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Formally Self-Adjoint Systems of Differential Operators*

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1. INTRODUCTION

Let

$$A_1(t)\,u' + A_0(t)\,u = \lambda B(t)\,u \tag{1.1}$$

be a system of a matrix differential equation on the real line, $\mathbb{R}$, where u is an n-dimensional column vector function of t, $' = d/dt$ and λ belongs to the set $\mathbb{C}$ of complex numbers. We assume that A_1, A_0, B are $n \times n$ continuous complex-valued matrix function of $t \in \mathbb{R}$ and satisfies the following conditions: (i) A_1 is a continuously differentiable function of t with each $A_1(t)$ being nonsingular, and $B(t) \not\equiv 0$; (ii) The system (1.1) is self-adjoint in that $A_1{}^* = -A_1$, $(A_1{}^*)' = -A_0 + A_0{}^*$, $B = B^*$; (iii) The system is definite, i.e., B is nonnegative definite.

We remark here that instead of taking the system (1.1) satisfying (i)–(iii), we may as well take a system (1.1) which is symmetrizable under a nonsingular transformation $u(t) = C(t)\,v(t)$. However, in this case the system is equivalent to a self-adjoint system [15, p. 444].

In Section 2 we shall find a lower bound for the number of linearly independent "integrable square" solutions of (1.1) and a relationship among the maximum numbers of "integrable square" solutions of (1.1) corresponding to the real line, the left-half line, and the right-half line (Theorem 2.1). The problem of finding lower bounds for the number of integrable square solutions of self-adjoint nth order differential operators has been considered by Glazman [6], Everitt [5], Kimura and Takahasi [10], and many others. The similar problem for the self-adjoint system (1.1) in the case when A_1 is a constant matrix has been considered by Atkinson [1]. He uses the technique of matrix theories. Here we use the technique of algebraic geometry developed by Kadaira [11].

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In Section 3 we assume an additional condition, a normality condition. This section is based on a short communication by author [12]. In this section we develop basic operators and find explicitly all possible generalized resolvents for the operators. These resolvents correspond to all possible self-adjoint extensions of the operators in larger Hilbert spaces. Here we do not assume that the operators have equal deficiency indices. The second part of Theorem 2.1 is reproved using another simple method.

Two-point boundary value problems on finite intervals have been considered by Reid [15]. Singular boundary value problems have been considered by Atkinson [1] in the case when A_1 is a constant matrix, by Brauer [3] in the case when the associated differential operators have equal deficiency indices, by Berman [2] in the case when B is the identity matrix, and by Kim [9] in the case when the order of matrices are an even integer.

2. Integrable Square Solutions

A solution $u(t, \lambda)$ of (1.1) is said to be "integrable square" if $\int_{-\infty}^{\infty} u^*(t, \lambda)\, B(t)\, u(t, \lambda)\, dt < \infty$. Let c be an arbitrary, but a fixed point in $\mathbb{R}$ throughout in this paper, and $U(t, \lambda)$ the fundamental matrix solution of (1.1) with $U(c, \lambda) = I_n$, where I_k denote the $k \times k$ identity matrix. Then any solution (1.1) has the form $U(t, \lambda) f$ for some $n \times 1$ column vector f. Suppose $u_i(t, \lambda_i)$ is a solution of (1.1) for $\lambda = \lambda_i$. Then

$$\begin{aligned}(\lambda_1 - \bar{\lambda}_2) \int_a^b u_2^*(t, \lambda_2)\, B(t)\, u_1(t, \lambda_1)\, dt \\ = u_2^*(b, \lambda_2)\, A_1(b)\, u_1(b, \lambda_1) - u_2^*(a, \lambda_1)\, A_1(a)\, u_1(a, \lambda_1)\end{aligned} \tag{2.1}$$

for every a, b with $-\infty < a < b < \infty$. Indeed, defining $(u \mid v) = \int_a^b v^* B u\, dt$,

$$\begin{aligned}(\lambda_1 u_1(\lambda_1) \mid u_2(\lambda_2)) &= \int_a^b u_2^*(\lambda_2)\,(A_1 u_1'(\lambda_1) + A_0 u_1(\lambda_1))\, dt \\ &= u_2^*(t, \lambda_2)\, A_1(t)\, u_1(t, \lambda_1)|_a^b \\ &\quad - \int_a^b (u_2^*(\lambda_2)\, A_1)'\, u_1(\lambda_1)\, dt + \int_a^b u_2^*(\lambda_2)\, A_0 u_1(\lambda_1)\, dt,\end{aligned}$$

and

$$(u_1(\lambda_1) \mid \lambda_2 u_2(\lambda_2)) = \int_a^b (A_1 u_2'(\lambda_2))^*\, u_1(\lambda_1)\, dt.$$

Hence

$$(\lambda_1 - \bar{\lambda}_2)\,(u_1(\lambda_1) \mid u_2(\lambda_2)) = u_2^*(t, \lambda_2)\, A_1(t)\, u_1(t, \lambda_1)|_a^b$$

using (ii).

For $0 \neq f \in \mathbb{C}^n$ we define

$$\Gamma(f; t, \lambda) = f^* U^*(t, \lambda) A_1(t) U(t, \lambda) f / (\lambda - \bar{\lambda}) \tag{2.2}$$

for $\operatorname{Im} \lambda \neq 0$. Then in view of (2.1)

$$-\Gamma(f; a, \lambda) + \Gamma(f; b, \lambda) = \int_a^b f^* U^*(t, \lambda) B(t) U(t, \lambda) f \, dt \tag{2.3}$$

for $a < b$. Since $A_1(t)$ is skew-hermitian for each t, $A_1(t)(\lambda - \bar{\lambda})$ is hermitian. Thus $\Gamma(f; t, \lambda)$ is a hermitian form of $f \in \mathbb{C}^n$. Since $A_1(t)/i$ is a nonsingular hermitian matrix which is continuous in t, the maximum number d of positive eigenvalues of $A_1(t)/i$ is invariant for each t. Hence the nonsingularity of $A_1(t)/i$ yields that $n - d$ is the maximum number of negative eigenvalues of $A_1(t)/i$. Thus, using properties of hermitian matrices [13, p. 84] we have

LEMMA 2.1. *There exists a $n \times n$ complex matrix function $M(t)$ of t such that*

$$M^*(t) A_1(t) M(t) = i \begin{pmatrix} I_d & 0 \\ 0 & -I_{n-d} \end{pmatrix} \tag{2.4}$$

for every $t \in \mathbb{R}$.

For an integer k let $\mathfrak{P}^k$ denote the k-dimensional complex projective space of equivalence classes (f) with $f \in \mathbb{C}^k$. Following Kodaira [11], for $\operatorname{Im} \lambda \neq 0$, we define:

$$\mathfrak{M}_{-\infty}(\lambda) = \left\{ (f) \in \mathfrak{P}^{n-1} : \int_{-\infty}^c f^* U^*(t, \lambda) B(t) U(t, \lambda) f \, dt < \infty \right\},$$

$$\mathfrak{M}_{\infty}(\lambda) = \left\{ (f) \in \mathfrak{P}^{n-1} : \int_c^{\infty} f^* U^*(t, \lambda) B(t) U(t, \lambda) f \, dt < \infty \right\},$$

$$\mathfrak{P}^+(t, \lambda) = \{(f) \in \mathfrak{P}^{n-1} : \Gamma(f; t, \lambda) > 0\},$$

$$\mathfrak{P}^-(t, \lambda) = \{(f) \in \mathfrak{P}^{n-1} : \Gamma(f; t, \lambda) < 0\},$$

$$\mathfrak{P}^0(t, \lambda) = \{(f) \in \mathfrak{P}^{n-1} : \Gamma(f; t, \lambda) = 0\},$$

$$\mathfrak{k}_{-\infty}(\lambda) = \bigcap_t \mathfrak{P}^-(t, \lambda), \qquad \mathfrak{k}_{\infty}(\lambda) = \bigcap_t \mathfrak{P}^+(t, \lambda).$$

For a subset $\mathfrak{S}$ of $\mathfrak{P}^{n-1}$, the closure of $\mathfrak{S}$ in the strong topology will be denoted by $[\mathfrak{S}]$. We say that $\mathfrak{S} \subset \mathfrak{P}^{n-1}$ is a linear subspace of dimension r if there exists $r + 1$ linearly independent vectors $f_1, \ldots, f_{r+1}$ in $\mathbb{C}^n$ such that $\mathfrak{S}$ is the set of the form $(\sum_1^{r+1} \alpha_i f_i)$ where the α_j are complex numbers not all zero. By the (-1)-dimensional linear subspace of $\mathfrak{P}^{n-1}$ we shall mean the empty set, $\varnothing$. First we have

LEMMA 2.2. *Let* $\operatorname{Im}\lambda > 0$. *Then for each* $t \in \mathbb{R}$, $[\mathfrak{P}^+(t, \lambda)]$, $[\mathfrak{P}^-(t, \bar{\lambda})]$ *contain* $(d - 1)$*-dimensional linear subspaces of* $\mathfrak{P}^{n-1}$. *On the other hand,* $[\mathfrak{P}^-(t, \lambda]$ *and* $\mathfrak{P}^+(t, \bar{\lambda})$ *contain* $(n - d - 1)$*-dimensional linear subspaces of* $\mathfrak{P}^{n-1}$.

Proof. We shall prove the result for $[\mathfrak{P}^+(t, \lambda)]$ and $[\mathfrak{P}^-(t, \bar{\lambda})]$ only as the proof for the rest of the lemma is similar. Let $\operatorname{Im}\lambda > 0$ and let x be any point. Let $Y_x(t, \lambda)$ be the fundamental matrix of (1.1) with the initial condition at $t = x$ given by $Y_x(x, \lambda) = M(x)$ where $M(x)$ satisfies (2.4) for $t = x$. Then, since $Y_x(t, \lambda) = U(t, \lambda)\, Y_x(c, \lambda)$, we have

$$\Gamma(f; x, \lambda) = \left(\sum_{1}^{d} | b_k |^2 - \sum_{d+1}^{n} | b_k |^2\right) \Big/ 2 \operatorname{Im}\lambda \tag{2.5}$$

where the b_k is the kth row of the $n \times 1$ column vector $Y_x^{-1}(c, \lambda) f$. The equation (2.5) shows that $\mathfrak{P}^+(x, \lambda)$ (resp. $\mathfrak{P}^-(x, \lambda)$) contains d (resp. $n - d$) linearly independent elements. However, it is easily seen that $[\mathfrak{P}^+(t, \lambda)] = \mathfrak{P}^+(t, \lambda) \cup \mathfrak{P}^0(t, \lambda)$, $[\mathfrak{P}^-(t, \lambda)] = \mathfrak{P}^-(t, \lambda) \cup \mathfrak{P}^0(t, \lambda)$. Thus $[\mathfrak{P}^+(x, \lambda)]$ and $[\mathfrak{P}^-(x, \lambda)]$ contain $(d - 1)$- and $(n - d - 1)$-dimensional linear subspaces of $\mathfrak{P}^{n-1}$, respectively.

LEMMA 2.3. *Let* $\operatorname{Im}\lambda > 0$. *Then* $\mathfrak{k}_{-\infty}(\bar{\lambda})$ *and* $\mathfrak{k}_{\infty}(\lambda)$ *contain* $(d - 1)$*-dimensional linear subspaces of* $\mathfrak{P}^{n-1}$. *The sets* $\mathfrak{k}_{-\infty}(\lambda)$ *and* $\mathfrak{k}_{\infty}(\bar{\lambda})$ *contain* $(n - d - 1)$*-dimensional linear subspaces of* $\mathfrak{P}^{n-1}$.

Proof. We shall prove our assertion for $\mathfrak{k}_{-\infty}(\bar{\lambda})$ and $\mathfrak{k}_{\infty}(\lambda)$ as the proof for the rest is similar. We shall consider the following three cases:

Case 1. $d = 0$ or n.

Case 2. $d = 1$.

Case 3. $2 \leqslant d \leqslant n - 1$.

Case 1. If $d = 0$, then, in view of (2.5), $[\mathfrak{P}^+(x, \lambda)] = [\mathfrak{P}^-(x, \bar{\lambda})] = \phi$, and $[\mathfrak{P}^+(x, \bar{\lambda})] = [\mathfrak{P}^-(x, \lambda)] = \mathfrak{P}^{n-1}$ for each $x \in \mathbb{R}$. Since it is easily seen that

$$\mathfrak{k}_{-\infty}(l) = \bigcap_x [\mathfrak{P}^-(x, l)], \qquad \mathfrak{k}_{\infty}(l) = \bigcap_x [\mathfrak{P}^+(x, l)]$$

for $l \in \mathbb{C}$ with $\operatorname{Im} l \neq 0$, we have $\mathfrak{k}_{\infty}(\lambda) = \mathfrak{k}_{-\infty}(\bar{\lambda}) = \phi$ and $\mathfrak{k}_{-\infty}(\lambda) = \mathfrak{k}_{\infty}(\bar{\lambda}) = \mathfrak{P}^{n-1}$. Similarly, when $d = n$, $\mathfrak{k}_{-\infty}(\bar{\lambda}) = \mathfrak{k}_{\infty}(\lambda) = \mathfrak{P}^{n-1}$, $\mathfrak{k}_{-\infty}(\lambda) = \mathfrak{k}_{\infty}(\bar{\lambda}) = \phi$.

Case 2. In this case, by Lemma 2.2, $[\mathfrak{P}^+(t, \lambda)]$ and $[\mathfrak{P}^-(t, \bar{\lambda})]$ are not void. Since $[\mathfrak{P}^+(t, \lambda)]$ and $[\mathfrak{P}^-(t, \bar{\lambda})]$ are closed subsets of $\mathfrak{P}^{n-1}$ which is compact in the strong topology, they are also compact subsets of $\mathfrak{P}^{n-1}$.

But, in view of (2.3), $\{[\mathfrak{P}^+(t, \lambda)]: t \in \mathbb{R}\}$ and $\{[\mathfrak{P}^-(t, \bar\lambda)]: t \in \mathbb{R}\}$ are nested families and hence they have nonempty intersections, i.e., $\mathfrak{k}_{-\infty}(\bar\lambda) \neq \phi$, $\mathfrak{k}_\infty(\lambda) \neq \phi$.

Case 3. Let $\mathfrak{N}(d-1)$ denote the set of $(d-1)$-dimensional linear subspaces of $\mathfrak{P}^{n-1}$, and let $\mathfrak{N}(d-1; t, \bar\lambda)$ be the set of $\mathscr{N} \in \mathfrak{N}(d-1)$ such that $\mathscr{N} \subset [\mathfrak{P}^-(t, \bar\lambda)]$. Then $\{\mathscr{N} \in \mathfrak{N}(d-1): \mathscr{N} \subset \mathfrak{k}_{-\infty}(\bar\lambda)\} = \bigcap_t \mathfrak{N}(d-1; t, \bar\lambda)$. Let $N = \binom{n}{d}$ and define a map $\Pi: \mathfrak{N}(d-1) \to \mathfrak{P}^{N-1}$ by letting $\Pi(\mathscr{N})$ be the Grassmann coordinate of $\mathscr{N}$. It is known that Π is one-to-one and the range set $\Pi(\mathfrak{N}(d-1)$ is a projective variety (or Grassmann variety) in $\mathfrak{P}^{N-1}$ [8, Chap. VII, Sects. 6–7]. Hence $\Pi(\mathfrak{N}(d-1))$ is a compact subset of $\mathfrak{P}^{N-1}$ in the natural topology [18, Chap. VII, Sects. 3–4]. Since $\mathfrak{N}(d-1; t, \bar\lambda)$ is a closed subset of $\mathfrak{N}(d-1)$ in the natural topology, it is a compact subset (cf. [11, p. 509]). Since $1 \leqslant d-1$ by assumption, each $\mathfrak{N}(d-1; t, \bar\lambda)$ is not empty. We now consider $\mathfrak{N}(d-1)$ as a topological space with the topology being induced by Π. Hence $\mathfrak{N}(d-1)$ is a compact space and $\mathfrak{N}(d-1, t, \bar\lambda)$ is a nonempty compact subset of $\mathfrak{N}(d-1)$. Since $[\mathfrak{P}^-(t, \bar\lambda)]$ decreases as t increases, so does $\mathfrak{N}(d-1; t, \bar\lambda)$ as t decreases. Therefore $\{\mathfrak{N}(d-1; t, \bar\lambda): t \in \mathbb{R}\}$ is a nested family of nonempty compact subsets of $\mathfrak{N}(d-1)$, and hence it has a nonempty intersection. Hence $\mathfrak{k}_{-\infty}(\bar\lambda)$ contains a $(d-1)$-dimensional linear subspace of $\mathfrak{P}^{n-1}$. Similarly we can show that $\mathfrak{k}_{+\infty}(\lambda)$ contains a $(d-1)$-dimensional linear subspace of $\mathfrak{P}^{n-1}$. Combining Cases 1–3, we see that $\mathfrak{k}_{-\infty}(\bar\lambda)$ and $\mathfrak{k}_{+\infty}(\lambda)$ contain $(d-1)$-dimensional linear subspaces of $\mathfrak{P}^{n-1}$. This completes the proof.

Let us denote by $\alpha(\lambda)$ the maximum number of linearly independent solutions $u(x, \lambda)$ of (1.1) such that $\int_{-\infty}^{c} u^*Bu\, dt < \infty$. Let us denote by $\beta(\lambda)$ the maximum number of linearly independent solutions $u(t, \lambda)$ of (1.1) such that $\int_c^\infty u^*Bu\, dt < \infty$. Finally $\gamma(\lambda)$ will denote the maximum number of linearly independent solutions $u(t, \lambda)$ of (1.1) such that $\int_{-\infty}^{\infty} u^*Bu\, dt < \infty$. Then, since $\mathfrak{k}_{-\infty}(\lambda) \subset \mathfrak{M}_{+\infty}(\lambda)$, $\mathfrak{k}_\infty(\lambda) \subset \mathfrak{M}_{-\infty}(\lambda)$, and $\mathfrak{k}_{-\infty}(\lambda) \cap \mathfrak{k}_\infty(\lambda) = \phi$, Lemma 2.3 yields the following.

THEOREM 2.1. *Let* $\operatorname{Im} \lambda > 0$. *Then*

(I) $\alpha(\lambda) \geqslant d,\ \beta(\bar\lambda) \geqslant d,\ \alpha(\bar\lambda) \geqslant n-d,\ \beta(\lambda) \geqslant n-d.$

(II) $\alpha(\lambda) + \beta(\lambda) = \gamma(\lambda) + n.$

The last equality is also true when λ *is replaced by* $\bar\lambda$.

Remark 2.1. Theorem 2.1 remains true if $-\infty$ or ∞ is replaced by a regular point. The second part of the theorem will be proved again in Section 3 using a different method. Atkinson [1] proved (I), using a matrix technique, in the special case when A_1 is a constant matrix (see [1, Sect. 9.11]). The relation (II) is also true when the differential operator under consideration is a formally self-adjoint nth order differential operator with complex coefficients. This has been proved by Kimura and Takahasi [10] using the similar technique used in this section. (I) of Theorem 2.1 has been proved by Everitt [5] in the case when the differential operators under consideration are formally self-adjoint nth order differential operators with complex coefficients.

3. Generalized Resolvents

In this section we assume the normality condition, i.e., if $A_1 u' + A_0 u = 0$ and $Bu = 0$ on a subinterval of $\mathbb{R}$, then u becomes identically zero on that interval. Let $\mathscr{C}_0{}^1(\mathbb{R})$ denote the set of continuously differentiable $n \times 1$ column vector functions with compact support on $\mathbb{R}$. For $u(t)$, $v(t) \in \mathscr{C}_0{}^1(\mathbb{R})$ we set $(u \mid v) = \int_{-\infty}^{\infty} v^*(t)\, B(t)\, u(t)\, dt$. Then $\mathscr{C}_0(\mathbb{R})$ is an inner product space and its Hilbert space completion, $\mathfrak{H}$, is the set of all $n \times 1$ column vector functions $u(t)$ which are measurable and $\int_{-\infty}^{\infty} u^*(t)\, B(t)\, u(t)\, dt < \infty$ [3, p. 19]. Let us denote by D^* the set of $u \in \mathfrak{H}$ such that u is locally absolutely continuous on $\mathbb{R}$, and there exists a $v \in \mathfrak{H}$ such that

$$A_1(t)\, u' + A_0(t)\, u = B(t)\, v \tag{3.1}$$

for almost all $t \in \mathbb{R}$. It is shown that such v exists uniquely [3]. It is easy to see that D^* is a vector subspace of $\mathfrak{H}$. Let T^* be the operator in $\mathfrak{H}$ with a domain D^*, and $T^* u = v$ for $u \in D^*$ where v satisfies (3.1). For $u, v \in D^*$ we set $\langle u \mid v \rangle = (T^* u \mid v) - (u \mid T^* v)$, which again can be written

$$\langle u \mid v \rangle = (v^* A_1 u)(\infty) - (v^* A_1 u)(-\infty) \tag{3.2}$$

using a method similar to the one used to derive (2.1). Let D denote the set of $u \in D^*$ such that $\langle u \mid v \rangle = 0$ for every $v \in D^*$. Then, since $\mathscr{C}_0{}^1(\mathbb{R}) \subset D \subset D^*$, $T \subset T^*$. The following lemma is a direct consequence of the basic solvability theorem for two-point boundary problems on a compact interval (cf. [16, Theorem 6.2, p. 135]):

Lemma 3.1. *Let* $-\infty < a < b < \infty$ *and* $\lambda \in \mathbb{C}$, $v \in \mathfrak{H}$. *Let* $A_1 u' + A_0 u = \lambda B u + B v$, $u(a) = 0$. *Then* $u(b) = 0$ *if and only if* $\int_a^b w^* B v\, dt = 0$ *for every* w *with* $A_1 w' + A_0 w = \lambda B w$, $a \leqslant t \leqslant b$.

Using Lemma 3.1 and the exact same method as in [6, Sect. 4] we have

Theorem 3.1. *The operator* T *is symmetric, closed and* T^* *is its adjoint. Moreover* $T = T^{**}$.

We shall call $T(T^*)$ the minimal symmetric (maximal) operator, respectively, on the interval $(-\infty, \infty)$. The following theorem has been proved in Section 2. However we shall give an elementary proof.

THEOREM 3.2. *Let* $\operatorname{Im}\lambda \neq 0$. *Then*

$$\alpha(\lambda) + \beta(\lambda) = \gamma(\lambda) + n.$$

Proof. Let T_- and T_+ be the minimal symmetric operators on the interval $(-\infty, c]$ and $[c, \infty)$, respectively. $D(T_c)$ denotes the set of $u \dotplus v$ with $u \in D(T_-)$, $v \in D(T_+)$ where

$$\begin{aligned}(u \dotplus v)(t) &= u(t) && \text{for } -\infty < t < c,\\ &= v(t) && \text{for } c < t < \infty,\\ &= (u(t) + v(t))/2 && \text{for } t = c.\end{aligned}$$

Then, since c is a regular point for T_- and T_+, we see that $u \in D(T_c)$ iff $u \in D$ and $u(c) = 0$. Let T_c be an operator in $\mathfrak{H}$ with domain $D(T_c)$, and $T_c(u \dotplus v) = T_-u \dotplus T_+v$. It is easily seen that T_c is a closed symmetric operator in $\mathfrak{H}$ with deficiency indices $\{\alpha(l) + \beta(l), \alpha(l) + \beta(l)\}$ ($\operatorname{Im} l > 0$), and $T_c \subset T$. Since $\{\gamma(\bar{l}), \gamma(l)\}$ ($\operatorname{Im}\ l > 0$) are the deficiency indices of T, there exists a nonnegative integer P such that $\gamma(l) + P = \alpha(l) + \beta(l)$ and $\dim D/D(T_c) = p$ for every l with $\operatorname{Im} l \neq 0$ (cf. [14, p. 35]). To complete the proof, we must show that $p = n$. Let $u_j(t, \lambda)$ $(1 \leqslant j \leqslant n)$ be the jth column vector of $U(t, \lambda)$ where $U(t, \lambda)$ is defined in Section 2. Let $-\infty < a < c < b < \infty$. Let $\tilde{u}_j(t)$ $(1 \leqslant j \leqslant n)$ be a vector function in D with compact support on $\mathbb{R}$ such that $\tilde{u}_j(t) = u_j(t, \lambda)$ for $a \leqslant t \leqslant b$. We shall show that the n cosets $\tilde{u}_j + D(T_c)$ form a basis for $D/D(T_c)$. Suppose $c_1\tilde{u}_1 + \cdots + c_n\tilde{u}_n \in D(T_c)$ for some complex numbers c_j. Then $c_1\tilde{u}_1(c) + \cdots + c_n\tilde{u}_n(c) = c_1u_1(c, \lambda) + \cdots + c_nu_n(c, \lambda) = 0$, so that the c_j are zero since $U(c, \lambda) = I_n$, proving that $\tilde{u}_1 + D(T_c), \ldots, \tilde{u}_n + D(T_c)$ are linearly independent. Take any vector $u \in D$. Then $u(c) - b_j(c)\,\tilde{u}_j(c) - \cdots - b_n(c)\,\tilde{u}_n(c) = u(c) - b_1(c)\,u_1(c, \lambda) - \cdots - b_n(c)\,u_n(c, \lambda) = 0$ where the $b_j(c)$ is the jth row of $u(c)$. Hence $u - (b_1(c)\,\tilde{u}_1 + \cdots + b_n(c)\,\tilde{u}_n) \in D(T_c)$. Therefore the $\tilde{u}_j + D(T_c)$ span $D/D(T_c)$. Consequently $D/D(T_c) = n$.

The Theorem 3.1 will play an important role to compute all possible generalized resolvents of T. Let $\mathfrak{A} = \{A(\lambda)\colon \operatorname{Im}\lambda > 0\}$ be a family of contraction operator $A(\lambda)$ in $\mathfrak{H}$ taking $\mathfrak{E}(-i)$ into $\mathfrak{E}(i)$ such that $A(\lambda)$ is analytic for $\operatorname{Im}\lambda < 0$ where $\mathfrak{E}(-i)$ is the eigenspace of T^* corresponding to $-i$ and $\mathfrak{E}(i)$ is the one corresponding to $+i$.

For $A(\lambda) \in \mathfrak{A}$, set

$$D(\lambda) = D + (I - A(\lambda))\,\mathfrak{E}(-i)$$

where I denote the identity operator on $\mathfrak{C}(-i)$. Define an operator $T_{A(\lambda)}$ in $\mathfrak{H}$ with domain $D(\lambda)$ by $T_{A(\lambda)}u = T^*u$ for $u \in D(\lambda)$. According to Straus [17], the operator R:

$$\begin{aligned} R(\lambda) &= (T_{A(\lambda)} - \lambda)^{-1} \qquad &&\text{for } \operatorname{Im} \lambda > 0, \\ &= [R(\bar{\lambda})]^* &&\text{for } \operatorname{Im} \lambda < 0, \end{aligned} \tag{3.3}$$

is a generalized resolvent of T, and every generalized resolvent for T arises in this way from $\mathfrak{A}$. We note that $\Delta R(\lambda) = D(T_{A(\bar{\lambda})})^*$ for $\operatorname{Im} \lambda < 0$, and $\Delta R(\lambda) = D(T_{A(\lambda)})$ for $\operatorname{Im} \lambda > 0$, where for an operator W, $D(W)$ and $\Delta(W)$ denote the domain and the range of W, respectively. By definition dim $\mathfrak{C}(\lambda)$ equals $\gamma(\lambda)$ for $\operatorname{Im} \lambda \neq 0$. Let $u_1(t, i),\ldots, u_{\gamma(i)}(t, i)$ be an orthonormal vector functions for $\mathfrak{C}(i)$, and let $u_1(t, -i),\ldots, u_{\gamma(-i)}(t, i)$ be one for $\mathfrak{C}(-i)$. For $\operatorname{Im} \lambda \neq 0$, define $\gamma(\lambda)$ vector functions $v_j(t, \lambda)$ $(1 \leqslant j \leqslant \gamma(\lambda))$ in $\mathfrak{H}$ by $v_j(t, \lambda) = u_j(t, i) - A(\bar{\lambda})\, u_j(t, i)$ for $\operatorname{Im} \lambda > 0$, and $v_j(t, \lambda) = u_j(t, -i) - A(\bar{\lambda})\, u_j(t, -i)$ for $\operatorname{Im} \lambda < 0$, where, for $\operatorname{Im} \lambda < 0$, $A(\lambda)$ denotes $(A(\bar{\lambda}))^*$. First we have

THEOREM 3.3. *Let R be an arbitrary generalized resolvent of T defined in* (3.3). *Then*

$$\Delta R(\lambda) = \{u \in D^* \colon \langle u \mid v_j(\lambda)\rangle = 0,\ 1 \leqslant j \leqslant \gamma(\lambda)\}$$

for $\operatorname{Im} \lambda \neq 0$.

The proof of this theorem can be carried out in the exact same way as in [4]. Thus we omit the proof.

The following lemma is an immediate consequence of basic properties of linearly independent sets of elements in a linear space and Theorem 3.1 (cf. [14, p. 91]):

LEMMA 3.2. *Let* $\operatorname{Im} \lambda \neq 0$. *We can choose a fundamental matrix* $\Psi(t, \lambda)$ *of* (1.1) *such that its* jth *column* $\Psi_j(t, \lambda)$ $(1 \leqslant j \leqslant n)$ *satisfies:*

$$\int_{-\infty}^{\infty} \Psi_j^* B \Psi_j\, dt < \infty \qquad \textit{for} \quad 1 \leqslant j \leqslant \gamma(\lambda);$$

$$\int_{-\infty}^{c} \Psi_j^* B \Psi_j\, dt < \infty \qquad \textit{for } \gamma(\lambda) < j \leqslant \alpha(\lambda);$$

$$\int_{c}^{\infty} \Psi_j^* B \Psi_j\, dt < \infty \qquad \textit{for } \alpha(\lambda) < j \leqslant n.$$

Remark. If one of $-\infty$ and ∞ is replaced by a regular point, then above lemma is a direct consequence of basic properties of linearly independent solutions. However, in our case we need Theorem 3.1 to conclude our assertion.

Throughout this paper $\Psi(t, \lambda)$ ($\operatorname{Im} \lambda \neq 0$) will denote a fundamental matrix of (1.1) whose column vectors $\Psi_j(t, \lambda)$ ($1 \leqslant j \leqslant n$) satisfy the condition in Lemma 3.2, and let $R(\lambda)$ be an arbitrary, but fixed generalized resolvent of T defined in (3.3). We shall compute $R(\lambda)$ ($\operatorname{Im} \lambda \neq 0$). Let u be an arbitrary vector function in $\mathfrak{H}$ with compact support on $(-\infty, \infty)$. Then, since $(T^* - \lambda) R(\lambda) u = u$, we see that $A_1(R(\lambda) u)' + A_0(R(\lambda) u) = \lambda B(R(\lambda) u) + B(R(\lambda) u)$. Hence, using the method of variation of parameters (cf. also [7, p. 48]),

$$(R(\lambda) u)(x) = \sum_1^n \Psi_j(x, \lambda) \left(a_j + \int_{-\infty}^x \phi_j^*(t, \lambda) A_1^{-1}(t) B(t) u(t)\, dt\right) \tag{3.4}$$

where the a_j are complex constants depending on u and $\phi_j(t, \lambda)$ is the $n \times 1$ column vector function such that the ith row of $\phi_j(t, \lambda) \det \Psi(t, \lambda)$ is the cofactor of the (i, j)-element of $\det \Psi(t, \lambda)$. Since $R(\lambda) u \in \mathfrak{H}$, using the definition for $\alpha(\lambda)$, $\beta(\lambda)$, $\gamma(\lambda)$ and the linearly independence of solutions Ψ_j, the (3.4) can be written again

$$(R(\lambda) u)(x) = \sum_1^{\gamma(\lambda)} a_j \Psi_j(x, \lambda) + \int_{-\infty}^{\infty} K_0(x, t, \lambda) u(t)\, dt, \tag{3.5}$$

where $K_0(x, t, \lambda)$ is a $n \times n$ matrix function of x, t, λ which is given by $\sum_{\gamma(\lambda)+1}^{\alpha(\lambda)} \Psi_j(x, \lambda) \phi_j^*(t, \lambda) A_1^{-1}(t) B(t)$ for $x \leqslant t$ and

$$\sum_1^{\gamma(\lambda)} \Psi_j(x, \lambda) \phi_j^*(t, \lambda) A_1^{-1}(t) B(t) + \sum_{\alpha(\lambda)+1}^{n} \Psi_j(x, \lambda) \phi_j^*(t, \lambda) A_1^{-1}(t) B(t)$$

for $x > t$. $R(\lambda) u$ belongs to $D(\lambda)$ and so by Theorem 3.3, we have $\langle R(\lambda) u \mid v_j(\lambda)\rangle = 0$ for $j = 1, 2, \ldots, \gamma(\lambda)$. Let $\Delta(\lambda; \gamma(\lambda))$ denote the determinant of the $\gamma(\lambda) \times \gamma(\lambda)$ matrix $(\langle \Psi_i(\lambda) \mid v_j(\lambda)\rangle)$ where i and j denote the ith row and jth column respectively. Then, since $\Delta(\lambda; \gamma(\lambda) \neq 0$ (cf. [4, p. 385]), using the similar method used in Naimark [14, p. 91], the form (3.5) has the form:

$$(R(\lambda)) u = \int_{-\infty}^{\infty} K(x, t, \lambda) u(t)\, dt \tag{3.6}$$

where

$$K(x, t, \lambda) = K_0(x, t, \lambda) + \sum_1^{\gamma(\lambda)} \Psi_k(x, \lambda) \theta_k^*(t, \lambda)$$

and

$$\theta_k(t, \lambda) = -\sum_{j=1}^{\gamma(\lambda)} \langle v_j(\bar{\lambda}) \mid K_0(t, \lambda)\rangle \bar{\Delta}_k^j(\lambda; \gamma(\lambda))/\bar{\Delta}(\lambda; \gamma(\lambda))$$

with $\Lambda_k^j(\lambda; \gamma(\lambda))$ denoting the cofactor of (j, k)-element of $\Lambda(\lambda; \gamma(\lambda))$. We have seen that (3.6) hold for every u in $\mathfrak{H}$ with compact support on $(-\infty, \infty)$. This form is not true for an arbitrary u in $\mathfrak{H}$ as we will see later. To see this, first we prove the following.

LEMMA 3.3. *For* $\operatorname{Im} \lambda \neq 0$.

(I) $B(x)\, K(x, t, \lambda) = K^*(t, x, \bar{\lambda})\, B(t)$ *for* $x \neq t$;

(II) *the column vectors of* $K(x, t, \lambda)$ *belong to* $\mathfrak{H}$ *as a function of* x;

(III) *for each* x *fixed and for every* $u \in \mathfrak{H}$, *the entries of the* $n \times 1$ *column vector* $B(x)\, K(x, t, \lambda)\, u(t)$ *are absolutely integrable on* $(-\infty, \infty)$.

Proof. Since $R^*(\lambda) = R(\bar{\lambda})$, $\langle R(\lambda)\, u \mid v\rangle = \langle u \mid R(\bar{\lambda})\, v\rangle$ for every $u, v \in \mathfrak{H}$. This relation in view of (3.6) yields (I). (II) is an immediate consequence of the definition of $K(x, t, \lambda)$. We now prove (III). Let $k_j(t, x, \bar{\lambda})$ denote the jth column vector of $K(t, x, \bar{\lambda})$. Then $k_j^*(t, x, \bar{\lambda})\, B(t)\, u(t)$ is the jth column vector of $K^*(t, x, \bar{\lambda})\, B(t)\, u(t)$. But by (II) the $k_j(t, x, \bar{\lambda})$ belongs to $\mathfrak{H}$ as a function of t. Hence, for $u \in \mathfrak{H}$, $k_j^*(t, x, \bar{\lambda})\, B(t)\, u(t)$ is absolutely integrable on $(-\infty, \infty)$. Thus together with (I) we have (III). This completes the proof.

We now have the following.

THEOREM 3.4. *Let* $\operatorname{Im} \lambda \neq 0$. *Then*

$$B(x)\, (R(\lambda)\, u)\, (x) = \int_{-\infty}^{\infty} B(x)\, K(x, t, \lambda)\, u(t)\, dt$$

for every $u \in \mathfrak{H}$.

Proof. Take any $u \in \mathfrak{H}$. There exists a sequence (u_n) in $\mathfrak{H}$ with compact support on $(-\infty, \infty)$ converging to u in $\mathfrak{H}$. Since $R(\lambda)$ is a bounded operator, for every $v \in \mathfrak{H}$ with compact support on $\mathbb{R}$,

$$(R(\lambda)\, u \mid v) = \lim (R(\lambda)\, u_n \mid v) = \lim \int_{-\infty}^{\infty} v^*(x)\, B(x)\, (R(\lambda)\, u_n)\, (x)\, dx$$

$$= \lim \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v^*(x)\, B(x)\, K(x, t, \lambda)\, u_n(t)\, dt\, dx.$$

By (III) of Lemma 3.3, for each x fixed, $B(x)\, K(x, t, \lambda)\, u(t)$ is absolutely integrable on $(-\infty, \infty)$. Hence

$$\int_{-\infty}^{\infty} v^*(x)\, B(x)\, (R(\lambda)\, u)\, (x)\, dx = \int_{-\infty}^{\infty} v^*(x) \int_{-\infty}^{\infty} B(x)\, K(x, t, \lambda)\, u(t)\, dt\, dx,$$

from which the result follows.

According to Theorem 2.1, $\beta(\lambda) + \beta(\bar{\lambda}) \geqslant n$ for $\operatorname{Im} \lambda \neq 0$. We shall give a necessary and sufficient condition that the equality actually holds:

THEOREM 3.5. *The following are equivalent:*

(I) $\dim D(T_+^*)/D(T_+) = n$.

(II) $\beta(i) + \beta(-i) = n$.

(III) $(u^*(t)\, A_1(t)\, u(t))\,(\infty) = 0$ *for every* $u \in D(T_+^*)$.

Proof. That (I) is equivalent to (II) is an immediate consequence of the decomposition theorem which states that $D(T_+^*) = D(T_+) + \mathfrak{E}(i) + \mathfrak{E}(-i)$. Let us construct n vector functions $u_1(t),..., u_n(t)$ in $D(T_+^*)$ such that each u_j has compact support on $[0, \infty)$ and the ith row of $u_j(c)$ is δ_{ij} (δ = Kronecker delta). Since c is a regular point for T_+, we can see easily that the u_j are linearly independent Mod $D(T_+)$. Suppose (II) holds. Then $u_1 + D(T_+),..., u_n + D(T_+)$ are a basis for $D(T_+^*)/D(T_+)$. Hence any vector $v \in D(T_+^*)$ is a linear combination of the u_j plus some vector function in $D(T_+)$. Therefore $(v^*(t)\, A_1(t)\, v(t))\,(\infty) = 0$ for every $v \in D(T_+^*)$. Hence (II) implies (III). Suppose (III) holds. First we define two spaces D_c and D_∞ as follows: $u \in D_c$ if and only if $u \in D(T_+^*)$ and $u(c) = 0$; $u \in D_\infty$ if and only if $(u^*(t)\, A_1(t)\, u(t))\,(\infty) = 0$ for every $u \in D(T_+^*)$. Then $\dim D(T_+^*)/D_c + \dim D(T_+)/D_c = \dim D(T_+^*)/D(T_+)$ (cf. [11, p. 520]). By our assumption $D(T_+) = D_c$, so that $\dim D(T_+)/D_c = 0$. We note that the u_j constructed above are in $D(T_+)$ and linearly independent Mod D_c. Let v be arbitrary vector function in $D(T_+^*)$. Then $v(c) - (a_1 u_1(c) + \cdots + a_n u_n(c)) \in D_c$ where a_i is the ith row of $v(c)$. This shows that the n elements $u_1 + D_c,...,$ $u_n + D_c$ span $D(T_+)/D_c$. Therefore $\dim D(T_+)/D_c = n$. Consequently $\dim D(T_+^*)/D(T_+) = n$, and so we have proved that (III) implies (II). This completes the proof.

Similar consideration discussed above together with Theorem 2.1 can lead us the following

COROLLARY 3.1. $\gamma(i) = \gamma(-i) = 0$ *if and only if*

$$(u^*(t)\, A_1(t)\, v(t))\,(-\infty) = (u^*(t)\, A_1(t)\, v(t))\,(\infty) = 0$$

for every $u, v \in D(T_+^*)$.

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