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On preprojective and preinjective partitions of a Δ-good module category ☆

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1. Introduction

Let Λ be an artin algebra. It is shown in [4] that the module category mod Λ has preprojective and preinjective partitions and that Λ is of finite representation type if and only if either all indecomposable Λ-modules are preprojective or all indecomposable Λ-modules are preinjective. Further, in [5] it is proved that for a finite-dimensional algebra over an infinite perfect field of infinite representation type there always exists an indecomposable module which is neither preprojective nor preinjective. More generally, Skowronski and Smalø [16] proved that Λ is of finite representation type if and only if each Λ-module is either preprojective or preinjective.

In the study of a quasi-hereditary algebra Λ, instead of the complete module category mod Λ, one is mainly interested in the Δ-good module category $\mathcal{F}(\Delta)$ which consists of Λ-modules which have a filtration by standard modules. It is proved by Ringel [14] that $\mathcal{F}(\Delta)$ is functorially finite in mod Λ. Thus, from [3] it follows that $\mathcal{F}(\Delta)$ has both preprojective and preinjective partitions. The main purpose of the present paper is to study the finiteness of $\mathcal{F}(\Delta)$ in terms of preprojective and preinjective partitions of $\mathcal{F}(\Delta)$. More precisely, by defining the degree of a relative irreducible map in $\mathcal{F}(\Delta)$ in a similar way as in [10], we prove that, if Λ is quasi-hereditary and each module in $\mathcal{F}(\Delta)$ is either preprojective or preinjective, then $\mathcal{F}(\Delta)$ does not satisfy the second Brauer–Thrall conjecture.

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In view of Ringel's work [13], this implies particularly that, if Λ is a finite-dimensional quasi-hereditary algebra over an infinite field, then $\mathcal{F}(\Delta)$ is finite, that is, up to isomorphism, there are only finitely many indecomposable modules in $\mathcal{F}(\Delta)$, if and only if each module in $\mathcal{F}(\Delta)$ is either preprojective or preinjective.

2. Preprojective and preinjective partitions

Let Λ be an artin algebra over a commutative artin ring R. By $\operatorname{mod}\Lambda$ we denote the category of finitely generated left Λ-modules and by $\operatorname{ind}\Lambda$ a full subcategory of $\operatorname{mod}\Lambda$ of the chosen representatives of the isomorphism class of the indecomposable Λ-modules. Similarly, for a subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$, by $\operatorname{ind}\mathcal{C}$ we denote a full subcategory of the chosen representatives of the isomorphism class of the indecomposable modules in $\mathcal{C}$.

Definition 2.1. The preprojective partition of $\operatorname{ind}\mathcal{C}$ is a partition $\mathcal{P}_i$, $i \in \mathbf{N} \cup \{\infty\}$, of objects of $\operatorname{ind}\mathcal{C}$ satisfying the following properties:

(i) $\mathcal{P}_i$ is finite for each $i < \infty$,
(ii) setting $\mathcal{P}^i = \bigcup_{j<i} \mathcal{P}_j$ for $i \in \mathbf{N} \cup \{\infty\}$ and $P_i = \coprod_{X \in \mathcal{P}_i} X$ for $i \in \mathbf{N}$, we have that for each $i < \infty$ and each $X \in \operatorname{ind}\mathcal{C} \setminus \mathcal{P}^i$, the induced map $\operatorname{Hom}(P_i, X) \otimes P_i \to X$ is surjective, and
(iii) each $\mathcal{P}_i$ is minimal with the property in (ii).

The preinjective partition $\mathcal{I}_i$, $i \in \mathbf{N} \cup \{\infty\}$, is defined dually. The modules in $\mathcal{P}^\infty$ are called preprojective and those in $\mathcal{I}^\infty$ are called preinjective. In [4, Theorems 1.2, 1.3] it is proved that both preprojective and preinjective partitions are unique.

In this paper, we always assume that Λ is quasi-hereditary with a fixed ordering $E(1), \ldots, E(n)$ of the isomorphism classes of the simple Λ-modules and where $\Delta(1), \ldots, \Delta(n)$ are the corresponding standard modules, and $T(1), \ldots, T(n)$ are the characteristic modules (see [14]). Let $\mathcal{F}(\Delta)$ be the Δ-good module category of Λ which by definition consists of modules having a Δ-good filtration. It is proved in [14] that $\mathcal{F}(\Delta)$ is functorially finite (i.e., every Λ-module has a right $\mathcal{F}(\Delta)$-approximation and a left $\mathcal{F}(\Delta)$-approximation). So it follows from [3, Theorem 3.3] that $\mathcal{F}(\Delta)$ admits both preprojective and preinjective partitions, denoted by $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_n, \ldots, \mathcal{P}_\infty$ and $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n, \ldots, \mathcal{I}_\infty$, respectively. From the definition, we have the following proposition.

Proposition 2.1. *$\mathcal{P}_0$ consists of all indecomposable projective modules, and $\mathcal{I}_0$ consists of all indecomposable* Ext-*injective modules, that is, the characteristic modules $T(1), \ldots, T(n)$.*

For two modules A, B in $\mathcal{F}(\Delta)$, we define

$$\operatorname{Hom}_\Delta(A, B) = \operatorname{Hom}_\Lambda(A, B),$$

$$\Re_\Delta(A,B)=\big\{f\in \operatorname{Hom}_\Delta(A,B)\mid \text{for every module } X\in\mathcal{F}(\Delta),\ g:X\to A,\ h:B\to X,\ hfg \text{ is not an isomorphism}\big\},$$
$$\Re^n_\Delta(A,B)=\big\{f\in \operatorname{Hom}_\Delta(A,B)\mid \text{there exist } X\in\mathcal{F}(\Delta),\ g\in\Re_\Delta(A,X), \text{ and } h\in\Re^{n-1}_\Delta(X,B) \text{ such that } f=hg\big\},$$

where $n \geqslant 1$. Thus, we get a chain

$$\operatorname{Hom}_\Delta(A,B)\supseteq\Re_\Delta(A,B)\supseteq\Re^2_\Delta(A,B)\supseteq\cdots\supseteq\Re^n_\Delta(A,B)\supseteq\cdots.$$

Given Λ-modules A, B in $\mathcal{F}(\Delta)$, a morphism $f:A\to B$ is to said to be relative irreducible in $\mathcal{F}(\Delta)$ if f is neither a split monomorphism nor a split epimorphism, and for any factorization $f=f_2f_1$ in $\mathcal{F}(\Delta)$, then either f_1 is a split monomorphism, or f_2 is a split epimorphism. In case A, B are indecomposable, we get a bimodule of relative irreducible maps $\operatorname{Irr}_{\mathcal{F}(\Delta)}(A,B)=\Re_\Delta(A,B)/\Re^2_\Delta(A,B)$.

By [14], $\mathcal{F}(\Delta)$ has relative almost split sequence, that is, for any non-projective indecomposable module A in $\mathcal{F}(\Delta)$, there exists a relative almost split sequence

$$0\to B\to M\to A\to 0.$$

In this case, we define B as $\tau_\Delta A$, and A as $\tau_\Delta{}^{-1}B$. We denote by $\mathcal{O}(A)$ the τ_Δ-orbit of a module A in $\mathcal{F}(\Delta)$.

The Auslander–Reiten quiver $\Gamma_{\mathcal{F}(\Delta)}$ of $\mathcal{F}(\Delta)$ is a valued translation quiver defined as follows [15]: its vertices are the isomorphism classes $[A]$ of indecomposable Λ-modules A in $\mathcal{F}(\Delta)$ (sometimes we use A directly for the corresponding vertex). There is an arrow $[A]\to[B]$ provided there exists a relative irreducible map $A\to B$ in $\mathcal{F}(\Delta)$, that is, $\operatorname{Irr}_{\mathcal{F}(\Delta)}(A,B)\neq 0$.

3. Relative irreducible maps and their degrees

Lemma 3.1.

(a) *A map $f:X\to Y$ in $\mathcal{F}(\Delta)$ is irreducible if and only if there exists a map $f':X\to Y'$ in $\mathcal{F}(\Delta)$ such that $(f,f')^t:X\to Y\oplus Y'$ is a minimal left almost split map in $\mathcal{F}(\Delta)$, where $(f,f')^t$ denotes the transpose of (f,f').*

(b) *Dually, a map $f:X\to Y$ in $\mathcal{F}(\Delta)$ is irreducible if and only if there exists a map $f':X'\to Y$ in $\mathcal{F}(\Delta)$ such that $(f,f'):X\oplus X'\to Y$ is a minimal right almost split map $\mathcal{F}(\Delta)$.*

The proof of the lemma is a complete analogue of [2, V, Theorem 5.3].

By [15, Theorem 4.3], $\mathcal{F}(\Delta)$ is resolving (i.e., $\mathcal{F}(\Delta)$ contains all the projective Λ-modules, is closed under extension and closed under kernels of surjective maps). This fact gives the following lemma (see [1, Proposition 3.7]).

Proposition 3.2. *Let* $0 \to X \to B \to Y \to 0$ *be an exact sequence in* $\operatorname{mod} \Lambda$. *If* A_X, A_Y *are minimal right* $\mathcal{F}(\Delta)$*-approximations of* X *and* Y*, respectively. Then the minimal right* $\mathcal{F}(\Delta)$*-approximation of* B *is a summand of an extension of* A_Y *by* A_X.

For each Λ-module A, by $l(A)$ we denote the length of A as an R-module. Let $D(i) \to E(i)$ be the minimal right $\mathcal{F}(\Delta)$-approximation of $E(i)$ for $1 \leqslant i \leqslant n$.

Corollary 3.3. *If* A_B *is the minimal right* $\mathcal{F}(\Delta)$*-approximation of* B*, then* $l(A_B) \leqslant N \cdot l(B)$*, where* $N = \max\{l(D(i)) \mid 1 \leqslant i \leqslant n\}$.

Lemma 3.4 [8, Proposition 9.10]. *For* $A \in \mathcal{F}(\Delta)$*, let* $0 \to \tau A \to M \to A \to 0$ *be the almost split sequence in* $\operatorname{mod} \Lambda$*, and* $X \to \tau A$ *be the minimal right* $\mathcal{F}(\Delta)$*-approximation of* τA. *Then* $X \cong \tau_\Delta A \oplus T_X$*, where* $T_X \in \operatorname{add} T$.

Theorem 3.5. *There exists a constant* b*, depending only on* Λ*, such that if* $f : X \to Y$ *is a relative irreducible morphism in* $\mathcal{F}(\Delta)$ *between indecomposable modules* X *and* Y*, then* $l(X) \leqslant b \cdot l(Y)$.

Proof. According to [12, Lemma 2.1], there exists a constant b_1, which only depends on Λ, such that for any indecomposable Λ-modules A, B and an irreducible map $f : A \to B$, we have $l(A) \leqslant b_1 \cdot l(B)$. In particular, if the indecomposable module B is non-projective, then $l(\tau_\Lambda B) \leqslant b_1^2 \cdot l(B)$.

Let X and Y be indecomposable modules in $\mathcal{F}(\Delta)$ and $f : X \to Y$ be a relative irreducible map. If Y is non-projective, we get $l(\tau_\Delta Y) \leqslant N \cdot b_1^2 \cdot l(Y)$ from Proposition 3.2 and Corollary 3.3. If Y is indecomposable projective module, then X is the minimal right $\mathcal{F}(\Delta)$-approximation of a summand of radical of Y, and $l(X) \leqslant N \cdot l(\operatorname{rad} Y) \leqslant N \cdot l(Y)$. Finally, let $b = \max\{N, N \cdot b_1^2\}$, we conclude that $l(X) \leqslant b \cdot l(Y)$. □

Theorem 3.6.

(a) *Let* A *be an indecomposable preprojective module in* $\mathcal{F}(\Delta)$. *Then there exist indecomposable modules* $A = M_1, M_2, \ldots, M_k$*, and relative irreducible maps* $M_{i+1} \to M_i$ *in* $\mathcal{F}(\Delta)$*,* $i = 1, 2, \ldots, k-1$*, where* M_i*,* $i = 2, \ldots, k-1$*, are preprojective, and* M_k *is projective.*

(b) *Let* A *be an indecomposable preinjective module in* $\mathcal{F}(\Delta)$. *Then there exist indecomposable modules* $A = M_1, M_2, \ldots, M_k$*, and relative irreducible maps* $M_i \to M_{i+1}$ *in* $\mathcal{F}(\Delta)$*,* $i = 1, 2, \ldots, k-1$*, where* M_i*,* $i = 2, \ldots, k-1$*, are preinjective, and* M_k *is* Ext-*injective.*

Proof. (a) Let $A \in \mathcal{P}_i$. We proceed by induction on i. If $i = 0$, that is, A is projective, this is clear. Let $A \in \mathcal{P}_1$ and consider the relative almost split sequence $0 \to \tau_\Delta A \to Y \to A \to 0$. If Y does not contain an indecomposable projective summand, then $0 \to \tau_\Delta A \to Y \to A \to 0$ is split since $A \in \mathcal{P}_1$. This is a contradiction. Hence, Y admits an indecomposable projective summand P' and with an irreducible map $P' \to A$ according to Lemma 3.1. Let $m > 1$ and suppose that the statement holds for each module $B \in \mathcal{P}_{m-1}$. Now let

$A \in \mathcal{P}_m$. We then have a right almost split map $C \to A$ which is an epimorphism, but not a split epimorphism. Then C admits a summand in $\mathcal{P}^m = \bigcup_{i=1}^{m-1} \mathcal{P}_i$, that is, there is an irreducible map $B \to A$ with $B \in \mathcal{P}^m$. By induction hypothesis, we have indecomposable modules $B = M'_1, \ldots, M'_k$ and irreducible maps $M'_{j+1} \to M'_j$, $j = 1, \ldots, k-1$, with M'_k is projective and M'_i, $i = 2, \ldots, k-1$, are preprojective. Set $M_1 = A$ and $M_j = M'_{j-1}$ for $j = 2, \ldots, k+1$, as required.

(b) It is the dual of (a). □

The following theorem will be useful. However, we omit its proof since it is similar to that of [5, Lemma 3.1].

Theorem 3.7. *Let X be an indecomposable module in $\mathcal{F}(\Delta)$.*

(a) *If X is preprojective, then there exists a sectional path*

$$P = X_0 \to X_1 \to \cdots \to X_t = {\tau_\Delta}^n X, \quad n \geqslant 0$$

from an indecomposable projective module P to a positive power of the relative translate of X such that

(1) *X_i is left stable for all $i > 0$, and*

(2) *if $X_i \in \mathcal{O}(X_j)$ for $j < i$, then $X_i = {\tau_\Delta}^l X_j$ for some $l > 0$.*

(b) *If X is preinjective, then there exists a sectional path*

$${\tau_\Delta}^n X = X_t \to X_{t-1} \to \cdots \to X_0 = I, \quad n \leqslant 0$$

from a negative power of the relative translate of X to an indecomposable Ext*-injective module I such that*

(1) *X_i is right stable for all $i > 0$, and*

(2) *if $X_i \in \mathcal{O}(X_j)$ for $j < i$ then $X_i = {\tau_\Delta}^l X_j$ for some $l < 0$.*

In order to study the properties of stable components of $\Gamma_{\mathcal{F}(\Delta)}$, we now define the degrees of relative irreducible maps in $\mathcal{F}(\Delta)$ as Liu has done in [10, Definition 1.1].

Definition 3.1. Let $f : X \to Y$ be a relative irreducible map in $\mathcal{F}(\Delta)$. It then induces a natural transformation for each $n \geqslant 0$

$$l_n(f) : \Re^n_\Delta(-, X)/\Re^{n+1}_\Delta(-, X) \to \Re^{n+1}_\Delta(-, Y)/\Re^{n+2}_\Delta(-, Y).$$

We defined the left degree $d_l(f)$ of f to be ∞ if all $l_n(f)$, $n \geqslant 0$, are monomorphisms, otherwise, to be the least integer m such that $l_m(f)$ is not a monomorphism.

Remark 3.8. Note that $d_l(f) = m$, where $f : X \to Y$ is a relative irreducible map in $\mathcal{F}(\Delta)$, means that there exists $p \notin \Re^{m+1}_\Delta(Z, X)$ such that $fp \in \Re^{m+2}_\Delta$. If the composition of some relative irreducible maps is zero, then at least one of these maps has finite left degree.

Proposition 3.9. *Let $f : Y \to Z$ be a relative irreducible map with Z indecomposable non-projective and $d_l(f) = m$. Let further*

$$0 \to \tau_\Delta Z \xrightarrow{(g,g')^t} Y \oplus Y' \xrightarrow{(f,f')} Z \to 0$$

be the relative almost split sequence. If there exists $p : X \to Y \notin \Re_\Delta^{m+1}$ such that $fp \in \Re_\Delta^{m+2}$, then there exist a map $q : X \to \tau_\Delta Z \notin \Re_\Delta^m$ satisfying $p + gq \in \Re_\Delta^{m+1}$ and $g'q \in \Re_\Delta^{m+1}$.

Proof. Since $fp \in \Re_\Delta^{m+2}$, we have a factorization $fp = ts$ with $s : X \to W \in \Re_\Delta^{m+1}$, $t : W \to Z \in \Re_\Delta$. Then t factors through (f, f'), say $t = (f, f')\binom{u}{u'}$, then $(f, f')\binom{us-p}{u's} = 0$. This implies

$$\operatorname{Im}\begin{pmatrix} us - p \\ u's \end{pmatrix} \subseteq \operatorname{Im}\begin{pmatrix} g \\ g' \end{pmatrix} = \operatorname{Ker}\bigl(f, f'\bigr).$$

So there exists a $q : X \to \tau_\Delta Z$ such that $\binom{us-p}{u's} = \binom{g}{g'} \cdot q$, i.e.,

$$\begin{pmatrix} us \\ u's \end{pmatrix} = \begin{pmatrix} p + gq \\ g'q \end{pmatrix} \in \Re_\Delta^{m+1}.$$

From $p \notin \Re_\Delta^{m+1}$ we conclude that $q \notin \Re_\Delta^m$. □

Corollary 3.10. *Assume a relative irreducible map $f : Y \to Z$ satisfies $d_l(f) = m < \infty$ with Z indecomposable non-projective. If $Y \oplus Y'$ is a summand of the whole middle term of the relative almost split sequence ending at Z and $Y' \neq 0$. Then there is an irreducible map $g' : \tau_\Delta Z \to Y'$ with $d_l(g') < d_l(f)$. Consequently, if $d_l(f) = 1$, then f is a surjective map.*

Proof. According to Proposition 3.9, there exists $q : X \to \tau_\Delta Z \notin \Re_\Delta^m$ such that $g'q \in \Re_\Delta^{m+1}$, so $d_l(g') \leqslant m - 1 < d_l(f) = m$. If $d_l(f) = 1$ and $Y' \neq 0$, then $g' : \tau_\Delta Z \to Y'$ has left degree 0. This implies that there exists an isomorphism t such that $g't \in \Re_\Delta^2$. This is a contradiction. Hence, $Y' = 0$ and f is a surjective map. □

Proposition 3.11. *Let $f : X \to Y$ be a relative irreducible map of finite left degree in $\mathcal{F}(\Delta)$ with Y indecomposable. Assume that*

$$Y_m \to Y_{m-1} \to \cdots \to Y_1 \to Y_0 = Y$$

is a sectional path in a left stable connected component with $m \geqslant 0$. If $X \oplus Y_1$ is a summand of the whole middle term of the relative almost split sequence ending at Y, then for each $1 \leqslant i \leqslant m$, there is a relative irreducible map $f_i : \tau_\Delta Y_{i-1} \to Y_i$ such that $d_l(f_m) < d_l(f_{m-1}) < \cdots < d_l(f_1) < d_l(f)$. In particular, $d_l(f) > m$.

Proof. From Corollary 3.10, we have $d_l(f_1) < d_l(f)$, thus

$$d_l(f_{k+1}) < d_l(f_k), \quad 1 \leqslant k \leqslant m-1$$

by an inductive argument. Therefore,

$$d_l(f_m) < d_l(f_{m-1}) < \cdots < d_l(f_1) < d_l(f)$$

and $d_l(f) > m$. □

Corollary 3.12. *Let $f : Y \to X$ be a relative irreducible map with X indecomposable non-projective. Assume that there is an infinite sectional path*

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$$

in a left stable component of $\Gamma_{\mathcal{F}(\Delta)}$ such that $Y \oplus X_1$ is a summand of the whole middle term of the relative almost split sequence ending at X, then $d_l(f) = \infty$.

The proof follows directly from Proposition 3.11.

The following proposition and corollary will be used to define the degree of an arrow in $\Gamma_{\mathcal{F}(\Delta)}$. The result follows from those of [10, Lemma 1.7] and its corollary.

Proposition 3.13. *Let $[X] \to [Y]$ be an arrow in $\Gamma_{\mathcal{F}(\Delta)}$ with valuation $(\alpha_{X,Y}, \alpha'_{X,Y})$, and with Y not projective. If a relative irreducible map $f : X \to Y$ has finite left degree, then at least one of $\alpha_{X,Y}$ and $\alpha'_{X,Y}$ is equal to* 1.

Corollary 3.14. *Let $[X] \to [Y]$ be an arrow in $\Gamma_{\mathcal{F}(\Delta)}$, with Y non-projective. If $f : X \to Y$, $g : X \to Y$ are both relative irreducible maps in $\mathcal{F}(\Delta)$, then $d_l(f) = d_l(g)$.*

So in a left stable component of $\Gamma_{\mathcal{F}(\Delta)}$, we may define the left degree of an arrow $[X] \to [Y]$ to be the left degree of a relative irreducible map $X \to Y$.

Lemma 3.15. *Let $f = (f_1, f_2) : X \to Y = Y_1 \oplus Y_2$ be a relative irreducible map in $\mathcal{F}(\Delta)$ with Y_1, Y_2 indecomposable non-projective. Let*

$$0 \to \tau_\Delta Y_i \xrightarrow{(g_i, g'_i)^t} X \oplus X' \xrightarrow{(f_i, f'_i)} Y_i \to 0$$

be relative almost split sequences in $\mathcal{F}(\Delta)$, $i = 1, 2$. Then $d_l(g) < d_l(f)$, where $g = (g_1, g_2)^t : \tau_\Delta Y_1 \oplus \tau_\Delta Y_2 \to X$.

Proof. Since $d_l(f) = m < \infty$, there exists a map $p : M \to X \in \Re^m_\Delta$ with $p \notin \Re^{m+1}_\Delta$ such that $fp \in \Re^{m+2}_\Delta$. This implies $f_i p \in \Re^{m+2}_\Delta$, $i = 1, 2$. Then there exist relative almost split sequences in $\mathcal{F}(\Delta)$

$$0 \to \tau_\Delta Y_i \xrightarrow{(g_i, g'_i)^t} X \oplus X' \xrightarrow{(f_i, f'_i)} Y_i \to 0, \quad i = 1, 2.$$

According to Proposition 3.9, there exist $q_i : M \to \tau_\Delta Y_i \notin \Re_\Delta^m$ such that $g_i q_i + p \in \Re_\Delta^{m+1}$, $i = 1, 2$. So $\binom{q_1}{-q_2} \notin \Re_\Delta^m$ and $(g_1, g_2)\binom{q_1}{-q_2} \in \Re_\Delta^{m+1}$. Thus, $d_l((g_1, g_2)) \leqslant m - 1 < m$. □

Theorem 3.16. *Let $X_0 \to X_1 \to \cdots \to X_{m-1} \to X_m$ be a sectional path in a left stable component of $\Gamma_{\mathcal{F}(\Delta)}$. Then there are relative irreducible maps $f_i : X_{i-1} \to X_i$ in $\mathcal{F}(\Delta)$ such that the composite $f_m f_{m-1} \cdots f_1$ is not in $\Re_\Delta^{m+1}$. In particular, $f_m f_{m-1} \cdots f_1 \neq 0$.*

Proof. It is sufficient to prove that for every $1 \leqslant j \leqslant m$ there exists a relative irreducible map $(f_j, g_j) : X_{j-1} \oplus \tau_\Delta X_{j+1} \to X_j$ such that for every $p_{j-1} : X_0 \to \tau_\Delta X_{j-1}$, we have $f_j f_{j-1} \cdots f_1 + g_j p_{j-1} \notin \Re_\Delta^{j+1}$ ($\tau_\Delta X_{m+1} = 0$ by convention).

Let $(f_1, g_1) : X_0 \oplus \tau_\Delta X_2 \to X_1$ be a relative irreducible map, then for any map $p_0 : X_0 \to \tau_\Delta X_2$, we have $f_1 + g_1 p_0 \notin \Re_\Delta^2$ since $g_1 \in \Re_\Delta$ and $p_0 \in \Re_\Delta$. We make an induction on j. Let $1 < j < m$ and suppose that we have a relative irreducible map (f_j, g_j) satisfying that for every $p_{j-1} : X_0 \to \tau_\Delta X_{j+1}$, it holds $f_j \cdots f_1 + g_j p_{j-1} \notin \Re_\Delta^{j+1}$. Thus $f_j \cdots f_1 \notin \Re_\Delta^{j+1}$ by taking $p_{j-1} = 0$.

If $\tau_\Delta X_{j+1} \neq 0$, and $X_j \oplus \tau_\Delta X_{j+2}$ is a summand of the middle term of the relative almost split sequence ending at X_{j+1}, we get irreducible maps $\binom{g_j}{h_j} : \tau_\Delta X_{j+1} \to X_j \oplus \tau_\Delta X_{j+2}$ and $(f_{j+1}, g_{j+1}) : X_j \oplus \tau_\Delta X_{j+2} \to X_{j-1}$. If there exists $p_j : X_0 \to \tau_\Delta X_{j+2}$ such that $f_{j+1} f_j \cdots f_1 + g_{j+1} p_j = (f_{j+1}, g_{j+1})\binom{f_j f_{j-1} \cdots f_1}{p_j} \in \Re_\Delta^{j+2}$, then there exists $p_{j-1} : X_0 \to \tau_\Delta X_{j-1}$ satisfying $\binom{f_j f_{j-1} \cdots f_1}{p_j} + \binom{g_j}{h_j} p_{j-1} \in \Re_\Delta^{j+1}$ by Proposition 3.9, so $f_j f_{j-1} \cdots f_1 + g_j p_{j-1} \in \Re_\Delta^{j+1}$, which contradicts the induction hypothesis. If $\tau_\Delta X_{j+1} = 0$, then $j = m$ because the component is left stable. This implies $f_m \cdots f_1 \in \Re_\Delta^{m+1}$. By Proposition 3.9, there exist $p_{m-2} : X_0 \to \tau X_m$ such that $f_{m-1} \cdots f_1 + g_{m-1} p_{m-2} \in \Re_\Delta^m$. This also contradicts the induction hypothesis. Hence, $f_m f_{m-1} \cdots f_1 \notin \Re_\Delta^{m+1}$. □

4. Left-stable and stable components

In this section, we include some useful lemmas and theorems concerning left stable and stable components in $\Gamma_{\mathcal{F}(\Delta)}$. The proofs are similar to those in the Auslander–Reiten quiver Γ_Λ of Λ.

Lemma 4.1 [11, Lemma 2.1]. *Let Γ be a left stable component of $\Gamma_{\mathcal{F}(\Delta)}$. If there is a path from X to Y in Γ, then either $X = \tau_\Delta{}^r Y$ for some $r > 0$ or there is a sectional path in Γ from X to $\tau_\Delta{}^r Y$ for some $r \geqslant 0$.*

Lemma 4.2 [10, Proposition 1.13]. *Let*

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X \tag{1}$$

be an infinite sectional path in $\Gamma_{\mathcal{F}(\Delta)}$ with all X_i left stable. If the path (1) *contains infinitely many arrows with finite left degree, then the relative almost split sequence ending at X_n has at most two left stable summands as middle terms.*

Theorem 4.3 [11, Theorem 2.3]. *Let Γ be a stable component of $\Gamma_{\mathcal{F}(\Delta)}$, containing no τ_Δ-periodic module. If there is an oriented cycle in Γ, then Γ contains only finitely many τ_Δ-orbits.*

Proposition 4.4. *Suppose there exists a relative irreducible map between two indecomposable modules X and Y in $\mathcal{F}(\Delta)$. If Y is τ_Δ-periodic, then either X is τ_Δ periodic or there are non-negative integers n and m such that ${\tau_\Delta}^n X$ is projective and ${\tau_\Delta}^{-m} X$ is* Ext-*injective.*

Proof. Since Y is τ_Δ-periodic, we get ${\tau_\Delta}^k Y \cong Y$ for some $k > 0$. Let $f : X \to Y$ be a relative irreducible map. If X is not in the τ_Δ-orbit of a projective, then ${\tau_\Delta}^n X$ exist for all $n > 0$. So there are relative irreducible maps ${\tau_\Delta}^k X \to {\tau_\Delta}^k Y \cong Y$. This implies that there exist relative irreducible maps from $X, {\tau_\Delta}^k X, \ldots, {\tau_\Delta}^{mk} X, \ldots$ to Y. Because $\Gamma_{\mathcal{F}(\Delta)}$ is locally finite, there is an $l > 1$ such that ${\tau_\Delta}^{lk} X \cong {\tau_\Delta}^k X$, thus, ${\tau_\Delta}^{lk-k} X \cong X$, i.e., X is τ_Δ-periodic. In a similar way, we can show that, if X is not in a τ_Δ-orbit of an Ext-injective module, then X is τ_Δ-periodic. $\square$

Corollary 4.5. *Assume that $\mathcal{C}$ is a left or right stable component in $\Gamma_{\mathcal{F}(\Delta)}$. If there is a τ_Δ-periodic module in $\mathcal{C}$, then all modules in $\mathcal{C}$ are τ_Δ-periodic* (*such a component $\mathcal{C}$ will be called periodic*).

In the following, we consider the stable part $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$ of $\Gamma_{\mathcal{F}(\Delta)}$, that is, the maximal full sub-quiver of $\Gamma_{\mathcal{F}(\Delta)}$ by deleting all τ_Δ-orbits of projective and Ext-injective modules.

Theorem 4.6 [9,15,17]. *A periodic component of $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$ is of the form $\mathbb{Z}\mathcal{Q}/G$, where $\mathcal{Q}$ is $Dynkin$ quiver or a quiver of the form A_∞, and G is a non-trivial group of automorphism of $\mathbb{Z}\mathcal{Q}$. A non-periodic component of $\Gamma_{\mathcal{F}(\Delta)}$ is of the form $\mathbb{Z}\mathcal{Q}$, where $\mathcal{Q}$ is a connected valued quiver without cyclic paths.*

Definition 4.1. Let Γ be a connected component of $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$, and $Y, Z \in \mathcal{F}(\Delta)$ be indecomposable modules with $Y \notin \Gamma$, $Z \in \Gamma$. If there is a relative irreducible map $f : Y \to Z$ or $Z \to Y$, we call Z a frontier of Γ.

A frontier Y of a component must lie in the orbit of a projective or Ext-injective module. Thus, there are only finitely many indecomposable module $Z_1, Z_2, \ldots, Z_q \in \Gamma$, such that all the frontiers in Γ lie in the orbits of $Z_1, Z_2, \ldots, Z_q$, that is, in $\bigcup_{i=1}^{q} \mathcal{O}(Z_i)$.

Lemma 4.7 [5, Lemma 3.2]. *Let Γ be a non-periodic connected component of $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$. Assume Γ is of the form $\mathbb{Z}\mathcal{Q}$, where $\mathcal{Q}$ is an infinite connect valued quiver without oriented cycles, and $Z_1, Z_2, \ldots, Z_q$ are chosen as above, then*

(a) *for each $1 \leqslant i \leqslant q$, there exists a non-negative integer n_i such that ${\tau_\Delta}^j Z_i$ belongs to $\mathcal{P}_\infty$ for all $j > n_i$,*
(b) *for each $1 \leqslant i \leqslant q$, there exists a non-negative integer m_i such that ${\tau_\Delta}^{-j} Z_i$ belongs to $\mathcal{I}_\infty$ for all $j > m_i$.*

Lemma 4.8 [5, Lemma 3.3]. *Let Γ be as above, and D a module in Γ. Then*

(a) *there exists a non-negative integer r such that if X is a preprojective module in Γ, then there exists a path in Γ from ${\tau_\Delta}^r D$ to X,*
(b) *there exists a non-negative integer s such that if X is a preinjective module in Γ, then there exists a path in Γ from X to ${\tau_\Delta}^{-s} D$.*

Theorem 4.9. *Let Γ be a non-periodic connected component of $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$. If Γ is of the form $\mathbb{Z}\mathcal{Q}$, with $\mathcal{Q}$ being an infinite quiver without oriented cycles, then there are infinitely many modules in Γ which are neither preprojectives nor preinjectives.*

Proof. Let $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ be such that $\mathcal{Q}_0 = \{a_i\colon i \in \mathbb{N}\}$ and that there is a path from a_0 to every point in $\mathcal{Q}$. Clearly, $(\mathbb{Z}\mathcal{Q})_0 = \{(a_i, j)\colon i \in \mathbb{N}, j \in \mathbb{Z}\}$.

Let D be the module corresponding to $(a_0, 0)$ and r and s be chosen as in Lemma 4.8. Then we have the set of irreducible successors of ${\tau_\Delta}^r D$ in Λ is

$$\mathrm{Sc}\big({\tau_\Delta}^r D\big)_\Gamma = \big\{(a_i, j),\ j \leqslant r\big\}$$

and the set of irreducible predecessors of ${\tau_\Delta}^{-s} D$ in Γ is

$$\mathrm{Pr}\big({\tau_\Delta}^{-s} D\big)_\Gamma = \big\{(a_i, j),\ j \geqslant -s + n(a_i)\big\},$$

where $n(a_i)$ denotes the length of the shortest path from $(a_0, 0)$ to $(a_i, 0)$. Since $\mathcal{Q}_0$ is infinite, for each $m \geqslant 0$ there exists a vertex b_m such that $n(b_m) \geqslant m$, that is, there are infinitely many $c_i, i \in N$ such that $n(c_i) > s + r + 1$ for every $i \in N$. Consider now for each i the module C_i in Γ corresponding to the vertex $(c_i, r + 1)$. Note that $C_i \notin \mathrm{Sc}(\tau^r D)$ and then C_i is not preprojective according to Lemma 4.8. On the other hand, $C_i \notin \mathrm{Pr}(\tau^{-s} D)$ since $-s + n(c_i) > -s + (s + r + 1) = r + 1$. This implies that C_i is not preinjective either. Therefore, for each $i \in N$, $C_i \in P_\infty \cap I_\infty$. □

Proposition 4.10 [5, Proposition 4.2]. *Let $X_1, \ldots, X_n$ be indecomposable modules in a connected component Γ of $\Gamma_{\mathcal{F}(\Delta)}$. Let Γ' be the full sub-quiver of Γ without the vertices corresponding to the modules in the union $\bigsqcup_{i=1}^n \mathcal{O}(X_i)$. Then Γ' contains only finitely many non-trivial connected components. Moreover, all the trivial components of Γ', if any, belong to a finite number of τ_Δ-orbits.*

5. Components containing finitely many τ_Δ-orbits only

Definition 5.1. Let Γ be a sub-quiver of $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$. We say that Γ satisfies Brauer–Thrall-II or, shorter, BT-II, if there are infinitely many natural numbers $\{d_i\}_{i\in\mathbb{N}}$ such that for each $i \in \mathbb{N}$ there are infinitely many modules in Γ of dimension d_i.

Theorem 5.1. *Let Γ be a connected component of $\Gamma_{\mathcal{F}(\Delta)}$ with only finitely many τ_Δ-orbits. Then Γ does not satisfy* BT-II.

In order to prove the theorem we need the following definition and lemmas.

Definition 5.2 [10]. Let Ω be a connected value quiver without oriented cycles. A sectional subgraph Σ of type Ω in $\Gamma_{\mathcal{F}(\Delta)}$ is a value quiver with a value quiver isomorphism $\phi : \Omega \to \Sigma$ such that the following conditions hold:

(1) For each vertex i of Ω, $X_i := \phi(i)$ is a vertex of $\Gamma_{\mathcal{F}(\Delta)}$.
(2) If $i \to j$ and $j \to k$ are arrows in Ω, then $X_i \neq \tau_\Delta X_j$.
(3) If $i \to j$ is an arrow with valuation(β, β') in Ω, then $X_i \to X_j$ is an arrow with valuation (α, α') in $\Gamma_{\mathcal{F}(\Delta)}$ satisfying $(\alpha, \alpha') \geqslant (\beta, \beta')$.
(4) If i and j are different immediate predecessors or successors of a vertex of Ω, then X_i and X_j are different vertices of $\Gamma_{\mathcal{F}(\Delta)}$.

If, moreover, for all arrows in Ω, we have $(\alpha, \alpha') = (\beta, \beta')$, then we say that Σ is fully valued.

Lemma 5.2 [10, Lemma 3.4]. *Let Ω be a Euclidean quiver and Σ a sectional subgraph of type Ω in $\Gamma_{\mathcal{F}(\Delta)}$. If Σ contains only left stable modules, then $l({\tau_\Delta}^m X) \to \infty$ as $m \to \infty$ for each vertex $X \in \Sigma$. Dually, if Σ contains only right stable modules, then $l({\tau_\Delta}^{-m} X) \to \infty$ as $m \to \infty$ for each vertex $X \in \Sigma$.*

Lemma 5.3 [10, Lemma 3.3]. *Let Ω be a Dynkin quiver and Σ a sectional subgraph of type Ω in $\Gamma_{\mathcal{F}(\Delta)}$. If Σ contains only left stable modules, then either Σ consists of τ_Δ-periodic modules or there is some integer $m \geqslant 0$ such that ${\tau_\Delta}^m \Sigma$ is properly contained in a sectional subgraph in $\Gamma_{\mathcal{F}(\Delta)}$ which contains only left stable modules.*

Lemma 5.4. *Let Γ be a left stable sub-quiver of $\Gamma_{\mathcal{F}(\Delta)}$. Assume that there exists a module X in Γ such that $l({\tau_\Delta}^m X)$ does not tend to infinity as m tending to infinity. Then there is no sectional subgraph of Euclidean type in Γ.*

Proof. Suppose there is an arrow $Y \to X$ or $X \to Y$ in Γ, so there exists $b > 0$ such that $l(Y) \leqslant bl(X)$ or $l(Y) \leqslant bl(\tau_\Delta X)$ according to Theorem 3.5. Because $l({\tau_\Delta}^m X)$ does not tend to infinity when m tends to infinity, there exists b_X such that $l({\tau_\Delta}^m X) < b_X$ for infinitely many m's. Then $l({\tau_\Delta}^s Y) < b \cdot b_X$, for $s = m$ or $m + 1$, that is, $l({\tau_\Delta}^s X)$ does not tend to infinity as s tends to infinity. Since Γ is connected, every module in Γ has the same property. Hence, there is no sectional subgraph of Euclidean type in Γ according to Lemma 5.2. $\square$

Lemma 5.5. *Let Γ be a connected left stable sub-quiver of $\Gamma_{\mathcal{F}(\Delta)}$. Assume that there is a module X in Γ such that $l({\tau_\Delta}^m X)$ does not tend to infinity as m tends to infinity. If there is a sectional path in Γ which meets a τ_Δ-orbit twice, then Γ contains τ_Δ-periodic modules.*

Proof. According to Lemma 5.4, for every module Y in Γ, there is a constant b_Y such that $l({\tau_\Delta}^m Y) \leqslant b_Y$ for infinitely many m's. If there is an arrow $Z \to Z'$ in $\mathcal{F}(\Delta)$, then

$l(Z) \leqslant b \cdot l(Z')$ because of Theorem 3.5. Hence, for a given positive integer r, there are infinitely many m's such that $l({\tau_\Delta}^{m \cdot r} Y) \leqslant b^r \cdot b_Y$. If there is a sectional path

$$ {\tau_\Delta}^r Y_0 = Y_t \to Y_{t-1} \to \cdots \to Y_1 \to Y_0 $$

in Γ, we may suppose Y_{t-1} and Y_1 are not in the same τ_Δ-orbit. Since $\Gamma_{\mathcal{F}(\Delta)}$ has no sectional cyclic paths [12, Theorem 2], we have $r \neq 0$. Now assume $t > 2$. If $r > 0$, then there exists an infinite path

$$ \cdots \to {\tau_\Delta}^{2r} Y_0 \to {\tau_\Delta}^r Y_{t-1} \to \cdots \to {\tau_\Delta}^r Y_1 \to {\tau_\Delta}^r Y_0 \to Y_{t-1} \to \cdots \to Y_1 \to Y_0. \quad (*) $$

Since $l({\tau_\Delta}^{sr} Y_0) \leqslant b^r \cdot b_{Y_0}$ holds for infinitely many s, and in a left stable sub-quiver, the composition of the sectional path is not zero (Theorem 3.16), $(*)$ is not a sectional path. Thus, $1 \leqslant t \leqslant 2$. In case $t = 1$, ${\tau_\Delta}^{2r-1} Y_0 = Y_0$. In case $t = 2$, $r > 1$, $Y_1 = {\tau_\Delta}^{r-1} Y_1$. So there is a τ_Δ-periodic module in Γ. Because Γ is a left stable connected sub-quiver, Γ contains only τ_Δ-periodic modules. The case $r < 0$ can be treated in a similar way. □

Lemma 5.6 [10, Lemma 3.5]. *Let Γ be a maximal connected left stable sub-quiver of $\Gamma_{\mathcal{F}(\Delta)}$. Assume that there is no sectional subgraph of Euclidean type in Γ and that each sectional path in Γ meets each τ_Δ-orbit in Γ at most once. Then either Γ consists of τ_Δ-periodic modules or, for each module X, there is an infinite sectional path in Γ which ends at ${\tau_\Delta}^m X$ for some $m \geqslant 0$.*

Proof of Theorem 5.1. If, for some b, there are infinitely many modules with the length b, then there must exist a τ_Δ-orbit of a module X such that $l({\tau_\Delta}^m X) = b$ for infinitely many m's. Hence, $l({\tau_\Delta}^n X)$ does not tend to infinity (as $m \to \infty$). Therefore, there exists an $r > 0$ such that ${\tau_\Delta}^r X$ is in Γ', the maximal left stable connected sub-quiver of Γ. Since X is not τ_Δ-periodic, Γ' is not periodic. According to Lemmas 5.5 and 5.6, Γ' has infinitely many orbits. This is a contradiction. □

Definition 5.3. A quasi-periodic component is a connected component of $\Gamma_{\mathcal{F}(\Delta)}$ with infinitely many τ_Δ-orbits such that at most finitely many of them do not contain a τ_Δ-period module.

Theorem 5.7. *A quasi-periodic component of $\Gamma_{\mathcal{F}(\Delta)}$ does not satisfy* BT-II.

Proof. Let Γ be a quasi-periodic component of $\Gamma_{\mathcal{F}(\Delta)}$, and Γ' the full sub-quiver of Γ without the vertices corresponding to the τ_Δ-orbits of non-periodic modules. It follows from Proposition 4.9 that Γ' is a finite union of non-trivial connected sub-quivers of Γ and by construction each of them contains only τ_Δ-periodic modules, that is, they are periodic.

Let $\mathcal{C}$ be a connected sub-quiver of Γ'. If $\mathcal{C}$ is a finite sub-quiver of $\Gamma_{\mathcal{F}(\Delta)}$, then it does not satisfy BT-II. On the other hand, if $\mathcal{C}$ is not finite, then it is a stable tube and does not satisfy BT-II, either.

If Γ satisfies BT-II, then so is Γ/Γ'. Since Γ/Γ' has only finitely many τ_Δ-orbits, at least one of them should satisfy BT-II. Because Γ' is a finite union of periodic sub-quivers of Γ, there exists $m > 0$ such that $\mathrm{Pr}({\tau_\Delta}^m M) \subseteq \Gamma/\Gamma'$, where $\mathrm{Pr}({\tau_\Delta}^m M)$ should be maximal left stable. So according to Theorem 5.1, $\Pr(\tau_\Delta{}^m M)$ has only finite many τ_Δ-orbits, which does not satisfy BT-II. This is a contradiction and finishes the proof. □

6. Proof of the main result

In this section we use a relation between the category $\mathcal{F}(\Delta)$ and the subspace category given in [6] to prove the main theorem.

Lemma 6.1 [5, Lemma 6.1]. *Let $\mathcal{C}$ be a connected component of $\Gamma_{\mathcal{F}(\Delta)}$ with infinitely many τ_Δ-orbits. Suppose that $\mathcal{C}$ has no connected sub-quiver of the form $\mathbb{Z}\mathcal{Q}$ with $\mathcal{Q}$ an infinite quiver without oriented cycles. Then $\mathcal{C}$ is quasi-periodic.*

Theorem 6.2. *If $\mathcal{P}_\infty \cap \mathcal{I}_\infty = \emptyset$ in $\mathcal{F}(\Delta)$, then $\mathcal{F}(\Delta)$ does not satisfy* BT-II.

Proof. Let $\mathcal{P}_\infty \cap \mathcal{I}_\infty = \emptyset$. Then any indecomposable module in $\mathcal{F}(\Delta)$ is either a preprojective or a preinjective module. It follows from Theorems 3.6 and 3.7 that any connected component of $\Gamma_{\mathcal{F}(\Delta)}$ contains either a projective or an Ext-injective. In particularly, $\Gamma_{\mathcal{F}(\Delta)}$ has only finitely many connected components, say $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$. Since $\mathcal{P}_\infty \cap \mathcal{I}_\infty = \emptyset$, it follows from Theorem 4.9 that for each $1 \leqslant i \leqslant m$, Γ_i contains no connected sub-quiver of the form $\mathbb{Z}\mathcal{Q}$ with $\mathcal{Q}$ an infinite quiver without oriented cycles. Therefore, according to Lemma 6.1 all the components Γ_i containing infinitely many τ_Δ-orbits are quasi-periodic. To summarize, for each $1 \leqslant i \leqslant m$, either Γ_i is a quasi-periodic component or Γ_i has only finitely many τ_Δ-orbits. By Theorems 5.1 and 5.7, Γ_i does not satisfy BT-II. Thus, $\Gamma_{\mathcal{F}(\Delta)}$ does not satisfy BT-II. □

Definition 6.1 [6]. Let $\mathcal{K}$ be a Krull–Schmidt category over a commutative artin ring R, D a division ring over R which is finitely generated as an R-module, and $|\cdot| : \mathcal{K} \to \operatorname{mod} D$ an additive functor. We call the pair $(\mathcal{K}, |\cdot|)$ a vector space category and denote by $\mathcal{U}(\mathcal{K}, |\cdot|) =: \mathcal{X}$, called a subspace category of $(\mathcal{K}, |\cdot|)$, the category of all triples $V = (V_0, V_\omega, \gamma_V)$, where $V_\omega \in \operatorname{mod} D$, $V_0 \in \mathcal{K}$, and $\gamma_V : V_\omega \to |V_0|$ is a D-linear map. A morphism from V to V' by definition is a pair (f_0, f_ω), where $f_0 : V_0 \to V_0'$, $f_\omega : V_\omega \to V_\omega'$, such that $f_\omega \gamma_{V'} = \gamma_\omega |f_0|$.

Since $\Delta(n) = P(n)$, $D = \operatorname{End}_\Lambda(P(n))$ is a division ring. Let $\Lambda_0 = \Lambda/\Lambda e_n \Lambda$, where e_n is the idempotent corresponding to the indecomposable projective module $P(n) = \Lambda e_n$. Then Λ_0 is a quasi-hereditary algebra and $\mathcal{F}(\Delta_{\Lambda_0}) = \mathcal{F}(\Delta(1), \ldots, \Delta(n-1))$. Now we get a functor

$$\operatorname{Ext}^1_\Lambda\big(-, P(n)\big) : \mathcal{F}(\Delta_{\Lambda_0})^{\mathrm{op}} \to \operatorname{mod} D,$$

where $\mathcal{F}(\Delta_{\Lambda_0})^{\mathrm{op}}$ is the opposite category of $\mathcal{F}(\Delta_{\Lambda_0})$. So we have a vector space category $(\mathcal{F}(\Delta_{\Lambda_0})^{\mathrm{op}}, \operatorname{Ext}(-, P(n)))$ and the subspace category $\mathcal{U}(\mathcal{F}(\Delta^{\mathrm{op}}_{\Lambda_0}, \operatorname{Ext}(-, P(n)))$.

Lemma 6.3 [7, Proposition 2.2]. *There is a full and dense functor*

$$\eta : \mathcal{F}(\Delta)^{\mathrm{op}} \to \mathcal{U}\big(\mathcal{F}(\Delta_{\Lambda_0})^{\mathrm{op}}, \mathrm{Ext}^1_\Lambda\big(-, P(n)\big)\big),$$

such that the kernel of η is contained in the radical of $\mathcal{F}(\Delta)^{\mathrm{op}}$. So η induces a bijection between the isomorphism classes of indecomposable objects in $\mathcal{F}(\Delta)$ and those in $\mathcal{U}(\mathcal{F}(\Delta_{\Lambda_0})^{\mathrm{op}}, \mathrm{Ext}^1_\Lambda(-, P(n)))$.

Let D be a finite dimension division ring over a field k and $\mathcal{K}$ a k-additive category. Assume that the number of the isomorphism classes of indecomposable objects in $\mathcal{K}$ is finite. Moreover, let $|\cdot| : \mathcal{K} \to \operatorname{mod} D$ be a functor and $\mathcal{U}({}_D\mathcal{K})$ denote the subspace category $\mathcal{U}(\mathcal{K}, |\cdot|)$.

Proposition 6.4 [13, Proposition 3.1]. *Assume ${}_D\mathcal{K}$ is infinite, that is, there are infinitely many isomorphism classes of indecomposable objects in $\mathcal{U}({}_D\mathcal{K})$, then there exists a bimodule ${}_F M_G$ such that* $\dim {}_F M \cdot \dim M_G \geqslant 4$ *and that* $\operatorname{mod} \begin{pmatrix} F_F & M_G \\ 0 & G \end{pmatrix}$ *is equivalent to a full subcategory of $\mathcal{U}({}_D\mathcal{K})$, where F, G are finite-dimensional division rings over k.*

From now on, we assume Λ is a finite-dimensional quasi-hereditary algebra over an infinite field k.

Theorem 6.5. *If $\mathcal{F}(\Delta)$ is infinite, then $\mathcal{F}(\Delta)$ satisfies* BT-II.

Proof. According to Lemma 6.3, there exists a functor

$$\eta : \mathcal{F}(\Delta)^{\mathrm{op}} \to \mathcal{U}\big(\mathcal{F}(\Delta_{\Lambda_0})^{\mathrm{op}}, \mathrm{Ext}^1_\Lambda\big(-, P(n)\big)\big).$$

By an inductive argument, we may assume that $\mathcal{F}(\Delta)(\Delta_{\Lambda_0})$ is finite. Since $\mathcal{F}(\Delta)$ is infinite, so is the vector space category $(\mathcal{F}(\Delta)(\Delta_{\Lambda_0}), \mathrm{Ext}^1_\Lambda(-, P(n)))$. It follows from Proposition 6.4 that there is a bimodule ${}_F M_G$ such that a full subcategory $\mathcal{V}$ of $\mathcal{U}(\mathcal{F}(\Delta)(\Delta_{\Lambda_0}), \mathrm{Ext}^1_\Lambda(-, P(n)))$ is equivalent to $\operatorname{mod} \begin{pmatrix} F & {}_F M_G \\ 0 & G \end{pmatrix}$. Since k is an infinite field, the category $\operatorname{mod} \begin{pmatrix} F & {}_F M_G \\ 0 & G \end{pmatrix}$ satisfies BT-II (see [13, Example 2.6]). Hence, $\mathcal{F}(\Delta)$ satisfies BT-II. $\square$

Theorem 6.6. *$\mathcal{F}(\Delta)$ is finite if and only if $\mathcal{P}_\infty \cap \mathcal{I}_\infty = \emptyset$.*

Proof. If $\mathcal{F}(\Delta)$ is finite, then from the definition we have $\mathcal{P}_\infty = \mathcal{I}_\infty = \emptyset$, thus $\mathcal{P}_\infty \cap \mathcal{I}_\infty = \emptyset$. Conversely, let $\mathcal{P}_\infty \cap \mathcal{I}_\infty = \emptyset$. By Theorem 6.2, $\mathcal{F}(\Delta)$ does not satisfy BT-II. The finiteness of $\mathcal{F}(\Delta)$ follows from Theorem 6.5. $\square$

Corollary 6.7. *$\mathcal{F}(\Delta)$ is finite if and only if either all indecomposable modules in $\mathcal{F}(\Delta)$ are preprojective, or all indecomposable modules in $\mathcal{F}(\Delta)$ are preinjective.*

Corollary 6.8. *$\mathcal{F}(\Delta)$ is finite if and only if $\Gamma_{\mathcal{F}(\Delta)}$ has only finitely many τ_Δ-orbits.*

Proof. Let $\mathcal{F}(\Delta)$ be finite. Then, obviously, $\Gamma_{\mathcal{F}(\Delta)}$ has only finitely many τ_Δ-orbits. Conversely, assume that $\Gamma_{\mathcal{F}(\Delta)}$ has only finitely many τ_Δ-orbits. Then $\mathcal{F}(\Delta)$ does not satisfy BT-II according to Theorem 4.1. By Theorem 6.5, $\mathcal{F}(\Delta)$ is finite. □

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