ON WYLER'S TAUT LIFT THEOREM

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This paper proves Wyler's famous "Taut Lift Theorem" in a general categorical context and makes it applicable not only in categorical topology. Some well-known and important categorical results appear as corollaries.

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1. Introduction

One of the most important results in categorical topology (see Brümm [1], Herrlich [3], Hoffmann [7], Shukla [15], Wischnewsky [20] and Wyler [23, 24] and other references there) is Wyler's theorem on taut lifts of adjoint functors along top categories: Given a commutative square of functors

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{g} & \mathcal{B} \\
\downarrow r & & \downarrow q \\
\mathcal{A}' & \xrightarrow{g'} & \mathcal{B}'
\end{array}
\]

where \( P \) and \( Q \) are top categories and \( G \) sends \( P \)-initial cones to \( Q \)-initial cones ("\( G \) is a taut lift of \( G' \)"), then a left adjoint functor \( F' \) of \( G' \) can be lifted to a left adjoint functor \( F \) of \( G \), such that \( F \) is co-taut over \( F' \). Moreover the initial continuity of \( G \) is a necessary condition for this. One finds the nicest application of Wyler's theorem in topological algebra, namely the proof of the existence of free universal algebras over topological, uniform or limit spaces etc. (see [16], Wischnewsky [21, 22], Wyler [23] and other references there).

The purpose of this paper is to show, that Wyler's theorem is valid in a much more general context: The necessity of the condition, that \( G \) preserves initial
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cones, can be proved for arbitrary $P$ and $Q$, and if the condition of initial continuity is fulfilled, right adjointness of $G'$ implies right adjointness of $G$ for any functor $Q$, provided $P$ is an "$M$-functor". In general, however, the left adjoint functor $F$ is co-taut over $F'$ (up to a natural equivalence) only, if $P$ is absolutely topological (in the sense of Herrlich [3]).

"$M$-functors" admit certain factorizations of cones (cp. [18]) and generalize Herrlich's concept of $(E, M)$-topological functors (cp. [3]), which, for their part, generalize the older notions of topological functors. $M$-functors arise not only in topology but also in algebra; every monadic functor over $Sets$ is an $M$-functor. They permit a general categorical treatment of "nice concrete categories" and are investigated in more detail in [19]. So the use of the following generalization of Wyler's theorem is by no means restricted to categorical topology. By special choice of the functors in (*) one gets some general categorical results. For instance, the "Adjoint Functor Theorem" for adjoint triangles (see [17]) and the "Colimit Theorem" (see [17] and Manes [10]) are corollaries of the theorem presented here.

2. Functors preserving initiality arise naturally

For any functor $P : \mathcal{A} \to \mathcal{A}'$ a cone $\alpha : \Delta A \to D$ in $\mathcal{A}$ (with $A \in \text{Ob} \mathcal{A}$, $D : \mathcal{B} \to \mathcal{A}$, $\mathcal{B}$ may be empty or large) is called $P$-initial, iff for every cone $\beta : \Delta B \to D$ in $\mathcal{A}$ and every morphism $a' : PB \to PA$ in $\mathcal{A}'$ with $(P \circ a)(\Delta a') = P \circ \beta$ there is precisely one $a : B \to A$ in $\mathcal{A}$ with $Pa = a'$ and $\alpha(\Delta a) = \beta$.

Let us now assume that there is given a commutative square (*) and that $G, G'$ have left adjoints $F, F'$ with units $\eta, \eta'$ respectively. Then there exists a natural comparison transformation

$$\kappa : F' \circ Q \to P \circ F$$

which is unique determined by the equation

$$(G' \circ \kappa)(\eta' \circ Q) = Q \circ \eta;$$

Explicitly one has

$$\kappa = (\varepsilon' \circ P \circ F)(F' \circ Q \circ \eta),$$

where $\varepsilon'$ is the co-unit of $G'$.

**Proposition 2.1** (cp. [16, Proposition 14.1]). *If the natural comparison transformation is an isomorphism, then $G$ sends $P$-initial cones to $Q$-initial cones.*
Proof. Let $\alpha : \Delta A \to D$ be $P$-initial and let $\xi : \Delta X \to G \circ D$ be any cone in $\mathcal{A}$ and $x' : QX \to QGA$ be any morphism in $\mathcal{A}'$ rendering the following diagram commutative:

\[
\begin{array}{ccc}
\Delta G'PA & G' \circ P \circ D \\
\downarrow \alpha_{\circ P\circ D} & \downarrow \\
\Delta QGA & Q \circ G \circ D \\
\downarrow Q\xi & \downarrow \\
\Delta QX & \\
\end{array}
\]

$\xi$ and $x'$ determine a cone $\beta : \Delta FX \to D$ in $\mathcal{A}$ and a morphism $a' : F'OX \to PA$ in $\mathcal{A}'$:

\[
\begin{array}{ccc}
\Delta X & \Delta GFX \\
\downarrow \xi & \downarrow G\beta \\
G \circ D & QX & G'F'QX \\
\downarrow x' & \downarrow \alpha a' \\
G'PA & \\
\end{array}
\]

Using the $P$-initiality of $\alpha$ we get a unique morphism $a' : FX \to A$, such that $(Pa)(\kappa X) = a'$ and $\alpha(\Delta a) = \beta$:

\[
\begin{array}{ccc}
A & \Delta PA & P \circ D \\
\downarrow \alpha & \downarrow \Delta a' & \downarrow r_{P} \\
\Delta F'QX & \Delta PFX \\
\downarrow \Delta(\kappa X)^{-1} & \\
FX & \\
\end{array}
\]

Now one easily shows that $x : = (Ga)(\eta X) : X \to GA$ is the desired morphism in $\mathcal{A}$ with $Qx = x'$ and $(G \circ \alpha)(\Delta x) = \xi$.

Taking the identical functor for $G'$ we get the following corollary, which was already proved by Pumplün [12]:

**Corollary 2.2.** Let

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\eta} & \mathcal{A}' \\
\downarrow r & \downarrow q \\
\mathcal{B} & \\
\end{array}
\]

be a commutative triangle of functors, where $G$ has a left adjoint with unit $\eta$, such that $Q \circ \eta$ is an isomorphism. Then $G$ sends $P$-initial cones to $Q$-initial cones.

$Q \circ \eta$ is an isomorphism, if for instance either the left adjoint of $G$ is full and
faithful (i.e. \( \eta \) is an isomorphism) or if \( \mathcal{A}' \) is the one point category \( \mathcal{1} \); in this case \( P \)-initial cones are just limit cones (cp. Hoffman [7]):

**Corollary 2.3.** (1) If in (**), \( G \) has a full and faithful left adjoint, then \( G \) sends \( P \)-initial cones to \( Q \)-initial cones.

(2) Every right adjoint functor preserves limits.

3. \( M \)-functors

Let \( P : \mathcal{A} \to \mathcal{A}' \) be a functor and let \( M \) be a class of cones in \( \mathcal{A} \), which is closed under composition with isomorphisms from the right, i.e. if \( \mu : \Delta A \to D \) is in \( M \) and \( i : B \to A \) is an isomorphism, then \( \mu (\Delta i) \) is in \( M \).

**Definition 3.1.** \( P \) is called an \( M \)-functor, if all cones in \( M \) are \( P \)-initial and if for all cones \( \alpha ' : \Delta A' \to P \circ D \) in \( \mathcal{A}' \) with \( A' \in \text{Ob} \; \mathcal{A}' \), \( D : \mathcal{B} \to \mathcal{A} \) (\( \mathcal{B} \) any category) there is an object \( A \) in \( \mathcal{A} \), a \( P \)-epimorphism \( e' : A' \to PA \) in \( \mathcal{A}' \) (i.e. \( a, b : A \to B \), \( (Pa)e' = (Pb)e' \) implies \( a = b \)) and a cone \( \mu : \Delta A \to D \) in \( M \), such that the diagram

\[
\begin{array}{ccc}
\Delta PA & \xrightarrow{P} & P \circ D \\
\Delta A & \xrightarrow{\Delta \alpha'} & P \circ D
\end{array}
\]

commutes. It moreover \( e' \) is always orthogonal to all \( M \)-cones \( \nu : \Delta B \to \tilde{D} \), \( \tilde{D} : \mathcal{B} \to \mathcal{A} \), i.e. for all commutative squares

\[
\begin{array}{ccc}
\Delta A & \xrightarrow{\Delta \nu} & \Delta PA \\
\Delta B & \xrightarrow{\Delta \beta} & P \circ \tilde{D}
\end{array}
\]

with \( b' : A' \to PB \) and \( \beta : \Delta A \to \tilde{D} \) the Isbell condition is fulfilled, i.e. there is a "diagonal" \( d : A \to B \) such that \( (Pd)e' = b' \) (and hence \( \nu (\Delta d) = \beta \)) holds, we call \( P \) an \( M \)-orthogonal functor. \( \mathcal{A} \) is called an \( M \)-(orthogonal) category, iff the identical functor on \( \mathcal{A} \) is an \( M \)-(orthogonal) functor.

**Examples 3.2.** (1) If \( P : \mathcal{A} \to \mathcal{A}' \) is \((\mathcal{C}', M')\)-topological in the sense of Herrlich [3], then \( P \) is \( M \)-orthogonal, where \( M \) denotes the class of all \( P \)-initial cones \( \mu \) in \( \mathcal{A} \) with \( P \circ \mu \) in \( M' \). (Note that we can use cones instead of sources, since an \((\mathcal{C}', M')\)-topological functor is faithful; it's only a question of categorical taste whether to use cones or sources.)

Especially, every absolutely topological functor (cp. [3]) and so every top category \( P \) is \( M \)-orthogonal, where \( M \) denotes the class of all \( P \)-initial cones.
Because of these examples it could seem adequate to speak of "weakly $M$-topological functors" instead of "$M$-(orthogonal) functors". The reason why we do not do this is simply this: These functors arise in algebra as well as in topology, as we shall see now.

(2) If $P : \mathcal{A} \rightarrow \mathcal{A}'$ is a regular functor over a regular category $\mathcal{A}'$ in the sense of Herrlich [4], then $P$ is $M$-orthogonal, where $M$ denotes the class of all mono-cones of $\mathcal{A}$. Especially, the underlying functor of any full replete regular epi-reflective subcategory of a monadic category over $Sets$ is $M$-orthogonal.

(3) If $P : \mathcal{A} \rightarrow \mathcal{A}'$ is an inclusion functor of a full reflective subcategory, then $P$ is $M$-orthogonal, where $M$ denotes the class of all cones in $\mathcal{A}$ (every cone in $\mathcal{A}$ is $P$-initial!). Of course, it is even possible to take any full and faithful right adjoint functor.

Remark 3.3. In [18] functors admitting factorizations of cones for a fixed diagram base $\mathcal{D}$ are investigated. Some of the results there yield also interesting results in the present context. If $P : \mathcal{A} \rightarrow \mathcal{A}'$ is an $M$-functor (not necessarily $M$-orthogonal), then we have for instance:

(1) If $\mathcal{A}'$ is $\mathcal{D}$-complete, then so is $\mathcal{A}$ (and $P$ is $\mathcal{D}$-continuous; cp. [18, Theorem (16)]).

(2) If $\mathcal{A}'$ is $\mathcal{D}$-co-complete, then so is $\mathcal{A}$ (cp. Corollary 5.1).

(3) $P$ has a left adjoint functor (cp. Corollary 5.2).

(4) $P$ is faithful. This can be proved in the same way as for $(\mathcal{C}', M')$-topological functors in [3].

Further investigations will appear in [19]. There the following characterization of $M$-(orthogonal) functors (which generalizes Theorem 7.1 of [3]) is proved:

Let $M'$ be a class of cones in $\mathcal{A}'$ and $M$ denote the class of all $P$-initial cones $\mu$ with $P \circ \mu$ in $M'$.

Then the following assertions are equivalent:

(i) $P$ is an $M$-(orthogonal) functor.

(ii) $P$ has a left adjoint and $\mathcal{A}$ is an $M$-(orthogonal) category.

(iii) $P$ has a left adjoint and there exists a class $N$ of cones in $\mathcal{A}$, such that

(1) $\mathcal{A}$ is an $N$-(orthogonal) category,

(2) $P$ sends $N$-cones to $M'$-cones,

(3) for all $A \in \text{Ob } \mathcal{A}$ the co-unit morphism $eA$ is orthogonal to all $N$-cones.

$M$-functors can be easily constructed using more classical categorical notions as the following lemma shows, which is also proved in [19]. This enables us to derive many well-known categorical results from the theory of $M$-functors.

Lemma 3.4. Let $P : \mathcal{A} \rightarrow \mathcal{A}'$ be faithful and right adjoint and let $\mathcal{A}$ have (small) products. If there are classes $\mathcal{E}$ of epimorphisms and $M$ of $P$-initial morphisms, such that every $f$ in $A$ has a factorization $f = me$, $e \in \mathcal{E}$, $m \in M$ and if $\mathcal{A}$ is $\mathcal{E}$-co-well-powered, then $P$ is an $M$-functor for a suitable class $\widetilde{M}$ of $P$-initial cones.
Proof. Since $P$ has a left adjoint and is faithful it obviously suffices to show, that every source, i.e. discrete cone $(a_i: A_i \to A_i)$, in $\mathcal{A}$ has a factorization $a_i = m_i e$, where $e$ is an epimorphism and $(m_i: B \to A_i)_i$ is $P$-initial. Let us assume at first, that $I$ is a set.

Then one gets the desired factorization from an $(\mathcal{E}, \mathcal{M})$-factorization of the induced morphism

$$a: A \to \prod_i A_i.$$ 

If $I$ is a proper class, one has to factorize at first all $a_i = m_i e$, and then to take a representative set $e_i, i \in j \subset I$, which can be factorized in the way described above.

4. A general lifting theorem for left adjoint functors

Theorem 4.1. Let $(\ast)$ be a commutative square of functors, such that the following conditions are satisfied:

(a) $P$ is an $\mathcal{M}$-functor for some class $\mathcal{M}$ of $P$-initial cones,
(b) $G'$ sends all $\mathcal{M}$-cones to $Q$-initial cones,
(c) $G'$ has a left adjoint functor.

Then $G$ has a left adjoint functor.

Proof. Let $X$ be any fixed object in $\mathcal{A}$ and let $\mathcal{D}$ be the comma-category $(X, G)$, whose objects are pairs $(x, B)$ with $B \in \text{Ob } \mathcal{A}$ and $x: X \to GB$ in $\mathcal{A}$ and whose morphisms $f: (x, B) \to (y, C)$ are induced by morphisms $f: B \to C$ in $\mathcal{A}$ with $(Gf)x = y$. Then there is a canonical functor $D: \mathcal{D} \to \mathcal{A}$ and a cone

$$\xi: \Delta X \to G \circ D.$$ 

Since $G'$ has a left adjoint, $\xi$ induces a cone $\alpha'$:

$$\Delta QX \xrightarrow{\Delta^t} \Delta G'F'QX \xrightarrow{\Delta F'QX} \Delta F'QX$$

$$\phi \downarrow \quad \phi \downarrow \quad \phi \downarrow$$

$$Q \circ G \circ D = G' \circ P \circ D \quad P \circ D$$

$\alpha'$ has an $\mathcal{M}$-factorization:

$$\Delta PA \xrightarrow{\Delta^t \Delta F'QX} P \circ D$$

Because $G \circ P$ is $Q$-initial, there is a unique morphism $e: X \to G \circ P$ with
\( Q e = (G'e')(\eta'QX) \) and \( (G \circ \mu)(\Delta e) = \xi \):

\[
\begin{array}{ccc}
GA & \rightarrow & Q \circ G \circ D \\
\Delta QG & \downarrow & \Delta QG \\
\Delta G'\eta'QX & \rightarrow & \Delta QX \\
\end{array}
\]

Now for all \( x : X \rightarrow GB \) we have a factorization \( x = (Ga)e \) with \( a := \mu(x, B) : A \rightarrow B \), and for every \( b : A \rightarrow B \) with \( x = (Gb)e \) one gets

\[ (G'(Pa)e')(\eta'QX) = (G'(Pb)e')(\eta'QX) \]

and hence \( a = b \), since \( \eta'QX \) is \( G' \)-epimorphic and \( e' \) is \( P \)-epimorphic.

**Remark 4.2.** The morphism \( e' : F'QX \rightarrow PFX \) for all \( X \) is just the comparison morphism \( \kappa X \). If \( P \) is in fact \( (\mathcal{E}', \mathcal{M}') \)-topological, \( e' = \kappa X \) must be an \( \mathcal{E}' \)-morphism, and if \( P \) even is absolutely topological, \( \kappa X \) is an isomorphism.

From this observation, together with Proposition 2.1, one gets immediately:

**Corollary 4.3.** (Taut Lift Theorem). Let \((\ast)\) be a commutative square of functors, such that \( P \) is absolutely topological and \( G' \) has a left adjoint. Then the following two assertions are equivalent:

(i) \( G \) has a left adjoint and the natural comparison transformation is an isomorphism.

(ii) \( G \) sends \( P \)-initial cones to \( Q \)-initial cones.

The corollary shows, that in Wyler's theorem the condition, that \( Q \) also is absolutely topological, is superfluous. Note that in this case Brümmer and Hoffmann [2] have given an interesting external characterization of condition (ii).

Theorem 4.1 and Corollary 4.3 can be applied especially to the situation \((\ast\ast)\), where \( G' \) is the identical functor. But we omit here the formulation of the corresponding corollaries. At last we only show, that the "Adjoint Functor Theorem" for adjoint triangles, which is proved in [17] is also an immediate consequence of Theorem 4.1. For that, let \( P \) in \((\ast)\) be the identity and consider the commutative triangle

\[
\begin{array}{ccc}
\mathcal{A} & \rightarrow & \mathcal{B} \\
\sigma \downarrow & \downarrow & \sigma \\
\mathcal{B}' & \rightarrow & \mathcal{B}'
\end{array}
\]
Corollary 4.4 (Sandwich Theorem). Assume that for some class \( M \) of cones \( \mathcal{A} \) is an \( M \)-category and that \( G \) sends all \( M \)-cones to \( Q \)-initial cones. Then \( G \) has a left adjoint, if \( G' \) has.

The exact formulation in [17] is obtained from Corollary 4.4 by applying Lemma 3.4 with \( P \) the identity.

5. Applications to \( M \)-functors and their initial completions

Some properties of \( M \)-functors and their initial completions are consequences of Theorem 4.1. First we prove, that \( M \)-functors lift the existence of colimits (cp. Remark 3.3(2)):

Corollary 5.1 (Colimit Theorem). Let \( P : \mathcal{A} \to \mathcal{A}' \) be an \( M \)-functor and let \( \mathcal{A}' \) be \( D \)-co-complete (for some small \( D \)). Then \( \mathcal{A} \) is \( D \)-co-complete.

Proof. Looking at the diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\Delta} & [D, \mathcal{A}] \\
\downarrow{P} & & \downarrow{[P, P]} \\
\mathcal{A}' & \xrightarrow{\Delta} & [D, \mathcal{A}']
\end{array}
\]

one sees at once, that the canonical functor \( \Delta \) from \( \mathcal{A} \) into the functor category \([D, \mathcal{A}]\) sends \( P \)-initial cones to \([D, P] \)-initial cones.

Together with Lemma 3.4, one gets from Corollary 5.1 exactly the general lifting theorem for existence of colimits formulated in [13] and [17]. A similar result has been proved by Manes [10, p. 274].

Corollary 5.2. \( M \)-functors are right adjoint.

Proof. Apply Theorem 4.1 to the square

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{P} & \mathcal{A}' \\
\downarrow{P} & & \downarrow{=} \\
\mathcal{A}' & = & \mathcal{A}'.
\end{array}
\]

For small concrete categories \((\mathcal{A}, U)\) Herrlich constructs in [5] initial completions \( E^n : (\mathcal{A}, U) \to (\mathcal{A}_n, U^n) \), \( n = \pm 1, \pm 2, \pm 3, \pm 4 \), i.e. commutative triangles of functors.
where $E^n$ is a full embedding and $U^n$ is absolutely topological. He gives universal characterizations for these extensions (see also Porst [11]). For $n = -1, -2, \pm 3, \pm 4$, $E^n$ sends $U$-initial cones to $U^n$-initial cones. So applying Theorem 4.1 we get:

**Corollary 5.3.** If $U: \mathcal{A} \to \mathcal{K}$ is an $M$-functor, $\mathcal{A}$ is a full reflective subcategory of its initial completion $\mathcal{A}^n$ for $n = -1, -2, \pm 3, \pm 4$ (i.e. the completion exists).

**Added in proof.** For $M$ being the class of all initial cones Y.H. Hong has introduced the notion of an $M$-functor earlier using the name "topologically algebraic functor" (Studies on Categories of Universal Algebras, thesis, McMaster University, Hamilton 1974). In this case it can be shown that $M$-functor is equal to $M$-orthogonal functor. A proof will appear elsewhere.

**References**


