Projective and inductive limits in locally convex cones

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Abstract

We define and study the projective and inductive limit notions for locally convex cones. We use convex quasiuniform structure method for this purpose. Also we study the barreledness in the locally convex cones and introduce the notion upper-barreled cones and prove that the inductive limit of upper-barreled cones is upper-barreled.

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1. Introduction

The general theory of locally convex cones as developed in [2] deals with preordered cones. We review some of the main concepts and refer to [2] for details.

A cone is a set $P$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is associative and commutative, and there is a neutral element $0 \in P$. For the scalar multiplication the usual associative and distributive properties hold. We have $1a = a$ and $0a = 0$ for all $a \in P$. A preordered cone (ordered cone) is a cone with a preorder, that is a reflexive transitive relation $\leq$ which is compatible with the algebraic operations.

The extended real numbers $\mathbb{R} = \mathbb{R} \cup \{+\infty\}$ is a natural example of an ordered cone with the usual order and algebraic operations in $\mathbb{R}$, in particular $0 \cdot (+\infty) = 0$.

A linear functional on an ordered cone $P$ is a mapping $\mu : P \to \mathbb{R}$ such that $\mu(a + b) = \mu(a) + \mu(b)$ and $\mu(\alpha a) = \alpha \mu(a)$ for all $a, b \in P$ and $\alpha \geq 0$. More generally, for cones $P$ and

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Q a mapping \( t : \mathcal{P} \to \mathcal{Q} \) is called a \textit{linear operator} if \( t(a + b) = t(a) + t(b) \) and \( t(\alpha a) = \alpha t(a) \) hold for \( a, b \in \mathcal{P} \) and \( \alpha \geq 0 \).

A subset \( \mathcal{V} \) of the preordered cone \( \mathcal{P} \) is called an \textit{(abstract) 0-neighborhood system}, if the following properties hold:

(i) \( 0 < v \) for all \( v \in \mathcal{V} \);
(ii) for all \( u, v \in \mathcal{V} \) there is \( w \in \mathcal{V} \) with \( w \leq u \) and \( w \leq v \);
(iii) \( u + v \in \mathcal{V} \) and \( \alpha v \in \mathcal{V} \) whenever \( u, v \in \mathcal{V} \) and \( \alpha > 0 \).

For every \( a \in \mathcal{P} \) and \( v \in \mathcal{V} \) we define
\[
 v(a) = \{ b \in \mathcal{P} \mid b \leq a + v \}, \quad \text{respectively} \quad (a)v = \{ b \in \mathcal{P} \mid a \leq b + v \},
\]
to be a neighborhood of \( a \) in the \textit{upper}, respectively \textit{lower} topologies on \( \mathcal{P} \). Their common refinement is called \textit{symmetric topology}. We denote the neighborhoods of the symmetric topology as \( v(a) \cap (a)v \) or \( v(a)v \) for \( a \in \mathcal{P} \) and \( v \in \mathcal{V} \). We call \((\mathcal{P}, \mathcal{V})\) a \textit{full locally convex cone}, and each subcone of \( \mathcal{P} \), not necessarily containing \( \mathcal{V} \), is called a \textit{locally convex cone}. For technical reasons we require the elements of a locally convex cone to be \textit{bounded below}, i.e. for every \( a \in \mathcal{P} \) and \( v \in \mathcal{V} \) we have \( 0 \leq a + \rho v \) for some \( \rho > 0 \). An element \( a \) of \((\mathcal{P}, \mathcal{V})\) is called bounded if it is also \textit{upper bounded}, i.e. for every \( v \in \mathcal{V} \) there is \( \rho > 0 \) such that \( a \leq \rho v \). On \( \mathcal{P} \) we define the \textit{global preorder} \( \preceq \) as follows: \( a \preceq b \) if and only if \( a \leq b + v \) for all \( v \in \mathcal{V} \).

Let \( \mathcal{P} \) be a cone. A collection \( \mathcal{U} \) of convex subsets \( U \subseteq \mathcal{P}^2 = \mathcal{P} \times \mathcal{P} \) is called a \textit{convex quasiuniform structure} on \( \mathcal{P} \), if the following properties hold:

\[ (U1) \Delta \subseteq U \text{ for every } U \in \mathcal{U} (\Delta = \{(a,a) : a \in \mathcal{P}\}); \]
\[ (U2) \text{ for all } U, V \in \mathcal{U} \text{ there is } W \in \mathcal{U} \text{ such that } W \subseteq U \cap V; \]
\[ (U3) \lambda U \circ \mu U \subseteq (\lambda + \mu)U \text{ for all } U \in \mathcal{U} \text{ and } \lambda, \mu > 0; \]
\[ (U4) \lambda U \in \mathcal{U} \text{ for all } U \in \mathcal{U} \text{ and } \lambda > 0. \]

Here, for \( U, V \subseteq \mathcal{P}^2 \), by \( U \circ V \) we mean the set of all \((a,b) \in \mathcal{P}^2 \) such that there is \( c \in \mathcal{P} \) with \( (a,c) \in U \) and \( (c,b) \in V \). We call \((\mathcal{P}, \mathcal{U})\) is \textit{convex quasiuniform cone}.

To every convex quasiuniform structure \( \mathcal{U} \) on \( \mathcal{P} \) we associate a preorder defined by \( a \preceq b \) if and only if \((a,b) \in U \) for all \( U \in \mathcal{U} \) and, two topologies: The neighborhood bases for an element \( a \) in the upper and lower topologies are given by the sets
\[
 U(a) = \{ b \in \mathcal{P} : (b, a) \in U \}, \quad \text{respectively} \quad (a)U = \{ b \in \mathcal{P} : (a, b) \in U \}, \quad U \in \mathcal{U}.
\]

The topology associated with the uniform structure \( \mathcal{U}_c = \{ U \cap U^{-1} : U \in \mathcal{U} \} \) is the common refinement of the upper and lower topologies, where \( U^{-1} = \{(b, a) : (a, b) \in U\} \).

Let \((\mathcal{P}, \mathcal{U})\) and \((\mathcal{Q}, \mathcal{V})\) be two convex quasiuniform cones and \( t : \mathcal{P} \to \mathcal{Q} \) be a linear mapping. We say that \( t \) is \textit{uniformly continuous} if for each \( V \in \mathcal{V} \), there is \( U \in \mathcal{U} \) such that \((a,b) \in U \) implies \((t(a), t(b)) \in V \) or \( T(U) \subseteq V \), \( T = t \times t \). Let \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) be convex quasiuniform structures on \( \mathcal{P} \). Following N. Bourbaki [1, II, 2.2], we say that \( \mathcal{U}_1 \) is \textit{finer} than \( \mathcal{U}_2 \) if the identity mapping \( i : (\mathcal{P}, \mathcal{U}_1) \to (\mathcal{P}, \mathcal{U}_2) \) is uniformly continuous.

The notions of an (abstract) 0-neighborhood system \( \mathcal{V} \) and a convex quasiuniform structure \( \mathcal{U} \) for a cone \( \mathcal{P} \) are equivalent in the following sense:

Let \( \mathcal{P} \) be a preordered cone and \( \mathcal{V} \) an (abstract) 0-neighborhood system for \( \mathcal{P} \). For each \( v \in \mathcal{V} \), we put
\[
 \tilde{v} = \{ (a,b) \in \mathcal{P} \times \mathcal{P} : a \leq b + v \}.
\]
The collection \( \widetilde{V} = \{ \tilde{v} : v \in V \} \) is a convex quasiuniform structure on \( P \), which induces the global preorder on \( P \) and the same upper, lower and symmetric topologies. Furthermore if \( (P, V) \) is a locally convex cone, i.e. each element of \( P \) is bounded below, we have:

for all \( a \in P \) and \( \tilde{v} \in \widetilde{V} \) there is some \( \rho > 0 \) such that \( (0, a) \in \rho \tilde{v} \).

On the other hand:

If \( P \) is a cone with a convex quasiuniform structure \( \Gamma \), then one can find a preorder and an (abstract) 0-neighborhood system \( V \) such that the convex quasiuniform structure \( \widetilde{V} \) is equivalent to \( \Gamma \) (see [2, I.5.5]). In this case if we also have

\[(U) \quad \text{for all } a \in P \text{ and } U \in \Gamma \text{ there is some } \rho > 0 \text{ such that } (0, a) \in \rho U,\]

then by the equivalency of \( \Gamma \) and \( \widetilde{V} \), all elements of \( P \) would be bounded below. \((U1)\)–\(U5\) make \((P, \Gamma)\) into a locally convex cone.

By the above consideration if \( V \) and \( W \) are (abstract) 0-neighborhood systems on \( P \) and \( Q \), respectively, \( t \) is u-continuous if and only if for every \( w \in W \) there is \( v \in V \) such that \( (a, b) \in \tilde{v} \) implies \( (t(a), t(b)) \in \tilde{w} \), or equivalently, \( t(a) \leq t(b) + w \) whenever \( a \leq b + v \). Uniform continuity implies continuity with respect to the upper, lower and symmetric topologies on \( P \) and \( Q \).

Endowed with the (abstract) 0-neighborhood system \( V = \{ \varepsilon \in \mathbb{R} : \varepsilon > 0 \}, \mathbb{R} \) is a full locally convex cone. The u-continuous linear functionals on a locally convex cone \((P, V)\) (into \( \mathbb{R} \)) form a cone with the usual addition and scalar multiplication of functions. This cone is called the dual cone of \( P \) and denoted by \( P^* \).

For a locally convex cone \((P, V)\), the polar \( v^o \) of \( v \in V \) consists of all linear functionals \( \mu \) on \( P \) satisfying \( \mu(a) \leq \mu(b) + 1 \) whenever \( a \leq b + v \) for \( a, b \in P \). We have \( \bigcup \{ v^o : v \in V \} = P^* \).

Hahn–Banach type and uniform boundedness type theorems for locally convex cones have been studied in [5] and [4]. Some other concepts also have been studied for locally convex cones in several papers. Here we present and study the inductive and projective limit notions.

In Section 2 we introduce projective limit in locally convex cones and study some of the properties of projective limits and give some examples.

In Section 3 we define and study inductive limit in locally convex cones with some examples.

In Section 4 we study the barreledness, and define upper-barreled cones and show that the inductive limit of upper-barreled cones is upper-barreled.

2. Projective limits

Suppose that \( (P_\gamma, V_\gamma)_{\gamma \in \Gamma} \) is a family of locally convex cones, \( P \) be a cone and for each \( \gamma \in \Gamma \), \( g_\gamma \) be a linear mapping of \( P \) into \( P_\gamma \). We want to define an (abstract) 0-neighborhood system for \( P \) by \( (V_\gamma) \) such that \( P \) becomes a locally convex cone. We use the equivalence of the (abstract) 0-neighborhood system, i.e. the convex quasiuniform structure.

**Theorem 2.1.** For each \( \gamma \in \Gamma \), let \( P_\gamma \) be a cone with a convex quasiuniform structure \( \Omega_\gamma = \{ U_{\gamma \delta} : \delta \in D_\gamma \} \). Let \( P \) be a cone and, for each \( \gamma \in \Gamma \), \( g_\gamma \) be a linear mapping of \( P \) into \( P_\gamma \). Then there is a coarsest convex quasiuniform structure \( \Omega \) on \( P \) under which all the \( g_\gamma \) are u-continuous. If all \( P_\gamma \) are locally convex, then \( P \) is also locally convex.

**Proof.** Put \( G_\gamma = g_\gamma \times g_\gamma \), that is, \( G_\gamma(a, b) = (g_\gamma(a), g_\gamma(b)) \) for \( (a, b) \in P \times P \). Let \( \Omega \) be the finite intersections of the sets \( G_\gamma^{-1}(U_{\gamma \delta}) \) \( (U_{\gamma \delta} \in \Omega_\gamma, \delta \in D_\gamma, \gamma \in \Gamma) \).
\[\Omega\] is the required convex quasiuniform structure on \(P\):

\((U1)\) and \((U4)\) are trivial.

\((U2)\) Let \(U, V \in \Omega\), by the definition of \(\Omega\), \(W = U \cap V \in \Omega\).

\((U3)\) Let \(U = \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} G^{-1}_{\gamma_i} (U_{\gamma_i \delta_j}) \in \Omega\) and \(\lambda, \mu > 0\). Let \((a, b) \in \lambda U \circ \mu U\). Then, there exists \(c \in P\) such that \((a, c) \in \lambda \bar{U}\), \((c, b) \in \mu U\). Hence

\[(g_{\gamma_i} (a), g_{\gamma_i} (c)) \in \lambda U_{\gamma_i \delta_j} \text{ and } (g_{\gamma_i} (c), g_{\gamma_i} (b)) \in \mu U_{\gamma_i \delta_j},\]

where \(i = 1, 2, \ldots, n\) and \(j = 1, 2, \ldots, m\). Therefore

\[G_{\gamma_i} (a, b) \in \lambda U_{\gamma_i \delta_j} \circ \mu U_{\gamma_i \delta_j} \subseteq (\lambda + \mu) U_{\gamma_i \delta_j}, \quad i = 1, 2, \ldots, n\]

hence

\[(a, b) \in (\lambda + \mu) \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} G^{-1}_{\gamma_i} (U_{\gamma_i \delta_j}) = (\lambda + \mu) U.\]

Now let all \(P_{\gamma}\)s are locally convex. We prove condition \((U5)\) for \(P\). For \(a \in P\) and \(U = \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} G^{-1}_{\gamma_i} (U_{\gamma_i \delta_j}) \in \Omega\), there exist strictly positive numbers \(\rho_{ij}\) where \(1 \leq i \leq n\) and \(1 \leq j \leq m\), such that for each \(i = 1, 2, \ldots, n\),

\[(0, g_{\gamma_i}(a)) \in \rho_{ij} U_{\gamma_i \delta_j}, \quad j = 1, 2, \ldots, m.\]

Put \(\rho = \max \{\rho_{ij}: 1 \leq i \leq n\} \text{ and } 1 \leq j \leq m\}. Then by convexity of each \(U_{\gamma_i \delta_j}\), linearity of \(G_{\gamma_i}\) and that \((0, 0) \in U_{\gamma_i \delta_j}\), we have \((0, a) \in \rho U\).

Clearly each \(g_{\gamma}\) is u-continuous, and \(\Omega\) is the coarsest convex quasiuniform structure making each \(g_{\gamma}\) u-continuous. \(\square\)

The locally convex cone \(P\) with the preorder and (abstract) 0-neighborhood system induced by this convex quasiuniform structure is called the \textit{projective limit} of the locally convex cones \(P_{\gamma}\) by the mappings \(g_{\gamma}\).

For \(a \in (P, V)\), we define \(\bar{a} = \bigcap \{v(a): v \in V\}\), and we call \(P\) \textit{separated} if \(\bar{a} = \bar{b}\) implies \(a = b\) for all \(a, b \in P\).

**Proposition 2.2.** The locally convex cone \(P\) is separated if and only if the symmetric topology on \(P\) is Hausdorff.

\textbf{Proof.} See [2, I.3.9]. \(\square\)

Let \(\{g_{\gamma}: \gamma \in \Gamma\}\) be a family of functions on a cone \(P\). We say that \(\{g_{\gamma}: \gamma \in \Gamma\}\) is a separating family of functions over \(P\), if whenever \(x_1 \neq x_2\), there is \(g_{\gamma} (\gamma \in \Gamma)\) such that \(g_{\gamma} (x_1) \neq g_{\gamma} (x_2)\).

**Proposition 2.3.** Let \((P, V)\) be the projective limit of the locally convex cones \((P_{\gamma}, V_{\gamma})\) by the mappings \(g_{\gamma}, \gamma \in \Gamma\). If each \(P_{\gamma}\) is separated and \(\{g_{\gamma}: \gamma \in \Gamma\}\) is separating, then \(P\) is separated.

\textbf{Proof.} Let \(a, b \in P, a \neq b\). By the hypothesis there is \(\gamma \in \Gamma\) such that \(g_{\gamma} (a) \neq g_{\gamma} (b)\). Since \(P_{\gamma}\) is separated, by Proposition 2.2 there exists \(v_{\gamma} \in V_{\gamma}\) such that \(v_{\gamma}(g_{\gamma}(a))v_{\gamma}\) does not intersect \(v_{\gamma}(g_{\gamma}(b))v_{\gamma}\). By the u-continuity of \(g_{\gamma}\) and definition of the 0-neighborhood system, there exists \(v \in V\) such that \(g_{\gamma}(v(a))v \subseteq v_{\gamma}(g_{\gamma}(a))v_{\gamma}\) and \(g_{\gamma}(v(b))v \subseteq v_{\gamma}(g_{\gamma}(b))v_{\gamma}\). Hence \(v(a)v\) does not intersect \(v(b)v\). Using Proposition 2.2 once more we see that \(P\) is separated. \(\square\)
**Proposition 2.4.** Let \((Q, W)\) be a locally convex cone and \(h\) a linear mapping of \(Q\) into the projective limit \((P, V)\) of the locally convex cones \((P_\gamma, V_\gamma)\) by the mappings \(g_\gamma, \gamma \in \Gamma\). Then \(h\) is u-continuous if and only if \(g_\gamma \circ h\) is u-continuous of \(Q\) into \(P_\gamma\), for all \(\gamma \in \Gamma\).

**Proof.** \(h\) is u-continuous if and only if \(H = h \times h\) is u-continuous. Therefore, turning on convex quasiuniform structures \(\mathcal{W}, \mathcal{V}\) and \(\mathcal{V}'\), we see that \(H\) is u-continuous if and only if, for each \(\gamma\) and each \(U_\gamma \in \mathcal{V}'\), \(W \subseteq H^{-1}(G^{-1}_\gamma(U_\gamma)) = (G_\gamma \circ H)^{-1}(U_\gamma)\), for some \(W \in \mathcal{W}\). And this is the case that \(G_\gamma \circ H\) should be u-continuous. \(\square\)

We define a subset \(A \subseteq P = (P, \mathcal{W})\) to be **bounded** if for every \(U \in \mathcal{W}\) there is \(\lambda_U > 0\) such that

\[(0, a), (a, 0) \in \lambda_U U \quad \text{for all } a \in A.\]

Walter Roth in [4, p. 1975] has defined \(A \subseteq (P, \mathcal{V})\) to be bounded if for every \(v \in \mathcal{V}\) there is \(\lambda_v > 0\) such that

\[a \leq \lambda_v v \quad \text{and} \quad 0 \leq a + \lambda_v v.\]

This is equivalent to: For every \(\tilde{v} \in \mathcal{V}\) there is \(\lambda_{\tilde{v}} > 0\) such that

\[(0, a), (a, 0) \in \lambda_{\tilde{v}} \tilde{v} \quad \text{for all } a \in A,\]

i.e. two definitions are equivalent.

If \(t\) is a linear u-continuous mapping from \(P\) into locally convex cone \((Q, \mathcal{W})\), and \(A\) is a bounded subset of \(P\), then \(t(A)\) is bounded in \(Q\). For if \(W \in \mathcal{W}\) is arbitrary, there is \(U \in \mathcal{W}\) such that \(T(U) \subseteq W (T = t \times t)\). Since \(A\) is bounded, there is \(\lambda_U > 0\) such that

\[(0, a), (a, 0) \in \lambda_U U \quad \text{for all } a \in A,\]

and so

\[(0, t(a)), (t(a), 0) \in \lambda_U T(U) \subseteq \lambda_U W \quad \text{for all } t(a) \in t(A).\]

A subset \(A\) of locally convex cone \((P, \mathcal{V})\) is called **precompact** with respect to the symmetric topology if for every \(v \in \mathcal{V}\), there are \(a_1, \ldots, a_n \in A\) such that \(A \subseteq \bigcup_{i=1}^n v(a_i)\). If \(t\) is a u-continuous linear mapping of \(P\) into locally convex cone \((Q, \mathcal{W})\) and \(A \subseteq P\) is precompact, then \(t(A)\) is also precompact. For if \(w \in \mathcal{W}\) is arbitrary, there is \(v \in \mathcal{V}\) such that \(a \leq b + v\) implies \(t(a) \leq t(b) + w\). There are \(a_1, \ldots, a_n \in A\) such that \(A \subseteq \bigcup_{i=1}^n v(a_i)\). Since \(t(v(a_i)v) \subseteq w(t(a_i))w\), then \(t(A) \subseteq \bigcup_{i=1}^n w(t(a_i))w\), and \(t(a_i) \in t(A)\), i.e. \(t(A)\) is precompact.

**Proposition 2.5.** Let \((P, \mathcal{V})\) be the projective limit of the locally convex cones \((P_\gamma, \mathcal{V}_\gamma)\) by the mappings \(g_\gamma\). Then the subset \(A\) of \(P\) is bounded, or precompact, if and only if each \(g_\gamma(A)\) has the same property.

**Proof.** If \(A\) is bounded, or precompact, then each \(g_\gamma(A)\) has the same property since \(g_\gamma\) is u-continuous.

Let \(g_\gamma(A)\) be bounded for each \(\gamma \in \Gamma\). We show that \(A\) is bounded. Let \(\tilde{v} \in \mathcal{V}\) and \(\tilde{v} = \bigcap_{i=1}^n \bigcap_{j=1}^m G^{-1}_{\gamma_i}(\tilde{v}_{\gamma_i \delta_j}) (G_\gamma = g_\gamma \times g_\gamma \text{ and } \tilde{v}_{\gamma_i \delta_j} \in \mathcal{V}_{\gamma_i}).\) Since \(g_{\gamma_i}(A)\) is bounded, for each \(j = 1, 2, \ldots, m\) there is \(\lambda_{ij} > 0\) such that

\[(g_{\gamma_i}(a), 0) \in \lambda_{ij} \tilde{v}_{\gamma_i \delta_j} \quad \text{and} \quad (0, g_{\gamma_i}(a)) \in \lambda_{ij} \tilde{v}_{\gamma_i \delta_j}, \quad \text{for all } g_{\gamma_i}(a) \in g_{\gamma_i}(A).\]
Set $\lambda = \max\{\lambda_{ij}: 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$. Then

$$(a, 0) \in \lambda \tilde{v} \quad \text{and} \quad (0, a) \in \lambda \tilde{v}, \quad \text{for all } a \in A.$$ This shows that $A$ is bounded.

Let $g_\gamma(A)$ be precompact for each $\gamma$. We show that $A$ is precompact. Let $v \in \mathcal{V}$ be arbitrary, and $\tilde{v} = \bigcap_{i=1}^n \bigcap_{j=1}^m G^{-1}_\gamma (\tilde{v}_\gamma \delta_j)$. We have $a \leq b + v$, i.e. $(a, b) \in \tilde{v}$ if and only if

$$(a, b) \in G^{-1}_\gamma (\tilde{v}_\gamma \delta_j) \quad \text{for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, m,$$

or

$$g_\gamma(A) \leq g_\gamma(b) + v_\gamma \delta_j \quad \text{for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, m.$$ Since $g_\gamma(A)$ is precompact, then there are $a_{ij1}, \ldots, a_{ijm} \in A$ such that

$$g_\gamma(A) \subseteq \bigcup_{k=1}^s v_\gamma \delta_j (g_\gamma(a_{ijk})) v_\gamma \delta_j \quad \text{for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, m.$$

We have $A \subseteq \bigcup_{i=1}^n \bigcup_{j=1}^m \bigcup_{k=1}^s v(a_{ijk}) v$. □

**Example 2.6.** For a locally convex cone $(\mathcal{P}, \mathcal{V})$, let $\mathcal{P}^*$ be the dual cone of $\mathcal{P}$. Recall that for each $\mu \in \mathcal{P}^*$, there exists $v \in \mathcal{V}$ such that

$$a \leq b + v \quad \text{implies} \quad \mu(a) \leq \mu(b) + 1.$$ Now the coarsest topology on $\mathcal{P}$ making all $\mu \in \mathcal{P}^*$ u-continuous is projective limit on $\mathcal{P}$ induced by $\mathcal{P}^*$ denoted by $w(\mathcal{P}, \mathcal{P}^*)$. A typical neighborhood for $x \in \mathcal{P}$ in $w(\mathcal{P}, \mathcal{P}^*)$ is given via a finite subset $A = \{\mu_1, \mu_2, \ldots, \mu_n\}$ of $\mathcal{P}^*$ by

$$\omega_A(x) = \left\{ y \in \mathcal{P} : \begin{array}{ll} |\mu_i(x) - \mu_i(y)| \leq 1 & \text{if } \mu_i(x) < +\infty, \\ \mu_i(y) \geq 1 & \text{if } \mu_i(x) = +\infty \end{array} \right\}.$$ If $\mathcal{P}^*$ separates the points of $\mathcal{P}$, then $w(\mathcal{P}, \mathcal{P}^*)$ will be separated by Proposition 2.3.

**Example 2.7.** Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and $\mathcal{Q}$ a subcone of $\mathcal{P}$. The induced topology on $\mathcal{Q}$ is the coarsest topology making the identity mapping $i : \mathcal{Q} \to \mathcal{P}$ u-continuous. That is, the induced topology on subcones is precisely the projective limit topology by the identity mapping on the subcone into the cone itself.

**Example 2.8.** For each $\gamma \in \Gamma$ let $\mathcal{P}_\gamma$ be a cone and $\mathcal{P} = \times_{\gamma \in \Gamma} \mathcal{P}_\gamma$ the product of these cones. $\mathcal{P}$ can be made into a cone by defining $(x_\gamma) + (y_\gamma) = (x_\gamma + y_\gamma)$ and $\lambda(x_\gamma) = (\lambda x_\gamma)$, for all $(x_\gamma), (y_\gamma) \in \mathcal{P}$ and $\lambda > 0$. If each $\mathcal{P}_\gamma$ is a locally convex cone, then $\mathcal{P}$ can be made into a locally convex cone by regarding it as the projective limit of the cones $\mathcal{P}_\gamma$ by the projections $\pi_\gamma : \mathcal{P} \to \mathcal{P}_\gamma$, $\pi_\gamma(x_\gamma) = x_\gamma$. Since the set $\{\pi_\gamma: \gamma \in \Gamma\}$ separates the points of $\mathcal{P}$, if each $\mathcal{P}_\gamma$ is separated, then $\mathcal{P}$ is separated by Proposition 2.3.

**Proposition 2.9.** Let $\mathcal{P}$ be the projective limit of the locally convex cones $(\mathcal{P}_\gamma, \mathcal{V}_\gamma) \ (\gamma \in \Gamma)$ by the mappings $g_\gamma$. If $\{g_\gamma : \gamma \in \Gamma\}$ is a separating family, then $\mathcal{P}$ is isomorphic to a subcone of the $\times_{\gamma \in \Gamma} \mathcal{P}_\gamma$. 


Proof. We define a linear mapping $g$ of $P$ into $\times P_\gamma$ by putting $g(x) = (g_\gamma(x))$; then since $\pi_\gamma \circ g = g_\gamma$ for each $\gamma$ and each $g_\gamma$ is u-continuous, $g$ is u-continuous. $g$ is $1 - 1$; for if $x, y \in P$ and $g(x) = g(y)$, then $\pi_\gamma \circ g(x) = \pi_\gamma \circ g(y)$, for all $\gamma \in \Gamma$. Hence $g_\gamma(x) = g_\gamma(y)$ for all $\gamma \in \Gamma$. Since $\{g_\gamma\}$ is separating, then we have $x = y$. Let $g^{-1}$ be the inverse of $g$ on $g(P)$. Then $g_\gamma \circ g^{-1} = \pi_\gamma$ for each $\gamma \in \Gamma$ and so by Proposition 2.4, $g^{-1}$ is also u-continuous. □

3. Inductive limits

For each $\gamma \in \Gamma$ let $(P_\gamma, V_\gamma)$ be a locally convex cone and $P$ be a cone. We topologize $P$ by the convex quasiuniform structure generated by $V_\gamma$.

Theorem 3.1. For each $\gamma \in \Gamma$ let $P_\gamma$ be a cone with a convex quasiuniform structure $U_\gamma$. Let $P$ be a cone and, for each $\gamma \in \Gamma$, $f_\gamma : P_\gamma \to P$ is a linear mapping such that $P = \text{span} \bigcup_{\gamma \in \Gamma} f_\gamma(P_\gamma)$. Let $\mathcal{U}$ be the set of all convex subsets of $P^2$ such that:

(i) for each $U \in \mathcal{U}$, and each $\gamma \in \Gamma$, we have $F_\gamma^{-1}(U) \in \mathcal{U}_\gamma$.
(ii) Each $U \in \mathcal{U}$ satisfies $(U3)$.
(iii) For every $U_1, \ldots, U_n \in \mathcal{U}$ we have $U_1 \cap \cdots \cap U_n \in \mathcal{U}$.

Then $\mathcal{U}$ is the finest quasiuniform structure on $P$ which makes each $f_\gamma$ u-continuous. If all $P_\gamma$s are locally convex, then $P$ is also locally convex.

Here $F_\gamma = f_\gamma \times f_\gamma$; also, by linearity and span we mean the linearity and span on nonnegative scalars only.

Proof. Let $a \in P$, $a = \sum_{i=1}^{n} \lambda_{\gamma_i} f_\gamma(a_{\gamma_i})$, $a_{\gamma_i} \in P_{\gamma_i}$, $\lambda_{\gamma_i} \geq 0$. For each $U \in \mathcal{U}$, since $F_\gamma^{-1}(U) \in \mathcal{U}_{\gamma_i}$ and $\lambda_{\gamma_i} U_{\gamma_i} \in \mathcal{U}_{\gamma_i}$ for each $\lambda_{\gamma_i} > 0$, $U_{\gamma_i} \in \mathcal{U}_{\gamma_i}$, we have

$$(a_{\gamma_i}, a_{\gamma_i}) \in \left( \sum_{i=1}^{n} \lambda_{\gamma_i} \right)^{-1} F_\gamma^{-1}(U).$$

Hence

$$\left( \sum_{i=1}^{n} \lambda_{\gamma_i} f_\gamma(a_{\gamma_i}), \sum_{i=1}^{n} \lambda_{\gamma_i} f_\gamma(a_{\gamma_i}) \right) \in \frac{\sum_{i=1}^{n} \lambda_{\gamma_i}}{\sum_{i=1}^{n} \lambda_{\gamma_i}} F_\gamma^{1}(U) \subseteq \sum_{i=1}^{n} \lambda_{\gamma_i} U.$$

Therefore

$$(a, a) = \left( \sum_{i=1}^{n} \lambda_{\gamma_i} f_\gamma(a_{\gamma_i}), \sum_{i=1}^{n} \lambda_{\gamma_i} f_\gamma(a_{\gamma_i}) \right) \in U.$$

This proves (U1).

(U2) and (U3) satisfy by hypothesis and (U4) is trivial.

For (U5), where all $P_\gamma$s are locally convex, let $a \in P$ and $U \in \mathcal{U}$. Let $a = \sum_{i=1}^{n} \lambda_{\gamma_i} f_\gamma(a_{\gamma_i})$, $a_{\gamma_i} \in P_{\gamma_i}$, $\lambda_{\gamma_i} > 0$ and $n \in \mathbb{N}$. For each $i = 1, 2, \ldots, n$, there is $\rho_i > 0$ such that $(0, a_{\gamma_i}) \in \rho_i F_\gamma^{-1}(U)$. Now if $\rho = \sum_{i=1}^{n} \lambda_{\gamma_i} \rho_i$, then

$$\left( 0, \sum_{i=1}^{n} \lambda_{\gamma_i} f_\gamma(a_{\gamma_i}) \right) \in \sum_{i=1}^{n} \rho_i \lambda_{\gamma_i} U = \left( \sum_{i=1}^{n} \rho_i \lambda_{\gamma_i} \right) U = \rho U.$$
If $U$ is an entourage in any convex quasiuniform structure on $\mathcal{P}$ making all the $f_\gamma$ u-continuous, then each $F_{\gamma}^{-1}(U)$ is an entourage in $\mathcal{P}_\gamma$ and so $U \in \mathcal{U}$, that is, $\mathcal{U}$ is the finest convex quasiuniform structure on $\mathcal{P}$ which makes each $f_\gamma$ u-continuous. \hfill \Box

The locally convex cone $\mathcal{P}$ with the preorder and (abstract) 0-neighborhood system induced by this convex quasiuniform structure is called the inductive limit of the locally convex cones $\mathcal{P}_\gamma$ by the mappings $f_\gamma$.

**Remark 3.2.**

(a) It is clear that $\{\mathcal{P}^2\}$ is a convex quasiuniform structure on each $\mathcal{P}$, which is coarsest one. And for each convex quasiuniform structure $\mathcal{U}$ on $\mathcal{P}$, $\mathcal{U}$ and $\mathcal{U} \cup \{\mathcal{P}^2\}$ are equivalent, that is, $\mathcal{U}$ is finer than $\mathcal{U} \cup \{\mathcal{P}^2\}$ and $\mathcal{U} \cup \{\mathcal{P}^2\}$ is finer than $\mathcal{U}$. Therefore, without lose of the generality, we can suppose $\mathcal{P}^2$ is a member of every convex quasiuniform structure $\mathcal{U}$ on $\mathcal{P}$. So, we can consider $\mathcal{U}$ in the proof of Theorem 3.1 to be nonempty.

(b) We cannot omit the condition (iii), or even replace finite intersection by intersection of two elements only, because in that case for $U, V \in \mathcal{U}$ and each $\gamma \in \Gamma$, there exists $W_\gamma \in \mathcal{U}_\gamma$ such that $W_\gamma \subseteq F_\gamma^{-1}(U) \cap F_\gamma^{-1}(V) = F_\gamma^{-1}(U \cap V)$, which implies that $F_\gamma(W_\gamma) \subseteq U \cap V$. But $F_\gamma(W_\gamma)$ may not satisfy condition (U3).

**Proposition 3.3.** Let $(\mathcal{P}, \mathcal{V})$ be the inductive limit of the locally convex cones $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ by the mappings $f_\gamma, \gamma \in \Gamma$. Let $f$ be a linear mapping of $\mathcal{P}$ into a locally convex cone $(\mathcal{Q}, \mathcal{W})$. Then $f$ is u-continuous if and only if for each $\gamma \in \Gamma$, $f \circ f_\gamma$ is a u-continuous mapping of $\mathcal{P}_\gamma$ into $\mathcal{Q}$.

**Proof.** One side is obvious, for the other side let $\mathcal{U}, \mathcal{U}_\gamma$ and $\mathcal{W}$ be the convex quasiuniform structures on $\mathcal{P}, \mathcal{P}_\gamma$ and $\mathcal{Q}$, respectively, and let $f \circ f_\gamma$ be u-continuous for all $\gamma \in \Gamma$. $F^{-1}(\mathcal{W}) = \{F^{-1}(W) : W \in \mathcal{W}\}$ is clearly a convex quasiuniform structure on $\mathcal{P}$. For each $W \in \mathcal{W}$, and each $\gamma \in \Gamma$,

$$F_\gamma^{-1}(F^{-1}(W)) = (f \circ f_\gamma)^{-1}(W) \in \mathcal{U}.$$ 

That is, for each $\gamma \in \Gamma$, $f_\gamma$ is u-continuous with respect to $\mathcal{U}$ and $F^{-1}(\mathcal{W})$. But $\mathcal{U}$ is the finest convex quasiuniform structure on $\mathcal{P}$ which makes each $f_\gamma$ u-continuous. Hence for each $F^{-1}(W) \in F^{-1}(\mathcal{W})$ there is $U \in \mathcal{U}$ such that $U \subseteq F^{-1}(W)$. This shows that $f$ is u-continuous with respect to $\mathcal{U}$ and $\mathcal{W}$. \hfill \Box

**Example 3.4.** An extreme case of an inductive limit topology is the quotient topology studied in [3]. For if $(\mathcal{P}, \mathcal{V})$ is a locally convex cone, $\mathcal{Q}$ is a subcone of $\mathcal{P}$ and $k : \mathcal{P} \to \mathcal{P}/\mathcal{Q}$ is the canonical mapping, then $\hat{\mathcal{V}} = \{k(v) = \hat{v} : v \in \mathcal{V}\}$ is the finest (abstract) 0-neighborhood system on $\mathcal{P}/\mathcal{Q}$ making $k$ u-continuous.

**Example 3.5.** For each $\gamma \in \Gamma$, let $\mathcal{P}_\gamma$ be a subcone of $\mathcal{P}$ such that $\text{span}\bigcup_{\gamma \in \Gamma} \mathcal{P}_\gamma = \mathcal{P}$, and the linear mappings $f_\gamma$ are all the identity mapping from $\mathcal{P}$ restricted to $\mathcal{P}_\gamma$. If each $\mathcal{P}_\gamma$ is a locally convex cone, then the convex quasiuniform structure on $\mathcal{P}$, as inductive limit of the locally convex cones $\mathcal{P}_\gamma$ by the mappings $f_\gamma$, is the finest convex quasiuniform structure on $\mathcal{P}$ which induces on each $\mathcal{P}_\gamma$ a convex quasiuniformity coarser than the given convex quasiuniformity.
Example 3.6. For each $γ ∈ Γ$, let $P_γ$ be a locally convex cone. Then $\sum_{γ ∈ Γ} P_γ = \text{span} \bigcup_{γ ∈ Γ} P_γ$, which is called the direct sum of the cones $P_γ$, is a subcone of $\times_{γ ∈ Γ} P_γ$. It is the set of those elements of $\times P_γ$ with only a finite number of nonzero coordinates. If $j_γ$ is the injection mapping of $P_γ$ into $\times P_γ$, then considering $P = \sum P_γ$ as the inductive limit of the locally convex cones $P_γ$ by the injection mappings $j_γ$, $P$ is a locally convex cone. The convex quasiuniform structure $U$ on $P$ is the finest convex quasiuniform structure on $P$ which induces the original convex quasiuniformity on each $P_γ$.

A linear mapping $t : (P, U) → (Q, W)$ is called bounded if $t(A)$ is bounded in $Q$ for each bounded set $A$ in $P$. As we saw in Section 2 if $t$ is $u$-continuous, then $t$ is bounded.

We call the locally convex cone $(P, V)$ bornological if each bounded linear mapping from $P$ to an arbitrary locally convex cone is $u$-continuous.

Proposition 3.7. An inductive limit of bornological cones is bornological.

Proof. Let $P$ be the inductive limit of the bornological cones $\{P_γ\}_{γ ∈ Γ}$ by the mappings $j_γ$. Let $Q$ be an arbitrary locally convex cone and $t : P → Q$ a linear and bounded mapping. Each $t \circ j_γ : P_γ → Q$ is bounded, hence $u$-continuous. Therefore $t$ is $u$-continuous by Proposition 3.3. □

4. Barreledness

In [4] a barrel has been defined as follows:

Definition 4.1. Let $(P, V)$ be a locally convex cone. A barrel is a convex subset $B$ of $P^2$ with the following properties:

(B1) For every $b ∈ P$ there is $v ∈ V$ such that for every $a ∈ v(b)v$ there is $λ > 0$ such that $(a, b) ∈ λB$.

(B2) For all $a, b ∈ P$ such that $(a, b) \notin B$ there is $μ ∈ P^*$ such that $μ(c) ≤ μ(d) + 1$ for all $(c, d) ∈ B$ and $μ(a) > μ(b) + 1$.

Theorem 4.2. In a locally convex cone $(P, V)$, the set of all barrels $B$ is a convex quasiuniform structure on $P$.

Proof. (U1) If $(a, a) \notin B$ for some $a ∈ P$ and $B ∈ B$, by (B2), we can find some $μ ∈ P^*$ such that $μ(a) > μ(a) + 1$, which is impossible. For (U2) let $B_1, B_2 ∈ B$. We show that $B = B_1 \cap B_2$ is a barrel. For an arbitrary $b ∈ P$, there are $v_1, v_2 \in V$ such that for every $a ∈ v_1(b)v_1$, there is $λ_1 > 0$ with $(a, b) ∈ λ_1B_1$, and for each $a' ∈ v_2(b)v_2$ there is $λ_2 > 0$ with $(a', b) ∈ λ_2B_2$. Then if $v ≤ v_1, v_2$ and $λ ≥ λ_1, λ_2$, we have $(a, b) ∈ λB$ for all $a \in v(b)v$. Now let $(a, b) \notin B$, and let $(a, b) \notin B_1$ say. Then there exists $μ ∈ P^*$ such that $μ(c) ≤ μ(d) + 1$ for all $(c, d) ∈ B_1$ (hence for all $(c, d) ∈ B$) and $μ(a) > μ(b) + 1$.

Let $B ∈ B$ and $α > 0$. For (U4) we have to show that $αB ∈ B$. For every $b ∈ P$ there is $v ∈ V$ such that for every $a ∈ v(b)v$, there is $λ_a$ such that $(a, b) ∈ λ_aB$. Take $λ'_a > 0$ such that $αλ'_a ≥ λ_a$, then for each $a ∈ v(b)v$ we have $λ'_a > 0$ such that $(a, b) ∈ λ'_a(αB)$. Now let $(a, b) \notin αB$, that is $(\frac{a}{α}, \frac{b}{α}) \notin B$. Hence by (B2) there is $μ ∈ P^*$ such that $μ(\frac{a}{α}) > \frac{b}{α} + 1$ and $μ(\frac{c}{α}) ≤ μ(d) + 1$ for all $(c, d) ∈ B$. Now $μ' = \frac{1}{α} μ ∈ P^*$ satisfies in (B2) for $αB$ in place of $B$. 

For $(U3)$, let $B \in \mathcal{B}$, $\lambda_1, \lambda_2 > 0$ and $(a, b) \in \lambda_1 B \circ \lambda_2 B$. We have to show that $(a, b) \in (\lambda_1 + \lambda_2) B$. Let $(a, b) \notin (\lambda_1 + \lambda_2) B$. By $(U4)$ and $(B2)$ there exists $\mu' \in \mathcal{P}^*$ such that $\mu'(a) > \mu(b) + 1$ or $\mu(a) > \mu(b) + \lambda_1 + \lambda_2$ for $\mu = (\lambda_1 + \lambda_2) \mu' \in \mathcal{P}^*$ and $\mu(c) \leq \mu(d) + \lambda_1 + \lambda_2$ for all $(c, d) \in (\lambda_1 + \lambda_2) B$. Since $(a, b) \notin \lambda_1 B \circ \lambda_2 B$, there is $c \in \mathcal{P}$ such that $(a, c) \in \lambda_1 B$ and $(c, b) \in \lambda_2 B$. Hence $(a + c, c + b) = (a, c) + (c, b) \in (\lambda_1 + \lambda_2) B$. Hence $\mu(a + c) \leq \mu(c + b) + \lambda_1 + \lambda_2$ or $\mu(a) \leq \mu(b) + \lambda_1 + \lambda_2$ ($\mu(c)$ is finite) which contradicts with $\mu(a) > \mu(b) + \lambda_1 + \lambda_2$. \hfill \square

We recall Definition II.2.13 of [2].

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is said to be tightly covered by its bounded elements if for all $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ and $a \notin v(b)$ (or $a \notin b + v$) there is some bounded element $a' \in \mathcal{P}$ such that $a' \leq a$ and $a' \notin v(b)$.

If $(\mathcal{P}, \mathcal{V})$ is tightly covered by its bounded elements and $a \notin b + v$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$, there is $\mu \in \mathcal{V}^0$ such that $\mu(a) > \mu(b) + 1$. For there is a bounded element $a' \in \mathcal{P}$ such that $a' \leq a$ with $a' \notin b + \rho v$ for some $\rho > 1$. By Lemma II.11 of Chapter II of [2], there is $\mu \in \mathcal{V}^0$ such that $\mu(a') > \mu(b) + 1$. Since $\mu$ is monotone, we have $\mu(a) \geq \mu(a')$, i.e. $\mu(a) > \mu(b) + 1$.

This renders:

**Proposition 4.3.** If locally convex cone $(\mathcal{P}, \mathcal{V})$ is tightly covered by its bounded elements, then $\tilde{v} = \{(a, b) \in \mathcal{P}^2; \text{ } a \leq b + v\}$ is a barrel for every $v \in \mathcal{V}$.

**Proof.** Let $b \in \mathcal{P}$. For each $a \in v(b)v$, $v \in \mathcal{V}$, we have $(a, b) \in \tilde{v}$, hence $(B1)$. If $(a, b) \notin \tilde{v}$, that is $a \notin b + v$, there is $\mu \in \mathcal{V}^0$ such that $\mu(a) > \mu(b) + 1$, and $\mu(c) \leq \mu(d) + 1$ for all $(c, d) \in \tilde{v}$, hence $(B2)$. \hfill \square

**Lemma 4.4.** The inverse image of a barrel under a $u$-continuous linear mapping is a barrel.

**Proof.** Let $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ be two locally convex cones, $t : \mathcal{P} \rightarrow \mathcal{Q}$ a $u$-continuous linear mapping of $\mathcal{P}$ into $\mathcal{Q}$, and $B$ be a barrel in $\mathcal{Q}$. Let $T = t \times t$ and $A = T^{-1}(B)$. We show that $A$ is barrel. Let $b \in \mathcal{P}$, then $t(b) \in \mathcal{Q}$ and there is $w \in \mathcal{W}$ such that for each $a' \in w(t(b))w$ there is $\lambda > 0$ such that $(a', t(b)) \in \lambda B$. Since $t$ is $u$-continuous, there is $v \in \mathcal{V}$, such that $t(v(b)v) \subseteq w(t(b))w$. Hence for each $a \in v(b)v$, $(a, t(b)) \in \lambda B$; that is $(a, b) \in \lambda A$, and $(B1)$ is satisfied.

Now let $a, b \in \mathcal{P}$, $(a, b) \notin A$. Then $(t(a), t(b)) \notin B$, so there is $\mu \in \mathcal{Q}^*$ such that $\mu(t(a)) > \mu(t(b)) + 1$ and $\mu(t(c')) \leq \mu(t(d')) + 1$ for all $(c', d') \in B$. Clearly $\mu \circ t \in \mathcal{P}^*$, and for each $(c, d) \in A$ we have $t(c), t(d) \in t(A) = B$, hence $\mu(t(c)) \leq \mu(t(d)) + 1$, or $(\mu \circ t)(c) \leq (\mu \circ t)(d) + 1$. \hfill \square

A barreled cone is defined in [4] as follows:

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is said to be barreled if for every barrel $B \subseteq \mathcal{P}^2$ and every $b \in \mathcal{P}$ there is a neighborhood $v \in \mathcal{V}$ and $\lambda > 0$ such that $(a, b) \in \lambda B$ for all $a \in v(b)v$.

**Definition 4.5.** Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and $\tilde{V}$ be the convex quasiuniform structure generated by $\mathcal{V}$. $\mathcal{P}$ is called upper-barreled if for every barrel $B \subseteq \mathcal{P}^2$, there is $\tilde{v} \in \tilde{V}$ such that $\tilde{v} \subseteq B$. 
Remark 4.6. We saw in Theorem 4.2 that the set of all barrels \( \mathcal{B} \) for the locally convex cone \( (\mathcal{P}, \mathcal{V}) \) is a convex quasiuniform structure. Now if \( (\mathcal{P}, \mathcal{V}) \) is upper-barreled, for each \( B \in \mathcal{B} \) there is \( \tilde{v} \in \tilde{V} \) such that \( \tilde{v} \subseteq B \). This means that \( \tilde{V} \) is finer than \( \mathcal{B} \). On the other hand, since \( (\mathcal{P}, \mathcal{V}) \) is a locally convex cone, i.e. each element of \( v \in \mathcal{N} \). Bourbaki, Elements of Mathematics, General Topology, Springer-Verlag, 1989.

Example 4.7. [4, p. 1975] Every full locally convex cone \((\mathcal{P}, \mathcal{V})\) is upper-barreled. For, if \( B \subseteq \mathcal{P}^2 \) is a barrel, for \( 0 \in P \) there is \( v \in V \) such that for every \( a \in v(0)v \) there is \( \lambda > 0 \) such that \( (a, 0) \in \lambda B \). Since \( v \in \mathcal{P} \) and \( v \in v(0)v \), there is \( \lambda > 0 \) such that \( (v, 0) \in \lambda B \). Now let \( (a, b) \in \tilde{v} \) and \( (a, b) \notin \lambda B \). Then there is \( \mu \in P^* \) such that \( \mu(a) > \mu(b) + \lambda \) and \( \mu(v) \leq \lambda \). But \( a \leq b + v \) implies \( \mu(a) \leq \mu(b) + \mu(v) \leq \mu(b) + \lambda \), which is a contradiction. Hence \( \tilde{v} \subseteq \lambda B \) or \( (1/\lambda) \tilde{v} \subseteq B \).

On the other hand, it is clear that each upper-barreled cone is barreled. In fact for every \( b \in \mathcal{P} \) and every \( a \in v(b)v \subseteq v(b) \), where \( \tilde{V} \subseteq B \), we have \( (a, b) \in B \). In particular every full locally convex cone is barreled.

We do not know if there is a barreled locally convex cone which is not upper-barreled.

Theorem 4.8. An inductive limit of upper-barreled cones is upper-barreled.

Proof. Let \((\mathcal{P}, \mathcal{V})\) be the inductive limit of the upper-barreled cones \((\mathcal{P}_\gamma, \mathcal{V}_\gamma)\) by the mappings \( f_\gamma \). Let \( B \subseteq \mathcal{P}^2 \) be a barrel for \( \mathcal{P} \). By Lemma 4.4, \( F_\gamma^{-1}(B) \) is a barrel for \( \mathcal{P}_\gamma \) (\( F_\gamma = f_\gamma \times f_\gamma \)). Since \( \mathcal{P}_\gamma \) is upper-barreled, there is \( v_\gamma \in \mathcal{V}_\gamma \) such that \( \tilde{v}_\gamma \subseteq F_\gamma^{-1}(B) \). In fact each \( f_\gamma \) is \( u \)-continuous under the convex quasiuniform structure which the barrels generate on \( \mathcal{P} \). But \( \tilde{V} = \{ \tilde{v}: v \in \mathcal{V} \} \) is the finest convex quasiuniform structure which makes each \( f_\gamma \) \( u \)-continuous, so there is \( \tilde{v} \in \tilde{V} \) such that \( \tilde{v} \subseteq B \). □

Corollary 4.9. A quotient of an upper-barreled cone is upper-barreled.

Proposition 4.10. Let \((\mathcal{P}, \mathcal{V})\) and \((\mathcal{Q}, \mathcal{W})\) be locally convex cones and \( t \) be a linear mapping from \( \mathcal{P} \) into \( \mathcal{Q} \). If \((\mathcal{P}, \mathcal{V})\) is upper-barreled and \((\mathcal{Q}, \mathcal{W})\) is tightly covered by its bounded elements, then \( t \) is \( u \)-continuous.

Proof. Let \( w \in \mathcal{W} \) and \( T = t \times t \), \( \tilde{w} \) is a barrel for \( \mathcal{Q} \) by Proposition 4.3. Then \( T^{-1}(\tilde{w}) \) is a barrel for \( \mathcal{P} \) by Lemma 4.4. Since \((\mathcal{P}, \mathcal{V})\) is upper-barreled, there is a \( v \in \mathcal{V} \) such that \( \tilde{v} \subseteq T^{-1}(\tilde{w}) \). □

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