Examples of non-Kähler Hamiltonian circle manifolds with the strong Lefschetz property

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Abstract

In this paper we construct six-dimensional compact non-Kähler Hamiltonian circle manifolds which satisfy the strong Lefschetz property themselves but nevertheless have a non-Lefschetz symplectic quotient. This provides the first known counterexamples to the question whether the strong Lefschetz property descends to the symplectic quotient. We also give examples of Hamiltonian strong Lefschetz circle manifolds which have a non-Lefschetz fixed point submanifold. In addition, we establish a sufficient and necessary condition for a finitely presentable group to be the fundamental group of a strong Lefschetz manifold. We then use it to show the existence of Lefschetz four-manifolds with non-Lefschetz finite covering spaces.
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1. Introduction

Brylinski defined in [2] the notion of symplectic harmonic forms. He further conjectured that on a compact symplectic manifold every cohomology class has a harmonic representative and proved that this is the case for compact Kähler manifolds and certain other examples.

A symplectic manifold (M, ω) of dimension $2m$ is said to have the strong Lefschetz property or equivalently to be a strong Lefschetz manifold if and only if for any $0 \leqslant k \leqslant m$, the Lefschetz type map

$$L_{[\omega]}^k : H^{m-k}(M) \to H^{m+k}(M), \quad [\alpha] \to \left[\alpha \wedge \omega^k\right] \tag{1}$$

is onto. Mathieu [13] proved the remarkable theorem that Brylinski conjecture is true for a symplectic manifold (M, ω) if and only if it has the strong Lefschetz property. This result was strengthened by Merkulov [16] and Guillemin [6], who independently established the symplectic d, δ-lemma for compact symplectic manifolds with the strong Lefschetz property. As a consequence of the symplectic d, δ-lemma, they showed that strong Lefschetz manifolds are formal in a certain sense.

We obtained an equivariant version of the above results jointly with Sjamaar in [12]. In particular, it was proved in [12] that for a compact Hamiltonian G-manifold with the strong Lefschetz property every cohomology class has a canonical equivariant extension. In a subsequent paper [11] the author extended the main results in [12] to equivariant differential forms with generalized coefficients on Hamiltonian manifolds with the strong Lefschetz property.

Kaoru Ono and Reyer Sjamaar raised the question whether the strong Lefschetz property descends to the symplectic quotient. Obviously, this question has an affirmative answer in the category of equivariant Kähler geometry. It is then a very natural question to ask whether this is still the case in the symplectic category.

The main result of this paper is first known counterexamples which show, in contrast with the equivariant Kähler case, that the strong Lefschetz property does not survive symplectic reduction in general. The difficulty constructing such examples comes largely from the lack of general examples of non-Kähler Hamiltonian symplectic manifolds which have the strong Lefschetz property. Historically, a lot of examples of non-Kähler symplectic manifolds have been constructed. However, as the strong Lefschetz property is commonly used as a tool to detect the existence of Kähler structure, not many known examples of non-Kähler symplectic manifolds have the strong Lefschetz property.

By Mathieu's theorem [13] for a symplectic manifold with the strong Lefschetz property the symplectic harmonic groups always coincide with the de Rham cohomology groups. It is noteworthy that Dong [4] showed that there exist compact symplectic four-manifolds which admit a family ω_t of symplectic forms such that the dimension of the third symplectic harmonic group varies. Dong's construction depends heavily on the following result in Gompf's path-breaking paper [5].

Theorem 1.1.[1] [5] *Let G be any finitely presentable group. Then there is a closed, symplectic four-manifold (M, ω) such that*

(1) $\pi_1(M) = G$,
(2) *the Lefschetz map $L_{[\omega]} : H^1(M) \to H^3(M)$ is trivial.*

Our construction of counterexamples is inspired by the above-mentioned work of Dong [4], and by Karshon's example [9] of a Hamiltonian circle six-manifold with a non-log concave Duistermaat–Heckman function, which in turn is a piece of a manifold constructed by Mcduff [14]. Let us give a brief account of the main ideas of our construction here. First, we show that any finitely presentable group G with a certain structure can be realized as the fundamental group of a four-manifold N which supports a family of symplectic forms $\omega_t, t \in \mathbb{R}$, such that (N, ω_0) does not have the strong Lefschetz property. Second, we prove that for such a manifold N there exists a six-dimensional strong Lefschetz compact Hamiltonian S^1-manifold

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[1] The first assertion of Theorem 1.1 is contained in the statement of Theorem 4.1 of [5]; the second assertion follows from the discussion following the proof of Observation 7.4 in the same paper.

M which is fibred over N with the fibre S^2; furthermore, the symplectic quotient of M taken at a certain value will be exactly N with the reduced form ω_0. As G varies, we actually obtain infinitely many topologically inequivalent six-dimensional compact Hamiltonian S^1-manifolds, each of which has the strong Lefschetz property itself but nevertheless admits a non-Lefschetz symplectic quotient. This also gives us new examples of compact non-Kähler Hamiltonian manifolds. (Cf. [10,18].)

The same ideas also allow us to construct Hamiltonian strong Lefschetz manifolds which have non-Lefschetz fixed point submanifold. For a compact Hamiltonian manifold, one interesting question is what the relationship is between the symplectic harmonic theory of the manifold itself and that of its fixed point submanifold. For instance it remains an open question whether a compact Hamiltonian circle manifold with isolated fixed points has to satisfy the strong Lefschetz property. And one may further ask whether the strong Lefschetz property for a Hamiltonian circle manifold and its fixed point submanifolds will imply each other. Our examples give a negative answer to the latter question.

As an aside, we give a sufficient and necessary condition for a finitely presentable group to be the fundamental group of a compact symplectic four-manifold with the strong Lefschetz property. It suggests that the fundamental groups of strong Lefschetz manifolds and that of Kähler manifolds may have quite different behavior. In fact, it enables us to construct examples of compact strong Lefschetz manifolds which have non-Lefschetz finite covering spaces.

It is an important question to which extent the symplectic manifolds are more general than Kähler manifolds. The examples constructed in this paper show clearly that the category of strong Lefschetz manifolds with Hamiltonian circle actions is much larger than the category of Kähler manifolds with compatible Hamiltonian circle actions.

This paper is organized as follows. Section 2 modifies Dong Yan's methods [4] to prove the existence of the symplectic four-manifolds with certain properties we want. Section 3 records a sufficient and necessary condition for a finitely presentable group G to be the fundamental group of a compact strong Lefschetz four-manifold. As an immediate application of this observation, Section 3 also gives us examples of strong Lefschetz manifolds with non-Lefschetz finite covering spaces. Section 4 shows how to construct compact Hamiltonian strong Lefschetz circle manifolds with a non-Lefschetz symplectic quotient. In addition, Section 4 also explains how to obtain examples of Hamiltonian strong Lefschetz circle manifolds with a non-Lefschetz fixed point submanifold.

2. Symplectic four-manifolds with certain properties

In this section, we establish the existence of symplectic four-manifolds with certain properties which we need in Section 4 for our construction of counterexamples. This is stated precisely in Proposition 2.3, which has appeared in different guises in [4] and [5] and depends on an idea of Johnson and Rees [8].

Definition 2.1. Let G be a discrete group. A non-degenerate skew structure on G is a non-degenerate skew bilinear form

$$\langle\,,\,\rangle : H^1(G, \mathbb{R}) \times H^1(G, \mathbb{R}) \to \mathbb{R}$$

which factors through the cup product, that is, there exists a linear functional $\sigma : H^2(G, \mathbb{R}) \to \mathbb{R}$ so that $\langle a, b \rangle = \sigma(a \cup b)$, for all $a, b \in H^1(G, \mathbb{R})$.

A finitely presentable group G is called a Kähler group if it is the fundamental group of a closed Kähler manifold; otherwise it is a non-Kähler group. It was proved in [8] that any Kähler group and any of its finite index subgroups must admit a non-degenerate skew structure.

Lemma 2.2. *Let (N, ω) be a closed, symplectic four-manifold so that $\pi_1(N)$ is a finitely presentable group which admits a non-degenerate skew structure. Then there exists an integral class c such that the map $L_c: H^1(N) \to H^3(N)$ is an isomorphism.*

Proof. By elementary homotopy theory there is a natural map $f: N \to K(G, 1)$ such that the induced homomorphism

$$f^*: H^*(G, \mathbb{R}) \to H^*(N, \mathbb{R})$$

is an isomorphism in dimension 1 and injective in dimension 2. Let $\langle , \rangle$ be a non-degenerate skew structure on G and σ be the corresponding functional on $H^2(G, \mathbb{R})$. Since $H^2(G, \mathbb{R})$ is a subspace of $H^2(N, \mathbb{R})$, σ extends to a functional $\tilde{\sigma}$ on $H^2(N, \mathbb{R})$. By Poincaré duality, there exists a class c such that

$$\tilde{\sigma}(a) = \bigl(a \wedge c, [N]\bigr),$$

where $a \in H^2(N, \mathbb{R})$ and $[N]$ is the fundamental class of N. Suppose $x \in H^1(N, \mathbb{R})$ such that $L_c(x) = x \wedge c = 0 \in H^3(N, \mathbb{R})$. Then for any $y \in H^1(N, \mathbb{R})$ we have $\tilde{\sigma}(y \wedge x) = ((y \wedge x) \wedge c, [N]) = (y \wedge (x \wedge c), [N]) = 0$. Note $\tilde{\sigma}(y \wedge x) = \sigma(y \wedge x) = \langle y, x \rangle$ we conclude that $\langle y, x \rangle = 0$ for any $y \in H^1(N, R)$. It then follows from the non-degeneracy of $\langle , \rangle$ that $x = 0$. This shows that L_c is injective. Then by Poincaré duality L_c must be an isomorphism indeed. Finally note that the set

$$\bigl\{\alpha \in H^2(N) \mid L_\alpha: H^1(N) \to H^3(N) \text{ is an isomorphism}\bigr\}$$

is an open subset of $H^2(N)$. Without the loss of generality, we may assume that the class c we obtained above is rational. Replace c by nc for some sufficiently large integer n if necessary, we get an integral class c such that the map $L_c: H^1(N) \to H^3(N)$ is an isomorphism. $\quad\square$

Combining Lemma 2.2 and Theorem 1.1, we get the existence of symplectic four-manifolds with the desired properties as stated in the following proposition.

Proposition 2.3. *Let G be a finitely presentable group which admits a non-degenerate skew structure. Then there is a closed, symplectic four-manifold (N, ω) with $\pi_1(N) = G$ such that the following two conditions are satisfied:*

(1) *the Lefschetz map $L_{[\omega]}: H^1(N) \to H^3(N)$ is identically zero;*
(2) *there exists an integral class $c \in H^2(N)$ such that the map $L_c: H^1(N) \to H^3(N)$ is an isomorphism.*

3. A remark on the fundamental groups of strong Lefschetz four-manifolds

As an application of Proposition 2.3, we record in this section an interesting observation on the fundamental groups of strong Lefschetz four-manifolds.

Using the hard Lefschetz theorem, Johnson and Rees proved in [8] that if a finitely presentable group G is the fundamental group of a compact Kähler manifold, then G has to admit a non-degenerate skew structure. We note that the fundamental groups of strong Lefschetz manifolds also have to admit a non-degenerate skew structure, and Johnson and Rees's argument applies verbatim to our situation. On the other hand, if G is a finitely presentable group which supports a non-degenerate skew structure, then by Proposition 2.3 there exists a compact symplectic four-manifold (N, ω_0) and a closed two-form c on N such that the Lefschetz map $L_{[\omega_0]} : H^1(N) \to H^3(N)$ is identically zero and such that the map $L_c : H^1(N) \to H^3(N)$ is an isomorphism. For a sufficiently small constant $\epsilon > 0$, set $\omega' = \omega_0 + \epsilon c$. It is easy to see that ω' is symplectic and satisfies the strong Lefschetz property. In summary we have the following result.

Theorem 3.1. *Suppose G is a finitely presentable group. Then the following statements are equivalent*:

 (i) *G admits a non-degenerate skew structure.*
(ii) *G can be realized as the fundamental group of a compact strong Lefschetz four-manifold.*

Theorem 3.1 raises a natural question whether there exist finitely presentable non-Kähler groups which support a non-degenerate skew structure. This question is answered affirmatively in Lemma 3.3. However, to prove this lemma we will need a non-trivial fact concerning non-Kähler groups which is due to Johnson and Rees.

Theorem 3.2. [8] *Let G_1, G_2 be groups which both have at least one non-trivial finite quotient, and let H be any group. Assume that $G = (G_1 * G_2) \times H$ admits a non-degenerate skew structure. Then G has a subgroup of finite index which does not support any non-degenerate skew structure, and consequently is not a Kähler group itself.*

Lemma 3.3. *For any positive composite numbers m, n, the group $G_{m,n} = (\mathbb{Z}_m * \mathbb{Z}_n) \times (\mathbb{Z} \times \mathbb{Z})$ admits a non-degenerate skew structure; furthermore, $G_{m,n}$ has a subgroup of finite index which does not admit any non-degenerate skew structure, and therefore is not a Kähler group.*

Proof. Since m, n are composite numbers, both $\mathbb{Z}_m$ and $\mathbb{Z}_n$ have non-trivial finite quotient. It follows from Theorem 3.2 that the group $G_{m,n}$ has a subgroup of finite index which does not support any non-degenerate skew structure. Note that by Corollary 6.2.10 and Exercise 6.2.5 of [19], $H^i(\mathbb{Z}_m * \mathbb{Z}_n, \mathbb{R}) = H^i(\mathbb{Z}_n, \mathbb{R}) \oplus H^i(\mathbb{Z}_m, \mathbb{R}) = 0$ for $i \geqslant 1$. Then it follows from the Künneth formula in group cohomology (see for instance Exercise 6.1.10 of [19]) that $H^i(G_{m,n}, \mathbb{R}) = H^i(\mathbb{Z} \times \mathbb{Z}, \mathbb{R})$ for $i \geqslant 1$. Since $(\mathbb{Z} \times \mathbb{Z})$ is a Kähler group, $(\mathbb{Z} \times \mathbb{Z})$ must have a non-degenerate skew structure. It follows that $G_{m,n}$ also has such a structure. $\square$

Example 3.4. Let m, n be two composite natural numbers and let $G_{m,n}$ be defined as in Lemma 3.3. Since $G_{m,n}$ does support a non-degenerate skew structure itself, by Theorem 3.1 it can be realized as the fundamental group of some symplectic four-manifold N. By Lemma 3.3 $G_{m,n}$ must have a subgroup K of finite index which does not support any non-degenerate skew

structure at all. Let $\widetilde{N}$ be the finite covering space of N with fundamental group K. Then by Theorem 3.1 again we have that $\widetilde{N}$ does not support any symplectic form ω such that $(\widetilde{N}, \omega)$ has the strong Lefschetz property.

Gompf proved in [5] the remarkable result that any finitely presentable group can be realized as the fundamental group of a symplectic four-manifold. In contrast, Theorem 3.1 imposes a rather stringent restriction on the fundamental groups of compact strong Lefschetz four-manifolds. For example, any non-trivial finitely presentable free group cannot be the fundamental group of a compact strong Lefschetz four-manifold. (Cf. pages 592–593 of [5].) In addition, Theorem 3.1 also asserts that, different from the fundamental groups of compact Kähler manifolds to which far more rich restrictions apply (see, e.g., [1]), the fundamental groups of compact strong Lefschetz four-manifolds have only one restriction as we stated in Theorem 3.1. Therefore, as suggested by Example 3.4, fundamental groups may serve as effective tools to distinguish strong Lefschetz manifolds from Kähler manifolds.

4. Examples that the strong Lefschetz property is not preserved by symplectic reduction

Since in this section we are going to make an extensive use of the Leray–Hirsch theorem, we first give its precise statement here and refer to [3] for details.

Theorem 4.1 (*Leray–Hirsch theorem*)*. Let E be a fiber bundle over M with fiber F. Suppose M has a finite good cover.*[2] *If there are global cohomology classes $e_1, e_2, \ldots, e_r$ which when restricted to each fiber freely generate the cohomology of the fiber, then $H^*(E)$ is a free module over $H^*(M)$ with basis $\{e_1, e_2, \ldots, e_r\}$, i.e.,*

$$H^*(E) \simeq H^*(M) \otimes \mathbb{R}\{e_1, e_2, \ldots, e_r\} \simeq H^*(M) \otimes H^*(F).$$

The following proposition enables us to construct six-dimensional Hamiltonian symplectic manifolds which have the strong Lefschetz property from the symplectic four-manifolds with properties stated in Proposition 2.3.

Proposition 4.2. *Suppose (N, ω_0) is a four-dimensional compact symplectic manifold such that*:

(i) *the Lefschetz map $L_{[\omega_0]} : H^1(N) \to H^3(N)$ is not an isomorphism*;
(ii) *there exists an integral cohomology class $[c] \in H^2(N)$ such that the map $L_{[c]} : H^1(N) \to H^3(N)$ is an isomorphism.*

Then there exists a S^2 bundle $\pi_M : M \to N$ which satisfies the following conditions:

(i) *there is a symplectic form ω on M such that (M, ω) has the strong Lefschetz property*;
(ii) *there is an S^1 action on M such that (M, ω, S^1) is a compact Hamiltonian manifold which has a non-Lefschetz symplectic quotient.*

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[2] An open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of an n-dimensional manifold M is called a good cover if all non-empty finite intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ is diffeomorphic to $\mathbb{R}^n$. It is well known that every compact manifold has a finite good cover. See, e.g., [3].

Proof. Let S^2 be the set of unit vectors in R^3. In cylindrical polar coordinates (θ, h) away from the poles, where $0 \leqslant \theta < 2\pi$, $-1 \leqslant h \leqslant 1$, the standard symplectic form on S^2 is the area form given by $\sigma = \theta \wedge dh$. The circle S^1 acts on (S^2, σ) by rotations

$$e^{it}(\theta, h) = (\theta + t, h).$$

This action is Hamiltonian with the moment map given by $\mu = h$, i.e., the height function.

Let $\pi_P : P \to N$ be the principle S^1-bundle with Euler class $[c]$, let Θ be the connection one-form such that $d\Theta = \pi_P^* c$, and let M be the associated bundle $P \times_{S^1} S^2$. Then $\pi_M : M \to N$ is a symplectic fibration over the compact symplectic four-manifold N. The standard symplectic form σ on S^2 gives rise to a symplectic form σ_x on each fibre $\pi_M^{-1}(x)$, where $x \in N$. The S^1-action on S^2 that we described above induces a fibrewise S^1-action on M. Furthermore, there is a globally defined function H on M such that the restriction of H to each fiber S^2 is just the height function h.

Next, we resort to minimal coupling construction to get a closed two-form η on M which restricts to the forms σ_x on the fibres. Let us give a sketch of this construction here and refer to [7,17,20] for technical details. Consider the closed two-form $-d(t\Theta) = -t d\Theta - dt \wedge \Theta$ defined on $P \times \mathbb{R}$. It is easy to see the S^1-action on $P \times \mathbb{R}$ given by

$$e^{it}(p, t) = \left(e^{it} p, t\right)$$

is Hamiltonian with the moment map t. Thus the diagonal action of S^1 on $(P \times \mathbb{R}) \times S^2$ is also Hamiltonian, and M is just the reduced space of $(P \times \mathbb{R}) \times S^2$ at the zero level. Moreover, the closed two-form $(-d(t\Theta) + \sigma)|_{\text{zero level}}$ descends to a closed two-form η on M with the desired property.

It is useful to have the following explicit description of η. Observe that $\theta - \Theta$ is a basic form on $(P \times \mathbb{R}) \times S^2$. Its restriction to the zero level of $(P \times \mathbb{R}) \times S^2$ descends to a one form $\tilde{\theta}$ on M whose restriction to each fibre S^2 is just θ. It is easy to see that on the associated bundle $P \times_{S^1} (S^2 - \{\text{two poles}\})$ we actually have $\eta = H\pi_M^* c + dH \wedge \tilde{\theta}$.

For any real number t_0, note that the restriction of $\eta - t_0 \pi_M^* c$ to fibres are symplectic forms σ_x. By an argument due to Thurston [15], for sufficiently large constant $K > 0$ the form $K\pi_M^* \omega_0 - t_0 \pi_M^* c + \eta$ is symplectic. Equivalently, define $\omega = \pi_M^* \omega_0 - \epsilon t_0 \pi_M^* c + \epsilon \eta$ for sufficiently small constant $\epsilon > 0$. Then ω is a symplectic form on M; furthermore, the fibrewise S^1-action on (M, ω) is Hamiltonian with the moment map $H : M \to \mathbb{R}$.

Choose some $\min_{x \in M} H(x) < t_0 < \max_{x \in M} H(x)$ and have it fixed. If we perform symplectic reduction at $H = t_0$, the symplectic reduced space is N with the reduced form ω_0. Clearly, (N, ω_0) does not satisfy the strong Lefschetz property since the Lefschetz map $L_{[\omega_0]} : H^1(N) \to H^3(N)$ is not an isomorphism.

It remains to check that for sufficiently small constant $\epsilon > 0$, (M, ω) has the strong Lefschetz property.

Consider the closed two-form η on M. Its restriction to each fibre S^2 generates the second cohomology group $H^2(S^2)$. Write $H(S^2) = \mathbb{R}[x]/(x^2)$, where $\mathbb{R}[x]$ is the real polynomial ring and (x^2) is the ideal of $\mathbb{R}[x]$ generated by the quadratic polynomial x^2. By the Leray–Hirsch theorem there is an additive isomorphism

$$H(N) \otimes \mathbb{R}[x]/\left(x^2\right) \to H(M), \quad [\alpha] \otimes x^i \to \left[\pi_M^* \alpha \wedge \eta^i\right], \quad i = 0, 1.$$

As a result we have $[\eta^2] = [\pi_M^* \beta_2 \wedge \eta] + [\pi_M^* \beta_4]$, where β_2 and β_4 are closed forms on N of degree two and four, respectively.

Choose an $\epsilon > 0$ which is sufficiently small such that

$$[\omega_0 - t_0 \epsilon c]^2 \neq -\epsilon^2 [\beta_4] + \epsilon \big[(\omega_0 - t_0 \epsilon c) \wedge \beta_2 \big]. \tag{2}$$

We claim for the ϵ chosen above, the symplectic manifold $(M, \pi_M^* \omega_0 - \epsilon t_0 \pi_M^* c + \epsilon \eta)$ will satisfy the strong Lefschetz property. By Poincaré duality it suffices to show the two Lefschetz maps

$$L_{[\omega]}^2 : H^1(M) \to H^5(M), \tag{3}$$

$$L_{[\omega]} : H^2(M) \to H^4(M) \tag{4}$$

are injective. We will give a proof in two steps below.

(i) It follows from the Leray–Hirsch theorem that

$$H^1(N) \xrightarrow[\pi_M^*]{\simeq} H^1(M).$$

Thus to show map (3) is injective we need only to show for any $[\lambda] \in H^1(N)$ if $L_{[\omega]}^2(\pi_M^*[\lambda]) = 0$ then we have $[\lambda] = 0$. Since $\omega = \pi_M^*(\omega_0 - t_0 \epsilon c) + \epsilon \eta$, $[\eta^2] = [\pi_M^* \beta_2 \wedge \eta] + [\pi_M^* \beta_4]$ and any forms on N with degree greater than 4 vanishes, we have

$$0 = L_{[\omega]}^2 \big([\pi^* \lambda]\big) = \pi_M^* \big(2\epsilon[\omega_0 - t_0 \epsilon c] + \epsilon^2 [\beta_2]\big) \wedge \big[\pi_M^* \lambda\big] \wedge [\eta]. \tag{5}$$

Since by the Leray–Hirsch theorem $H(M)$ is free over 1 and $[\eta]$, we get that

$$0 = \pi_M^* \big(2\epsilon[\omega_0] - 2t_0 \epsilon^2 [c] + \epsilon^2 [\beta_2]\big) \wedge \pi_M^*[\lambda]$$
$$= \pi_M^* \big(\big([2\epsilon \omega_0 - 2t_0 \epsilon^2 c + \epsilon^2 \beta_2]\big) \wedge [\lambda]\big).$$

Since $L_{[c]} : H^1(N) \to H^3(N)$ is an isomorphism, the determinant of the linear map $L_{[2\epsilon \omega_0 - 2t_0 \epsilon^2 c + \epsilon^2 \beta_2]} : H^1(N) \to H^3(N)$ is a polynomial in t_0 of positive degree. Therefore $L_{[2\epsilon \omega_0 - 2t_0 \epsilon^2 c + \epsilon^2 \beta_2]} : H^1(N) \to H^3(N)$ is an isomorphism except for finitely many possible values of t_0. If necessary, replace H and t_0 by $H + C$ and $t_0 + C$ respectively for some suitable small constant $C > 0$. We conclude that map (3) is an isomorphism.

(ii) By the Leray–Hirsch theorem, to show that map (4) is injective it suffices to show if $L_{[\omega]}(\pi_M^*[\varphi] + k[\eta]) = 0$ for arbitrarily chosen scalar k and second cohomology class $[\varphi] \in H^2(N)$, then we have $[\varphi] = 0$ and $k = 0$. Since $\omega = \pi_M^*(\omega_0 - t_0 \epsilon c) + \epsilon \eta$ and $[\eta^2] = [\pi_M^* \beta_2 \wedge \eta] + [\pi_M^* \beta_4]$, we have

$$0 = L_{[\omega]} \big(\pi_M^*[\varphi] + k[\eta]\big)$$
$$= \big(\pi_M^* \big[(\omega_0 - t_0 \epsilon c) \wedge \varphi\big] + \epsilon k \pi_M^*[\beta_4]\big)$$
$$+ \big(k \pi_M^*[\omega_0 - t_0 \epsilon c] + \epsilon \pi_M^*[\varphi] + \epsilon k \pi_M^*[\beta_2]\big) \wedge \eta. \tag{6}$$

By the Leray–Hirsch theorem $H(M)$ is a free module over 1 and $[\eta]$. So we have that

$$\pi_M^*\big[(\omega_0 - t_0\epsilon c) \wedge \varphi\big] + \epsilon k\pi_M^*[\beta_4] = 0, \tag{7}$$

$$k\pi_M^*[\omega_0 - t_0\epsilon c] + \epsilon\pi_M^*[\varphi] + \epsilon k\pi_M^*[\beta_2] = 0. \tag{8}$$

If $k = 0$, it follows easily from Eq. (8) that $[\varphi] = 0$. Assume $k \neq 0$. Substitute $\pi_M^*[\varphi] = -\frac{1}{\epsilon}k\pi_M^*[\omega_0 - t_0\epsilon c] - k\pi_M^*[\beta_2]$ into Eq. (7) we get

$$\pi_M^*[\omega_0 - t_0\epsilon c] \wedge \big(-k\pi_M^*[\omega_0 - t_0\epsilon c] - \epsilon k\pi_M^*[\beta_2]\big) + \epsilon^2 k\pi_M^*[\beta_4] = 0.$$

Since $k \neq 0$, we get

$$\pi_M^*\big([\omega_0 - t_0\epsilon c]\big)^2 = -\epsilon^2\pi_M^*[\beta_4] + \epsilon\pi_M^*\big[(\omega_0 - t_0\epsilon c) \wedge \beta_2\big].$$

This contradicts Eq. (2). $\square$

Now we are in a position to construct examples that the strong Lefschetz property does not survive symplectic reduction.

Example 4.3. Since the torus is a Kähler manifold, $G = \mathbb{Z} \times \mathbb{Z}$ is a Kähler group and thus admits a non-degenerate skew structure. Clearly, by Proposition 2.3 there is a closed, symplectic four-manifold (N, ω_0) which satisfies the following conditions:

(i) $\pi_1(N) = \mathbb{Z} \times \mathbb{Z}$.
(ii) The Lefschetz map $L_{[\omega]} : H^1(N) \to H^3(N)$ is trivial.
(iii) There is an integral class $[c] \in H^2(N)$ such that the map $L_{[c]} : H^1(N) \to H^3(N)$ is an isomorphism.

Then it follows easily from Proposition 4.2 that there exists a compact six-dimensional Hamiltonian circle manifold (M, ω) which has the strong Lefschetz property itself but admits a non-Lefschetz symplectic quotient.

Since the six-dimensional Hamiltonian S^1-manifold (M, ω) constructed in Example 4.3 has a non-Lefschetz symplectic quotient, ω cannot be an invariant Kähler form. But in general we do not know whether M supports any Kähler form or not. To get examples which do not admit any Kähler structure, we observe that by our construction $M \to N$ is a fibration with fiber S^2 and so $\pi_1(M) = \pi_1(N)$. Instead of choosing $G = \mathbb{Z} \times \mathbb{Z}$, we may well choose $G = G_{m,n}$, where m, n are any composite numbers and $G_{m,n}$ is defined as in Lemma 3.3. For any such a group G, the corresponding Hamiltonian manifold M has a non-Kähler fundamental group and therefore is not homotopy equivalent to any compact Kähler manifold. Thus we have proved the following theorem:

Theorem 4.4. *There exist infinitely many topologically inequivalent six-dimensional compact Hamiltonian symplectic S^1-manifolds which satisfy the following conditions*:

(i) *the strong Lefschetz property,*

(ii) *admitting a non-Lefschetz symplectic quotient,*
(iii) *not homotopy equivalent to any compact Kähler manifold.*

Finally we observe that the fixed point set of the Hamiltonian symplectic manifold (M, ω, S^1) constructed in Proposition 4.2 has two components on which the moment map takes maximum and minimum, respectively; furthermore, in the proof of Proposition 4.2, if we choose t_0 to be the minimum value of the moment map, then the minimal component as a symplectic submanifold can be identified with (N, ω_0) which clearly does not have the strong Lefschetz property. This observation, together with Lemma 3.3, leads to the following result.

Theorem 4.5. *There exist infinitely many topologically inequivalent six-dimensional compact Hamiltonian symplectic S^1-manifolds which satisfy the following conditions*:

 (i) *the strong Lefschetz property,*
 (ii) *admitting a non-Lefschetz fixed point submanifold,*
(iii) *not homotopy equivalent to any compact Kähler manifold.*

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