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Model theoretic reformulation of the Baum–Connes and Farrell–Jones conjectures ☆

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Abstract

The Isomorphism Conjectures are translated into the language of homotopical algebra, where they resemble Thomason's descent theorems.

Keywords: Model categories; K-theory; Codescent

1. Introduction and statement of the results

In [8], Thomason establishes that algebraic K-theory satisfies Zariski and Nisnevich *descent*. This is now considered a profound algebraico-geometric property of K-theory. In [1,2], we have introduced the sister notion of *codescent*. Here, we prove that each one of the so-called Isomorphism Conjectures (see [3,5]) among

(1) the Baum–Connes Conjecture,
(2) the real Baum–Connes Conjecture,
(3) the Bost Conjecture,
(4) the Farrell–Jones Conjecture in K-theory,
(5) the Farrell–Jones Conjecture in L-theory

is equivalent to the codescent property for a suitable K- or L-theory functor.

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For a (discrete) group G, these conjectures aim at computing, in geometrical and topological terms, the groups $K_*^{\mathrm{top}}(C_r^*G)$, $KO_*^{\mathrm{top}}({}_{\mathbb{R}}C_r^*G)$, $K_*^{\mathrm{top}}(\ell^1G)$, $K_*^{\mathrm{alg}}(RG)$ and $L_*^{\mathrm{alg}}(\Lambda G)$, respectively, where R and Λ are associative rings with units, and Λ is equipped with an involution. Davis and Lück [4] express these conjectures as follows (the equivalence with the original statements is due to Hambleton–Pedersen [6]). First, fix one of the Conjectures (1)–(5) and denote by $K_*(G)$ the corresponding K- or L-group among the five listed above (for (4) and (5), R and Λ are understood). Denote by $\mathscr{C} := \mathrm{Or}(G)$ the orbit category of G, whose objects are the quotients G/H with H running among the subgroups of G, and the morphisms are the left-G-maps. Let $\mathscr{D} := \mathrm{Or}(G, \mathscr{V}\mathscr{C})$ be the full subcategory of $\mathrm{Or}(G)$ on those objects G/H for which H is virtually cyclic. We sometimes write $\mathscr{C}_G$ and $\mathscr{D}_G$ to stress the dependence on the group G. Then, a suitable functor $X_G\colon \mathscr{C} \to \mathscr{S}$ is constructed, where $\mathscr{S}$ denotes the usual stable model category of spectra (of compactly generated Hausdorff spaces), for which the weak equivalences are the stable ones. This functor X_G has the property that $\pi_*(X_G(G/H))$ is canonically isomorphic to $K_*(H)$ for all $H \leqslant G$. Then, the fixed Isomorphism Conjecture for G amounts to the statement that the following composition, called *assembly map*, is a weak equivalence in $\mathscr{S}$:

$$\mu^G\colon \quad \underset{\mathscr{D}}{\mathrm{hocolim}}\ \mathrm{res}_{\mathscr{D}}^{\mathscr{C}} X_G \to \underset{\mathscr{C}}{\mathrm{hocolim}}\ X_G \xrightarrow{\sim} \underset{\mathscr{C}}{\mathrm{colim}}\ X_G \xrightarrow{\cong} X_G(G/G).$$

We turn to homotopical algebra. First, we denote by $\mathscr{U}_{\mathscr{S}}(\mathscr{C}, \mathscr{D})$ the model category $\mathscr{S}^{\mathscr{C}}$ of functors $\mathscr{C} \to \mathscr{S}$, where the weak equivalences and fibrations are the $\mathscr{D}$-*weak equivalences* and $\mathscr{D}$-*fibrations*, respectively, i.e. they are defined $\mathscr{D}$-objectwise. See details in [1, Section 3], for instance. For a diagram $X \in \mathscr{S}^{\mathscr{C}}$, we let $\xi_X\colon QX \to X$ be the cofibrant replacement of X in $\mathscr{U}_{\mathscr{S}}(\mathscr{C}, \mathscr{D})$. As in [1, Section 4], we say that X *satisfies* $\mathscr{D}$-*codescent* if the map $\xi_X(c)$ is a weak equivalence in $\mathscr{S}$ for every $c \in \mathscr{C}$; if this is only fulfilled at some $c_0 \in \mathscr{C}$, we say that X *satisfies* $\mathscr{D}$-*codescent at* c_0. For a conceptual approach to codescent and a parallel with descent, see [1, Sections 1 and 5]. Let $\mathscr{U}_{\mathscr{S}}(\mathscr{C})$ be the model structure on $\mathscr{S}^{\mathscr{C}}$ with the $\mathscr{C}$-weak equivalences and $\mathscr{C}$-fibrations; we define $\mathscr{U}_{\mathscr{S}}(\mathscr{D})$ on $\mathscr{S}^{\mathscr{D}}$ similarly. We denote by $\mathrm{Ho}_{\mathscr{S}}(\mathscr{C})$ and $\mathrm{Ho}_{\mathscr{S}}(\mathscr{D})$ the homotopy category of $\mathscr{U}_{\mathscr{S}}(\mathscr{C})$ and $\mathscr{U}_{\mathscr{S}}(\mathscr{D})$, respectively. As in [1, Proposition 13.2], we have the derived adjunction of the Quillen adjunction $\mathrm{ind}_{\mathscr{D}}^{\mathscr{C}}\colon \mathscr{U}(\mathscr{D}) \rightleftarrows \mathscr{U}(\mathscr{C})\colon \mathrm{res}_{\mathscr{D}}^{\mathscr{C}}$, namely

$$L\mathrm{ind}_{\mathscr{D}}^{\mathscr{C}}\colon \mathrm{Ho}_{\mathscr{S}}(\mathscr{D}) \rightleftarrows \mathrm{Ho}_{\mathscr{S}}(\mathscr{C})\colon \mathrm{Res}_{\mathscr{D}}^{\mathscr{C}}.$$

For the sequel, fix a group G and one of the Isomorphism Conjectures (1–5); let $X_G \in \mathscr{S}^{\mathscr{C}}$ be the corresponding functor. Keep the other notations as above.

Theorem 1.1. *The following statements are equivalent*:

(1) *G satisfies the considered Isomorphism Conjecture*;
(2) *the corresponding functor $X_G \in \mathscr{U}_{\mathscr{S}}(\mathscr{C}, \mathscr{D})$ satisfies $\mathscr{D}$-codescent at $G/G \in \mathscr{C}$.*

Theorem 1.2. *For subgroups $L \leqslant H \leqslant G$, the following statements are equivalent*:

(1) $X_H \in \mathcal{U}_{\mathcal{S}}(\mathcal{C}_H, \mathcal{D}_H)$ *satisfies* $\mathcal{D}_H$*-codescent at* $H/L \in \mathcal{C}_H$;
(2) $X_G \in \mathcal{U}_{\mathcal{S}}(\mathcal{C}_G, \mathcal{D}_G)$ *satisfies* $\mathcal{D}_G$*-codescent at* $G/L \in \mathcal{C}_G$.

In fact, by general results of [1] (without invoking 1.1 above), if X_G satisfies $\mathcal{D}_G$-codescent, then X_H satisfies $\mathcal{D}_H$-codescent for every subgroup $H \leqslant G$.

Main Theorem. *The following statements are equivalent*:

(1) *every subgroup H of G satisfies the considered Isomorphism Conjecture*;
(2) *the corresponding functor* $X_G \in \mathcal{U}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$ *satisfies* $\mathcal{D}$*-codescent*;
(3) *up to isomorphism, the image of* X_G *in* $\mathrm{Ho}_{\mathcal{S}}(\mathcal{C})$ *belongs to* $\mathrm{Lind}_{\mathcal{D}}^{\mathcal{C}}(\mathrm{Ho}_{\mathcal{S}}(\mathcal{D}))$.

Note that the usual Baum–Connes and Bost Conjectures are stated with *finite* subgroups instead of virtually cyclic ones, but this is known to be equivalent. So, in these cases, we could as well set $\mathcal{D}_G \coloneqq \mathrm{Or}(G, \mathcal{F}in)$ instead of $\mathrm{Or}(G, \mathcal{VC})$.

Remark 1.3. Let $X \in \mathcal{S}^{\mathcal{C}}$ be a diagram and let $\zeta_X \colon \mathcal{Q}X \to X$ be an arbitrary *cofibrant approximation* of X in $\mathcal{U}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$, namely, ζ_X is merely a $\mathcal{D}$-weak equivalence and $\mathcal{Q}X$ is cofibrant in $\mathcal{U}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$. Then, X satisfies $\mathcal{D}$-codescent at some object $c \in \mathcal{C}$ if and only if $\zeta_X(c)$ is a weak equivalence in $\mathcal{S}$, see [1, Proposition 6.5]. This illustrates the flexibility of the codescent-type reformulation of the Isomorphism Conjectures, namely, every such cofibrant approximation of X_G yields a possibly very different assembly map that can be used to test the considered conjecture.

2. The proofs

Let $\mathcal{G}\mathrm{pds}^{\mathrm{f}}$ be the category of groupoids with *faithful* functors. For the considered conjecture, by [4,7], there exists a *homotopy functor* $\mathcal{X} \colon \mathcal{G}\mathrm{pds}^{\mathrm{f}} \to \mathcal{S}$, i.e. $\mathcal{X}$ takes equivalences of groupoids to weak equivalences, such that X_G is the composite

$$X_G \colon \quad \mathcal{C} = \mathrm{Or}(G) \xrightarrow{\imath} \mathcal{G}\mathrm{pds}^{\mathrm{f}} \xrightarrow{\mathcal{X}} \mathcal{S}.$$

The functor $\imath$ takes G/H to its *G-transport groupoid* $\overline{G/H}^G$ with the set G/H as objects and with $\{g \in G \mid gg_1H = g_2H\}$ as morphisms from g_1H to g_2H. Moreover, the functor $\mathcal{X}$ takes values in cofibrant spectra, so that X_G is $\mathcal{C}$-objectwise cofibrant.

Let $\mathcal{C}\mathrm{at}$ be the category of small categories and $\mathrm{s}\mathcal{S}\mathrm{ets}$ that of simplicial sets. Denote by $\otimes_{\mathcal{D}} \colon \mathrm{s}\mathcal{S}\mathrm{ets}^{\mathcal{D}^{\mathrm{op}}} \times \mathcal{S}^{\mathcal{D}} \to \mathcal{S}$ the tensor product over $\mathcal{D}$ induced by the simplicial model structure on $\mathcal{S}$, where $K \in \mathrm{s}\mathcal{S}\mathrm{ets}$ "acts" on $E \in \mathcal{S}$ by $|K|_+ \wedge E$.

Proof of Theorem 1.1. *A priori*, to test whether X_G satisfies $\mathcal{D}$-codescent at some $c \in \mathcal{C}$ requires a thorough understanding of the usually mysterious cofibrant *replacement* of X_G. A key point here is the freedom to use *any* cofibrant *approximation* instead, see Remark 1.3. We provide in [2, Section 6] a general construction of cofibrant approximations in $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$, one of which is exactly suited for our present purposes [2, Corollary 6.9]. Evaluated at the terminal object $G/G \in \mathcal{C}$, this cofibrant approximation $\zeta_{X_G}\colon \mathcal{Q}X_G \to X_G$ is a certain map (described at the end of the proof)

$$\zeta_{X_G}(G/G)\colon \quad \mathcal{Q}X_G(G/G) = B(?\searrow\mathcal{D})^{\mathrm{op}} \bigotimes_{?\in\mathcal{D}} \mathrm{res}^{\mathcal{C}}_{\mathcal{D}}\, X_G(?) \to X_G(G/G).$$

Indeed, using the notations of [2, Notation 6.1], this follows from the canonical identification $(?\searrow\mathcal{D} \overset{\mathcal{C}}{\searrow} G/G)^{\mathrm{op}} = (?\searrow\mathcal{D})^{\mathrm{op}}$ of diagrams in $\mathcal{C}\mathrm{at}^{\mathcal{D}^{\mathrm{op}}}$ and from the fact that X_G is $\mathcal{C}$-objectwise cofibrant. By definition of the homotopy colimit, we have

$$B(?\searrow\mathcal{D})^{\mathrm{op}} \bigotimes_{?\in\mathcal{D}} \mathrm{res}^{\mathcal{C}}_{\mathcal{D}}\, X_G(?) = \mathrm{hocolim}_{\mathcal{D}}\, \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X_G.$$

So, it suffices to show that $\zeta_{X_G}(G/G)$ coincides with the assembly map μ^G. In the notations of [2, Notation 5.1], we have $\underline{\mathrm{mor}_{\mathcal{D},\mathcal{C}}(?,G/G)} = *$ in $\mathrm{s}\mathcal{S}\mathrm{ets}^{\mathcal{D}^{\mathrm{op}}}$ (the constant diagram with value the point). By [2, Lemma 5.3], the spectrum $*\otimes_{\mathcal{D}}\mathrm{res}^{\mathcal{C}}_{\mathcal{D}}X_G$ identifies with $\mathrm{ind}^{\mathcal{C}}_{\mathcal{D}}\,\mathrm{res}^{\mathcal{C}}_{\mathcal{D}}X_G(G/G)$. Letting ε denote the counit of the adjunction $(\mathrm{ind}^{\mathcal{C}}_{\mathcal{D}}, \mathrm{res}^{\mathcal{C}}_{\mathcal{D}})$, it is routine to verify that there is a canonical commutative diagram

$$\begin{array}{ccccc}
\mathrm{hocolim}_{\mathcal{D}}\, \mathrm{res}^{\mathcal{C}}_{\mathcal{D}}\, X_G & & = & & B(?\searrow\mathcal{D})^{\mathrm{op}} \otimes_{\mathcal{D}} \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X_G(?) \\
\| & & & & \downarrow \\
\mathrm{hocolim}_{\mathcal{D}}\, \mathrm{res}^{\mathcal{C}}_{\mathcal{D}}\, X_G & \longrightarrow & \mathrm{colim}_{\mathcal{D}}\, \mathrm{res}^{\mathcal{C}}_{\mathcal{D}}\, X_G & \xrightarrow{\cong} & *\otimes_{\mathcal{D}} \mathrm{res}^{\mathcal{C}}_{\mathcal{D}}\, X_G \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle \varepsilon_{X_G}(G/G)} \\
\mathrm{hocolim}_{\mathcal{C}}\, X_G & \xrightarrow{\sim} & \mathrm{colim}_{\mathcal{C}}\, X_G & \xrightarrow{\cong} & X_G(G/G)
\end{array}$$

The composition of the first column followed by the last row is the assembly map μ^G. The composition in the last column is $\zeta_{X_G}(G/G)$, see [2, Corollary 6.9]. □

More generally, one can prove that the "$(X,\mathcal{F},G)$-Isomorphism Conjecture" of [4, Definition 5.1] is equivalent to X satisfying $\mathrm{Or}(G,\mathcal{F})$-codescent at G/G, for any objectwise cofibrant diagram $X\in\mathcal{S}^{\mathrm{Or}(G)}$ and any family $\mathcal{F}$ of subgroups of G.

For $g\in G$ and $H\leqslant G$, we write ${}^gH := gHg^{-1}$. In the orbit category $\mathrm{Or}(G) = \mathcal{C}_G$, for an element $g\in G$ such that ${}^gH\leqslant K$ for some subgroups H and K of G, we designate by the right coset Kg the morphism $G/H\to G/K$ taking $\tilde{g}H$ to $\tilde{g}g^{-1}K$.

Proof of Theorem 1.2. Consider the functor $\Phi\colon \mathcal{C}_H\to\mathcal{C}_G$ taking a coset $H/L\in\mathcal{C}_H$ to G/L. For any $L\leqslant H$, we have canonical equivalences of groupoids in $\mathcal{G}\mathrm{pds}^{\mathrm{f}}$

$$\overline{H/L}^H \xleftarrow{\sim} \bar{L} \xrightarrow{\sim} \overline{G/L}^G,$$

where $\bar{L}$ is L viewed as a one-object groupoid. Since $\mathscr{X}$ is a homotopy functor, one checks that there is a canonical zig-zag of two $\mathscr{C}_H$-weak equivalences between X_H and $\Phi^* X_G = X_G \circ \Phi$ in $\mathscr{U}_{\mathscr{S}}(\mathscr{C}_H)$. By weak invariance of codescent [1, Proposition 6.10], X_H and $\Phi^* X_G$ satisfy $\mathscr{D}_H$-codescent at exactly the same objects H/L of $\mathscr{C}_H$.

Fix an object $H/K \in \mathscr{D}_H$. Let $E_{H/K} \subset G$ be a set of representatives for the quotient $H \backslash \{g \in G \mid {}^g K \leqslant H\}$. Let $M\gamma$: $\Phi(H/K) = G/K \to G/M = \Phi(H/M)$ be a morphism in $\mathscr{C}_G$ with $M \leqslant H$ (and $\gamma \in G$). It is straightforward that there is a unique pair (g, Mh) with $g \in E_{H/K}$ and $Mh \in \mathrm{mor}_{\mathscr{C}_H}(H/{}^g K, H/M)$ (namely characterized by $Hg = H\gamma$ and $Mh = M\gamma g^{-1}$) such that $M\gamma$ decomposes in $\mathscr{C}_G$ as

$$G/K \xrightarrow{{}^g K g} G/{}^g K \xrightarrow{Mh} G/M, \qquad M\gamma = Mh \circ {}^g K g.$$

Since $\Phi(\mathscr{D}_H) \subset \mathscr{D}_G$, this precisely says that Φ is a *left glossy morphism of pairs of small categories* in the sense of [1, Definitions 7.3 and 8.1]. By left glossy invariance of codescent [1, Theorem 9.14], $\Phi^* X_G$ satisfies $\mathscr{D}_H$-codescent at some $H/L \in \mathscr{C}_H$ if and only if X_G satisfies $\mathscr{D}_1$-codescent at $G/L \in \mathscr{C}_G$, where $\mathscr{D}_1 := \Phi(\mathscr{D}_H)$. Set $\mathscr{D}_2 := \mathscr{D}_G$ and fix $H/L \in \mathscr{C}_H$. For $i = 1, 2$, consider the full subcategory $\mathscr{E}_i$ of $\mathscr{D}_i$ given by

$$\mathscr{E}_i := \{G/K \in \mathscr{D}_i \mid \mathrm{mor}_{\mathscr{C}_G}(G/K, G/L) \neq \emptyset\}.$$

By the Pruning Lemma [1, Theorem 11.5], X_G satisfies $\mathscr{D}_1$-codescent at G/L if and only if it satisfies $\mathscr{E}_1$-codescent at G/L. Since $L \leqslant H$, every object of $\mathscr{E}_1$ is isomorphic, inside $\mathscr{C}_G$, to some object of $\mathscr{E}_2$ and conversely; in other words, $\mathscr{E}_1$ and $\mathscr{E}_2$ are *essentially equivalent* in $\mathscr{C}_G$, in the sense of [1, Definition 3.12]. So, by [1, Proposition 10.1], X_G satisfies $\mathscr{E}_1$-codescent at G/L if and only if it satisfies $\mathscr{E}_2$-codescent at G/L. By the Pruning Lemma again, X_G satisfies $\mathscr{E}_2$-codescent at G/L if and only if it satisfies $\mathscr{D}_2$-codescent at G/L, i.e. $\mathscr{D}_G$-codescent at G/L.

In total, we have proven that X_H satisfies $\mathscr{D}_H$-codescent at an object $H/L \in \mathscr{C}_H$ if and only if X_G satisfies $\mathscr{D}_G$-codescent at G/L, as was to be shown. □

Proof of the Main Theorem. The equivalence between (1) and (2) follows from Theorems 1.1 and 1.2; (2) and (3) are equivalent by [1, Theorem 13.5]. □

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