



Weak Dirichlet processes with a stochastic control perspective

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Received 21 March 2003; received in revised form 27 March 2006; accepted 14 April 2006

Available online 3 May 2006

Abstract

The motivation of this paper is to prove verification theorems for stochastic optimal control of finite dimensional diffusion processes without control in the diffusion term, in the case where the value function is assumed to be continuous in time and once differentiable in the space variable ($C^{0,1}$) instead of once differentiable in time and twice in space ($C^{1,2}$), like in the classical results. For this purpose, the replacement tool of the Itô formula will be the Fukushima–Dirichlet decomposition for weak Dirichlet processes. Given a fixed filtration, a weak Dirichlet process is the sum of a local martingale M plus an adapted process A which is orthogonal, in the sense of covariation, to any continuous local martingale. The decomposition mentioned states that a $C^{0,1}$ function of a weak Dirichlet process with finite quadratic variation is again a weak Dirichlet process. That result is established in this paper and it is applied to the strong solution of a Cauchy problem with final condition.

Applications to the proof of verification theorems will be addressed in a companion paper.

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MSC: 60G05; 60G44; 60G48; 60H05; 60H10; 60J60; 35K15; 35K55; 35J15; 93E20

Keywords: Stochastic calculus via regularization; Weak Dirichlet processes; Stochastic optimal control; Cauchy problem for parabolic partial differential equations

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1. Introduction

In this paper we prepare a framework of stochastic calculus via regularization in order to apply it to the proof of verification theorems in stochastic optimal control in finite dimension. The application part will be implemented in the companion paper [17].

This paper has interest in itself and its most significant result is a generalized time dependent Fukushima–Dirichlet decomposition which is proved in Section 3. This will be the major tool for applications.

The proof of verification theorems for stochastic control problems under classical conditions is an application of the Itô formula. In fact, under good assumptions, the value function $V : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ associated with a stochastic control problem is of class C^1 in time and C^2 in space ($C^{1,2}$ in symbols). This allows one to apply it to the solution of a corresponding state equation (S_t) and differentiate $V(t, S_t)$ through the classical Itô formula, see e.g. [9, pp. 140, 163, 172]. The substitution tool of that formula will be a time dependent Fukushima–Dirichlet decomposition which will hold for functions $u : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ that are $C^{0,1}$ in symbols; so, our verification theorem will have the advantage of requiring less regularity on the value function V than the classical ones.

It is also possible to prove a verification theorem in the case where V is only continuous (see e.g. [20], [35, Section 5.2], [18]) in the framework of viscosity solutions: however such a result applied to our cases is weaker than ours, as it requires more assumptions on the candidate optimal strategy; see on this the last section of the companion paper [17] where also a comparison with other nonsmooth verification theorems is performed.

We come back to the Fukushima–Dirichlet decomposition as a replacement for the Itô formula. Roughly speaking, given a function u of class $C^{1,2}$, the classical Itô formula gives a decomposition of $u(t, S_t)$ into a martingale part, say M (which is thrown away taking expectation in the case of deterministic data and expected cost) plus an absolutely continuous process, say A . Then, in the case of deterministic data and expected cost, one uses the fact that u is a classical solution of a partial differential equation (PDE), which is in fact the Hamilton–Jacobi–Bellman (HJB) equation, to represent A in terms of the Hamiltonian function. If one wants to repeat the above arguments when u is not $C^{1,2}$, a natural way is to try to extend the decomposition $u(\cdot, S) = M + A$ and the representation of A via the HJB PDE. This is what we do in this paper in the case when $u \in C^{0,1}$ using a point of view that can also be applied to problems with stochastic data and pathwise cost, so the HJB becomes a stochastic PDE, see e.g. [3, 21, 22, 25]).

We propose in fact an extension of the classical Fukushima–Dirichlet decomposition. That decomposition is inspired by the theory of Dirichlet forms. A classical monograph concerning this theory is [13] where one can find classical references on the subject. In Theorem 5.2.2 of [13], given a “good” symmetric Markov process $(X_t^x)_{t \geq 0}$ and a function belonging to some suitable space (Dirichlet space), it is possible to write

$$u(X_t) = u(x) + M_t^u + A_t^u, \quad (1)$$

where M is a local martingale and $(A_t^u)_{t \geq 0}$ is a zero quadratic variation process, for quasi-everywhere x , i.e. for x belonging to a zero capacity set. For instance, if $X = W$ is a classical Brownian motion in \mathbb{R}^n then the Dirichlet space is $H^1(\mathbb{R}^n)$ and $M_t^u = \int_0^t \nabla u(W_u) dW_u$. We call (1) a Fukushima–Dirichlet decomposition. Our point of view is of pathwise nature: as in [10, 11], a process Y , as $u(X)$, which is the sum of a local martingale and a zero quadratic variation process, even without any link to Dirichlet forms, is called a Dirichlet process. However, in [10], the notion of quadratic variation, even if in the same spirit as ours, was defined through

discretizations, while we define it through regularizations. Therefore that notion of Dirichlet process is similar but not identical to ours.

The papers [29,12] reinterpret A in a “pathwise” way as the covariation process $[\nabla u(B), B]$ transforming (1) into a true Itô formula; the first work considers $u \in C^1(\mathbb{R}^n)$ and it extends the framework to reversible continuous semimartingales; the second work is connected with Brownian motion and $u \in H^1(\mathbb{R}^n)$. The literature on Itô’s formula for nonsmooth functions of semimartingales or diffusion processes has seen a lot of development in recent years, see for instance [23] for nondegenerate Brownian martingales, [8] for nondegenerate one dimensional diffusions with bounded measurable drift or [6] for the jump case.

In our applications, the fact of identifying the remainder process A^u as a covariation is not so important since the goal is to give the representation of it via the data of the HJB PDE. So we come back to the spirit of the Fukushima–Dirichlet decomposition. Besides our “pathwise” approach to Dirichlet processes, the true novelty of this approach is the time-inhomogeneous version of the decomposition; this is in particular motivated by non-autonomous problems in control theory.

This is based on the theory, under construction, of weak Dirichlet processes with respect to some fixed filtration (\mathcal{F}_t) . A weak Dirichlet process is the sum of a local martingale M and a process A which is adapted and $[A, N] = 0$ for any continuous (\mathcal{F}_t) -local martingale N . We will be able in particular to decompose $u(t, D_t)$ when $u \in C^{0,1}$ and D is a weak Dirichlet process with finite quadratic variation process, and so in particular if D is a semimartingale (even a diffusion process). This will be our time dependent Fukushima–Dirichlet decomposition: it will be the object of Proposition 3.10 and Corollary 3.11. In particular that result holds for semimartingales (and so for diffusion processes). The notion of weak Dirichlet process appears also in [5]. Our Fukushima–Dirichlet decomposition could be linked to the theory of “time dependent Dirichlet forms” developed for instance by [24,33,34] but we have not investigated that direction.

The paper is organized as follows. In Section 2 we introduce some notation for real analysis and we establish preliminary notions for calculus via regularization with some remarks on classical Dirichlet processes. Section 3 will be devoted to some basic facts about weak Dirichlet processes and to the above mentioned Fukushima–Dirichlet decomposition of process $(u(t, D_t))$, with a sufficient condition to guarantee that the resulting process is a true Dirichlet process. Section 4 will be concerned with application to the case where u is a strong $C^{0,1}$ solution of a Cauchy parabolic problem with initial condition; C^1 solutions of an elliptic problem are also represented probabilistically.

2. Preliminaries

2.1. Notation

Throughout this paper we will denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a given stochastic basis (where \mathcal{F} stands for a given filtration $(\mathcal{F}_s)_{s \geq 0}$ satisfying the usual conditions). Given a finite dimensional real Hilbert space E , W will denote a cylindrical Brownian motion with values in E and adapted to $(\mathcal{F}_s)_{s \geq 0}$. Given $0 \leq t \leq T \leq +\infty$ and setting $\mathcal{T}_T = [t, T] \cap \mathbb{R}$ the symbol $\mathcal{C}_{\mathcal{F}}(\mathcal{T}_T \times \Omega; E)$ will denote the space of all continuous processes adapted to the filtration \mathcal{F} with values in E . This is a Fréchet space if endowed with the topology of the uniform convergence in probability (*u.c.p.* from now on). To be more precise this means that, given a sequence $(X^n) \subseteq \mathcal{C}_{\mathcal{F}}(\mathcal{T}_T \times \Omega; E)$ and $X \in \mathcal{C}_{\mathcal{F}}(\mathcal{T}_T \times \Omega; E)$, we have

$$X^n \rightarrow X$$

if and only if for every $\varepsilon > 0, t_1 \in \mathcal{T}_t$,

$$\lim_{n \rightarrow +\infty} \sup_{s \in [t, t_1]} \mathbb{P}(|X_s^n - X_s|_E > \varepsilon) = 0.$$

Given a random time $\tau \geq t$ and a process $(X_s)_{s \in \mathcal{T}_t}$, we denote by X^τ the stopped process defined by $X_s^\tau = X_{s \wedge \tau}$. The space of all processes in $[t, T]$, adapted to \mathcal{F} and square integrable with values in E is denoted by $L^2_{\mathcal{F}}(t, T; E)$. S^n will denote the space of all symmetric matrices of dimension n . If Z is a vector or a matrix, then Z^* is its transposition.

Let $k \in \mathbb{N}$. As usual $C^k(\mathbb{R}^n)$ is the space of all functions $:\mathbb{R}^n \rightarrow \mathbb{R}$ that are continuous together with their derivatives up to the order k . This is a Fréchet space equipped with the seminorms

$$\sup_{x \in K} |u(x)|_{\mathbb{R}} + \sup_{x \in K} |\partial_x u(x)|_{\mathbb{R}^n} + \sup_{x \in K} |\partial_{xx} u(x)|_{\mathbb{R}^{n \times n}}$$

for every compact set $K \subset \subset \mathbb{R}^n$. This space will be denoted simply by C^k when no confusion may arise. The symbol $C^k_b(\mathbb{R}^n)$ will denote the Banach space of all continuous and bounded functions from \mathbb{R}^n to \mathbb{R} . This space is endowed with the usual sup norm. Passing to parabolic spaces we denote by $C^0(\mathcal{T}_t \times \mathbb{R}^n)$ the space of all functions

$$u : \mathcal{T}_t \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad (s, x) \mapsto u(s, x)$$

that are continuous. This space is a Fréchet space equipped with the seminorms

$$\sup_{(s,x) \in [t, t_1] \times K} |u(s, x)|_{\mathbb{R}}$$

for every $t_1 > 0$ and every compact set $K \subset \subset \mathbb{R}^n$. Moreover we will denote by $C^{1,2}(\mathcal{T}_t \times \mathbb{R}^n)$ (respectively $C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$) the space of all functions

$$u : \mathcal{T}_t \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad (s, x) \mapsto u(s, x)$$

that are continuous together with their derivatives $\partial_t u, \partial_x u, \partial_{xx} u$ (respectively $\partial_x u$). This space is a Fréchet space equipped with the seminorms

$$\begin{aligned} &\sup_{(s,x) \in [t, t_1] \times K} |u(s, x)|_{\mathbb{R}} + \sup_{(s,x) \in [t, t_1] \times K} |\partial_s u(s, x)|_{\mathbb{R}^n} \\ &+ \sup_{(s,x) \in [t, t_1] \times K} |\partial_x u(s, x)|_{\mathbb{R}^n} + \sup_{(s,x) \in [t, t_1] \times K} |\partial_{xx} u(s, x)|_{\mathbb{R}^{n \times n}} \end{aligned}$$

(respectively

$$\sup_{(s,x) \in [t, t_1] \times K} |u(s, x)|_{\mathbb{R}} + \sup_{(s,x) \in [t, t_1] \times K} |\partial_x u(s, x)|_{\mathbb{R}^n}$$

for every $t_1 > 0$ and every compact set $K \subset \subset \mathbb{R}^n$. This space will be denoted simply by $C^{1,2}$ (respectively $C^{0,1}$) when no confusion may arise.

Similarly, for $\alpha, \beta \in [0, 1]$ one defines $C^{\alpha,1+\beta}(\mathcal{T}_t \times \mathbb{R}^n)$ (or simply $C^{\alpha,1+\beta}$) as the subspace of $C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$ of functions $u : \mathcal{T}_t \times \mathbb{R}^n \mapsto \mathbb{R}$ such that $u(\cdot, x)$ is α -Hölder continuous and $\partial_x u(s, \cdot)$ is β -Hölder continuous (with the agreement that 0-Hölder continuity means just continuity).

We define similarly to $C^k_b(\mathbb{R}^n)$ the Banach spaces $C^0_b(\mathcal{T}_t \times \mathbb{R}^n), C^{1,2}_b(\mathcal{T}_t \times \mathbb{R}^n), C^{\alpha,1+\beta}_b(\mathcal{T}_t \times \mathbb{R}^n), C^{0,1}_b(\mathcal{T}_t \times \mathbb{R}^n)$.

2.2. The calculus via regularization

We will follow here a framework of calculus via regularizations started in [27]. At the moment many authors have contributed to this and we suggest the reader consult the recent survey paper [32] on it.

For simplicity, all the processes considered, except if stated otherwise, will be continuous processes. We first recall some one dimensional consideration. For two processes $(X_s)_{s \geq 0}$, $(Y_s)_{s \geq 0}$, we define the forward integral and the covariation as follows

$$\int_0^s X_r d^- Y_r = \lim_{\varepsilon \rightarrow 0} \int_0^s X_r \frac{Y_{r+\varepsilon} - Y_r}{\varepsilon} dr, \tag{2}$$

$$[X, Y]_s = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^s (X_{r+\varepsilon} - X_r)(Y_{r+\varepsilon} - Y_r) dr, \tag{3}$$

if those quantities exist in the sense of *u.c.p* with respect to s . This ensures that the forward integral defined in (2) and the covariation process defined in (3) are continuous processes. It can be seen that the covariation is a bilinear and symmetric operator.

We fix now, as above, $0 \leq t \leq T \leq +\infty$ and set $\mathcal{T}_t = [t, T] \cap \mathbb{R}$. A process $(X_s)_{s \in \mathcal{T}_t}$ can always be extended (if $T < +\infty$) to a process indexed by \mathbb{R}_+ by continuity. The corresponding extension will always denoted by the same symbol. Given two processes $(X_s)_{s \in \mathcal{T}_t}$, $(Y_s)_{s \in \mathcal{T}_t}$, we define the corresponding stochastic integrals and covariations by the integrals and covariation of the corresponding extensions. We define also integrals from t to s as follows.

$$\int_t^s X_r d^- Y_r = \int_0^s X_r d^- Y_r - \int_0^t X_r d^- Y_r.$$

If $\tau \geq t$ is a random (not necessarily) stopping time, the following equality holds:

$$[X, Y]^\tau = [X^\tau, Y^\tau]. \tag{4}$$

If (X^1, \dots, X^n) is a vector of continuous processes we say that it has all its mutual covariations (brackets) if $[X^i, X^j]$ exist for any $1 \leq i, j \leq n$. If X^1, \dots, X^n have all their mutual covariations then by polarization (i.e. writing a bilinear form as a sum/difference of quadratic forms) we know that $[X^i, X^j]$ ($1 \leq i, j \leq n$) are locally bounded variation processes.

If $[X, X]$ exists, then X is said to be a finite quadratic variation process; $[X, X]$ is called the quadratic variation of X . If $[X, X] = 0$ then X is said to be a zero quadratic variation process. A bounded variation process is a zero quadratic variation process. If S^1, S^2 are (\mathcal{F}_s) -semimartingales then $[S^1, S^2]$ coincides with the classical bracket $\langle S^1, S^2 \rangle$. If H is a (\mathcal{F}_s) -progressively measurable process then $\int_t^s H_r d^- S_r$ is the classical Itô integral $\int_t^s H_r dS_r$.

Remark 2.1. Let X (respectively A) be a finite (respectively zero) quadratic variation process. Then (X, A) has all its mutual covariations and $[X, A] = 0$. \square

We recall now an easy extension of stability results, see [7, th. 2.9], that will be used in Section 3.3.

Proposition 2.2. Let $V = (V^1, \dots, V^m)$ (respectively $X = (X^1, \dots, X^n)$) be a vector of continuous processes on \mathbb{R}_+ with bounded variation processes (respectively having all its mutual covariations). Let $f, g \in C_{loc}^{\frac{1}{2}+\gamma, 1}(\mathbb{R}^m \times \mathbb{R}^n)$ ($\gamma > 0$). Then $\forall s \geq 0$

$$[f(V, X), g(V, X)]_s = \sum_{i,j=1}^n \int_0^s \partial_{x_i} f(V, X) \partial_{x_j} g(V, X) d[X^i, X^j]_r.$$

Proof. We give a sketch of the proof as the arguments are similar to the ones used in [7, th. 2.9]. By localization $C_{loc}^{\frac{1}{2}+\gamma,1}$ can be replaced by $C^{\frac{1}{2}+\gamma,1}$. The case where f and g do not depend on V was treated for instance in [28,30]. Since the covariation is a bilinear operation, using polarization techniques we can take $g = f$. For simplicity we set here $m = n = 1$. For given $\varepsilon > 0$ we write, when $r \in \mathcal{T}_t$,

$$f(V_{r+\varepsilon}, X_{r+\varepsilon}) - f(V_r, X_r) = J_1(r, \varepsilon) + J_2(r, \varepsilon),$$

where

$$\begin{aligned} J_1(r, \varepsilon) &= f(V_{r+\varepsilon}, X_{r+\varepsilon}) - f(V_r, X_{r+\varepsilon}), \\ J_2(r, \varepsilon) &= f(V_r, X_{r+\varepsilon}) - f(V_r, X_r). \end{aligned}$$

Therefore, for $s \geq 0$

$$\frac{1}{\varepsilon} \int_0^s [f(V_{r+\varepsilon}, X_{r+\varepsilon}) - f(V_r, X_r)]^2 dr \leq \frac{2}{\varepsilon} \int_0^s J_1^2(r, \varepsilon) dr + \frac{2}{\varepsilon} \int_0^s J_2^2(r, \varepsilon) dr.$$

Now

$$\frac{2}{\varepsilon} \int_0^s J_1^2(r, \varepsilon) dr \leq c^2(s, f, V) \frac{2}{\varepsilon} \int_0^s (|V_{r+\varepsilon} - V_r|^{\frac{1}{2}+\gamma})^2 dr,$$

where $c(s, f, V)$ is a (random) Hölder constant of f such that

$$\begin{aligned} |f(v_1, x) - f(v_2, x)| &\leq c(s, f, V) |v_1 - v_2|^{\frac{1}{2}+\gamma}, \\ \forall v_1, v_2 \in \left[\inf_{r \in [0,s]} V_r, \sup_{r \in [0,s]} V_r \right], \quad \forall x \in \left[\inf_{r \in [0,s]} X_r, \sup_{r \in [0,s]} X_r \right]. \end{aligned}$$

Then we get

$$\frac{2}{\varepsilon} \int_0^s J_1^2(r, \varepsilon) dr \leq c^2(s, f, V) \frac{2}{\varepsilon} \int_0^s |V_{r+\varepsilon} - V_r|^{1+2\gamma} dr.$$

Since V is a bounded variation process, this term converges to zero in probability.

On the other hand,

$$J_2(r, \varepsilon) = \partial_x f(V_r, X_r) (X_{r+\varepsilon} - X_r) + J_3(r, \varepsilon),$$

where $J_3(r, \varepsilon)$ converges *u.c.p.* to zero, as in [7, th. 2.9]. Therefore, like in [28] we have

$$\frac{2}{\varepsilon} \int_0^t J_2^2(s, \varepsilon) ds \rightarrow \sum_{i,j=1}^n \int_0^t (\partial_x f(V_s, X_s))^2 d[X, X]_s$$

and so the claim. \square

For our purposes we need to express integrals and covariation in a multidimensional setting, in the spirit of [5].

If $X = (X^1, \dots, X^n)^*$ is a vector of continuous processes in \mathbb{R}_+ , Y is a $m \times n$ matrix of continuous processes in \mathbb{R}_+ , $(Y^{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$, then the symbol $\int_0^s Y d^- X$ denotes, whenever it

exists, the *u.c.p.* limit of the integral $\int_0^s Y_r \frac{X_{r+\varepsilon} - X_r}{\varepsilon} dr$ where the product is intended in the matrix sense. Similarly, if A is a $n \times d$ matrix $(A^{j,k})_{1 \leq j \leq n, 1 \leq k \leq d}$ then $[Y, A]_s$ is the $m \times d$ real matrix constituted by the following *u.c.p.* limit (if it exists)

$$\frac{1}{\varepsilon} \int_0^s (Y_{r+\varepsilon} - Y_r) (A_{r+\varepsilon} - A_r) dr.$$

Clearly the matrix operation cannot be commutative in general. Let now A, X, Y, C be real matrix valued processes which are successively compatible for the matrix product. We define

$$\int_0^s A_r d[X, Y]_r C_r = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^s A_r (X_{r+\varepsilon} - X_r) (Y_{r+\varepsilon} - Y_r) C_r dr,$$

where the limit is intended in the *u.c.p.* sense. The previous stability transformations (Proposition 2.2 above) can be extended to the case of vector valued functions. This is pointed out in the next remark.

Remark 2.3. Let $f \in C_{loc}^{\frac{1}{2}+\gamma, 1}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^p)$, $g \in C_{loc}^{\frac{1}{2}+\gamma, 1}(\mathbb{R}_+ \times \mathbb{R}^m; \mathbb{R}^q)$, $X = (X^1, \dots, X^n)^*$, $Y = (Y^1, \dots, Y^m)^*$ such that (X, Y) has all its mutual covariations. Let V^1, V^2 be bounded variation processes. Then

$$\left[f(V^1, X), g(V^2, Y) \right]_s = \int_0^s \partial_x f(V^1, X) d[X, Y^*] \partial_x g(V^2, Y)^*$$

for every $s \geq 0$. \square

One refined result is the vector Itô formula whose proof follows like in [28,7], where the stochastic integrals involved were scalar.

Proposition 2.4. Let $f \in C^{1,2}(\mathcal{T}_t \times \mathbb{R}^n)$. Let $X = (X^1, \dots, X^n)^*$, having all its mutual covariations, V be a bounded variation process indexed by \mathcal{T}_t . Then, for $0 \leq s \leq t$,

$$\begin{aligned} f(V_s, X_s) &= f(V_t, X_t) + \int_t^s \partial_x f(V_r, X_r) d^- X_r + \int_t^s \partial_v f(V_r, X_r) dV_r \\ &\quad + \frac{1}{2} \int_t^s \partial_{xx} f(V_r, X_r) d[X, X^*]. \end{aligned}$$

Remark 2.5. From the above statement it follows that, in particular, the integral $\int_t^s \partial_x f(V_r, X_r) d^- X_r$ automatically exists. \square

Remark 2.6. Let $W = (W^1, \dots, W^n)^*$ be an (\mathcal{F}_s) -Brownian motion. Then $[W, W^*]_s = (\delta_{i,j})_s$. \square

3. Fukushima–Dirichlet decomposition

3.1. Definitions and remarks

Throughout this section we fix, as above, $0 \leq t \leq T \leq +\infty$ and set $\mathcal{T}_t = [t, T] \cap \mathbb{R}$. Recall that all processes we consider are continuous except when explicitly stated.

Definition 3.1. A real process D is called an (\mathcal{F}_s) -Dirichlet process in \mathcal{T}_t if it is (\mathcal{F}_s) -adapted and can be written as

$$D = M + A, \tag{5}$$

where

- (i) M is an (\mathcal{F}_s) -local martingale,
- (ii) A is a zero quadratic variation process such that (for convenience) $A_0 = 0$.

A vector $D = (D^1, \dots, D^n)$ is said to be Dirichlet if it has all its mutual covariations and every D^i is Dirichlet.

Remark 3.2. An (\mathcal{F}_s) -semimartingale is an (\mathcal{F}_s) -Dirichlet process. \square

Remark 3.3. The decomposition (5) is unique, see for instance [31]. \square

The concept of Dirichlet process can be weakened for our purposes. We will make use of an extension of such processes, called *weak Dirichlet processes*, introduced in parallel in [5] and implicitly in [4]. Recent developments concerning the subject appear in [1,32,2].

Weak Dirichlet processes are not Dirichlet processes but they preserve a sort of orthogonal decomposition. In all the papers mentioned above however one deals with one dimensional weak Dirichlet processes while here we treat the multidimensional case.

Definition 3.4. A real process D is called an (\mathcal{F}_s) -**weak Dirichlet** process in \mathcal{T}_t if it can be written as

$$D = M + A, \tag{6}$$

where

- (i) M is an (\mathcal{F}_s) -local martingale,
- (ii) A is a process such that $[A, N] = 0$ for every (\mathcal{F}_s) -continuous local martingale N . (For convenience $A_0 = 0$.)

A will be said to be a **weak zero energy** process.

Remark 3.5. The decomposition (6) is unique. In fact, let

$$D = M^1 + A^1 = M^2 + A^2,$$

where M^1, M^2, A^1, A^2 fulfill properties (i) and (ii) of Definition 3.4. Then we have $M + A = 0$ where $M = M^1 - M^2$ and $A = A^1 - A^2$ is such that $[A, N] = 0$ for every (\mathcal{F}_s) -local martingale N .

It is now enough to evaluate the covariation of both members with M to get $[M, M] = 0$. Since $A_0 = 0$ then $M_0 = 0$ and consequently $M \equiv 0$. \square

Example 3.6. A simple example of a weak Dirichlet process is given by a process Z which is independent of \mathcal{F} , for instance a deterministic one! Clearly if Z is not at least a finite quadratic variation process, it cannot be Dirichlet. However it is possible to show that $[Z, N] = 0$ for any local \mathcal{F} -martingale. In fact

$$\begin{aligned} \int_0^s (Z_{r+\varepsilon} - Z_r)(N_{r+\varepsilon} - N_r)dr &= \int_0^s dr (Z_{r+\varepsilon} - Z_r) \int_r^{r+\varepsilon} \frac{1}{\varepsilon} dN_\lambda \\ &= \int_0^s \frac{dN_\lambda}{\varepsilon} \int_{(\lambda-\varepsilon)\vee 0}^\lambda (Z_{r+\varepsilon} - Z_r)dr \\ &= \int_0^s \frac{dN_\lambda}{\varepsilon} \int_{(\lambda-\varepsilon)\vee 0}^\lambda Z_{r+\varepsilon}dr - \int_0^s \frac{dN_\lambda}{\varepsilon} \int_{(\lambda-\varepsilon)\vee 0}^\lambda Z_r dr. \end{aligned}$$

The previous expression converges *u.c.p.* to zero since the two last terms converge *u.c.p.* to the Itô integral $\int_0^s Z dN$ since N is also a local martingale with respect to the filtration \mathcal{F} enlarged with Z .

Remark 3.7. [5] provides an example of a weak Dirichlet process coming from convolutions of local martingales. If for every $s > 0$, $(G(s, \cdot))$ is a continuous random field such that $(G(s, \cdot))$ is (\mathcal{F}_r) -progressively measurable and M is an (\mathcal{F}_r) -local martingale, then $X_s = \int_t^s G(s, r) dM_r$ defines a weak Dirichlet process. \square

Remark 3.8. [2] made the following observation. If the underlying filtration (\mathcal{F}_s) is the natural filtration associated with a Brownian motion (W_t) then condition (ii) in Definition 3.4 can be replaced with

(ii') A is a process with $[A, W] = 0$. \square

Definition 3.9. A vector $D = (D^1, \dots, D^n)$ is said to be an (\mathcal{F}_s) -**weak Dirichlet** process if every D^i is an (\mathcal{F}_s) -weak Dirichlet process. A vector $A = (A^1, \dots, A^n)$ is said to be an (\mathcal{F}_s) -**weak zero energy** process if every A^i is an (\mathcal{F}_s) -weak zero energy process.

The aim now is to study what happens to a Dirichlet process after a $C^{0,1}$ type transformation. It is well known, see for instance [31], that a C^1 function of a finite quadratic variation process (respectively Dirichlet process) is a finite quadratic variation (respectively Dirichlet) process. Here, motivated by applications to optimal control, see the introduction, Section 1, we look at the (possibly) inhomogeneous case showing two different results: the first result (Proposition 3.10 with Corollary 3.11) states that a function $C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^n)$ of a weak Dirichlet (vector) process having all its mutual covariations is again a weak Dirichlet process; the second (Proposition 3.14) gives stability for Dirichlet processes in the inhomogeneous case for functions in $C_{loc}^{\frac{1}{2}+\gamma, 1}(\mathcal{I}_t \times \mathbb{R}^n)$.

The first result comes from the need for treating optimal control problems where the state process is a semimartingale that solves a classical SDE, which is the case we treat in the companion paper [17].

3.2. The decomposition for $C^{0,1}$ functions

We now go on with a result concerning weak Dirichlet processes. Suppose $(D_s)_{s \in \mathcal{I}_t}$ to be an (\mathcal{F}_s) -Dirichlet process with decomposition (5) where A is a zero quadratic variation process. Given a $C^{0,1}$ function u of D , we cannot expect $Z = u(\cdot, D)$ to be a Dirichlet process. However one can hope that it is at least a weak Dirichlet process. Indeed this result is true even if D is a weak Dirichlet process with finite quadratic variation.

Proposition 3.10. *Suppose $(D_s)_{s \geq 0}$ to be an (\mathcal{F}_s) -weak Dirichlet (vector) process having all its mutual covariations. For every $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^n)$ we have, for $s \geq t$,*

$$u(s, D_s) = u(t, D_t) + \int_t^s \partial_x u(r, D_r) dM_r + \mathcal{B}^D(u)_s - \mathcal{B}^D(u)_t, \tag{7}$$

where $\mathcal{B}^D : C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^n) \rightarrow \mathcal{C}_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \mathbb{R}^n)$ is a linear map having the following properties:

- (a) \mathcal{B}^D is continuous;
- (b) if $u \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ then

$$\mathcal{B}^D(u)_s = \int_0^s \partial_s u(r, D_r) dr + \int_0^s \partial_{xx} u(r, D_r) d[M, M]_r + \int_0^s \partial_x u(r, D_r) d^- A_r;$$

- (c) if $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^n)$ then $(\mathcal{B}^D(u))_s$ is a weak zero energy process.

Corollary 3.11. *Suppose $(D_t)_{t \geq 0}$ to be an (\mathcal{F}_t) -weak Dirichlet (vector) process having all its mutual covariations. For every $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^n)$, $(u(t, D_t))$ is an (\mathcal{F}_t) -weak Dirichlet process with martingale part $M_t = \int_0^t \partial_x u(s, D_s) dM_s$.*

Remark 3.12. Given a bounded stopping time τ with values in \mathcal{T}_t , it is easy to see that decomposition (7) still holds for the stopped process D^τ . In fact given an (\mathcal{F}_s) -martingale N , and an (\mathcal{F}_s) -weak zero energy process A , we have $[N, A^\tau] = [N, A]^\tau = 0$. \square

Proof (Of The Proposition). Without restriction of generality we will set $t = 0$.

Property (a) follows simply by writing

$$\mathcal{B}^D(u)_s = u(s, D_s) - u(0, D_0) - \int_0^s \partial_x u(r, D_r) dM_r$$

and observing that the process defined on the right hand side has the required continuity property.

Property (b) follows from Proposition 2.4 applied reversely. Indeed, given $u \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$, Proposition 2.4 can be applied. In particular

$$\int_0^s \partial_x u(r, D_r) d^- D_r$$

exists; this implies that also

$$\int_0^s \partial_x u(r, D_r) d^- A_r$$

exists since

$$\int_0^s \partial_x u(r, D_r) d^- M_r = \int_0^s \partial_x u(r, D_r) dM_r$$

is the classical Itô integral.

It remains to prove point (c)

$$\left[u(\cdot, D) - \int_0^\cdot \partial_x u(r, D_r) dM_r, N \right] = 0$$

for every one dimensional (\mathcal{F}_r) -local martingale N .

For simplicity of notation, we will suppose that D is one dimensional. Therefore D will be a finite quadratic variation process. Since the covariation of semimartingales coincides with the classical covariation

$$\left[\int_0^\cdot \partial_x u(r, D_r) dM_r, N \right] = \int_0^\cdot \partial_x u(r, D_r) d[M, N]_r,$$

it remains to check that, for every $s \in [0, T]$,

$$[u(\cdot, D), N]_s = \int_0^s \partial_x u(r, D_r) d[M, N]_r.$$

For this, we have to evaluate the *u.c.p.* limit of

$$\int_0^s [u(r + \varepsilon, D_{r+\varepsilon}) - u(r, D_r)] \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr$$

in probability. This can be written as the sum of two terms:

$$I_1(s, \varepsilon) = \int_0^s (u(r + \varepsilon, D_{r+\varepsilon}) - u(r + \varepsilon, D_r)) \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr,$$

$$I_2(s, \varepsilon) = \int_0^s (u(r + \varepsilon, D_r) - u(r, D_r)) \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr.$$

First we prove that $I_1(s, \varepsilon)$ goes to $\int_0^t \partial_x u(r, D_r) d[M, N]_r$. In fact

$$I_1(s, \varepsilon) = \int_0^s (u(r + \varepsilon, D_{r+\varepsilon}) - u(r + \varepsilon, D_r)) \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr$$

$$= \int_0^s \partial_x u(r + \varepsilon, D_r) (D_{r+\varepsilon} - D_r) \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr + R_1(s, \varepsilon), \tag{8}$$

where $R_1(s, \varepsilon) \rightarrow 0$ *u.c.p.* as $\varepsilon \rightarrow 0$. Indeed

$$R_1(s, \varepsilon) = \int_0^s \left[\int_0^1 [\partial_x u(r + \varepsilon, D_r + \lambda(D_{r+\varepsilon} - D_r)) - \partial_x u(r + \varepsilon, D_r)] d\lambda \right. \\ \left. \times \frac{N_{r+\varepsilon} - N_r}{\varepsilon} (D_{r+\varepsilon} - D_r) \right] dr$$

and the claim follows by the continuity of $\partial_x u$ and from the estimate

$$\frac{1}{\varepsilon} \int_0^T (N_{r+\varepsilon} - N_r) (D_{r+\varepsilon} - D_r) dr$$

$$\leq \left[\frac{1}{\varepsilon} \int_0^T (N_{r+\varepsilon} - N_r)^2 dr \cdot \frac{1}{\varepsilon} \int_0^T (D_{r+\varepsilon} - D_r)^2 dr \right]^{\frac{1}{2}} \xrightarrow{\varepsilon \rightarrow 0} ([N] \cdot [D])^{\frac{1}{2}}. \tag{9}$$

On the other hand the first term in (8) can be rewritten as

$$\int_0^s \partial_x u(r, D_r) (D_{r+\varepsilon} - D_r) \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr + R_2(s, \varepsilon), \tag{10}$$

where $R_2(s, \varepsilon) \rightarrow 0$ *u.c.p.* arguing as for $R_1(s, \varepsilon)$. The integral in (10) goes then *u.c.p.* to $\int_0^s \partial_x u(r, D_r) d[M, N]_r$, since by (9) the measures $\frac{(N_{r+\varepsilon} - N_r)(D_{r+\varepsilon} - D_r)}{\varepsilon} dr$ weakly converge to $d[N, D]$ as $\varepsilon \rightarrow 0$.

It remains to show that $I_2(s, \varepsilon) \rightarrow 0$ *u.c.p.* for every $s \in [0, T]$ as $\varepsilon \rightarrow 0$. By using suitable localization theorems (e.g. as usually done for instance in [26], section IV.1), it is enough to suppose u to be with compact support and N to be a square integrable martingale. Then we evaluate

$$\mathbb{E} \left(\sup_{0 \leq s \leq T} |I_2(s, \varepsilon)|^2 \right). \tag{11}$$

Now we have, exchanging integrals,

$$\begin{aligned} I_2(s, \varepsilon) &= \int_0^s [u(r + \varepsilon, D_r) - u(r, D_r)] \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr \\ &= \int_0^s [u(r + \varepsilon, D_r) - u(r, D_r)] dr \int_r^{r+\varepsilon} \frac{1}{\varepsilon} dN_\lambda \\ &= \int_0^s \frac{dN_\lambda}{\varepsilon} \int_{(\lambda-\varepsilon)\vee 0}^\lambda [u(r + \varepsilon, D_r) - u(r, D_r)] dr. \end{aligned}$$

The Doob inequality implies that (11) is smaller than

$$4\mathbb{E} \left\{ \int_0^T d[N]_\lambda \left(\frac{1}{\varepsilon} \int_{(\lambda-\varepsilon)\vee 0}^\lambda [u(r + \varepsilon, D_r) - u(r, D_r)] dr \right)^2 \right\}.$$

The fact that u is uniformly continuous on compact sets and the Lebesgue dominated convergence theorem imply the result. \square

A significant bracket evaluation, in the spirit of Proposition 2.2, but for $C^{0,1}$ functions of semimartingales, is the following. For simplicity we formulate the one dimensional case, even if it extends to the multidimensional case.

Corollary 3.13. *Let S be an (\mathcal{F}_s) -semimartingale in \mathbb{R}_+ , $u \in C^{0,1}(\mathcal{T}_t \times \mathbb{R})$. Then*

$$[u(\cdot, S), S]_t = \int_0^t \partial_x u(s, S_s) d[S]_s.$$

Proof. Let $S = M + V$ be the decomposition of S with M being a local martingale and V a finite variation process with $V_0 = 0$. Then

$$u(t, S_t) = u(0, S_0) + \int_0^t \partial_x u(s, S_s) dM_s + \tilde{A}_t,$$

where \tilde{A} is a weak zero energy process. In particular, a classical localization argument shows that $[\tilde{A}, M] = 0$. On the other hand, obviously $[\tilde{A}, V] = 0$; consequently, by linearity and since the covariation of local martingales is the classical convolution, the result follows. \square

3.3. The decomposition for $C^{\frac{1}{2}+\gamma,1}$ functions

If, in Proposition 3.10, D is a Dirichlet process and u is of class $C^{\frac{1}{2}+\gamma,1}$, $\gamma > 0$, then the results of Proposition 3.10 and Corollary 3.11 can be made more precise. In fact it is possible to show that Dirichlet processes are stable through $C^{\frac{1}{2}+\gamma,1}$, $\gamma > 0$, transformations.

Proposition 3.14. *Let $(D_s)_{s \geq 0}$ be an (\mathcal{F}_s) -Dirichlet process with decomposition (5). The statement of Proposition 3.10 holds with*

$$\mathcal{B}^D : C^{\frac{1}{2}+\gamma,1}(\mathbb{R}_+ \times \mathbb{R}^n) \rightarrow \mathcal{C}_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \mathbb{R}^n)$$

fulfilling properties (a), (b) and

(c) if $u \in C^{\frac{1}{2}+\gamma,1}(\mathbb{R}_+ \times \mathbb{R}^n)$ then $(\mathcal{B}^D(u)_s)$ is a zero quadratic variation process.

Proof. Points (a) and (b) follow like for the $C^{0,1}$ decomposition.

In order to establish Property (c) we proceed using the bilinearity of the covariation. We will in fact show that $\mathcal{B}^D(u)$ is a zero quadratic variation process. We operate with the bilinearity of the covariation process and we evaluate

- (i) $[u(\cdot, D), u(\cdot, D)],$
- (ii) $[u(\cdot, D), \int_0^\cdot \partial_x u(r, D_r) dM_r],$
- (iii) $[\int_0^\cdot \partial_x u(r, D_r) dM_r, \int_0^\cdot \partial_x u(r, D_r) dM_r],$

as follows.

- (i) We apply Proposition 2.2 to get that

$$[u(\cdot, D), u(\cdot, D)]_s = \int_0^s \partial_x u(s, D_r) d[D, D^*]_r \partial_x u(r, D_r)^*.$$

- (ii) Setting $N_t = \int_0^t \partial_x u(r, D_r) dM_r,$ Remark 2.1 implies that (N, D) has all its mutual brackets; therefore again Proposition 2.2 implies that

$$[u(\cdot, D), N^*]_s = \int_0^s \partial_x u(r, D_r) d[D, N^*]_r.$$

On the other hand, by Remark 2.1

$$[D, N^*]_t = [M, N^*]_s = \int_0^s \partial_x u(r, D_r) d[M, M^*]_r \partial_x u(r, D_r)^*.$$

- (iii) The fact that the covariation of semimartingales coincides with the classical covariation gives

$$\begin{aligned} & \left[\int_0^\cdot \partial_x u(r, D_r) dM_r, \int_0^\cdot \partial_x u(r, D_r) dM_r \right]_s \\ &= \int_0^s \partial_x u(r, D_r) d[M, M^*]_r \partial_x u(r, D_r)^*. \end{aligned}$$

Finally by Remark 2.1 and the decomposition we get that

$$[D, D^*] = [M, M^*].$$

The bilinearity of the covariation allows us now to conclude. \square

4. Representation of operator \mathcal{B} when u solves a suitable PDE

Here we want to develop the connection between suitable (deterministic) linear differential operators and our (stochastic) operators \mathcal{B} introduced in the previous section. This connection is well known and obvious when u is the $C^{1,2}$ solution of a second order PDE and D is a diffusion process. Our aim is to extend the validity of such a representation when u is only a $C^{0,1}$ solution (in a suitable sense that we will define below) and D is a weak Dirichlet process of a suitable kind (see below). This will be used as a key tool in the applications to optimal control.

4.1. Strong solutions of parabolic PDEs

Let $0 < T < +\infty,$ consider two continuous functions

$$b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \sigma : [0, T] \times \mathbb{R}^n \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

and the linear parabolic operator

$$\begin{aligned} \mathcal{L}_0 : D(\mathcal{L}_0) \subseteq C^0([0, T] \times \mathbb{R}^n) &\longrightarrow C^0([0, T] \times \mathbb{R}^n), \\ D(\mathcal{L}_0) &= C^{1,2}([0, T] \times \mathbb{R}^n), \\ \mathcal{L}_0 u(t, x) &= \partial_t u(t, x) + \langle b(t, x), \partial_x u(t, x) \rangle + \frac{1}{2} \text{Tr} [\sigma^*(t, x) \partial_{xx} u(t, x) \sigma(t, x)]. \end{aligned}$$

Defining

$$L_0(t) u(t, x) = \langle b(t, x), \partial_x u(t, x) \rangle + \frac{1}{2} \text{Tr} [\sigma^*(t, x) \partial_{xx} u(t, x) \sigma(t, x)],$$

we can write

$$\mathcal{L}_0 u(t, x) = \partial_t u(t, x) + L_0(t) u(t, x).$$

Recall that an operator $\mathcal{M} : D(\mathcal{M}) \subseteq F \rightarrow G$ (F, G suitable Fréchet spaces) is closable if, given any sequence $(u_n)_{n \in \mathbb{N}} \subseteq D(\mathcal{M})$, we have

$$\left. \begin{aligned} u_n &\longrightarrow 0, & \text{in } F \\ \mathcal{M}u_n &\longrightarrow \eta & \text{in } G \end{aligned} \right\} \implies \eta = 0.$$

When \mathcal{L}_0 is closable we denote its closure by \mathcal{L} and recall that

$$u \in D(\mathcal{L}) \iff \exists (u_n) \subset C^{1,2}([0, T] \times \mathbb{R}^n) : \begin{cases} u_n \longrightarrow u \\ \mathcal{L}_0 u_n \longrightarrow \mathcal{L}u \end{cases} \text{ in } C^0([0, T] \times \mathbb{R}^n).$$

Now, given $\phi \in C^0(\mathbb{R}^n)$ and $h \in C^0([0, T] \times \mathbb{R}^n)$ we consider the inhomogeneous backward parabolic problem

$$\begin{cases} \partial_t u(t, x) + \langle b(t, x), \partial_x u(t, x) \rangle + \frac{1}{2} \text{Tr} [\sigma^*(t, x) \partial_{xx} u(t, x) \sigma(t, x)] = h(t, x), \\ t \in [0, T], \quad x \in \mathbb{R}^n, \\ u(T, x) = \phi(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{12}$$

that can be rewritten as

$$\partial_t u(t, x) + L_0(t) u(t, x) = h(t, x), \quad u(T, x) = \phi(x),$$

or

$$\mathcal{L}_0 u(s, x) = h(s, x), \quad u(T, x) = \phi(x).$$

Definition 4.1. We say that $u \in C^0([0, T] \times \mathbb{R}^n)$ is a strict solution to the backward Cauchy problem (12) if $u \in D(\mathcal{L}_0)$ and (12) holds.

Definition 4.2. We say that $u \in C^0([0, T] \times \mathbb{R}^n)$ is a strong solution to the backward Cauchy problem (12) if there exists a sequence $(u_n) \subset D(\mathcal{L}_0)$ and two sequences $(\phi_n) \subseteq C^0(\mathbb{R}^n)$, $(h_n) \subseteq C^0([0, T] \times \mathbb{R}^n)$, such that

1. For every $n \in \mathbb{N}$ u_n is a strict solution of the problem

$$\mathcal{L}_0 u_n(t, x) = h_n(t, x), \quad u_n(T, x) = \phi_n(x).$$

2. The following limits hold

$$\begin{aligned} u_n &\longrightarrow u \quad \text{in } C^0([0, T] \times \mathbb{R}^n), \\ h_n &\longrightarrow h \quad \text{in } C^0([0, T] \times \mathbb{R}^n), \\ \phi_n &\longrightarrow \phi \quad \text{in } C^0(\mathbb{R}^n). \end{aligned}$$

4.2. The representation result

Let u be a strong solution of class $C^{0,1}$ of (12) and S be a weak Dirichlet process that can be written in the following form:

$$S_t = S_0 + \int_0^t \sigma(s, S_s) dW_s + A_t$$

where σ is as in the previous section and A is a weak zero energy process with finite quadratic variation.

We observe that for our applications to optimal control it would be enough to take S semimartingale. We deal with this more general case to prepare the field for a forthcoming paper in which we consider the optimal control of solutions of SDEs where the drift b is the derivative in space of a continuous function β , therefore a Schwartz distribution. One would have in that case $A_t = \int_0^t \partial_x \beta(s, x) ds$ in some specific sense. Equations of that type, when there is no dependence on time, appear for instance in [8]. Solutions are Dirichlet processes in the time-homogeneous case and weak Dirichlet in the general case.

We remark that the coefficient σ must coincide with the one appearing in the second order term of the operator.

We state first a technical lemma whose proof is elementary.

Lemma 4.3. *Let $T < +\infty$. Let $f_n, f : [0, T] \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, continuous such that $f_n \rightarrow f$ uniformly. For a fixed constant $K > 0$, we define*

$$\tau_n = \inf \{t \in [0, T] : |f_n(t)| \geq K\}, \quad \tau = \inf \{t \in [0, T] : |f(t)| \geq K\}$$

with the convention that $\inf \emptyset = T$. Then

$$\lim_{n \rightarrow +\infty} f_n^{\tau_n}(T) = f^\tau(T)$$

where f^τ (respectively $f_n^{\tau_n}$) is the stopped function defined by $f^\tau(t) = f(\tau \wedge t)$ (respectively $f_n^{\tau_n}(t) = f_n(\tau_n \wedge t)$).

From the above lemma we get the following proposition.

Proposition 4.4. *The set \mathcal{M}_{loc} of all (\mathcal{F}_t) -continuous local martingales is a closed subset of $\mathcal{C}_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R})$ endowed with the u.c.p. topology.*

Proof. Let $(M_n(t), t \in [0, T])_{n \in \mathbb{N}}$ be a sequence of local continuous martingales converging u.c.p. to a continuous process $M \in \mathcal{C}_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R})$. For $K > 0$ we define the following stopping times:

$$\begin{aligned} \tau^n &= \inf \{t \in [0, T] : |M_n(t)| \geq K\} \\ \tau &= \inf \{t \in [0, T] : |M(t)| \geq K\} \end{aligned}$$

with the convention that $\inf \emptyset = T$.

In order to conclude, it suffices to show that M^τ is a square integrable martingale. Lemma 4.3 implies that

$$M_n^{\tau_n}(T) \longrightarrow M^\tau(T) \quad a.e. \tag{13}$$

Using (13) above and Lebesgue dominated convergence theorem, we have

$$M_n^{\tau_n}(T) \longrightarrow M^\tau(T) \quad \text{in } L^2(\Omega).$$

The fact that M^τ is a square integrable martingale follows then from Proposition 5.23, Ch. 1 of [19]. \square

We are now able to state a useful representation result. Below we fix $t \in [0, T]$ and, since now $T < +\infty$, we have $\mathcal{T}_t = [t, T]$.

Theorem 4.5. *Let $T < +\infty$ and*

$$b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \text{resp. } \sigma : [0, T] \times \mathbb{R}^n \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

be continuous functions. Let $u \in C^{0,1}([0, T] \times \mathbb{R}^n)$ be a strong solution of the Cauchy problem (12).

Fix $t \in [0, T]$, $x \in \mathbb{R}^n$, and let $(S_s)_{s \in \mathcal{T}_t}$ be a process of the form

$$S_s = x + \int_t^s \sigma(r, S_r) dW_r + A_t - A_s$$

where $(A_s)_{s \in \mathcal{T}_t}$ is an (\mathcal{F}_s) -weak zero energy process having all its mutual covariations.

Then, provided that the following assumption is verified for every $s \in \mathcal{T}_t$:

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_t^s (\partial_x u_n(r, S_r) - \partial_x u(r, S_r)) d^- A_r \\ & - \int_t^s \langle \partial_x u_n(r, S_r) - \partial_x u(r, S_r), b(r, S_r) \rangle dr = 0, \quad u.c.p., \end{aligned} \tag{14}$$

we have

$$u(s, S_s) = u(t, S_t) + \int_t^s \partial_x u(r, S_r) \sigma(r, S_r) dW_r + \mathcal{B}^S(u)_s - \mathcal{B}^S(u)_t, \tag{15}$$

where, for $s \in \mathcal{T}_t$

$$\mathcal{B}^S(u)_s = \int_0^s h(r, S_r) dr + \int_0^s \partial_x u(r, S_r) d^- A_r - \int_0^s \langle \partial_x u(r, S_r), b(r, S_r) \rangle dr. \tag{16}$$

Proof. We set $t = 0$ for simplicity. The general case is obtained by additivity of different integrals. Let $u_n \longrightarrow u$ in $C^0([0, T] \times \mathbb{R}^n)$ be a sequence such that $\mathcal{L}_0 u_n = h_n \longrightarrow h$ in $C^0([0, T] \times \mathbb{R}^n)$. By Proposition 2.4, we get

$$\begin{aligned} u_n(s, S_s) &= u_n(0, S_0) + \int_0^s \mathcal{L}_0 u_n(r, S_r) dr - \int_0^s \langle \partial_x u_n(r, S_r), b(r, S_r) \rangle dr \\ &+ \int_0^s \partial_x u_n(r, S_r) \sigma(r, S_r) dW_r + \int_0^s \partial_x u_n(r, S_r) d^- A_r. \end{aligned}$$

From (14) we conclude that

$$M_s^n = u_n(s, S_s) - u_n(0, S_0) - \int_0^s \mathcal{L}_0 u_n(r, S_r) dr + \int_0^s \langle \partial_x u_n(r, S_r), b(r, S_r) \rangle dr - \int_0^s \partial_x u_n(r, S_r) d^- A_r$$

converges *u.c.p.* to

$$M_s = u(s, S_s) - u(0, S_0) - \int_0^s h(r, S_r) dr + \int_0^s \langle \partial_x u(r, S_r), b(r, S_r) \rangle dr - \int_0^s \partial_x u(r, S_r) d^- A.$$

Using Proposition 4.4 above we get that M is an (\mathcal{F}_s) -local martingale. The result follows by Proposition 3.10, where $dM = \sigma(s, S_s)ds$ and $D = S$, by identification of the weak zero energy processes. \square

For our applications in [17] we will need to consider a process A which is of bounded variation (so that S is solution of an SDE) but which is non-Markovian.

Corollary 4.6. *Let $T < +\infty$ and*

$$b_1 : \Omega \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

be a continuous progressively measurable field (continuous in (s, x)) and

$$b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \sigma : [0, T] \times \mathbb{R}^n \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

be continuous functions. Let $u \in C^{0,1}([0, T] \times \mathbb{R}^n)$ be a strong solution of the Cauchy problem (12).

Fix $t \in [0, T]$, $x \in \mathbb{R}^n$ and let (S_s) be a solution to the SDE

$$dS_s = b_1(s, S_s) ds + \sigma(s, S_s) dW_s; \quad S_t = x.$$

Then, provided that the following assumption be verified for every $s \in \mathcal{T}_t$

$$\lim_{n \rightarrow +\infty} \int_t^s \langle \partial_x u_n(r, S_r) - \partial_x u(r, S_r), b_1(r, S_r) - b(r, S_r) \rangle dr = 0, \quad u.c.p., \tag{17}$$

(15) holds with

$$B^S(u)_s = \int_0^s h(r, S_r) dr + \int_0^s \langle \partial_x u(r, S_r), b_1(r, S_r) \rangle dr - \int_0^s \langle \partial_x u(r, S_r), b(r, S_r) \rangle dr.$$

Proof. The result follows setting

$$A_s = \int_0^s b_1(r, S_r) dr$$

in the previous Theorem 4.5. \square

The above result depends on the extra Assumption (17) which is essential but not easy to check. We give first a special (but useful) case where it holds and then an improvement for the nondegenerate case. We have the following.

Remark 4.7. If

$$\lim_{n \rightarrow +\infty} \partial_x u_n = \partial_x u \quad \text{in } C^0([0, T] \times \mathbb{R}^n)$$

then Assumption (14) is verified. This means that the result of Proposition 4.4 above applies if we know that u is a strong solution in a more restrictive sense, i.e. replacing point 2 of Definition 4.2 with

$$\begin{aligned} u_n &\longrightarrow u && \text{in } C^0([0, T] \times \mathbb{R}^n) \\ \partial_x u_n &\longrightarrow \partial_x u && \text{in } C^0([0, T] \times \mathbb{R}^n) \\ h_n &\longrightarrow h && \text{in } C^0([0, T] \times \mathbb{R}^n) \\ \phi_n &\longrightarrow \phi && \text{in } C^0(\mathbb{R}^n). \end{aligned}$$

This is a particular case of our setting and it is the one used e.g. in [14–16] to get the verification result. We can say that in these works a result like Theorem 4.5 is proved under the assumption that u is a strong solution in this more restrictive sense. It is worth noting that in such a simplified setting, the proof of Theorem 4.5 follows simply by using standard convergence arguments. In particular there one does not need to use the Fukushima–Dirichlet decomposition presented in Section 3. So, from the methodological point of view there is a serious difference from the result of Theorem 4.5, see [17], Section 8 for comments. \square

A more significant achievement concerns the nondegenerate case. It is illustrated in the following corollary.

Corollary 4.8. *We make the same assumptions as for Corollary 4.6 except (14) which we replace by the assumption that*

$$\sigma^{-1}(b_1 - b) \quad \text{is bounded} \tag{18}$$

where σ^{-1} stands for the pseudo-inverse of σ . Then the same conclusion as for Corollary 4.6 holds.

Proof. Setting $t = 0$ for simplicity we write

$$\beta_s = W_s + \int_0^s (\sigma^{-1}(b_1 - b))(r, S_r) dr$$

and, applying the Girsanov theorem, there is a probability \mathbb{Q} equivalent to \mathbb{P} on (Ω, \mathcal{F}_T) such that $(\beta_s)_{s \in [0, T]}$ is an (\mathcal{F}_s) -standard Brownian motion.

So, under \mathbb{Q} the process (S_s) fulfills the equation

$$dS_s = b(s, S_s) ds + \sigma(s, S_s) dW_s; \quad S_0 = x.$$

Under \mathbb{Q} , assumption (14) is trivially verified since $b = b_1$ and so we have (the tilde stands for operators under the new probability \mathbb{Q})

$$\tilde{B}^S(u)_s = \int_0^s h(r, S_r) dr$$

and

$$u(s, S_s) = u(0, S_0) + \int_0^s \partial_x u(r, S_r) \sigma(r, S_r) d\beta_r + \tilde{B}^S(u)_s.$$

Expressing β in terms of W we obtain the result. \square

4.3. Some useful consequences

The previous results have some important consequences.

Remark 4.9. From Remark 3.12 it follows that the conclusion of the above Corollary 4.8 holds also if we stop the processes $\mathcal{B}(u)$ with a stopping time $t \leq \tau \leq T$. More precisely we have

$$\mathcal{B}^S(u)_{s \wedge \tau} = \int_0^{s \wedge \tau} h(r, S_r) dr - \int_0^{s \wedge \tau} \langle \partial_x u(r, S_r), b_1(r, S_r) - b(r, S_r) \rangle dr.$$

This fact will be useful in [17], Section 6. \square

Remark 4.10. The results of the Corollary 4.6 and of Corollary 4.8 above still hold true with suitable modifications if we assume that, instead of having $u \in C^{0,1}([0, T] \times \mathbb{R}^n)$:

- (i) the strong solution u belongs to $C^0([0, T] \times \mathbb{R}^n) \cap C^{0,1}([\varepsilon, T] \times \mathbb{R}^n)$ for every small $\varepsilon > 0$;
- (ii) for some $\beta \in (0, 1)$ the map $(t, x) \rightarrow t^\beta \partial_x u(t, x)$ belong to $C^{0,1}([0, T] \times \mathbb{R}^n)$.

The proof of Corollary 4.6 in this case is a straightforward generalization of the one presented above: we do not give it here to avoid technicalities. In fact in proving the verification theorem in [17], Section 6, we will deal with initial data that are only continuous, so with solutions u satisfying (i) and (ii) above and possibly not $C^{0,1}$. This difficulty will be faced directly in the proof of verification Theorem 6.19 in [17] by approximating the initial data with C^1 ones, using (16) and passing to the limit. \square

Remark 4.11. From the above Corollary 4.6 it follows that the process $\mathcal{B}^S(u)_s$ is in fact a semimartingale (and also absolutely continuous). \square

4.4. The elliptic case

We devote the last part of this subsection to applying the same setting as above to elliptic problems. Consider the inhomogeneous elliptic problem

$$\lambda u(x) + L_0 u(x) + h(x) = 0, \quad \forall x \in \mathbb{R}^n \tag{19}$$

where $D(L_0) = C^2(\mathbb{R}^n)$ and

$$L_0 u(x) = \langle b(x), \partial_x u(s, x) \rangle + \frac{1}{2} \text{Tr} [\sigma^*(x) \partial_{xx} u(x) \sigma(x)].$$

Definition 4.12. We say that u is a strict solution to the elliptic problem (19) if $u \in D(L_0)$ and (19) holds.

Definition 4.13. We say that u is a strong solution to the elliptic problem (19) if there exists a sequence $(u_n) \subseteq D(L_0)$ and a sequence $(h_n) \subseteq C^0(\mathbb{R}^n)$, such that

1. For every $n \in \mathbb{N}$ u_n is a strict solution of the problem

$$\lambda u_n(x) - L_0 u_n(x) = h_n(x), \quad \forall x \in \mathbb{R}^n.$$

2. The following limits hold

$$\begin{aligned} u_n &\longrightarrow u && \text{in } C^0(\mathbb{R}^n), \\ h_n &\longrightarrow h && \text{in } C^0(\mathbb{R}^n). \end{aligned}$$

Note that if L is the closure of L_0 in $C^0(\mathbb{R}^n)$ then a strong solution u , by construction, belongs to $D(L)$. We now exploit the above setting to show that, for functions $u \in C^1(\mathbb{R}^n)$ that

are strong solutions of the elliptic problem (19), the following results hold; we omit the proof as it is completely similar to (and even simpler than) that of Theorem 4.5 for the parabolic case.

Theorem 4.14. *Let*

$$b_1 : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

be a continuous progressively measurable process (continuous in x) and

$$b : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \sigma : \mathbb{R}^n \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

be continuous functions. Let S_s be a solution to the SDE

$$dS_s = b_1(S_s) ds + \sigma(S_s) dW_s; \quad S_0 = x.$$

Let $u \in C^1(\mathbb{R}^n)$ be a strong solution of the elliptic problem (19). Assume that

$$\lim_{n \rightarrow +\infty} \int_0^s \langle \partial_x u_n(S_r) - \partial_x u(S_r), b_1(S_r) - b(S_r) \rangle dr = 0, \quad u.c.p., \quad (20)$$

or that (18) holds. Then we have

$$u(S_s) = u(S_t) + \int_t^s \partial_x u(S_r) \sigma(S_r) dW_r + \mathcal{B}^S(u)_s - \mathcal{B}^S(u)_t, \quad (21)$$

where

$$\mathcal{B}^S(u)_s = \int_0^s h(S_r) dr + \int_0^s \langle \partial_x u(S_r), b_1(S_r) - b(S_r) \rangle dr.$$

Acknowledgements

The authors wish to thank the referee and the Editor for careful reading and for motivating suggestions.

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