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Cauchy–Riemann equations and J-symplectic forms [☆]

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Abstract

Let (Σ, j) be a Riemann surface. The almost complex manifolds (M, J) for which the J-holomorphic curves $\phi: \Sigma \to M$ are of variational type, are characterized. This problem is related to the existence of a vertically non-degenerate closed complex 3-form on $\Sigma \times M$ (see Theorem 4.3 below), which determines a family of J-symplectic structures on (M, J) parametrized by Σ.

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1. Introduction and preliminaries

Below we study the global aspects related to the variational character of Cauchy–Riemann equations from the Hamiltonian point of view. In the local setting, the solution to the problem under consideration here is particularly simple (see [8]): Cauchy–Riemann equations for holomorphic maps $\phi: \mathbb{C} \to \mathbb{C}^k$ are variational if and only if k is even. The global version of this statement leads one to characterize when the space of J-holomorphic curves $\phi: (\Sigma, j) \to (M, J)$ from a Riemann surface into an almost complex manifold, coincides with the space of extremals of a variational principle, i.e. when the Cauchy–Riemann equations can be considered as the Euler-Lagrange equations of a Lagrangian density. We show (see Theorem 4.3 and Remark 4.4) that the solution to this problem is closely related to the existence of a holomorphic family of J-symplectic structures on (M, J) parametrized by Σ.

We recall that the inverse problem of the calculus of variations consists in the characterization of the systems of differential equations that are 'equivalent' to the Euler–Lagrange equations $E(L)$ of a Lagrangian L. From the early work [4] by J. Douglas (also see [9]), two ways of defining the notion of equivalence in dealing with such a characterization have been considered. The first one (e.g., see [2,10,12]), which is better formulated in terms of the Lagrangian formalism, is to consider the system of differential equations as being the components of a certain

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differential operator D and to ask whether a Lagrangian L exists such that $D = E(L)$. The second way, more adapted to the Hamiltonian formalism, is to consider linear combinations of the given equations with suitable variational multipliers and then, to ask whether the new differential equations coincide with the Euler–Lagrange equations of a Lagrangian; for example, see [1]. In this paper we apply the second method to the Cauchy–Riemann equations.

Next, we recall the definition of a module of variational type in the Hamiltonian formulation of the inverse problem of the variational calculus for first-order partial differential equations (e.g., see [7]); namely,

Definition 1.1. Let $p: M \to N$ be a fibred manifold, i.e., p is a surjective submersion. A $C^\infty(M)$-module $\mathcal{M} \subset \Omega^n(M)$ is said to be of *variational type* if there exists a closed $(n+1)$-form Ω on M such that $\mathcal{M}$ coincides with the image of the map $\mathfrak{X}^v(M) \to \Omega^n(M)$, $X \mapsto X \lrcorner \Omega$, where $\mathfrak{X}^v(M)$ denotes the space of p-vertical vector fields on M.

A differential 2-form on $\mathbb{C} \times \mathbb{C}^k$ is called a *Cauchy–Riemann form* if it belongs to the module spanned over $C^\infty(\mathbb{C} \times \mathbb{C}^k)$ by the forms

$$du^\alpha \wedge dx - dv^\alpha \wedge dy,\ du^\alpha \wedge dy + dv^\alpha \wedge dx, \quad \alpha = 1, \ldots, k,$$

where $z = x + iy$ is the complex coordinate in $\mathbb{C}$ and $w^\alpha = u^\alpha + iv^\alpha$ are the complex coordinates in $\mathbb{C}^k$. We denote this module by $CR(\mathbb{C}, \mathbb{C}^k)$.

From the identity

$$dw^\alpha \wedge dz = (du^\alpha \wedge dx - dv^\alpha \wedge dy) + i(du^\alpha \wedge dy + dv^\alpha \wedge dx),$$

we conclude that the Cauchy–Riemann forms are the real and imaginary parts of the forms $f_\alpha dw^\alpha \wedge dz$, $f_\alpha \in C^\infty(\mathbb{C} \times \mathbb{C}^k, \mathbb{C})$.

Taking the following equations:

$$\left(\frac{\partial \varphi^\alpha}{\partial x} - \frac{\partial \psi^\alpha}{\partial y}\right) dx \wedge dy = \phi^*(du^\alpha \wedge dy + dv^\alpha \wedge dx),$$
$$\left(\frac{\partial \varphi^\alpha}{\partial y} + \frac{\partial \psi^\alpha}{\partial x}\right) dx \wedge dy = \phi^*(-du^\alpha \wedge dx + dv^\alpha \wedge dy),$$

into account, we conclude that a smooth map $\phi: \mathbb{C} \to \mathbb{C}^k$, with components $\phi^\alpha = \varphi^\alpha + i\psi^\alpha$, is holomorphic if and only if $\phi^*(\omega) = 0$ for every Cauchy–Riemann form $\omega \in CR(\mathbb{C}, \mathbb{C}^k)$.

In the global setting, i.e., for J-holomorphic maps from a Riemann surface (Σ, j) into an almost complex manifold (M, J) the solution to the problem of the variational character of Cauchy–Riemann equations, is more complex. In fact, it is related to the existence of J-symplectic structures on M, as proved in Theorem 4.3 below.

If k is even, say $k = 2r$, then the module $CR(\mathbb{C}, \mathbb{C}^k)$ is proved to be variational by means of the 3-form on $\mathbb{C} \times \mathbb{C}^k$ given by (see [8]),

$$\Omega = \left(\sum_{\alpha=1}^{r} du^\alpha \wedge du^{r+\alpha} - \sum_{\alpha=1}^{r} dv^\alpha \wedge dv^{r+\alpha}\right) \wedge dx - \left(\sum_{\alpha=1}^{r} du^\alpha \wedge dv^{r+\alpha} + \sum_{\alpha=1}^{r} dv^\alpha \wedge du^{r+\alpha}\right) \wedge dy.$$

Clearly, $\Omega = d\Theta$, where

$$\Theta = \frac{\partial L}{\partial u_x^{r+\alpha}} \theta^{r+\alpha} \wedge dy - \frac{\partial L}{\partial u_y^{r+\alpha}} \theta^{r+\alpha} \wedge dx + \frac{\partial L}{\partial v_x^{r+\alpha}} \eta^{r+\alpha} \wedge dy - \frac{\partial L}{\partial v_y^{r+\alpha}} \eta^{r+\alpha} \wedge dx + L\, dx \wedge dy$$

is the Poincaré–Cartan form of the Lagrangian

$$L = \sum_{\alpha=1}^{r} (-u^\alpha u_y^{r+\alpha} + v^\alpha v_y^{r+\alpha} - u^\alpha v_x^{r+\alpha} - v^\alpha u_x^{r+\alpha}),$$

and $\theta^\alpha = du^\alpha - u_x^\alpha dx - u_y^\alpha dy$, $\eta^\alpha = dv^\alpha - v_x^\alpha dx - v_y^\alpha dy$ are the standard contact 1-forms on the first jet bundle $J^1(\mathbb{C}, \mathbb{C}^k)$. Also note that the Euler–Lagrange equations of L coincide with Cauchy–Riemann equations.

The main goal of this paper is to obtain the intrinsic properties of such a form Ω in the case of an arbitrary almost complex structure.

2. J-symplectic structures

Let V be an $\mathbb{R}$-vector space of dimension n endowed with a complex structure J; i.e., $J: V \to V$ is an endomorphism such that $J^2 = -1$. As usual (e.g., see [11, IX, §1]), we set $V^c = V \otimes_{\mathbb{R}} \mathbb{C}$, $V^{1,0} = \ker(J - i)$, $V^{0,1} = \ker(J + i)$, and, similarly, for the dual space, $V_{1,0} = (V^*)^{1,0}$, $V_{0,1} = (V^*)^{0,1}$. Taking account of the canonical decomposition of a complex-valued $\mathbb{R}$-linear form into a J-linear form and a J-antilinear form, we obtain (cf. [11, IX, Proposition 1.7]) $\bigwedge^r V^{*c} = \bigoplus_{p+q=r} \bigwedge^{p,q} V^{*c}$, where $\bigwedge^{p,q} V^{*c} = \bigwedge^p V_{1,0} \wedge \bigwedge^q V_{0,1}$ is the space of forms of type (p,q), or even, the space of forms p times J-linear, q times J-antilinear. In particular, the forms of type $(p,0)$ are the J-multilinear p-forms; that is, the elements in $\bigwedge^p V_{1,0}$.

Proposition 2.1. *We have*

(a) *If $\omega = \omega_1 + i\omega_2 \in \bigwedge^p V^{*c}$, with $\omega_1, \omega_2 \in \bigwedge^p V^*$, then ω is of type $(p,0)$ if and only if $x \lrcorner \omega_2 = -Jx \lrcorner \omega_1$, $\forall x \in V$.*
(b) *If $\omega \in \bigwedge^p V^{*c}$ is of type $(p,0)$ and $\omega_1 = \mathrm{Re}(\omega)$, then for every $x \in V$ we have $x \lrcorner \omega = x \lrcorner \omega_1 - i(Jx) \lrcorner \omega_1$. In particular,* $\ker(\omega) = \ker(\omega_1) \cap J\ker(\omega_1)$. *If $p > 1$, $J\ker(\omega_1) = \ker(\omega_1)$; hence* $\ker(\omega) = \ker(\omega_1)$.
(c) *If $\omega_1 \in \bigwedge^p V^*$, then an element $\omega \in \bigwedge^p V^{*c}$ of type $(p,0)$ exists such that $\omega_1 = \mathrm{Re}(\omega)$ if and only if $(Jx) \lrcorner x \lrcorner \omega_1 = 0$, $\forall x \in V$.*

Definition 2.2. A *J-symplectic structure* on (V, J) is a non-degenerate form $\omega \in \bigwedge^{2,0} V^{*c}$.

Remark 2.3. By virtue of Proposition 2.1, ω decomposes as $\omega = \omega_1 + i\omega_2$, where $\omega_1, \omega_2 \in \bigwedge^2 V^*$ and $\omega_2(x,y) = -\omega_1(Jx,y)$, $\omega_1(Jx,x) = 0$, $\forall x \in V$. Conversely, given $\omega_1 \in \bigwedge^2 V^*$ satisfying $\omega_1(Jx,x) = 0$, $\forall x \in V$, then the 2-form defined by $\omega(x,y) = \omega_1(x,y) - i\omega_1(Jx,y)$, is J-bilinear.

Moreover, as $\ker(\omega) = \ker(\omega_1)$, the form ω is non-degenerate if and only if ω_1 is non-degenerate.

Remark 2.4. If V admits a complex structure J, then $k = \dim_{\mathbb{R}} V = 2\dim_{\mathbb{C}}(V,J)$ is even. In addition, if (V,J) admits a J-symplectic structure ω, then as ω is a J-complex non-degenerate 2-form the complex dimension of (V,J) must also be even, i.e., $\dim_{\mathbb{C}}(V,J) = 2r$; hence $\dim_{\mathbb{R}} V = 4r$.

Let (M,J) be an almost-complex manifold. The sheaf of germs of smooth sections of the vector bundles $\bigwedge^p T^{*c}M$, $\bigwedge^{p,q} T^{*c}M$ are denoted by $\mathcal{E}^p_M$, $\mathcal{E}^{p,q}_M$, respectively.

Remark 2.5. Assume $\omega \in \mathcal{E}^p(M)$, with $\omega = \omega_1 + i\omega_2$, $\omega_1, \omega_2 \in \Omega^p(M)$. Then $d\omega = 0$ if and only if $d\omega_1 = d\omega_2 = 0$. Furthermore, if $\omega \in \mathcal{E}^{p,0}(M)$, then we have $\omega_2(X,Y,\ldots) = -\omega_1(JX,Y,\ldots)$, and hence ω is completely determined by ω_1. Taking this fact into account we could be led to think that $d\omega_1 = 0$ implies $d\omega = 0$, but this is not the case. For example, if we set $\omega = \bar{z}dz$, then $d\omega = d\bar{z} \wedge dz = 2i\, dx \wedge dy$; hence $d\omega_1 = 0$, but $d\omega \neq 0$.

Definition 2.6. A *J-symplectic form* on M is a non-degenerate closed 2-form $\omega \in \mathcal{E}^{2,0}(M)$.

Remark 2.7. If J is integrable, then every J-symplectic form is holomorphic, and we recover the definition of a complex symplectic structure given in [3, §14.14]. We also recall (see [3, Proposition 14.15]) that every hyperkählerian manifold admits a canonical J-symplectic form.

3. Two spaces of complex forms on $(\Sigma, j) \times (M, J)$

Let (M,J) be a almost-complex manifold and let (Σ, j) be a Riemann surface. We recall that j is integrable (e.g., see [5, p. 126]). We want to study the variational structure on the space of J-holomorphic curves, i.e., smooth maps $\phi: \Sigma \to M$ such that $\phi_* \circ j = J \circ \phi_*$ (e.g., see [6]).

Let $\pi_1: \Sigma \times M \to \Sigma$, $\pi_2: \Sigma \times M \to M$ be the canonical projections onto the factors. We identify the sections of π_1 to the maps $\Sigma \to M$. For the sake of simplicity, sometimes we denote by the same symbol a differential form on Σ or on M and its pull-back to $\Sigma \times M$.

For every pair of integers s, t such that $0 \leqslant s \leqslant 2$, $0 \leqslant t$, let $\mathcal{W}^{s,t}_{\Sigma\times M} \subset \mathcal{E}^{s+t}_{\Sigma\times M}$ be the subsheaf of germs of complex valued $(s+t)$-forms on $\Sigma \times M$, which are s-horizontal with respect to π_1 and t-horizontal with respect to π_2; precisely, $\mathcal{W}^{s,t}_{\Sigma\times M} = \pi_1^*\mathcal{E}^s_\Sigma \wedge \pi_2^*\mathcal{E}^t_M$. For every $\omega \in \mathcal{W}^{s,t}_{\Sigma\times M}$, there exist unique germs of differential forms $d_1\omega \in \mathcal{W}^{s+1,t}_{\Sigma\times M}$, $d_2\omega \in \mathcal{W}^{s,t+1}_{\Sigma\times M}$ such that, $d\omega = d_1\omega + d_2\omega$.

In what follows, we denote by $W^{s,t}$, $E^{p,q}$ the space of global sections of $\mathcal{W}^{s,t}$, $\mathcal{E}^{p,q}$, respectively; i.e., $W^{s,t} = \mathcal{W}^{s,t}(\Sigma \times M)$, $E^{p,q} = \mathcal{E}^{p,q}(\Sigma \times M)$.

3.1. The space $W^{0,p+1} \cap E^{p+1,0}$

Above we have considered two bigraduations in the space of complex valued differential forms on $\Sigma \times M$: The $\mathbb{C}$-linear-$\mathbb{C}$-antilinear bigraduation, denoted by $\mathcal{E}^r_{\Sigma\times M} = \bigoplus_{r=p+q} \mathcal{E}^{p,q}_{\Sigma\times M}$, and the π_1-horizontal-π_2-horizontal bigraduation, denoted by $\mathcal{E}^r_{\Sigma\times M} = \bigoplus_{r=s+t} \mathcal{W}^{s,t}_{\Sigma\times M}$. As $\dim_{\mathbb{C}} \Sigma = 1$, we have

$$\mathcal{E}^{p+1,0}_{\Sigma\times M} = \left(\mathcal{W}^{0,p+1}_{\Sigma\times M} \cap \mathcal{E}^{p+1,0}_{\Sigma\times M}\right) \oplus \left(\mathcal{W}^{1,p}_{\Sigma\times M} \cap \mathcal{E}^{p+1,0}_{\Sigma\times M}\right).$$

The elements in $W^{0,p+1} \cap E^{p+1,0}$ are the sections of the sheaf $\pi_2^*\mathcal{E}^{p+1}_M$, i.e., the J-multilinear $(p+1)$-forms on M with coefficients on $\Sigma \times M$.

For each $z \in \Sigma$, let $\iota_z : M \to \Sigma \times M$ be the immersion $\iota_z(x) = (z, x)$.

If $\alpha \in W^{0,p+1} \cap E^{p+1,0}$, then we set $\alpha_z = \iota_z^*\alpha \in \mathcal{E}^{p+1,0}(M)$. Hence, the form α can be viewed as a family $\alpha_z \in \mathcal{E}^{p+1,0}(M)$ of $(p+1)$-forms on each fiber $(\pi_1)^{-1}(z)$, depending smoothly on $z \in \Sigma$.

By taking the differential of $\alpha \in W^{0,p+1} \cap E^{p+1,0}$ we obtain $d\alpha = d_1\alpha + d_2\alpha$, with $d_1\alpha \in W^{1,p+1}$, $d_2\alpha \in W^{0,p+2}$, where d_1, d_2 are exterior differentials of factor Σ and M. Locally, we have

$$d_1\alpha = dz \wedge \frac{\partial\alpha}{\partial z} + d\bar{z} \wedge \frac{\partial\alpha}{\partial\bar{z}},$$
$$\frac{\partial\alpha}{\partial z}, \frac{\partial\alpha}{\partial\bar{z}} \in W^{0,p+1} \cap E^{p+1,0},$$

where $\partial\alpha/\partial z$, $\partial\alpha/\partial\bar{z}$ are uniquely determined by this property. Hence, $d_1\alpha = 0$ if and only if, $\partial\alpha/\partial z = \partial\alpha/\partial\bar{z} = 0$. In other words, $d_1\alpha = 0$ is equivalent to saying that α is independent of $z \in \Sigma$ and hence, $\alpha = (\pi_2)^*\alpha'$ for some $\alpha' \in \mathcal{E}^{p+1,0}(M)$.

As for $d_2\alpha$, we have $d\alpha_z = \iota_z^*(d\alpha) = \iota_z^*(d_2\alpha) + \iota_z^*(d_1\alpha) = \iota_z^*(d_2\alpha)$, because $\iota_z^* dz = \iota_z^* d\bar{z} = 0$, and since $d_2\alpha$ belongs to $W^{0,p+2}$, we conclude that $d_2\alpha = 0$ if and only if $d\alpha_z = 0$, $\forall z \in \Sigma$. Hence, $\alpha \in W^{0,p+1} \cap E^{p+1,0}$ is closed if and only if $\alpha = (\pi_2)^*\alpha'$ for some $\alpha' \in \mathcal{E}^{p+1,0}(M)$ with $d\alpha' = 0$.

3.2. The space $W^{1,p} \cap E^{p+1,0}$

Next, we consider the space $W^{1,p} \cap E^{p+1,0}$; i.e., the J-multilinear complex valued 1-horizontal $(p+1)$-forms on $\Sigma \times M$.

If (U, z) is a holomorphic chart on Σ and $\omega \in W^{1,p} \cap E^{p+1,0}$, for all $(z, x) \in U \times M$ we have $\omega|_{U\times M}(z, x) = \mu(z, x) \wedge dz$, where $\mu \in W^{0,p} \cap E^{p,0}$ is uniquely determined by this property. We refer to the expression above as the 'local representation of ω with respect to (U, z)', and we write it as $\omega|_U = \mu \wedge dz$.

Definition 3.1. A form $\omega \in W^{1,p} \cap E^{p+1,0}$ is said to be *vertically non-degenerate* if $\ker(\omega) \cap \ker(\pi_1)_* = 0$.

With the previous notations, the definition above is equivalent to saying $\ker \mu_z = 0$, $\forall z \in U$ and any chart in Σ.

Taking the differential of ω, and recalling that μ belongs to $W^{0,p} \cap E^{p,0}$, we have

$$d\omega|_U = d\mu \wedge dz = d_2\mu \wedge dz + d_1\mu \wedge dz = d_2\mu \wedge dz + \frac{\partial\mu}{\partial\bar{z}} d\bar{z} \wedge dz,$$

with $d_2\mu \in W^{0,p+1}$, $d_1\mu \in W^{1,p}$. Hence ω is closed if and only if for any chart (U, z) we have $d_2\mu = 0$, $\partial\mu/\partial\bar{z} = 0$, on U; or equivalently, $d\mu_z = 0$, $\partial\mu/\partial\bar{z} = 0$, $\forall z \in U$.

Remark 3.2. From the previous equations, it follows that a closed form ω in $\mathcal{W}^{1,p} \cap \mathcal{E}^{p+1,0}$ can be interpreted as a holomorphic (i.e., $\partial\mu/\partial\bar{z} = 0$) 1-parameter family of closed 2-forms (μ_z) on M. If we consider the local representations of ω with respect to two local charts (U, z) and (U', z') of Σ, then we obtain $\mu \wedge dz = \mu' \wedge dz' = (\partial z'/\partial z)\mu' \wedge dz$, and so, $\mu = (\partial z'/\partial z)\mu'$. Hence μ transforms itself as a section of a vector bundle over Σ with fiber $\mathcal{Z}^{p,0}(M) \subset \mathcal{E}^{p,0}(M)$, the set of closed p-forms of type $(p, 0)$ on M. If (M, J) is a compact complex manifold, this space is of finite dimension by virtue of the finiteness theorem for elliptic complexes (e.g., see [13, Example 5.5]) and the condition $\partial\mu/\partial\bar{z} = 0$ means that ω corresponds to a holomorphic section of this bundle.

Proposition 3.3. *For every $\omega \in \mathcal{W}^{1,p} \cap \mathcal{E}^{p+1,0}$ the following conditions are equivalent*:

(a) $d\omega = 0$,
(b) $d(\mathrm{Re}(\omega)) = 0$.

Proof. Obviously, (a) implies (b). Conversely, assume (b) holds. If (U, z) is a local coordinate domain on Σ and the local representation of ω with respect to (U, z), is $\omega|_U = \mu \wedge dz$, then we have

$$d\omega = d\mu \wedge dz = d_2\mu \wedge dz + \frac{\partial\mu}{\partial\bar{z}} \wedge d\bar{z} \wedge dz,$$

with $d_2\mu \in \mathcal{W}^{0,p+1}$ and $\partial\mu/\partial\bar{z} \in \mathcal{W}^{0,p} \cap \mathcal{E}^{p,0}$. Hence

$$\begin{aligned} 0 &= d\big(\mathrm{Re}(\omega|_U)\big) \\ &= \mathrm{Re}(d\omega|_U) \\ &= \mathrm{Re}(d_2\mu \wedge dz) + \mathrm{Re}\left(\frac{\partial\mu}{\partial\bar{z}} \wedge d\bar{z} \wedge dz\right). \end{aligned}$$

Hence $\mathrm{Re}(d_2\mu \wedge dz) = \mathrm{Re}(\partial\mu/\partial\bar{z} \wedge d\bar{z} \wedge dz) = 0$. Moreover, we have

$$0 = \mathrm{Re}(d_2\mu \wedge dz) = \mathrm{Re}(d_2\mu) \wedge dx - \mathrm{Im}(d_2\mu) \wedge dy.$$

Hence $\mathrm{Re}(d_2\mu) = \mathrm{Im}(d_2\mu) = 0$, and then, $d_2\mu = 0$. In addition, we have

$$\begin{aligned} 0 &= \mathrm{Re}\left(\frac{\partial\mu}{\partial\bar{z}} \wedge d\bar{z} \wedge dz\right) \\ &= -2\,\mathrm{Im}\left(\frac{\partial\mu}{\partial\bar{z}}\right) dx \wedge dy. \end{aligned}$$

Hence $\mathrm{Im}(\partial\mu/\partial\bar{z}) = 0$. As $\partial\mu/\partial\bar{z} \in \mathcal{E}^{2,0}$, this implies $\partial\mu/\partial\bar{z} = 0$. Hence, for every chart (U, z) we have $d\omega|_U = 0$ and consequently $d\omega = 0$. □

4. Cauchy–Riemann forms

Definition 4.1. The 2-forms in $CR(\Sigma, M) = \{\mathrm{Re}(\beta)\colon\ \beta \in W^{1,1} \cap E^{2,0}\}$ are called the *Cauchy–Riemann forms* for maps from Σ to M.

Proposition 4.2. *Let $\phi\colon \Sigma \to M$ be a smooth map. The following conditions are equivalent*:

(a) *ϕ is J-holomorphic.*
(b) $\phi^*(\alpha) \in \mathcal{E}^{1,0}(\Sigma)$, $\forall \alpha \in \mathcal{E}^{1,0}(M)$.
(c) $\phi^*(\beta) = 0$, $\forall \beta \in W^{1,1} \cap E^{2,0}$.

Hence, a smooth map $\phi\colon \Sigma \to M$ is J-holomorphic if and only if $\phi^*(\beta) = 0$, $\forall \beta \in CR(\Sigma, M)$.

Proof. (a) $\Longrightarrow$ (b). Let ϕ be J-holomorphic. For every $\beta \in \mathcal{E}^{1,0}(M)$, $X \in \mathfrak{X}(\Sigma)$, we have $\phi^*(\beta)(jX) = \beta(\phi_*(jX)) = \beta(J\phi_*(X)) = i\beta(\phi_*(X)) = i\phi^*(\beta)(X)$. Hence $\phi^*(\beta) \in \mathcal{E}^{1,0}(\Sigma)$.

(b) $\Longrightarrow$ (a). Assume ϕ is not J-holomorphic. Then, there exist $z \in \Sigma$, $X_z \in T_z\Sigma$ such that $Y = \phi_*(jX_z) - J_p\phi_*(X_z) \neq 0$. Accordingly, there exists $\beta_p \in (T_{1,0})_z\Sigma$ such that $\beta_z(Y) \neq 0$, and by extending the covector β_z to a form $\beta \in \mathcal{E}^{1,0}(M)$ and the tangent vector X_z to a vector field $X \in \mathfrak{X}(\Sigma)$, we have

$$\phi^*(\beta)(jX) - i\phi^*(\beta)(X) = \beta\big(\phi_*(jX) - J\phi_*(X)\big) \neq 0,$$

and therefore, $\phi^*(\beta) \notin \mathcal{E}^{1,0}(\Sigma)$.

(b) $\Longleftrightarrow$ (c). Let $\beta|_U = \mu \wedge dz$, $\mu \in \mathcal{W}^{0,1} \cap \mathcal{E}^{1,0}$, be the local representation of $\beta \in W^{1,1} \cap E^{2,0}$ with respect to a local coordinate domain (U, z) in Σ. We have $\phi|_U^*(\beta|_U) = 0$ if and only if $\phi|_U^*(\mu) \in \mathcal{E}^{1,0}(\Sigma)$. From this, it follows the equivalence between (b) and (c). □

Theorem 4.3. *The Cauchy–Riemann forms $CR(\Sigma, M)$ (see Definition 4.1) for maps from Σ into M are of variational type according to Definition 1.1, if and only if there exists a closed and vertically non-degenerate 3-form $\omega \in W^{1,2} \cap E^{3,0}$ (see Definition 3.1).*

Moreover, if ω is exact, say $\omega = d\Theta$ for certain $\Theta \in \Omega^2(\Sigma \times M)$, then the horizontal component of Θ is a Lagrangian density whose Euler–Lagrange equations are the Cauchy–Riemann equations.

Proof. If a closed and vertically non-degenerate form $\omega \in W^{1,2} \cap E^{3,0}$ exists, then we define $\Omega = \mathrm{Re}(\omega)$. Certainly Ω is closed. Furthermore, as ω is vertically non-degenerate, the complex vector-bundle homomorphism

$$V(\pi_1) \cong \pi_2^*T(M) \to \pi_1^*T_{1,0}(\Sigma) \otimes \pi_2^*T_{1,0}(M),$$
$$X \mapsto X \lrcorner \omega,$$

is injective and since $\mathrm{rk}(\pi_1^*T_{1,0}(\Sigma) \otimes \pi_2^*T_{1,0}(M)) = \mathrm{rk}_{\mathbb{C}}(TM)$, it is a isomorphism. Hence, for every $\beta \in W^{1,1} \cap E^{2,0}$ there exists a vertical vector field X_β such that $\beta = X_\beta \lrcorner \omega$. By taking the real part on both sides of this equation, we have $\mathrm{Re}(\beta) = X_\beta \lrcorner \mathrm{Re}(\omega) = X_\beta \lrcorner \Omega$, and we can conclude the first part of the proof.

Conversely, assume Ω is a variational form for the Cauchy–Riemann equations. If (U, z) is a chart in Σ, and $\beta \in W^{1,1} \cap E^{2,0}$, then we have $\beta|_U = \alpha \wedge dz$, with $\alpha \in W^{0,1} \cap E^{1,0}$.

Clearly the map $\Phi: \pi_2^*T_{1,0}M \to \bigwedge^2 T^*(U \times M)$, $\alpha \mapsto \mathrm{Re}(\alpha \wedge dz)$ is injective. From the assumption, the image of the map $\pi_2^*TM \to \bigwedge^2 T^*(U \times M)$, $X \mapsto X \lrcorner \Omega$ contains $\mathrm{im}\,\Phi$, and since $\mathrm{rk}(\mathrm{im}\,\Phi) = \mathrm{rk}(T_{1,0}M) = \mathrm{rk}_{\mathbb{C}}(TM)$, we conclude that the latter map must also be injective. In particular, the map $X \mapsto \alpha_X$, where α_X is uniquely characterized by $X \lrcorner \Omega = \mathrm{Re}(\alpha_X \wedge dz)$, is an isomorphism from the space of π_1-vertical vector fields onto $W^{0,1} \cap E^{1,0}$.

If X is a π_1-vertical vector field, letting $\alpha_{1X} = \mathrm{Re}(\alpha_X)$, $\alpha_{2X} = \mathrm{Im}(\alpha_X)$, we have $X \lrcorner \Omega = \mathrm{Re}(\alpha_X \wedge dz) = \alpha_{1X} \wedge dx - \alpha_{2X} \wedge dy$. We conclude that Ω can be written as $\Omega = \mu_1 \wedge dx - \mu_2 \wedge dy = \mathrm{Re}((\mu_1 + i\mu_2) \wedge dz)$. We claim that $\mu = \mu_1 + i\mu_2 \in W^{0,2} \cap E^{2,0}$ is, in fact, of type (2,0). Actually, for any π_1-vertical vector fields X, Y, we have $Y \lrcorner X \lrcorner \mu_2 = Y \lrcorner \alpha_{2X} = -(JY) \lrcorner \alpha_{1X} = -(JY) \lrcorner X \lrcorner \mu_1$; that is, $Y \lrcorner \mu_2 = -(JY) \lrcorner \mu_1$ and μ is of type $(2, 0)$ by virtue of Proposition 2.1.

Let $\omega|_U = \mu \wedge dz$ be the local representation of a form $\omega \in W^{1,2} \cap E^{3,0}$ with respect to (U, z). As $X \lrcorner \mu_1 = \alpha_{1X} = 0$ implies $\alpha_X = 0$ by virtue of Proposition 2.1-(a), and hence $X = 0$, we conclude that $\ker \mu = \ker \mu_1 = 0$. So, ω is vertically non-degenerate and, by Proposition 3.3, we obtain $d\omega = 0$.

Finally, the last part of the statement directly follows from the results in [7]. □

Remark 4.4. Let $\mathcal{Z}_J^{2,0}(M) = \{\omega \in \mathcal{Z}^{2,0}(M) : \omega(JX, Y) = i\omega(X, Y)\}$. The space $S_J(M)$ of J-symplectic structures on M is a—possibly empty—open subset in $\mathcal{Z}_J^{2,0}(M)$. In particular, a necessary condition for $CR(\Sigma, M)$ to be of variational type is that (M, J) should admit a J-symplectic structure. Assuming this and taking Remark 3.2 into account, we can interpret a closed and vertically non-degenerate 3-form $\omega \in W^{1,2} \cap E^{3,0}$ as a holomorphic section of a bundle $B(\Sigma, M)$ over Σ with fiber $S_J(M)$. Hence $CR(\Sigma, M)$ is of variational type if and only if this bundle admits a holomorphic section.

Example 4.5. 1) Suppose that Σ admits a non-vanishing holomorphic 1-form α, and that M admits a J-symplectic structure γ. Then $\alpha \wedge \gamma$ (or more precisely $\pi_1^*\alpha \wedge \pi_2^*\gamma$) satisfies the conditions of the theorem above and hence $CR(\Sigma, M)$ is of variational type.

2) Every hyperkählerian manifold admits a J-symplectic form; see Remark 2.7. Also see [3, 14.25, 14.28, 14.33].

3) If M is a compact, connected, complex manifold of complex dimension 2 admitting a holomorphic volume form, then $S_J(M) \cong \mathbb{C} \setminus 0$ and hence, $B(\Sigma, M)$ is the bundle of complex linear coframes of Σ. Accordingly, if Σ is compact, then $B(\Sigma, M)$ admits a holomorphic section if and only if Σ is a complex torus.

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