variable vector may be written in the form:

\[
\dot{Q} = H L(G, Q) \langle N : \dot{G} \rangle,
\]

where \( \langle \cdot \rangle \) stands for the Macauley bracket defined as:

\[
\langle x \rangle = \begin{cases} 
  x & \text{if } x > 0 \\
  0 & \text{if } x \leq 0.
\end{cases}
\]

and \( H \) stands for a scalar function of the state variables. The value of \( H \) must be positive at any inelastic
state and zero at any elastic one. Finally, \( L \) stands for a non - vanishing (tensorial) function of the state
variables which is associated with the properties of the phase transformation and \( N \) can be either the
outward or the inward normal to the loading surface – depending on the direction in which a phase
transformation is active – at the current state.

It is emphasized that Eq. (1) has been derived under the assumption of a smooth loading surface at the
current strain – temperature point, which implies that only one loading mechanism can be considered. On
the other hand, the phase transformations include multiple and sometimes interacting loading mechanisms
which may result in the appearance of a vertex or a corner at the current strain – temperature point. This
fact calls for an appropriate modification of the rate equation (1).

In order to accomplish this goal we assume that the loading surfaces are defined in the state space by a
number – say \( n \) – of smooth surfaces, which are defined by expressions of the form:

\[
\Phi_i(G, Q) = 0, \quad i = 1, 2, ..., n
\]

These surfaces can be either disjoint, or intersect in a possibly non – smooth fashion. Each of these
surfaces is associated with a particular transformation mechanism which may be active at the current
strain – temperature point. Then by assuming that each equation \( \Phi_i(G, Q) = 0 \) defines independent (non –
redundant) active surfaces at the current stress temperature point the rate equations for the evolution of
the internal variables in view of Eq. (1) can be stated in the following general form:

\[
\dot{Q} = \sum_{i=1}^{n} H L_i(G, Q) \langle N : \dot{G} \rangle,
\]
where \( H_i \), \( \mathbf{L}_i \), and \( \mathbf{N}_i \) are functions of the state variables defined as in Eq. (1) and each set of them – defined by the index \( i \) – refers to the specific transformation associated with the part of the loading surface defined by \( \Phi_i(\mathbf{G}, \mathbf{Q}) = 0 \). From Eq. (3) one can deduce directly the loading – unloading criteria for the proposed formulation as follows: Let us denote by \( n_{\text{adm}} \leq n \) the number of loading surfaces that may be active at an inelastic state i.e. \( H_i > 0 \), and let us denote by \( J_{\text{adm}} \) the set of \( n_{\text{adm}} \) indices associated with those surfaces, i.e.

\[
J_{\text{adm}} = \{ \alpha \in \{1, 2, \ldots, n\} / H_\alpha > 0 \}.
\]

Then Eq. (3) implies the following loading – unloading conditions:

- If \( J_{\text{adm}} = \emptyset \), then \( \dot{\mathbf{Q}} = 0 \).
- If \( J_{\text{adm}} \neq \emptyset \), then:
  - i. If \( \mathbf{N}_\alpha : \dot{\mathbf{G}} \leq 0 \) for all \( \alpha \in J_{\text{adm}} \) then \( \dot{\mathbf{Q}} = 0 \),
  - ii. If \( \mathbf{N}_\alpha : \dot{\mathbf{G}} > 0 \) for at least one \( \alpha \in J_{\text{adm}} \) then \( \dot{\mathbf{Q}} \neq 0 \).

Then, if we denote further by \( n_{\text{act}} \leq n_{\text{adm}} \) the number of parts for which (ii) holds, and we set:

\[
J_{\text{act}} = \{ \alpha \in J_{\text{adm}} / \mathbf{N}_\alpha : \dot{\mathbf{G}} > 0 \},
\]

the loading criteria in terms of the sets \( J_{\text{adm}} \) and \( J_{\text{act}} \) may be stated as:

\[
\begin{align*}
\text{If } J_{\text{adm}} = \emptyset : & \quad \text{elastic state} \\
\text{If } J_{\text{adm}} \neq \emptyset \text{ and } J_{\text{act}} = \emptyset : & \\
\quad \text{i. If } \mathbf{N}_\alpha : \dot{\mathbf{G}} < 0 \text{ for all } \alpha \in J_{\text{adm}} : & \quad \text{elastic unloading} \\
\quad \text{ii. If } \mathbf{N}_\alpha : \dot{\mathbf{G}} = 0 \text{ for at least one } \alpha \in J_{\text{adm}} : & \quad \text{neutral loading} \\
\text{If } J_{\text{adm}} \neq \emptyset \text{ and } J_{\text{act}} \neq \emptyset : & \quad \text{inelastic loading}
\end{align*}
\]

An equivalent formulation of the governing equations in the spatial description may be done on the basis of a push – forward operation (e.g. see, [5], pp. 67 – 68) and is presented in detail in [9]. We now make the following remarks:

**REMARK 1 (Time – dependent phenomena).** Rather recent experimental results (see, [6], [7]), on a NiTi shape memory alloy, show that some of the phase transformations depend on the rate of loading. Such a behavior can be accommodated by the proposed (geometrical) framework, by noting that generalized plasticity can be combined consistently with a rate – dependent (viscoplastic) theory. In this case the rate equations for the internal variables may be written in the form:

\[
\dot{\mathbf{Q}} = \sum_{i=1}^{n} H_i(\mathbf{L}_i(G, Q)) \langle \mathbf{N}_i : \dot{\mathbf{G}} \rangle + \mathbf{M}_i(G, Q),
\]
where the $\mathbf{M}_i$'s stand for additional functions of the state variables which enforce the rate–dependent properties of the transformation defined by the part of the loading surface associated by the index $i$. The crucial advantage of this approach lies on the compatibility of the two theories, in the sense that neither viscoplasticity, nor generalized plasticity employs the concept of the yield surface as a basic ingredient.

**REMARK 2 (Transformation induced plasticity).** From a further study of the experimental results of Nemat–Nasser et al. (2005 a, b) it is observed that after a stress cycle within the appropriate limits for pseudo-elastic behavior, permanent deformations appear, a fact which implies that a yielding behavior takes place within the martensitic transformations. A similar response is also observed in austenitic steels, where plastic strain is identified in the austenite phase because of the volume increase of the martensite phase, a phenomenon which is known as Greenwood–Johnson effect, [1]. Such a response can be also described within the proposed framework by introducing additional (plastic) loading surfaces which control the yielding characteristics of the material. These are assumed to be given by expressions of the form:

$$G_i(\mathbf{G}, \mathbf{Q}, \mathbf{P}) = 0, \quad i=1,2,...,m,$$

where $\mathbf{P}$ is an additional internal variable vector which stands for the description of plastic phenomena within the material. In turn, the rate equations for the evolution of the plastic variables within the generalized plasticity context (which includes classical plasticity as a special case (e.g. [4], [10], [11], [12])) may be stated as:

$$\dot{\mathbf{P}} = \sum_{i=1}^{m} K_i T_i (\mathbf{G}, \mathbf{Q}, \mathbf{P}) \langle \mathbf{R}_i, \dot{\mathbf{G}} \rangle,$$

where the functions $K_i, T_i$ and $\mathbf{R}_i$ have an identical meaning with the functions $H_i, L_i$ and $\mathbf{N}_i$ which appear in Eq. (3). A further observation of Eqs. (6) and (7) and their comparison with the basic Eqs. (2) and (3) reveals that both set of equations show exactly the same qualitative characteristics. Accordingly it is concluded, that from a geometrical standpoint the phase transformation loading surfaces are indistinguishable of the plastic loading surfaces which means that the internal variable vector $\mathbf{P}$ may be absorbed in $\mathbf{Q}$ so that the basic equations can simulate both phase transformation and plasticity phenomena in a unified format. This implies that plastic yielding can be dealt with, within the proposed framework, as a phase transition. Equations (1) to (7) constitute the generalized plasticity theory for phase transformations and for arbitrary deformations. These equations provide the general framework for the derivation of a three–dimensional thermomechanical material model for the description of shape memory alloys undergoing large deformations [9]. This model is therefore more general and powerful than a previous one derived by the authors [12].

2. Conclusions

In this work, we provide a new generalized plasticity theory suitable to describe and model phase transformations. The new theory is presented abstractly in its most general form and it leaves the number and the kind of the internal variables, as well as the kinematics of the deformation, entirely unspecified. The proposed theory can effectively accommodate rate effects, as well as transformation induced plastic deformations, within a general finite–deformation regime under isothermal and non–isothermal loadings.
References


