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Maximum norm error estimates of efficient difference schemes for second-order wave equations[☆]

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ABSTRACT

The three-level explicit scheme is efficient for numerical approximation of the second-order wave equations. By employing a fourth-order accurate scheme to approximate the solution at first time level, it is shown that the discrete solution is conditionally convergent in the maximum norm with the convergence order of two. Since the asymptotic expansion of the difference solution consists of odd powers of the mesh parameters (time step and spacings), an unusual Richardson extrapolation formula is needed in promoting the second-order solution to fourth-order accuracy. Extensions of our technique to the classical ADI scheme also yield the maximum norm error estimate of the discrete solution and its extrapolation. Numerical experiments are presented to support our theoretical results.

1. Introduction

In recent years, much attention has been paid to the development and analysis of efficient methods that directly discretizes the second order system for numerical approximations of second-order hyperbolic equations, see [1–7]. Mohanty et al. [6,7] developed new three-level ADI schemes for two- and three-dimensional linear hyperbolic equations. Dehghan et al. [1,2] applied the radial basis functions method and the collocation method to wave equations.

For approximating wave problems, explicit schemes [5,8–10] are popular because the solution at each mesh point is updated by combining information from its near neighbours at previous time levels without the need of solving a large system of algebraic equations. Standard Fourier analysis for second-order explicit schemes shows that the maximum stable time-step size is directly proportional to the space mesh size. But this restriction is not so bad since optimal results are obtained when the space and time resolution are comparable. Once implicit schemes are necessary, the ADI approaches [11–14] are preferable in various applications because they reduce the solution of a multi-dimensional problem to a set of independent one-dimensional problems and thus are more efficient than implicit schemes. To achieve high-order accuracy, global Richardson extrapolations are practical computational methods, see e.g. [15,13,16–20]. They obtain high-order resolutions by using certain linear combinations of discrete solutions with different grid parameters (time step and

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spacings). The main advantage of the global extrapolations is that they preserve the stability of lower order methods used initially.

To measure computational error especially the phase error of numerical solutions, maximum norm error is preferable in practice or numerical analysis. By the standard H^1 energy analysis, it is not difficult to prove that the difference solutions for linear hyperbolic problems are convergent in the H^1 norm, see e.g. [9]. But the H^1 error estimate does not imply the maximum norm estimate. In our previous work [16], maximum norm error estimates of ADI and compact ADI solutions together with their extrapolations for solving parabolic equations were obtained by using an H^2 energy technique. We now apply the technique to deal with the second-order hyperbolic problems. Although the idea is similar, a lot of difference are seen since we consider here the three-level schemes. Typically, consider the following equation in two space dimensions

$$u_{tt} - \Delta u = f(x, y, t), \quad (x, y) \in \Omega,\ 0 < t \le T, \tag{1.1}$$

$$u(x, y, t) = \alpha(x, y, t), \quad (x, y) \in \partial\Omega,\ 0 < t \le T, \tag{1.2}$$

$$u(x, y, 0) = \varphi(x, y), \qquad u_t(x, y, 0) = \psi(x, y), \quad (x, y) \in \bar{\Omega}, \tag{1.3}$$

where Δ is the Laplacian operator, $\Omega = (0, 1)^2$, $\partial\Omega$ is the boundary and $\bar{\Omega} = \Omega \cup \partial\Omega$. Assume that the initial values φ and ψ, the boundary value α and exterior force f are regular enough, and satisfy the initial-boundary compatibility conditions

$$\alpha(x, y, 0) = \varphi(x, y), \qquad \alpha_t(x, y, 0) = \psi(x, y), \quad (x, y) \in \partial\Omega,$$
$$\alpha_{tt}(x, y, 0) = \Delta\varphi(x, y) + f(x, y, 0), \qquad \alpha_{ttt}(x, y, 0) = \Delta\psi(x, y) + f_t(x, y, 0), \quad (x, y) \in \partial\Omega$$

such that the initial-boundary value problem (1.1)–(1.3) admits a smooth solution.

We show that numerical solutions and their Richardson extrapolations of two efficient methods, including the second-order explicit and ADI schemes, are convergent in the maximum norm. For nonhomogeneous boundary conditions, a fourth-order accurate approximation of the solution at first time level is necessary for the second-order convergence in the maximum norm. Although the explicit and ADI schemes are central difference discretizations, two-grid based extrapolation formula would not promote the second-order methods to fourth-order accuracy. Actually, since the asymptotic expansions of the difference solutions consist of odd powers of the mesh parameters, a three-grid based Richardson extrapolation formula will be needed.

The content will be organized as follows. In the next section, some notations and auxiliary lemmas are presented. Section 3 devotes to the error analysis of the second-order explicit solution and its extrapolation. Theoretical considerations of the ADI scheme is addressed in Section 4. Numerical experiments are presented in Section 5 to support our analysis. Some comments including the three-dimensional extensions are presented in the concluding section.

2. Notation and auxiliary lemmas

Let $\tau = T/N$ for a positive integer N; $t_n = n\tau$, $0 \le n \le N$; and $t_{n-\frac{1}{2}} = (t_n + t_{n-1})/2$, $1 \le n \le N$. Given mesh function $w_\tau = \{w^n \mid 0 \le n \le N\}$, denote $w^{n-\frac{1}{2}} = (w^n + w^{n-1})/2$, $\delta_t w^{n-\frac{1}{2}} = (w^n - w^{n-1})/\tau$,

$$\delta_t^2 w^n = (\delta_t w^{n+\frac{1}{2}} - \delta_t w^{n-\frac{1}{2}})/\tau, \qquad D_t w^n = (w^{n+1} - w^{n-1})/(2\tau).$$

For spatial approximation, let $h_1 = 1/M_1$, $h_2 = 1/M_2$ for positive integers M_1, M_2; $h = \max\{h_1, h_2\}$; $x_i = ih_1$, $0 \le i \le M_1$; and $y_j = jh_2$, $0 \le j \le M_2$. The discrete grid $\Omega_h = \{(x_i, y_j) \mid 1 \le i \le M_1 - 1,\ 1 \le j \le M_2 - 1\}$, $\partial\Omega_h$ is the discrete boundary of Ω_h, $\partial\Omega_h = \{(x_i, y_j) \mid i = 0 \text{ or } i = M_1 \text{ or } j = 0 \text{ or } j = M_2\}$, and $\bar{\Omega}_h = \Omega_h \cup \partial\Omega_h$. For any grid function $v_h = \{v_{ij} \mid (x_i, y_j) \in \bar{\Omega}_h\}$, let $\delta_x v_{i-\frac{1}{2},j} = (v_{ij} - v_{i-1,j})/h_1$, $\delta_x^2 v_{ij} = \big(\delta_x v_{i+\frac{1}{2},j} - \delta_x v_{i-\frac{1}{2},j}\big)/h_1$,

$$\delta_y\delta_x v_{i-\frac{1}{2},j-\frac{1}{2}} = \big(\delta_x v_{i-\frac{1}{2},j} - \delta_x v_{i-\frac{1}{2},j-1}\big)/h_2, \qquad \delta_y\delta_x^2 v_{i,j-\frac{1}{2}} = \big(\delta_x^2 v_{ij} - \delta_x^2 v_{i,j-1}\big)/h_2.$$

Similar notations $\delta_y v_{i,j-\frac{1}{2}}$, $\delta_y^2 v_{ij}$, $\delta_x\delta_y v_{i-\frac{1}{2},j-\frac{1}{2}}$, $\delta_x\delta_y^2 v_{i-\frac{1}{2},j}$, $\delta_x^2\delta_y^2 v_{ij}$ can also be defined and the discrete Laplacian operator $\Delta_h v_{ij} = (\delta_x^2 + \delta_y^2)v_{ij}$. We also denote

$$\|v\| = \sqrt{h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}|v_{ij}|^2}, \qquad \|\delta_x v\| = \sqrt{h_1h_2\sum_{i=1}^{M_1}\sum_{j=1}^{M_2-1}\big|\delta_x v_{i-\frac{1}{2},j}\big|^2},$$

$$\|\delta_x^2 v\| = \sqrt{h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}\big|\delta_x^2 v_{ij}\big|^2}, \qquad \|\delta_x\delta_y v\| = \sqrt{h_1h_2\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\big|\delta_x\delta_y v_{i-\frac{1}{2},j-\frac{1}{2}}\big|^2},$$

$$\|\delta_y\delta_x^2 v\| = \sqrt{h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2}\big|\delta_y\delta_x^2 v_{i,j-\frac{1}{2}}\big|^2}, \qquad \|\Delta_h v\| = \sqrt{h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}\big|\Delta_h v_{ij}\big|^2}$$

and $\|\delta_y v\|$, $\|\delta_y^2 v\|$, $\|\delta_y\delta_x v\|$, $\|\delta_x\delta_y^2 v\|$ similarly. For any grid function

$$v \in \mathcal{V}_h = \left\{ v \middle| v = \{v_{ij} | (x_i, y_j) \in \bar{\Omega}_h\} \text{ and } v_{ij} = 0 \text{ if } (x_i, y_j) \in \partial\Omega_h \right\},$$

we introduce

$$|v|_1 = \sqrt{\|\delta_x v\|^2 + \|\delta_y v\|^2}, \qquad |\delta_x\delta_y v|_1 = \sqrt{\|\delta_y\delta_x^2 v\|^2 + \|\delta_x\delta_y^2 v\|^2}, \qquad \|v\|_\infty = \max_{\substack{0\le i\le M_1\\ 0\le j\le M_2}} |v_{ij}|.$$

To obtain the error estimate in the maximum norm, we need the following lemmas. Throughout this paper c or $c(u)$ will denote a generic positive constant, not necessarily the same at different occurrences, which may be dependent on the solution and the given data but independent of the time-step size τ and the grid spacings h_1, h_2.

Lemma 2.1 ([16]). *For any grid function $v \in \mathcal{V}_h$, $\|v\|_\infty \le c\|\Delta_h v\|$.*

Lemma 2.2. *For any grid function $v \in \mathcal{V}_h$, it holds that $\|\Delta_h v\|^2 \le 4(h_1^{-2} + h_2^{-2})|v|_1^2$.*

Proof. Using the Chauchy–Schwarz inequality, we have

$$\begin{aligned}
\|\Delta_h v\|^2 &= h_1h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} (\delta_x^2 v_{ij} + \delta_y^2 v_{ij})^2 = h_1h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} (h_1^{-1}\cdot h_1\delta_x^2 v_{ij} + h_2^{-1}\cdot h_2\delta_y^2 v_{ij})^2 \\
&\le h_1h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} (h_1^{-2} + h_2^{-2})\left[(h_1\delta_x^2 v_{ij})^2 + (h_2\delta_y^2 v_{ij})^2\right] \\
&= (h_1^{-2} + h_2^{-2})\cdot h_1h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} \left[(\delta_x v_{i+\frac12,j} - \delta_x v_{i-\frac12,j})^2 + (\delta_y v_{i,j+\frac12} - \delta_y v_{i,j-\frac12})^2\right] \\
&\le 2(h_1^{-2} + h_2^{-2})\cdot h_1h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} \left[(\delta_x v_{i+\frac12,j})^2 + (\delta_x v_{i-\frac12,j})^2 + (\delta_y v_{i,j+\frac12})^2 + (\delta_y v_{i,j-\frac12})^2\right] \\
&\le 4(h_1^{-2} + h_2^{-2})|v|_1^2. \quad \square
\end{aligned}$$

Lemma 2.3. *For time sequences $\{w^0, w^1, \ldots, w^N\}$ and $\{g^0, g^1, \ldots, g^N\}$,*

$$2\tau\sum_{l=1}^{k} g^l(D_t w^l) \le \frac{1}{\epsilon}\left[(w^{\frac12})^2 + \tau\sum_{l=1}^{k-1}(w^{l+\frac12})^2 + (w^{k+\frac12})^2\right] + \epsilon\left[(g^1)^2 + \tau\sum_{l=1}^{k-1}\left(\delta_t g^{l+\frac12}\right)^2 + (g^k)^2\right],$$

for any $\epsilon > 0$.

Proof. Note that

$$\begin{aligned}
2\tau\sum_{l=1}^{k} g^l(D_t w^l) &= \sum_{l=1}^{k} g^l(w^{l+1} - w^{l-1}) = \sum_{l=1}^{k} g^l(w^{l+1} - w^l) + \sum_{l=1}^{k} g^l(w^l - w^{l-1}) \\
&= -g^1w^1 - \tau\sum_{l=1}^{k-1}\left(\delta_t g^{l+\frac12}\right) w^{l+1} + g^k w^{k+1} - g^1 w^0 - \tau\sum_{l=1}^{k-1}\left(\delta_t g^{l+\frac12}\right) w^l + g^k w^k \\
&= -2g^1 w^{\frac12} - 2\tau\sum_{l=1}^{k-1}\left(\delta_t g^{l+\frac12}\right) w^{l+\frac12} + 2g^k w^{k+\frac12}.
\end{aligned}$$

Thus, with the ϵ-inequality [9] $2ab \le \frac{1}{\epsilon}a^2 + \epsilon b^2$, it is easy to get the claimed result. $\square$

3. Error analysis of the explicit solution and its extrapolation

3.1. Construction of the explicit method and a priori estimation

Define the grid function $U_{ij}^n = u(x_i, y_j, t_n)$ for $(x_i, y_j) \in \bar{\Omega}_h$ and $0 \le n \le N$. Utilizing the Taylor expansion with integral remainder (see e.g. [16]), one has

$$u_{tt}(x_i, y_j, t_n) = \delta_t^2 U_{ij}^n - (R_t)_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \tag{3.1}$$

$$-u_{xx}(x_i, y_j, t_n) = -\delta_x^2 U_{ij}^n + (R_x)_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \tag{3.2}$$

$$-u_{yy}(x_i, y_j, t_n) = -\delta_y^2 U_{ij}^n + (R_y)_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \tag{3.3}$$

where the truncation errors

$$(R_t)_{ij}^n = \frac{\tau^2}{6}\int_0^1 \left[\frac{\partial^4 u(x_i, y_j, t_n - s\tau)}{\partial t^4} + \frac{\partial^4 u(x_i, y_j, t_n + s\tau)}{\partial t^4}\right](1-s)^3 \mathrm{d}s,$$

$$(R_x)_{ij}^n = \frac{h_1^2}{6}\int_0^1 \left[\frac{\partial^4 u(x_i - \lambda h_1, y_j, t_n)}{\partial x^4} + \frac{\partial^4 u(x_i + \lambda h_1, y_j, t_n)}{\partial x^4}\right](1-\lambda)^3 \mathrm{d}\lambda,$$

$$(R_y)_{ij}^n = \frac{h_2^2}{6}\int_0^1 \left[\frac{\partial^4 u(x_i, y_j - \lambda h_2, t_n)}{\partial y^4} + \frac{\partial^4 u(x_i, y_j + \lambda h_2, t_n)}{\partial y^4}\right](1-\lambda)^3 \mathrm{d}\lambda.$$

Adding up the Eqs. (3.1)–(3.3) and using the following equality

$$u_{tt}(x_i, y_j, t_n) - \Delta u(x_i, y_j, t_n) = f(x_i, y_j, t_n), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1,$$

we have

$$\delta_t^2 U_{ij}^n - \Delta_h U_{ij}^n = f(x_i, y_j, t_n) + R_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \tag{3.4}$$

where the truncation error

$$R_{ij}^n = (R_t)_{ij}^n - (R_x)_{ij}^n - (R_y)_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1.$$

From the boundary and initial conditions (1.2)–(1.3), one has

$$U_{ij}^n = \alpha(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h,\ 0 \le n \le N, \tag{3.5}$$

$$U_{ij}^0 = \varphi(x_i, y_j), \quad (x_i, y_j) \in \Omega_h. \tag{3.6}$$

To find the solution at first time level, we derive from the wave equation (1.1) that

$$u_{tt}(x_i, y_j, t_0) = \Delta u(x_i, y_j, t_0) + f(x_i, y_j, t_0), \quad (x_i, y_j) \in \Omega_h,$$
$$u_{ttt}(x_i, y_j, t_0) = \Delta u_t(x_i, y_j, t_0) + f_t(x_i, y_j, t_0), \quad (x_i, y_j) \in \Omega_h.$$

Let

$$\omega_{ij} = \frac{\tau^4}{6}\int_0^1 \frac{\partial^4 u(x_i, y_j, s\tau)}{\partial t^4}(1-s)^3 \mathrm{d}s, \quad (x_i, y_j) \in \Omega_h.$$

Thus using the method of Taylor expansion and the initial conditions (1.3), we get

$$\begin{aligned} U_{ij}^1 &= u(x_i, y_j, t_0) + \tau u_t(x_i, y_j, t_0) + \frac{\tau^2}{2} u_{tt}(x_i, y_j, t_0) + \frac{\tau^3}{6} u_{ttt}(x_i, y_j, t_0) + \omega_{ij} \\ &= \varphi_1(x_i, y_j, \tau) + \omega_{ij}, \quad (x_i, y_j) \in \Omega_h, \end{aligned} \tag{3.7}$$

where $\varphi_1(x, y, \tau) = \varphi(x, y) + \tau\psi(x, y) + \frac{\tau^2}{2}[\Delta\varphi(x, y) + f(x, y, t_0)] + \frac{\tau^3}{6}[\Delta\psi(x, y) + f_t(x, y, t_0)]$. Omitting the small terms R_{ij}^n and ω_{ij}, and replacing U_{ij}^n with its numerical approximation u_{ij}^n in the Eqs. (3.4)–(3.7), one gets the following explicit difference scheme

$$\delta_t^2 u_{ij}^n - \Delta_h u_{ij}^n = f(x_i, y_j, t_n), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \tag{3.8}$$

$$u_{ij}^n = \alpha(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h,\ 0 \le n \le N, \tag{3.9}$$

$$u_{ij}^0 = \varphi(x_i, y_j), \quad (x_i, y_j) \in \Omega_h, \tag{3.10}$$

$$u_{ij}^1 = \varphi_1(x_i, y_j, \tau), \quad (x_i, y_j) \in \Omega_h. \tag{3.11}$$

Compared with the classical approach, we consider a fourth-order accurate procedure (3.11) to compute the solution at the first time level. Traditionally, it is approximated by the following third-order accurate difference scheme, see e.g. [9],

$$u_{ij}^1 = \varphi(x_i, y_j) + \tau\psi(x_i, y_j) + \frac{\tau^2}{2}\left[\Delta\varphi(x_i, y_j) + f(x_i, y_j, t_0)\right], \quad (x_i, y_j) \in \Omega_h. \tag{3.12}$$

Now we turn to the theoretical consideration of the explicit scheme (3.8)–(3.11). We prove firstly the following lemma of *a priori* estimation. To simplify the notation, we define

$$\|g^n\|_{\Sigma(\delta)} = \sqrt{\|g^1\|^2 + \tau\sum_{l=1}^{n-1}\|\delta_t g^{l+\frac{1}{2}}\|^2 + \|g^n\|^2}.$$

Lemma 3.1. *Let function* $\{w_{ij}^n | (x_i, y_j) \in \bar{\Omega}_h,\ 0 \le n \le N\}$ *be the solution of the following explicit difference system*

$$\delta_t^2 w_{ij}^n - \Delta_h w_{ij}^n = g_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \tag{3.13}$$

$$w_{ij}^n = 0, \quad (x_i, y_j) \in \partial\Omega_h,\ 0 \le n \le N, \tag{3.14}$$

$$w_{ij}^0 = \varphi_{ij}, \qquad w_{ij}^1 = \psi_{ij}, \quad (x_i, y_j) \in \Omega_h. \tag{3.15}$$

Then under the condition $\sigma \equiv \sqrt{(\tau/h_1)^2 + (\tau/h_2)^2} < 1$, *it holds that*

$$(1-\sigma^2)\left|\delta_t w^{n+\frac{1}{2}}\right|_1^2 + \left\|\Delta_h w^{n+\frac{1}{2}}\right\|^2 \le E^n \le e^{t_n}\left(3E^0 + 4\left\|g^n\right\|_{\Sigma(\delta)}^2\right), \quad 0 \le n \le N-1,$$

where the energy norm

$$E^n = \left|\delta_t w^{n+\frac{1}{2}}\right|_1^2 + h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}\left(\Delta_h w_{ij}^{n+1}\right)\left(\Delta_h w_{ij}^n\right), \quad 0 \le n \le N-1.$$

Proof. Multiplying (3.13) by $-2\tau h_1 h_2 D_t \Delta_h w_{ij}^n$ and summing i, j for $(x_i, y_j) \in \Omega_h$, we have

$$E^n - E^{n-1} = -2\tau h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} g_{ij}^n\left(D_t \Delta_h w_{ij}^n\right), \quad 1 \le n \le N-1,$$

where the discrete Green's first inequality together with the zero-valued boundary condition (3.14) is applied. Summing the above equation for n from 1 to k, and then replacing k with n, one gets

$$\begin{aligned} E^n &= E^0 - 2\tau h_1 h_2 \sum_{l=1}^{n}\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} g_{ij}^l\left(D_t \Delta_h w_{ij}^l\right) \\ &\le E^0 + \frac{1}{2}\left\|\Delta_h w^{\frac{1}{2}}\right\|^2 + \frac{\tau}{2}\sum_{l=1}^{n-1}\left\|\Delta_h w^{l+\frac{1}{2}}\right\|^2 + \frac{1}{2}\left\|\Delta_h w^{n+\frac{1}{2}}\right\|^2 + 2\left\|g^n\right\|_{\Sigma(\delta)}^2, \quad 1 \le n \le N-1, \end{aligned} \tag{3.16}$$

where Lemma 2.3 with $\epsilon = 2$ is used (by multiplying the two sides of the inequality with $h_1 h_2$, summing i from 1 to $M_1 - 1$ and summing j from 1 to $M_2 - 1$). By using Lemma 2.2, we obtain

$$\begin{aligned} E^n &= \left|\delta_t w^{n+\frac{1}{2}}\right|_1^2 + \left\|\Delta_h w^{n+\frac{1}{2}}\right\|^2 - h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}\left[(\Delta_h w_{ij}^{n+\frac{1}{2}})^2 - (\Delta_h w_{ij}^{n+1})(\Delta_h w_{ij}^n)\right] \\ &= \left|\delta_t w^{n+\frac{1}{2}}\right|_1^2 + \left\|\Delta_h w^{n+\frac{1}{2}}\right\|^2 - \frac{\tau^2}{4}\left\|\Delta_h \delta_t w^{n+\frac{1}{2}}\right\|^2 \\ &\ge \left|\delta_t w^{n+\frac{1}{2}}\right|_1^2 + \left\|\Delta_h w^{n+\frac{1}{2}}\right\|^2 - \tau^2(h_1^{-2} + h_2^{-2})\left|\delta_t w^{n+\frac{1}{2}}\right|_1^2 \\ &= (1-\sigma^2)\left|\delta_t w^{n+\frac{1}{2}}\right|_1^2 + \left\|\Delta_h w^{n+\frac{1}{2}}\right\|^2, \quad 0 \le n \le N-1. \end{aligned} \tag{3.17}$$

Thus the energy norm E^n is positive definite if $\sigma < 1$. Consequently, the inequality (3.16) becomes

$$E^n \le 3E^0 + \tau\sum_{l=1}^{n-1} E^l + 4\left\|g^n\right\|_{\Sigma(\delta)}^2, \quad 1 \le n \le N-1.$$

Thus the well-known discrete Gronwall inequality [21] yields the claimed second inequality. It completes the proof. □

Lemma 3.2. *Let function* $\{w_{ij}^n | (x_i, y_j) \in \bar{\Omega}_h,\ 0 \le n \le N\}$ *be the solution of the difference system* (3.13)–(3.15)*. Then under the condition* $\sigma < 1$, *it holds that*

$$\left\|w^{n+1}\right\|_\infty^2 \le \frac{c^2 e^{t_n}}{1-\sigma^2}\left[3\left|\delta_t w^{\frac{1}{2}}\right|_1^2 + 3h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}\left(\Delta_h w_{ij}^1\right)\left(\Delta_h w_{ij}^0\right) + 4\left\|g^n\right\|_{\Sigma(\delta)}^2\right], \quad 0 \le n \le N-1.$$

Proof. Noticing

$$\Delta_h w_{ij}^{n+1} = \Delta_h w_{ij}^{n+\frac{1}{2}} + \frac{\tau}{2}\delta_t \Delta_h w_{ij}^{n+\frac{1}{2}},$$

we apply the ϵ-inequality and Lemma 2.2 to find that

$$\begin{aligned}\left\|\Delta_h w^{n+1}\right\|^2 &= \left\|\Delta_h w^{n+\frac{1}{2}} + \frac{\tau}{2}\delta_t \Delta_h w^{n+\frac{1}{2}}\right\|^2 \le (1+\epsilon)\left\|\Delta_h w^{n+\frac{1}{2}}\right\|^2 + \left(1+\frac{1}{\epsilon}\right)\left\|\frac{\tau}{2}\delta_t \Delta_h w^{n+\frac{1}{2}}\right\|^2 \\ &= (1+\epsilon)\left\|\Delta_h w^{n+\frac{1}{2}}\right\|^2 + \left(1+\frac{1}{\epsilon}\right)\frac{\tau^2}{4}\left\|\delta_t \Delta_h w^{n+\frac{1}{2}}\right\|^2 \\ &\le (1+\epsilon)\left\|\Delta_h w^{n+\frac{1}{2}}\right\|^2 + \left(1+\frac{1}{\epsilon}\right)\tau^2(h_1^{-2}+h_2^{-2})\left|\delta_t w^{n+\frac{1}{2}}\right|_1^2 \\ &= (1+\epsilon)\left\|\Delta_h w^{n+\frac{1}{2}}\right\|^2 + \left(1+\frac{1}{\epsilon}\right)\sigma^2\left|\delta_t w^{n+\frac{1}{2}}\right|_1^2.\end{aligned}$$

Taking $\epsilon = \frac{\sigma^2}{1-\sigma^2}$ in the above inequality, we get

$$\left\|\Delta_h w^{n+1}\right\|^2 \le \frac{1}{1-\sigma^2}\left(\left\|\Delta_h w^{n+\frac{1}{2}}\right\|^2 + (1-\sigma^2)\left|\delta_t w^{n+\frac{1}{2}}\right|_1^2\right) \le \frac{E^n}{1-\sigma^2},$$

where the inequality (3.17) is used for $\sigma < 1$. Thus it follows from Lemma 2.1 that

$$\left\|w^{n+1}\right\|_\infty^2 \le c^2\left\|\Delta_h w^{n+1}\right\|^2 \le \frac{c^2 E^n}{1-\sigma^2}.$$

Then Lemma 3.1 yields the claimed inequality. The proof is completed. □

3.2. Convergence and stability of the explicit method

Now we present the error analysis of smooth solutions.

Theorem 3.1. *Let $u(x,y,t) \in \mathcal{C}^{(4,5)}(\bar{\Omega}\times[0,T])$ be the exact solution of the hyperbolic problem (1.1)–(1.3). Then, under the restriction $\sigma < 1$, the numerical solution of the second-order explicit scheme (3.8)–(3.11) is convergent with an order of $O(\tau^2+h_1^2+h_2^2)$ in the maximum norm.*

Proof. Let the solution error $\tilde{u}_{ij}^n = U_{ij}^n - u_{ij}^n$. Subtracting (3.8)–(3.11) from (3.4)–(3.7) respectively, one has the error system

$$\delta_t^2\tilde{u}_{ij}^n - \Delta_h\tilde{u}_{ij}^n = R_{ij}^n, \quad (x_i,y_j)\in\Omega_h,\ 1\le n\le N-1, \tag{3.18}$$

$$\tilde{u}_{ij}^n = 0, \quad (x_i,y_j)\in\partial\Omega_h,\ 0\le n\le N, \tag{3.19}$$

$$\tilde{u}_{ij}^0 = 0, \qquad \tilde{u}_{ij}^1 = \omega_{ij}, \quad (x_i,y_j)\in\Omega_h. \tag{3.20}$$

Under the condition $\sigma < 1$, one has $\tilde{u}_{ij}^0 = 0$ for $(x_i,y_j)\in\bar{\Omega}_h$, and

$$\left\|\delta_x\delta_t\tilde{u}^{\frac{1}{2}}\right\| \le c(u)\tau^3 h_1^{-1} \le c(u)\tau^2, \qquad \left\|\delta_y\delta_t\tilde{u}^{\frac{1}{2}}\right\| \le c(u)\tau^3 h_2^{-1} \le c(u)\tau^2,$$

where the integral formulation of ω_{ij} is applied. Therefore we have

$$\Delta_h\tilde{u}_{ij}^0 = 0, \qquad (x_i,y_j)\in\Omega_h, \qquad \left|\delta_t\tilde{u}^{\frac{1}{2}}\right|_1 \le c(u)\tau^2.$$

Hence, Lemma 3.2 gives

$$\left\|\tilde{u}^{n+1}\right\|_\infty^2 \le \frac{c^2 e^{t_n}}{1-\sigma^2}\left(3c(u)\tau^4 + 4\left\|R^n\right\|_{\Sigma(\delta)}^2\right), \quad 0\le n\le N-1. \tag{3.21}$$

From the integral formulation of the truncation error R_{ij}^n, one obtains

$$\left\|R^n\right\| \le c(u)(\tau^2+h_1^2+h_2^2), \quad 1\le n\le N-1.$$

We now need to evaluate $\left\|\delta_t R^{n+\frac{1}{2}}\right\|$ for $1\le n\le N-2$. Note that

$$\delta_t R_{ij}^{n+\frac{1}{2}} = \delta_t(R_t)_{ij}^{n+\frac{1}{2}} - \delta_t(R_x)_{ij}^{n+\frac{1}{2}} - \delta_t(R_y)_{ij}^{n+\frac{1}{2}}, \quad (x_i,y_j)\in\Omega_h,\ 1\le n\le N-2.$$

Recalling the integral formula of $(R_t)_{ij}^n$, we derive

$$\begin{aligned}\delta_t(R_t)_{ij}^{n+\frac{1}{2}} &= \frac{1}{\tau}\left[(R_t)_{ij}^{n+1}-(R_t)_{ij}^{n}\right]\\ &= \frac{\tau}{6}\int_0^1\left[\frac{\partial^4 u(x_i,y_j,t_{n+1}-s\tau)}{\partial t^4}-\frac{\partial^4 u(x_i,y_j,t_n-s\tau)}{\partial t^4}\right](1-s)^3\mathrm{d}s\\ &\quad+\frac{\tau}{6}\int_0^1\left[\frac{\partial^4 u(x_i,y_j,t_{n+1}+s\tau)}{\partial t^4}-\frac{\partial^4 u(x_i,y_j,t_n+s\tau)}{\partial t^4}\right](1-s)^3\mathrm{d}s\\ &= \frac{\tau^2}{6}\int_0^1\int_0^1\left[\frac{\partial^5 u(x_i,y_j,t_n-s\tau+\mu\tau)}{\partial t^5}+\frac{\partial^5 u(x_i,y_j,t_n+s\tau+\mu\tau)}{\partial t^5}\right](1-s)^3\mathrm{d}\mu\mathrm{d}s.\end{aligned}$$

Then we have $\left\|\delta_t(R_t)^{n+\frac{1}{2}}\right\|\le c(u)\tau^2$ for $1\le n\le N-2$. Similarly, it follows that

$$\left\|\delta_t(R_x)^{n+\frac{1}{2}}\right\|\le c(u)h_1^2,\qquad \left\|\delta_t(R_y)^{n+\frac{1}{2}}\right\|\le c(u)h_2^2,\quad 1\le n\le N-2.$$

Thus, utilizing the triangle inequality we get $\left\|\delta_t R^{n+\frac{1}{2}}\right\|\le c(u)(\tau^2+h_1^2+h_2^2)$, and then

$$\left\|R^n\right\|_{\Sigma(\delta)}\le c(u)(\tau^2+h_1^2+h_2^2),\quad 1\le n\le N-1.$$

Combining it with (3.21), we have $\left\|\tilde{u}^n\right\|_\infty=O(\tau^2+h_1^2+h_2^2)$. It completes the proof. □

When the condition (1.2) is zero-valued, i.e., $\alpha(x,y,t)=0$, we can obtain the same result by replacing the fourth-order approximation of $u(x_i,y_j,\tau)$ with the third-order scheme (3.12).

Theorem 3.2. *Let function $u(x,y,t)\in\mathcal{C}^{(4,5)}(\bar{\Omega}\times[0,T])$ be the exact solution of the hyperbolic problem (1.1)–(1.3) with $\alpha(x,y,t)=0$. Then, under the restriction $\sigma<1$, the numerical solution of the explicit scheme (3.8)–(3.10) together with the third-order accurate approximation (3.12) is convergent with an order of $O(\tau^2+h_1^2+h_2^2)$ in the maximum norm.*

Proof. The zero-valued boundary conditions and the initial-boundary compatibility conditions imply that u_{ij}^1 vanishes along the boundary $\partial\Omega_h$. Noticing the third-order approximation (3.12) of $u(x_i,y_j,\tau)$, one can rewrite the error system (3.18)–(3.20) as

$$\begin{aligned}&\delta_t^2\tilde{u}_{ij}^n-\Delta_h\tilde{u}_{ij}^n=R_{ij}^n,\quad (x_i,y_j)\in\Omega_h,\ 1\le n\le N-1,\\ &\tilde{u}_{ij}^n=0,\quad (x_i,y_j)\in\partial\Omega_h,\ 2\le n\le N,\\ &\tilde{u}_{ij}^0=0,\qquad \tilde{u}_{ij}^1=\frac{\tau^3}{2}\int_0^1\frac{\partial^3 u(x_i,y_j,s\tau)}{\partial t^3}(1-s)^2\mathrm{d}s,\quad (x_i,y_j)\in\bar{\Omega}_h.\end{aligned}$$

A similar presentation of the proof for Theorem 3.1 will yield our claim. □

Obviously, Lemma 3.2 implies the conditional stability of the explicit scheme (3.8)–(3.11).

Theorem 3.3. *Let $\{u_{ij}^n|(x_i,y_j)\in\bar{\Omega}_h,\ 0\le n\le N\}$ be the solution of the explicit scheme (3.8)–(3.11) with $\alpha(x,y,t)\equiv 0$. Then under the condition $\sigma<1$, it holds that,*

$$\|u^{n+1}\|_\infty^2\le\frac{c^2e^{t_n}}{1-\sigma^2}\left[3\left|\delta_t u^{\frac{1}{2}}\right|_1^2+3h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}\left(\Delta_h u_{ij}^1\right)\left(\Delta_h u_{ij}^0\right)+4\left\|f^n\right\|_{\Sigma(\delta)}^2\right],\quad 0\le n\le N-1.$$

3.3. Richardson extrapolation of the explicit solution

This subsection is devoted to the Richardson extrapolation of the discrete solution generated by the explicit scheme (3.8)–(3.11).

Theorem 3.4. *Let $u(x,y,t)$ be the smooth solution of the hyperbolic problem (1.1)–(1.3) and $u_{ij}^n(\tau,h_1,h_2)$ be the solution of the explicit scheme (3.8)–(3.11). If $\sigma<1$, it holds that*

$$\left\|U^n-(u_E)^n\right\|_\infty=O(\tau^4+h_1^4+h_2^4+\tau^2h_1^2+\tau^2h_2^2+h_1^2h_2^2),\quad 1\le n\le N,$$

where the extrapolation solution

$$(u_E)_{ij}^n=\frac{32}{21}u_{4i,4j}^{4n}\left(\frac{\tau}{4},\frac{h_1}{4},\frac{h_2}{4}\right)-\frac{12}{21}u_{2i,2j}^{2n}\left(\frac{\tau}{2},\frac{h_1}{2},\frac{h_2}{2}\right)+\frac{1}{21}u_{i,j}^n\left(\tau,h_1,h_2\right).$$

Proof. We define the following auxiliary functions

$$f_p(x,y,t)=\frac{1}{12}\frac{\partial^4 u}{\partial t^4},\qquad f_q(x,y,t)=-\frac{1}{12}\frac{\partial^4 u}{\partial x^4},\qquad f_r(x,y,t)=-\frac{1}{12}\frac{\partial^4 u}{\partial y^4},\qquad \psi_w(x,y)=\frac{1}{120}\frac{\partial^5 u}{\partial t^5}\Big|_{t=0}.$$

From the derivation of the scheme (3.8)–(3.11) described above, it is not difficult to know that

$$R_{ij}^n=\tau^2 f_p(x_i,y_j,t_n)+h_1^2 f_q(x_i,y_j,t_n)+h_2^2 f_r(x_i,y_j,t_n)+\widehat{R}_{ij}^n,\quad (x_i,y_j)\in\Omega_h,\ 1\le n\le N-1,$$

$$\omega_{ij}=\frac{\tau^4}{2}f_p(x_i,y_j,t_0)+\tau^5\psi_w(x_i,y_j)+\widehat{\omega}_{ij},\quad (x_i,y_j)\in\Omega_h,$$

where $\widehat{R}_{ij}^n=(\widehat{R}_t)_{ij}^n-(\widehat{R}_x)_{ij}^n-(\widehat{R}_y)_{ij}^n$, and

$$(\widehat{R}_t)_{ij}^n=\frac{\tau^4}{5!}\int_0^1\left[\frac{\partial^6 u(x_i,y_j,t_n-s\tau)}{\partial t^6}+\frac{\partial^6 u(x_i,y_j,t_n+s\tau)}{\partial t^6}\right](1-s)^5\mathrm{d}s,$$

$$(\widehat{R}_x)_{ij}^n=\frac{h_1^4}{5!}\int_0^1\left[\frac{\partial^6 u(x_i-\lambda h_1,y_j,t_n)}{\partial x^6}+\frac{\partial^6 u(x_i+\lambda h_1,y_j,t_n)}{\partial x^6}\right](1-\lambda)^5\mathrm{d}\lambda,$$

$$(\widehat{R}_y)_{ij}^n=\frac{h_2^4}{5!}\int_0^1\left[\frac{\partial^6 u(x_i,y_j-\lambda h_2,t_n)}{\partial y^6}+\frac{\partial^6 u(x_i,y_j+\lambda h_2,t_n)}{\partial y^6}\right](1-\lambda)^5\mathrm{d}\lambda,$$

$$\widehat{\omega}_{ij}=\frac{\tau^6}{5!}\int_0^1\frac{\partial^6 u(x_i,y_j,s\tau)}{\partial t^6}(1-s)^5\mathrm{d}s.$$

Thus the error system (3.18)–(3.20) can be rewritten as

$$\begin{aligned}&\delta_t^2\tilde{u}_{ij}^n-\Delta_h\tilde{u}_{ij}^n=f_p(x_i,y_j,t_n)\tau^2+f_q(x_i,y_j,t_n)h_1^2+f_r(x_i,y_j,t_n)h_2^2+\widehat{R}_{ij}^n,\quad (x_i,y_j)\in\Omega_h,\ 1\le n\le N-1,\\&\tilde{u}_{ij}^n=0,\quad (x_i,y_j)\in\partial\Omega_h,\ 0\le n\le N,\\&\tilde{u}_{ij}^0=0,\qquad \tilde{u}_{ij}^1=\frac{\tau^4}{2}f_p(x_i,y_j,t_0)+\tau^5\psi_w(x_i,y_j)+\widehat{\omega}_{ij},\quad (x_i,y_j)\in\Omega_h.\end{aligned}\tag{3.22}$$

We consider the following nonhomogeneous problems with homogeneous initial and boundary conditions:

$$\begin{cases}p_{tt}-\Delta p=f_p(x,y,t),\quad (x,y)\in\Omega,\ 0<t\le T,\\ p(x,y,t)=0,\quad (x,y)\in\partial\Omega,\ 0<t\le T,\\ p(x,y,0)=0,\qquad p_t(x,y,0)=0,\quad (x,y)\in\bar{\Omega};\end{cases}\tag{3.23}$$

$$\begin{cases}q_{tt}-\Delta q=f_q(x,y,t),\quad (x,y)\in\Omega,\ 0<t\le T,\\ q(x,y,t)=0,\quad (x,y)\in\partial\Omega,\ 0<t\le T,\\ q(x,y,0)=0,\qquad q_t(x,y,0)=0,\quad (x,y)\in\bar{\Omega};\end{cases}\tag{3.24}$$

$$\begin{cases}r_{tt}-\Delta r=f_r(x,y,t),\quad (x,y)\in\Omega,\ 0<t\le T,\\ r(x,y,t)=0,\quad (x,y)\in\partial\Omega,\ 0<t\le T,\\ r(x,y,0)=0,\qquad r_t(x,y,0)=0,\quad (x,y)\in\bar{\Omega}.\end{cases}\tag{3.25}$$

It is easy to develop the following explicit schemes to approximate the problems (3.23)–(3.25), respectively,

$$\begin{cases}\delta_t^2 p_{ij}^n-\Delta_h p_{ij}^n=f_p(x_i,y_j,t_n),\quad (x_i,y_j)\in\Omega_h,\ 1\le n\le N-1,\\ p_{ij}^n=0,\quad (x_i,y_j)\in\partial\Omega_h,\ 0\le n\le N-1,\\ p_{ij}^0=0,\qquad p_{ij}^1=\dfrac{\tau^2}{2}f_p(x_i,y_j,t_0),\quad (x_i,y_j)\in\Omega_h;\end{cases}\tag{3.26}$$

$$\begin{cases}\delta_t^2 q_{ij}^n-\Delta_h q_{ij}^n=f_q(x_i,y_j,t_n),\quad (x_i,y_j)\in\Omega_h,\ 1\le n\le N-1,\\ q_{ij}^n=0,\quad (x_i,y_j)\in\partial\Omega_h,\ 0\le n\le N-1,\\ q_{ij}^0=0,\qquad q_{ij}^1=\dfrac{\tau^2}{2}f_q(x_i,y_j,t_0),\quad (x_i,y_j)\in\Omega_h;\end{cases}\tag{3.27}$$

$$\begin{cases} \delta_t^2 r_{ij}^n - \Delta_h r_{ij}^n = f_r(x_i, y_j, t_n), & (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \\ r_{ij}^n = 0, & (x_i, y_j) \in \partial\Omega_h,\ 0 \le n \le N-1, \\ r_{ij}^0 = 0, \qquad r_{ij}^1 = \dfrac{\tau^2}{2} f_r(x_i, y_j, t_0), & (x_i, y_j) \in \Omega_h. \end{cases} \tag{3.28}$$

Theorem 3.2 shows that

$$p(x_i, y_j, t_n) - p_{ij}^n = O(\tau^2 + h_1^2 + h_2^2), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \tag{3.29}$$

$$q(x_i, y_j, t_n) - q_{ij}^n = O(\tau^2 + h_1^2 + h_2^2), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \tag{3.30}$$

$$r(x_i, y_j, t_n) - r_{ij}^n = O(\tau^2 + h_1^2 + h_2^2), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N. \tag{3.31}$$

Next, we consider the following homogeneous problems with zero boundary conditions and nonhomogeneous initial conditions:

$$\begin{cases} \hat{q}_{tt} - \Delta \hat{q} = 0, & (x, y) \in \Omega,\ 0 < t \le T, \\ \hat{q}(x, y, t) = 0, & (x, y) \in \partial\Omega,\ 0 < t \le T, \\ \hat{q}(x, y, 0) = 0, \qquad \hat{q}_t(x, y, 0) = -\dfrac{1}{2} f_q(x, y, 0), & (x, y) \in \bar{\Omega}; \end{cases} \tag{3.32}$$

$$\begin{cases} \hat{r}_{tt} - \Delta \hat{r} = 0, & (x, y) \in \Omega,\ 0 < t \le T, \\ \hat{r}(x, y, t) = 0, & (x, y) \in \partial\Omega,\ 0 < t \le T, \\ \hat{r}(x, y, 0) = 0, \qquad \hat{r}_t(x, y, 0) = -\dfrac{1}{2} f_r(x, y, 0), & (x, y) \in \bar{\Omega}; \end{cases} \tag{3.33}$$

$$\begin{cases} w_{tt} - \Delta w = 0, & (x, y) \in \Omega,\ 0 < t \le T, \\ w(x, y, t) = 0, & (x, y) \in \partial\Omega,\ 0 < t \le T, \\ w(x, y, 0) = 0, \qquad w_t(x, y, 0) = \psi_w(x, y), & (x, y) \in \bar{\Omega} \end{cases} \tag{3.34}$$

One can construct the following explicit difference schemes to solve the second-order hyperbolic problems (3.32)–(3.34):

$$\begin{cases} \delta_t^2 \hat{q}_{ij}^n - \Delta_h \hat{q}_{ij}^n = 0, & (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \\ \hat{q}_{ij}^n = 0, & (x_i, y_j) \in \partial\Omega_h,\ 0 \le n \le N-1, \\ \hat{q}_{ij}^0 = 0, \qquad \hat{q}_{ij}^1 = -\dfrac{\tau}{2} f_q(x, y, 0), & (x_i, y_j) \in \Omega_h; \end{cases} \tag{3.35}$$

$$\begin{cases} \delta_t^2 \hat{r}_{ij}^n - \Delta_h \hat{r}_{ij}^n = 0, & (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \\ \hat{r}_{ij}^n = 0, & (x_i, y_j) \in \partial\Omega_h,\ 0 \le n \le N-1, \\ \hat{r}_{ij}^0 = 0, \qquad \hat{r}_{ij}^1 = -\dfrac{\tau}{2} f_r(x, y, 0), & (x_i, y_j) \in \Omega_h; \end{cases} \tag{3.36}$$

$$\begin{cases} \delta_t^2 w_{ij}^n - \Delta_h w_{ij}^n = 0, & (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \\ w_{ij}^n = 0, & (x_i, y_j) \in \partial\Omega_h,\ 0 \le n \le N-1, \\ w_{ij}^0 = 0, \qquad w_{ij}^1 = \tau \psi_w(x_i, y_j), & (x_i, y_j) \in \Omega_h. \end{cases} \tag{3.37}$$

Also, Theorem 3.2 shows that

$$\hat{q}(x_i, y_j, t_n) - \hat{q}_{ij}^n = O(\tau^2 + h_1^2 + h_2^2), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \tag{3.38}$$

$$\hat{r}(x_i, y_j, t_n) - \hat{r}_{ij}^n = O(\tau^2 + h_1^2 + h_2^2), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \tag{3.39}$$

$$w(x_i, y_j, t_n) - w_{ij}^n = O(\tau^2 + h_1^2 + h_2^2), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N. \tag{3.40}$$

Define grid function

$$e_{ij}^n = \tilde{u}_{ij}^n - \tau^2 p_{ij}^n - h_1^2 q_{ij}^n - h_2^2 r_{ij}^n - \tau h_1^2 \hat{q}_{ij}^n - \tau h_2^2 \hat{r}_{ij}^n - \tau^4 w_{ij}^n, \quad (x_i, y_j) \in \bar{\Omega}_h,\ 0 \le n \le N.$$

Multiplying the difference systems (3.26)–(3.28) and (3.35)–(3.37) by $\tau^2, h_1^2, h_2^2, \tau h_1^2, \tau h_2^2$ and τ^4 respectively, and subtracting the resulting systems from (3.22), we find that the grid function $\{e_{ij}^n\}$ satisfies

$$\begin{aligned} &\delta_t^2 e_{ij}^n - \Delta_h e_{ij}^n = \widehat{R}_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \\ &e_{ij}^n = 0, \quad (x_i, y_j) \in \partial\Omega_h,\ 0 \le n \le N, \\ &e_{ij}^0 = 0, \quad e_{ij}^1 = \widehat{\omega}_{ij}, \quad (x_i, y_j) \in \Omega_h. \end{aligned} \tag{3.41}$$

Applying the technique in the proof of Theorem 3.1, we know that

$$\left\|\widehat{R}^{n-\frac{1}{2}}\right\|_{\Sigma(\delta)} \le c(u)(\tau^4 + h_1^4 + h_2^4), \quad 1 \le n \le N-2.$$

Thus applying Theorem 3.3 to the error system (3.41), we get

$$\tilde{u}_{ij}^n - \tau^2 p_{ij}^n - h_1^2 q_{ij}^n - h_2^2 r_{ij}^n - \tau h_1^2 \hat{q}_{ij}^n - \tau h_2^2 \hat{r}_{ij}^n - \tau^4 w_{ij}^n = O(\tau^4 + h_1^4 + h_2^4).$$

Inserting the equalities (3.29)–(3.31) and (3.38)–(3.40) into the equality above, we have

$$\begin{aligned} u_{ij}^n(\tau, h_1, h_2) &= u(x_i, y_j, t_n) - \left[\tau^2 p(x_i, y_j, t_n) + h_1^2 q(x_i, y_j, t_n) + h_2^2 r(x_i, y_j, t_n)\right] - \left[\tau h_1^2 \hat{q}(x_i, y_j, t_n) + \tau h_2^2 \hat{r}(x_i, y_j, t_n)\right] \\ &\quad + O(\tau^4 + h_1^4 + h_2^4 + \tau^2 h_1^2 + \tau^2 h_2^2 + h_1^2 h_2^2), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N. \end{aligned} \tag{3.42}$$

Similarly,

$$\begin{aligned} u_{2i,2j}^{2n}\left(\frac{\tau}{2}, \frac{h_1}{2}, \frac{h_2}{2}\right) &= u(x_i, y_j, t_n) - \left[\left(\frac{\tau}{2}\right)^2 p(x_i, y_j, t_n) + \left(\frac{h_1}{2}\right)^2 q(x_i, y_j, t_n) + \left(\frac{h_2}{2}\right)^2 r(x_i, y_j, t_n)\right] \\ &\quad - \left[\frac{\tau}{2}\left(\frac{h_1}{2}\right)^2 \hat{q}(x_i, y_j, t_n) + \frac{\tau}{2}\left(\frac{h_2}{2}\right)^2 \hat{r}(x_i, y_j, t_n)\right] \\ &\quad + O(\tau^4 + h_1^4 + h_2^4 + \tau^2 h_1^2 + \tau^2 h_2^2 + h_1^2 h_2^2), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \end{aligned} \tag{3.43}$$

$$\begin{aligned} u_{4i,4j}^{4n}\left(\frac{\tau}{4}, \frac{h_1}{4}, \frac{h_2}{4}\right) &= u(x_i, y_j, t_n) - \left[\left(\frac{\tau}{4}\right)^2 p(x_i, y_j, t_n) + \left(\frac{h_1}{4}\right)^2 q(x_i, y_j, t_n) + \left(\frac{h_2}{4}\right)^2 r(x_i, y_j, t_n)\right] \\ &\quad - \left[\frac{\tau}{4}\left(\frac{h_1}{4}\right)^2 \hat{q}(x_i, y_j, t_n) + \frac{\tau}{4}\left(\frac{h_2}{4}\right)^2 \hat{r}(x_i, y_j, t_n)\right] \\ &\quad + O(\tau^4 + h_1^4 + h_2^4 + \tau^2 h_1^2 + \tau^2 h_2^2 + h_1^2 h_2^2), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N. \end{aligned} \tag{3.44}$$

Multiplying the equalities (3.42)–(3.44) by $1/21$, $-12/21$ and $32/21$, respectively, and adding up the resulting equalities, we find

$$u(x_i, y_j, t_n) - (u_E)_{ij}^n = O(\tau^4 + h_1^4 + h_2^4 + \tau^2 h_1^2 + \tau^2 h_2^2 + h_1^2 h_2^2), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N.$$

Thus the proof is completed. □

4. Error analysis of the ADI solution and its extrapolation

Utilizing the Taylor expansion with integral remainder, one has

$$u_{tt}(x_i, y_j, t_n) = \delta_t^2 U_{ij}^n - (R_t)_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \tag{4.1}$$

$$-u_{xx}(x_i, y_j, t_n) = -\delta_x^2 \frac{U_{ij}^{n+1} + U_{ij}^{n-1}}{2} + (R_x)_{ij}^n + (R_{tx})_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \tag{4.2}$$

$$-u_{yy}(x_i, y_j, t_n) = -\delta_y^2 \frac{U_{ij}^{n+1} + U_{ij}^{n-1}}{2} + (R_y)_{ij}^n + (R_{ty})_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \tag{4.3}$$

$$0 = \frac{\tau^4}{4} \delta_x^2 \delta_y^2 \delta_t^2 U_{ij}^n - (R_{txy})_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \tag{4.4}$$

where the errors $(R_t)_{ij}^n$, $(R_x)_{ij}^n$, $(R_y)_{ij}^n$ are defined in the above section, and

$$(R_{tx})_{ij}^n = \frac{\tau^2}{2} \int_0^1 \delta_x^2 \left[\frac{\partial^2 u(x_i, y_j, t_n - s\tau)}{\partial t^2} + \frac{\partial^2 u(x_i, y_j, t_n + s\tau)}{\partial t^2}\right](1-s)\mathrm{d}s,$$

$$(R_{ty})_{ij}^n = \frac{\tau^2}{2} \int_0^1 \delta_y^2 \left[\frac{\partial^2 u(x_i, y_j, t_n - s\tau)}{\partial t^2} + \frac{\partial^2 u(x_i, y_j, t_n + s\tau)}{\partial t^2}\right](1-s)\mathrm{d}s,$$

$$(R_{txy})_{ij}^n = \frac{\tau^4}{4} \int_0^1 \delta_x^2 \delta_y^2 \left[\frac{\partial^2 u(x_i, y_j, t_n - s\tau)}{\partial t^2} + \frac{\partial^2 u(x_i, y_j, t_n + s\tau)}{\partial t^2}\right](1-s)\mathrm{d}s.$$

Adding up the Eqs. (4.1)–(4.4) and using the following equality

$$u_{tt}(x_i, y_j, t_n) - \Delta u(x_i, y_j, t_n) = f(x_i, y_j, t_n), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1,$$

we have

$$\delta_t^2 U_{ij}^n - \Delta_h \frac{U_{ij}^{n+1} + U_{ij}^{n-1}}{2} + \frac{\tau^4}{4}\delta_x^2\delta_y^2\delta_t^2 U_{ij}^n = f(x_i, y_j, t_n) + S_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \tag{4.5}$$

where the truncation error

$$S_{ij}^n = (R_t)_{ij}^n - (R_x)_{ij}^n - (R_{tx})_{ij}^n - (R_y)_{ij}^n - (R_{ty})_{ij}^n + (R_{txy})_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1.$$

Omitting the small terms S_{ij}^n and ω_{ij}, and replacing U_{ij}^n with its numerical approximation u_{ij}^n in the Eqs. (4.5) and (3.5)–(3.7), one gets the following approximate factorization scheme

$$\delta_t^2 u_{ij}^n - \Delta_h \frac{u_{ij}^{n+1} + u_{ij}^{n-1}}{2} + \frac{\tau^4}{4}\delta_x^2\delta_y^2\delta_t^2 u_{ij}^n = f(x_i, y_j, t_n), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \tag{4.6}$$

$$u_{ij}^n = \alpha(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h,\ 0 \le n \le N, \tag{4.7}$$

$$u_{ij}^0 = \varphi(x_i, y_j), \quad (x_i, y_j) \in \Omega_h, \tag{4.8}$$

$$u_{ij}^1 = \varphi_1(x_i, y_j, \tau), \quad (x_i, y_j) \in \Omega_h. \tag{4.9}$$

This scheme can be split into various ADI methods such as the Douglas [22] scheme or the Douglas–Gunn [11] scheme. Using the identical operator I and the following equality

$$\frac{1}{2}(u_{ij}^{n+1} + u_{ij}^{n-1}) = u_{ij}^n + \frac{\tau^2}{2}\delta_t^2 u_{ij}^n,$$

we can write the interior scheme (4.6) as

$$\left(I - \frac{\tau^2}{2}\delta_x^2\right)\left(I - \frac{\tau^2}{2}\delta_y^2\right)\delta_t^2 u_{ij}^n = \Delta_h u_{ij}^n + f(x_i, y_j, t_n).$$

By introducing intermediate variables

$$u_{ij}^* = \left(I - \frac{\tau^2}{2}\delta_y^2\right)\delta_t^2 u_{ij}^n, \quad 0 \le i \le M_1,\ 1 \le j \le M_2 - 1,$$

the above scheme is decomposed into an ADI scheme of the Douglas–Gunn type

$$\begin{cases} \left(I - \dfrac{\tau^2}{2}\delta_x^2\right) u_{ij}^* = \Delta_h u_{ij}^n + f(x_i, y_j, t_n), \\ \left(I - \dfrac{\tau^2}{2}\delta_y^2\right) \delta_t^2 u_{ij}^n = u_{ij}^*. \end{cases}$$

To find the unknown solution $\left\{u_{ij}^{n+1} | (x_i, y_j) \in \Omega_h\right\}$, we can run the x-sweep and y-sweep procedures to get $\left\{u_{ij}^* | (x_i, y_j) \in \Omega_h\right\}$ and $\left\{\delta_t^2 u_{ij}^n | (x_i, y_j) \in \Omega_h\right\}$ respectively, then compute the wanted solution by

$$u_{ij}^{n+1} = 2u_{ij}^n - u_{ij}^{n-1} + \tau^2\delta_t^2 u_{ij}^n, \quad (x_i, y_j) \in \Omega_h.$$

Now we consider the following lemma of *a priori* estimation.

Lemma 4.1. *Let grid function $\{w_{ij}^n | (x_i, y_j) \in \bar{\Omega}_h,\ 0 \le n \le N\}$ be the discrete solution of the following implicit difference system*

$$\delta_t^2 w_{ij}^n - \Delta_h \frac{w_{ij}^{n+1} + w_{ij}^{n-1}}{2} + \frac{\tau^4}{4}\delta_x^2\delta_y^2\delta_t^2 w_{ij}^n = g_{ij}^n,\ (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \tag{4.10}$$

$$w_{ij}^n = 0, \quad (x_i, y_j) \in \partial\Omega_h,\ 0 \le n \le N, \tag{4.11}$$

$$w_{ij}^0 = \varphi_{ij}, \qquad w_{ij}^1 = \psi_{ij}, \quad (x_i, y_j) \in \Omega_h. \tag{4.12}$$

Then it holds that

$$F^n \le e^{t_n}\left(3F^0 + 4\|g^n\|_{\Sigma(\delta)}^2\right), \quad 0 \le n \le N-1,$$

where the energy norm

$$F^n = \left|\delta_t w^{n+\frac{1}{2}}\right|_1^2 + \frac{1}{2}\left(\left\|\Delta_h w^{n+1}\right\|^2 + \left\|\Delta_h w^n\right\|^2\right) + \frac{\tau^4}{4}\left|\delta_y\delta_x\delta_t w^{n+\frac{1}{2}}\right|_1^2, \quad 0 \le n \le N-1.$$

Proof. Multiplying (4.10) by $-2\tau h_1 h_2 D_t \Delta_h w_{ij}^n$ and summing i, j for $(x_i, y_j) \in \Omega_h$, we have

$$F^n - F^{n-1} = -2\tau h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} g_{ij}^n \left(D_t \Delta_h w_{ij}^n\right), \quad 1 \le n \le N-1,$$

where the discrete Green's first inequality is applied. Summing the above equation for n from 1 to k, and then replacing k with n, one gets

$$\begin{aligned} F^n &= F^0 - 2\tau h_1 h_2 \sum_{l=1}^{n} \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} g_{ij}^l \left(D_t \Delta_h w_{ij}^l\right) \\ &\le F^0 + \frac{1}{2}\left\|\Delta_h w^{\frac{1}{2}}\right\|^2 + \frac{\tau}{2}\sum_{l=1}^{n-1}\left\|\Delta_h w^{l+\frac{1}{2}}\right\|^2 + \frac{1}{2}\left\|\Delta_h w^{n+\frac{1}{2}}\right\|^2 + 2\left\|g^n\right\|_{\Sigma(\delta)}^2, \quad 1 \le n \le N-1, \end{aligned}$$

where Lemma 2.3 with $\epsilon = 2$ is used. Using the inequality

$$\left\|\Delta_h w^{n+\frac{1}{2}}\right\|^2 \le \frac{1}{2}\left(\left\|\Delta_h w^{n+1}\right\|^2 + \left\|\Delta_h w^n\right\|^2\right) \le F^n, \quad 0 \le n \le N-1,$$

one can obtain that

$$F^n \le 3F^0 + \tau \sum_{l=1}^{n-1} F^l + 4\left\|g^n\right\|_{\Sigma(\delta)}^2, \quad 1 \le n \le N-1.$$

Thus the Gronwall inequality [21] yields the claimed inequality. It completes the proof. □

It is to present the error analysis for the smooth solution. It is easy to find that the solution error of the ADI scheme (4.6)–(4.9) satisfies

$$\begin{aligned} &\delta_t^2 \tilde{u}_{ij}^n - \Delta_h \frac{\tilde{u}_{ij}^{n+1} + \tilde{u}_{ij}^{n-1}}{2} + \frac{\tau^4}{4}\delta_x^2\delta_y^2\delta_t^2 \tilde{u}_{ij}^n = S_{ij}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N-1, \\ &\tilde{u}_{ij}^n = 0, \quad (x_i, y_j) \in \partial\Omega_h,\ 0 \le n \le N, \\ &\tilde{u}_{ij}^0 = 0, \quad \tilde{u}_{ij}^1 = \omega_{ij}, \quad (x_i, y_j) \in \Omega_h. \end{aligned}$$

Under the condition $\tau \le ch$, it is easy to get that $\left\|\Delta_h \tilde{u}^0\right\| = 0$, $\left\|\Delta_h \tilde{u}^1\right\| \le c(u)\tau^2$, and

$$\left|\delta_t \tilde{u}^{\frac{1}{2}}\right|_1 \le c(u)\tau^2, \qquad \left\|\delta_y \delta_x^2 \delta_t \tilde{u}^{\frac{1}{2}}\right\| \le c(u), \qquad \left\|\delta_x \delta_y^2 \delta_t \tilde{u}^{\frac{1}{2}}\right\| \le c(u).$$

The same arguments in the proof of Theorem 3.1 show that

$$\left\|S^n\right\|_{\Sigma(\delta)} \le c(u)(\tau^2 + h_1^2 + h_2^2), \quad 1 \le n \le N-1.$$

Then we obtain from Lemmas 4.1 and 2.1 that

$$\left\|\tilde{u}^{n+1}\right\|_\infty \le c\left\|\Delta_h \tilde{u}^{n+1}\right\| \le c(u)(\tau^2 + h_1^2 + h_2^2), \quad 0 \le n \le N-1. \quad \square$$

Therefore one has the following theorem.

Theorem 4.1. *Let $u(x, y, t) \in \mathcal{C}^{(4,5)}(\bar{\Omega} \times [0, T])$ be the solution of the wave problem (1.1)–(1.3). Then the solution of the ADI scheme (4.6)–(4.9) is convergent with an order of $O(\tau^2 + h_1^2 + h_2^2)$ in the maximum norm provided the maximum spacing h is sufficiently small and the time-step size $\tau = O(h)$. Further, it is also valid for the difference scheme with the third-order accurate approximation (3.12) of $u(x_i, y_j, \tau)$ if the boundary conditions are zero-valued, i.e., $\alpha(x, y, t) = 0$.*

Obviously, Lemma 4.1 implies the unconditional stability of the ADI scheme.

Theorem 4.2. *Let $\{u_{ij}^n | (x_i, y_j) \in \bar{\Omega}_h,\ 0 \le n \le N\}$ be the solution of the approximate factorization scheme (4.6)–(4.9) with $\alpha(x, y, t) \equiv 0$. Then,*

$$F^n(u) \le e^{t_n}\left(3F^0(u) + 4\left\|f^n\right\|_{\Sigma(\delta)}^2\right), \quad 0 \le n \le N-1,$$

where $F^n(u) = \left|\delta_t u^{n+\frac{1}{2}}\right|_1^2 + \frac{1}{2}\left(\left\|\Delta_h u^{n+1}\right\|^2 + \left\|\Delta_h u^n\right\|^2\right) + \frac{\tau^4}{4}\left|\delta_y \delta_x \delta_t u^{n+\frac{1}{2}}\right|_1^2.$

We also consider the fourth order extrapolation of the discrete solution.

Theorem 4.3. *Let function $u(x, y, t)$ be the smooth solution of the wave problem (1.1)–(1.3) and $u_{ij}^n(\tau, h_1, h_2)$ be the numerical solution of the ADI scheme (4.6)–(4.9). Then it holds that*

$$\left\|U^n - (u_E)^n\right\|_\infty = O\left(\tau^4 + h_1^4 + h_2^4 + \tau^2 h_1^2 + \tau^2 h_2^2 + h_1^2 h_2^2\right), \quad (x_i, y_j) \in \Omega_h, \; 1 \le n \le N,$$

where the extrapolation solution

$$(u_E)_{ij}^n = \frac{32}{21} u_{4i,4j}^{4n}\left(\frac{\tau}{4}, \frac{h_1}{4}, \frac{h_2}{4}\right) - \frac{12}{21} u_{2i,2j}^{2n}\left(\frac{\tau}{2}, \frac{h_1}{2}, \frac{h_2}{2}\right) + \frac{1}{21} u_{i,j}^n(\tau, h_1, h_2).$$

Proof. We define the following smooth functions

$$F_{p1}(x, y, t) = \frac{1}{12}\frac{\partial^4 u}{\partial t^4}, \qquad F_{p2}(x, y, t) = -\frac{1}{2}\frac{\partial^4 u}{\partial x^2 \partial t^2} - \frac{1}{2}\frac{\partial^4 u}{\partial y^2 \partial t^2},$$

$$F_q(x, y, t) = -\frac{1}{12}\frac{\partial^4 u(x, y, t)}{\partial x^4}, \qquad F_r(x, y, t) = -\frac{1}{12}\frac{\partial^4 u(x, y, t)}{\partial y^4}, \qquad \Psi_w(x, y) = \frac{1}{120}\frac{\partial^5 u}{\partial t^5}\bigg|_{t=0}.$$

From the derivation of the ADI scheme (4.6)–(4.9), it is not difficult to know that

$$S_{ij}^n = F_{p1}(x_i, y_j, t_n)\tau^2 + F_{p2}(x_i, y_j, t_n)\tau^2 + F_q(x_i, y_j, t_n)h_1^2 + F_r(x_i, y_j, t_n)h_2^2 + \widehat{S}_{ij}^n,$$

$$\omega_{ij} = \frac{\tau^4}{2} F_{p1}(x, y, 0) + \tau^5 \Psi_w(x_i, y_j) + \widehat{\omega}_{ij},$$

where

$$\widehat{S}_{ij}^n = (\widehat{R}_t)_{ij}^n - (\widehat{R}_x)_{ij}^n - (\widehat{R}_{tx})_{ij}^n - (\widehat{R}_y)_{ij}^n - (\widehat{R}_{ty})_{ij}^n + (R_{txy})_{ij}^n, \quad (x_i, y_j) \in \Omega_h, \; 1 \le n \le N-1.$$

The small terms $(\widehat{R}_t)_{ij}^n$, $(\widehat{R}_x)_{ij}^n$, $(\widehat{R}_y)_{ij}^n$ are defined in the proof of Theorem 3.4, and

$$\begin{aligned}(\widehat{R}_{tx})_{ij}^n &= \frac{\tau^4}{2\cdot 3!}\int_0^1 \delta_x^2\left[\frac{\partial^4 u(x_i, y_j, t_n - s\tau)}{\partial t^4} + \frac{\partial^4 u(x_i, y_j, t_n + s\tau)}{\partial t^4}\right](1-s)^3 ds \\ &\quad + \frac{\tau^2 h_1^2}{2\cdot 3!}\int_0^1 \left[\frac{\partial^6 u(x_i - \lambda h_1, y_j, t_n)}{\partial x^4 \partial t^2} + \frac{\partial^6 u(x_i + \lambda h_1, y_j, t_n)}{\partial x^4 \partial t^2}\right](1-\lambda)^3 d\lambda \\ (\widehat{R}_{ty})_{ij}^n &= \frac{\tau^4}{2\cdot 3!}\int_0^1 \delta_y^2\left[\frac{\partial^4 u(x_i, y_j, t_n - s\tau)}{\partial t^4} + \frac{\partial^4 u(x_i, y_j, t_n + s\tau)}{\partial t^4}\right](1-s)^3 ds \\ &\quad + \frac{\tau^2 h_2^2}{2\cdot 3!}\int_0^1 \left[\frac{\partial^6 u(x_i, y_j - \lambda h_2, t_n)}{\partial y^4 \partial t^2} + \frac{\partial^6 u(x_i, y_j + \lambda h_2, t_n)}{\partial y^4 \partial t^2}\right](1-\lambda)^3 d\lambda.\end{aligned}$$

Then, with the aid of Theorems 4.1 and 4.2 together with Lemma 2.1, one can get the claimed result by presenting similar arguments for Theorem 3.4. □

5. Numerical experiments

To verify our theory, we solve the hyperbolic problem (1.1)–(1.3) numerically by the explicit scheme (3.8)–(3.11) and the ADI method (4.6)–(4.9). In the runs, we use the same spacing h in each spatial direction, $h_1 = h_2 = h$. All experiments were carried out on a PC with 1024 RAM using the student version of MATLAB.

The explicit scheme (3.8)–(3.11) updates the solution at time level t_{n+1} by the explicit formula

$$u_{ij}^{n+1} = 2u_{ij}^n - u_{ij}^{n-1} + \tau^2\left[\Delta_h u_{ij}^n + f(x_i, y_j, t_n)\right], \quad (x_i, y_j) \in \Omega_h, \; 1 \le n \le N-1;$$

but the time step is restricted by $\tau < h/\sqrt{2}$. The ADI method (4.6)–(4.9) can use a larger time-step size, but it computes the solution at time level t_{n+1} by solving tridiagonal linear systems. In the x-sweep, one would run the well-known Thomas algorithm to solve

$$A\mathbf{u}_j^* = \mathbf{b}_j, \quad 1 \le j \le M-1,$$

where

$$A = \begin{bmatrix} 1+2\sigma^2 & -\sigma^2 & & & \\ -\sigma^2 & 1+2\sigma^2 & -\sigma^2 & & \\ & \ddots & \ddots & \ddots & \\ & & -\sigma^2 & 1+2\sigma^2 & -\sigma^2 \\ & & & -\sigma^2 & 1+2\sigma^2 \end{bmatrix}$$

Table 1
Computational cost of explicit and ADI solutions with time-step $\tau = h/2$.

h	τ	Explicit method		ADI method	
		$e(\tau, h)$	CPU time (s)	$e(\tau, h)$	CPU time (s)
1/160	1/320	1.37e−06	5.484	9.78e−07	29.89
1/320	1/640	3.42e−07	45.94	2.45e−07	157.0
1/640	1/1280	8.56e−08	375.0	6.12e−08	1003

Table 2
Convergence of explicit solution using the fourth-order start with time-step $\tau = h/2$.

h	τ	$e(\tau, h)$	Rate γ	$e_E(\tau, h)$	Rate γ_E
1/4	1/8	2.11e−03	–	3.38e−06	–
1/8	1/16	5.35e−04	1.98	2.36e−07	3.84
1/16	1/32	1.36e−04	1.97	1.50e−08	3.97
1/32	1/64	3.42e−05	2.00	9.26e−10	4.02
1/64	1/128	8.56e−06	2.00	–	–
1/128	1/256	2.14e−06	2.00	–	–

$$\mathbf{u}_j^* = \begin{bmatrix} u_{1j}^* \\ u_{2j}^* \\ \vdots \\ u_{M-2,j}^* \\ u_{M-1,j}^* \end{bmatrix}, \quad \mathbf{b}_j^* = \begin{bmatrix} \Delta_h u_{1j}^n + f(x_1, y_j, t_n) + \sigma^2 \left(I - \frac{\tau^2}{2}\delta_y^2\right) \delta_t^2 u_{0j}^n \\ \Delta_h u_{2j}^n + f(x_2, y_j, t_n) \\ \vdots \\ \Delta_h u_{M-2,j}^n + f(x_{M-2}, y_j, t_n) \\ \Delta_h u_{M-1,j}^n + f(x_{M-1}, y_j, t_n) + \sigma^2 \left(I - \frac{\tau^2}{2}\delta_y^2\right) \delta_t^2 u_{Mj}^n \end{bmatrix}.$$

In the y-sweep, one would run the Thomas algorithm to solve

$$A\mathbf{w}_i = \mathbf{u}_i^*, \quad 1 \le i \le M-1,$$

where

$$\mathbf{w}_i = \begin{bmatrix} w_{i1} \\ w_{i2} \\ \vdots \\ w_{i,M-2} \\ w_{i,M-1} \end{bmatrix}, \quad \mathbf{u}_i^* = \begin{bmatrix} u_{i1}^* + \sigma^2 \delta_t^2 u_{i0}^n \\ u_{i2}^* \\ \vdots \\ u_{i,M-2}^* \\ u_{i,M-1}^* + \sigma^2 \delta_t^2 u_{iM}^n \end{bmatrix}.$$

Then the solution at time level t_{n+1} is obtained by

$$u_{ij}^{n+1} = 2u_{ij}^n - u_{ij}^{n-1} + \tau^2 w_{ij}^n, \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N-1.$$

We see that, to compute the solution at time level t_{n+1}, the ADI method sweeps the spatial grids three times while the explicit method needs one sweep only. It is to be expected that the ADI scheme would be computationally more expensive than the explicit scheme on the same grids. With an exact solution $u(x, y, t) = e^{x+y} \sin t$, we compute the maximum norm error of the discrete solution, $e(\tau, h) = \max_{1\le n\le N} \|U^n - u^n\|_\infty$, on the space–time domain $\Omega \times (0, 1]$. Table 1 reports the CPU cost of the explicit and ADI method on the halving grids with the coarsest grid ($h = 1/160$, $\tau = 1/320$). We observe that the ADI method is accurate as the same as the explicit method for this example, but the CPU cost is much more than that of the explicit scheme.

Now we examine the numerical accuracy of the explicit and ADI schemes with the solution $u(x, y, t) = e^{x+y} \sin t$. On the space–time domain $\Omega \times (0, 1]$, the maximum norm errors of the discrete solution and its Richardson extrapolation $e_E(\tau, h) = \max_{1\le n\le N} \|U^n - (u_E)^n\|_\infty$ are computed. For our comparisons, two different approximations of solution $u(x_i, y_j, \tau)$, including the fourth-order scheme (3.11) and the third-order scheme (3.12), are considered to start the two difference methods. Succinctly, the former is called the fourth-order start and the latter the third-order start.

In Tables 2 and 3, the solutions are approximated by the explicit method, using the fourth-order start and the third-order start respectively, on the halving grids. Setting time step size $\tau = h/2$, the experimental rate (listed as Rate in the tables) of convergence, in h, is computed by observing that $e(\tau, h) \approx ch^\gamma$, $e_E(\tau, h) \approx ch^{\gamma_E}$ and utilizing $\gamma \approx \log_2(e(2\tau, 2h)/e(\tau, h))$, $\gamma_E \approx \log_2(e_E(2\tau, 2h)/e_E(\tau, h))$. We observe that, as predicted by Theorems 3.1 and 3.4, the explicit scheme (3.8)–(3.11) generates a second order accurate solution and one Richardson extrapolation produces a fourth order approximation. The solution of the explicit method using the third-order start is also second-order convergent but the extrapolated solution is only approximately third-order accurate.

Table 3
Convergence of explicit solution using the third-order start with time-step $\tau = h/2$.

h	τ	$e(\tau, h)$	Rate γ	$e_E(\tau, h)$	Rate γ_E
1/4	1/8	3.03e−03	–	1.03e−04	–
1/8	1/16	8.91e−04	1.77	1.36e−05	2.92
1/16	1/32	2.42e−04	1.88	2.21e−06	2.62
1/32	1/64	6.35e−05	1.93	3.35e−07	2.72
1/64	1/128	1.63e−05	1.96	–	–
1/128	1/256	4.15e−06	1.98	–	–

Table 4
Convergence of ADI solution using the fourth-order start with time-step $\tau = h$.

h	τ	$e(\tau, h)$	Rate γ	$e_E(\tau, h)$	Rate γ_E
1/8	1/8	3.42e−03	–	9.21e−06	–
1/16	1/16	8.66e−04	1.98	5.86e−07	3.97
1/32	1/32	2.15e−04	2.01	3.94e−08	3.90
1/64	1/64	5.38e−05	2.00	2.57e−09	3.94
1/128	1/128	1.35e−05	2.00	–	–
1/256	1/256	3.36e−06	2.00	–	–

Table 5
Convergence of ADI solution using the third-order start with time-step $\tau = h$.

h	τ	$e(\tau, h)$	Rate γ	$e_E(\tau, h)$	Rate γ_E
1/8	1/8	6.67e−03	–	1.99e−04	–
1/16	1/16	1.69e−03	1.98	2.31e−05	3.10
1/32	1/32	3.55e−04	2.25	3.22e−06	2.85
1/64	1/64	8.76e−05	2.02	4.93e−07	2.71
1/128	1/128	2.18e−05	2.00	–	–
1/256	1/256	5.45e−06	2.00	–	–

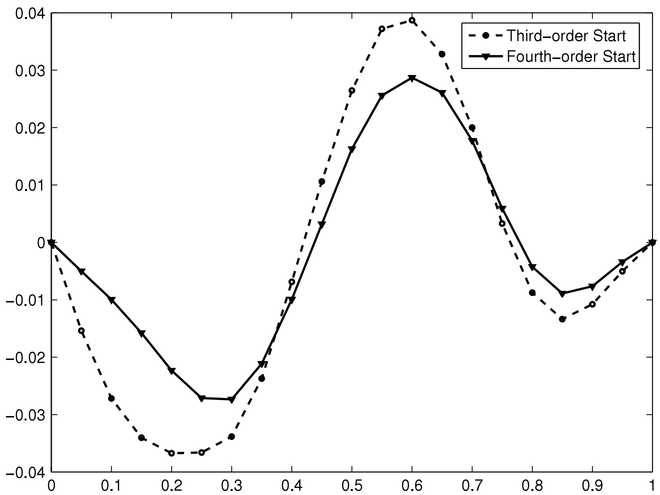

Fig. 1. Errors at $y = 1/2$, $T = 1$ of the explicit scheme with $h = 1/20$ and $\tau = 1/40$.

We also apply the ADI scheme of the Douglas–Gunn type to solve the above model. Data in Tables 4 and 5 are obtained in a similar way to those in Tables 2 and 3. Again, similar numerical phenomena are seen. In particular, data in Table 4 support the results of Theorems 4.1 and 4.3.

At last, we show the difference between the third and fourth order start of the two methods in approximating a traveling wave $u(x, y, t) = \sin(10(x + y + \sqrt{2}t))$ at different time. Given the $h = 1/20$ and $\tau = 1/40$, we compute the explicit approximation, at $T = 1$ and $T = 10$ respectively, as shown in Figs. 1 and 2, in which numerical errors at $y = 1/2$ are plotted. Observation shows that the numerical error of the fourth-order start is smaller than that of the third-order start. Similar phenomena are also seen in Figs. 3 and 4, where the solution is approximated by the ADI scheme.

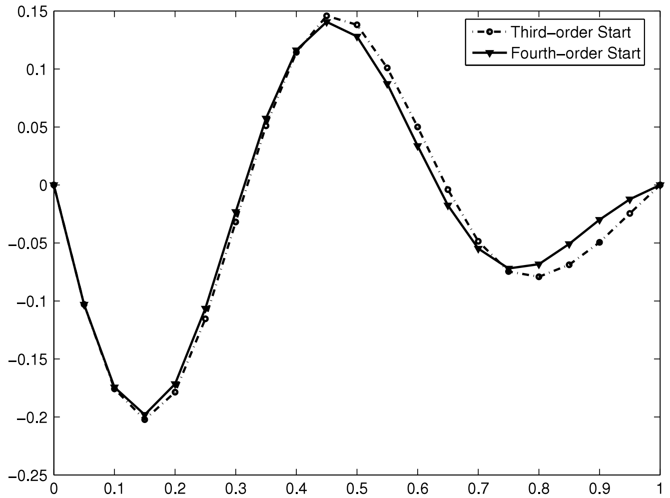

Fig. 2. Errors at $y = 1/2$, $T = 10$ of the explicit scheme with $h = 1/20$ and $\tau = 1/40$.

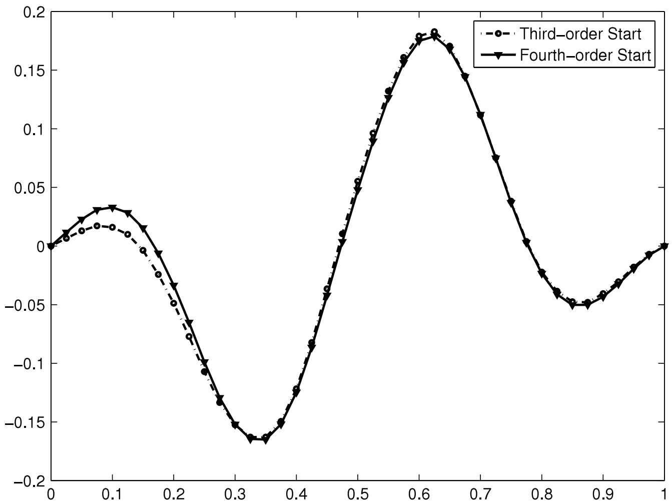

Fig. 3. Errors at $y = 1/2$, $T = 1$ of the ADI scheme with $h = 1/40$ and $\tau = 1/40$.

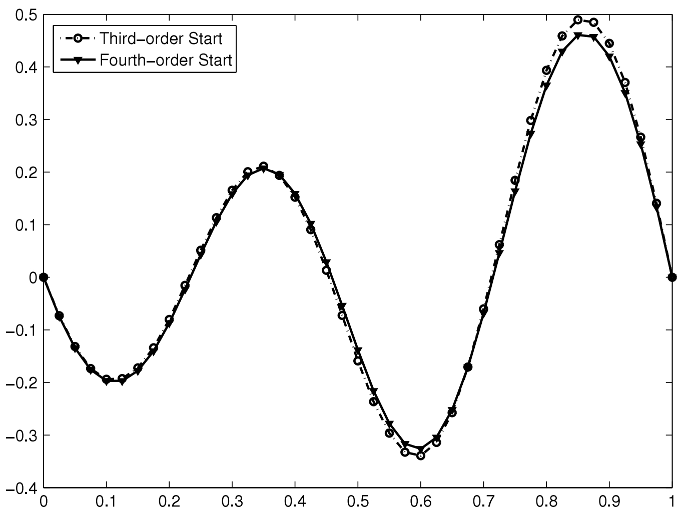

Fig. 4. Errors at $y = 1/2$, $T = 10$ of the ADI scheme with $h = 1/40$ and $\tau = 1/40$.

6. Concluding remarks

The recently suggested H^2 energy analysis is applied to theoretical considerations of the well-known second-order explicit and ADI methods for second-order hyperbolic problems. It has been shown that the explicit and ADI solutions are convergent in the maximum norm. Even though the centered schemes are employed for temporal discretizations, the asymptotic expansion of the explicit or ADI solution consists of odd powers of the time-step due to the inconsistency between the global scheme and the start procedure at the first time level. Thus an unusual Richardson extrapolation formula is needed in promoting the second-order solution to fourth-order accuracy. Numerical tests are included to verify our results. Our experiments show that the explicit scheme is more efficient than the ADI method in the sense of computational cost.

Extensions of the explicit and ADI approaches to three-dimensional wave equation are straightforward. As noted in [16], Lemma 2.1 are valid on the three-dimensional cuboidal domain. Thus, by applying the fourth-order start at the first time level and the suggested H^2 energy technique, it is easy to obtain the maximum norm error estimates of the numerical solutions and their Richardson extrapolations. Since the three-dimensional ADI method would sweep the entire grids more times than the two-dimensional version, the computational cost would be more expensive than the explicit method in getting certain accuracy. Future work is planned to improve the resolving efficiency of ADI approach and reduce the computational count of extrapolation.

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