Superposition and Constructions of Graphs Without Nowhere-zero $k$-flows

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Using multi-terminal networks we build methods on constructing graphs without nowhere-zero group- and integer-valued flows. In this way we unify known constructions of snarks (nontrivial cubic graphs without edge-3-colorings, or equivalently, without nowhere-zero 4-flows) and provide new ones in the same process. Our methods also imply new complexity results about nowhere-zero flows in graphs and state equivalences of Tutte’s 3- and 5-flow conjectures with formally weaker statements.

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1. INTRODUCTION

Nowhere-zero flows in graphs have been introduced by Tutte [38–40]. Primarily he showed that a planar graph is face-$k$-colorable if and only if it admits a nowhere-zero $k$-flow (its edges can be oriented and assigned values $±1, \ldots, ±(k − 1)$ so that the sum of the incoming values equals the sum of the outcoming ones for every vertex of the graph). Tutte also proved the classical equivalence result that a graph admits a nowhere-zero $k$-flow if and only if it admits a flow whose values are the nonzero elements of a finite abelian group of order $k$. Seymour [35] has proved that every bridgeless graph admits a nowhere-zero 6-flow, thereby improving the 8-flow theorem of Jaeger [16] and Kilpatrick [20].

There are three celebrated unsolved conjectures dealing with nowhere-zero flows in bridgeless graphs, all due to Tutte. The first is the 5-flow conjecture of [38], that every such graph admits a nowhere-zero 5-flow. The 4-flow conjecture of Tutte [40] suggests that if the graph does not contain a subgraph contractible to the Petersen graph, then it has a nowhere-zero 4-flow. Finally, the 3-flow conjecture is that if the graph does not contain a 3-edge cut, then it has a nowhere-zero 3-flow.

Graphs which do not admit nowhere-zero $k$-flows will be called $k$-snarks. Note that it is an easy problem to recognize 2- and $k$-snarks for $k \geq 6$, because they are the graphs having a vertex of odd valency and the graphs having a bridge, respectively. On the other hand, from results of Tutte [38], Holyer [13] and Garey et al. [9] it follows that the problems to recognize 3- and 4-snarks are co-NP-complete. By [25], the same holds for $k = 5$ if the 5-flow conjecture is false.

Until now, little has been known about constructions of $k$-snarks in general. The only exceptions are snarks, which are nontrivial cubic 4-snarks (or, equivalently, 3-regular graphs without an edge-3-coloring). By nontrivial we mean cyclically 4-edge-connected and with girth at least 5. Snarks are studied very intensively and several methods have been developed for their constructions. The main reason for interest in them is that among snarks must be the smallest counterexamples to the 5-flow conjecture and the cycle double-cover conjecture (every bridgeless graph has a family of circuits which together cover each edge twice). Furthermore, by Tait [36], the four-color theorem is equivalent to the statement that there exists no planar snark. But construction of snarks is not an easy task. For instance the first nontrivial infinite family of them was constructed in 1975 by Isaacs [15], though the first snark, the Petersen graph, depicted in Figure 1.1, was known late in the 19th century (see [19, 31]). More details about the history of snarks can be found in [8, 41, 42].

In this paper we build general methods on constructing graphs without nowhere-zero $k$-flows. First we study flows in multi-terminal networks and generalize some classical results which
have been known for flows in graphs. This enables us to develop several methods on constructing $k$-snarks, some of them having roots in constructions of snarks. We also study $k$-retractance of graphs, a parameter expressing how far a graph is from admitting a nowhere-zero $k$-flow. Furthermore, we show that the 3- and 5-flow conjectures of Tutte are equivalent to formally weaker (but also stronger) statements and obtain results about potential counterexamples to these conjectures. In order to demonstrate the versatility and power of our techniques we construct several families of snarks with special properties.

2. Graphs, Networks and Flows

The graphs considered in this paper are all finite and unoriented. Multiple edges and loops are allowed. If $G$ is a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges of $G$, respectively. By a multi-terminal network, briefly a network, we mean a pair $(G, U)$ where $G$ is a graph and $U = \langle u_1, \ldots, u_n \rangle$ is an ordered set of pairwise distinct vertices of $G$. If no confusions can occur, we denote by $U$ also the set $\{u_1, \ldots, u_n\}$ (we apply this convention writing formulas $u \in U$, $U \cap W$, or $W \setminus U$ for $W \subseteq V(G)$). The vertices from $U$ and $V(G) \setminus U$ are called the outer and inner vertices of the network $(G, U)$, respectively. We allow $n = 0$, i.e., $U = \emptyset$.

We postulate that with each edge of $G$ there are associated two distinct arcs. Arcs on distinct edges are distinct. If an arc on an edge is denoted by $x$ the other is denoted by $x^{-1}$. If the ends of an edge $e$ of $G$ are vertices $u$ and $v$, one of the arcs on $e$ is said to be directed from $u$ to $v$ and the other one is directed from $v$ to $u$. The two arcs on a loop, though distinct, are directed to the same vertex. (In other words, each edge of $G$ is duplicated and the two resulting edges are directed oppositely.)

Let $D(G)$ denote the set of arcs of $G$. Then $|D(G)| = 2|E(G)|$. If $X \subseteq D(G)$, then denote by $X^{-1} = \{x^{-1}; x \in X\}$. By an orientation of $G$ we mean every $X \subseteq D(G)$ such that $X \cup X^{-1} = D(G)$ and $X \cap X^{-1} = \emptyset$. If $W \subseteq V(G)$, then $\omega^+_G(W)$ denotes the set of the arcs of $G$ directed from $W$ to $V(G) \setminus W$. We write $\omega^+_G(u)$ instead of $\omega^+_G(\{u\})$.

In this paper, every abelian group is additive and has order at least two. If $G$ is a graph and $A$ is an abelian group, then an $A$-chain in $G$ is a mapping $\varphi : D(G) \rightarrow A$ such that $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$. Furthermore, the mapping $\partial \varphi : V(G) \rightarrow A$ such that

$$
\partial \varphi(v) = \sum_{x \in \omega^+_G(v)} \varphi(x) \quad (v \in V(G))
$$

is called the boundary of $\varphi$. The set of edges associated with the arcs of $G$ having nonzero values in $\varphi$ is called the support of $\varphi$. An $A$-chain $\varphi$ in $G$ is called nowhere-zero if its support equals $E(G)$. If $(G, U)$ is a network, then an $A$-chain $\varphi$ in $G$ is called an $A$-flow in $(G, U)$ if $\partial \varphi(v) = 0$ for every inner vertex $v$ of $(G, U)$.
By a \textit{(nowhere-zero)} $A$-flow in a graph $G$ we mean a (nowhere-zero) $A$-flow in the network $(G, \emptyset)$. (A graph $G$ is usually identified with the network $(G, \emptyset)$ in this paper.) Our concept of nowhere-zero flows in graphs coincides with the usual definition of nowhere-zero flows as presented in \cite{17, 43, 44}. The only difference is that instead of a fixed (but arbitrary) orientation of a graph $G$ we use the set $D(G)$ as a domain for a flow.

**Proposition 2.1.** Let $\varphi$ be an $A$-flow in a network $(G, U)$ and $W \subseteq V(G)$. Then

$$\sum_{x \in \omega^+_U(W)} \varphi(x) = \sum_{v \in U \cap W} \partial \varphi(v).$$

**Proof.** Since $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$ and $\partial \varphi(v) = 0$ for every $v \in W \setminus U$, we have

$$\sum_{x \in \omega^+_U(W)} \varphi(x) = \sum_{v \in W} \sum_{x \in \omega^+_U(v)} \varphi(x) = \sum_{v \in W} \partial \varphi(v) = \sum_{v \in U \cap W} \partial \varphi(v).$$

**Proposition 2.2.** Let $\varphi$ be an $A$-flow in a network $(G, U)$. Then $\sum_{u \in U} \partial \varphi(u) = 0$. 

**Proof.** Follows directly from Proposition 2.1 after setting $W = V(G)$. 

In order to generalize the classical equivalence results of Tutte we need to introduce the following notion. If $k$ is an integer $\geq 2$, then by a \textit{(nowhere-zero)} integral $k$-flow $\varphi$ in a network $(G, U)$ we mean a (nowhere-zero) $\mathbb{Z}$-flow $\varphi$ in $(G, U)$ such that $|\varphi(x)| < k$ for every $x \in D(G)$ and $|\partial \varphi(u)| < k$ for every $u \in U$ (flows not satisfying the latter condition are discussed in Remark 12.4). This notion coincides with the usual definition of (nowhere-zero) $k$-flows in graphs (see, e.g., \cite{17}).

With every $A$-flow in a network $(G, U)$, where $U = \langle u_1, \ldots, u_n \rangle$, is associated a characteristic vector $\chi(\varphi) = \langle z_1, \ldots, z_n \rangle$ so that $z_i = 0$ if $\partial \varphi(u_i) = 0$ and $z_i = 1$ otherwise. The $A$-characteristic set $\chi_A(G, U)$ (resp. $k$-characteristic set $\chi_k(G, U)$) of the network $(G, U)$ is the set of all characteristic vectors $\chi(\varphi)$ where $\varphi$ is a nowhere-zero $A$-flow in $(G, U)$ (resp. a nowhere-zero integral $k$-flow in $(G, U)$).

For a network $(G, U)$, $U = \langle u_1, \ldots, u_n \rangle$, and a binary vector $z = \langle z_1, \ldots, z_n \rangle$, construct the graph $(G, U)^z$ as follows: add a new vertex $v_0$ to $G$ and join it with every vertex $u_i \in U$ such that $z_i = 1$ (the valency of $v_0$ in $(G, U)^z$ is equal to the number of nonzero coordinates from $z$).

**Proposition 2.3.** Let $(G, U)$ be a network, $U = \langle u_1, \ldots, u_n \rangle$, and $z = \langle z_1, \ldots, z_n \rangle$ be a binary vector. Then $z \in \chi_A(G, U)$ (resp. $z \in \chi_k(G, U)$) if and only if $(G, U)^z$ admits a nowhere-zero $A$-flow (resp. a nowhere-zero integral $k$-flow).

**Proof.** If $\varphi$ is a nowhere-zero integral $k$-flow in $(G, U)^z$, then $\varphi'$ is the restriction of $\varphi$ to $D(G)$, then $\varphi'$ is a nowhere-zero integral $k$-flow in $(G, U)$ and $\chi(\varphi') = z$, whence $z \in \chi_A(G, U)$. On the other hand if $z = \langle z_1, \ldots, z_n \rangle \in \chi_k(G, U)$, then there exists a nowhere-zero integral $k$-flow $\psi$ in $(G, U)$ so that $\chi(\psi) = z$. Thus $|\partial \psi(u)| < k$ for every $u \in U$ and, by Proposition 2.2, $\sum_{u \in U} \partial \psi(u) = 0$. Then $\psi$ can be extended into a nowhere-zero integral $k$-flow $\psi'$ in $(G, U)^z$ after setting $\psi'(x_i) = -\psi'(x_i^{-1}) = \partial \psi(u_i)$ where $x_i$ denotes the arc directed from $v_0$ to $u_i$ (for every $i$ with $z_i \neq 0$).

The statement can be proved analogously for $A$-flows. 

**Theorem 2.4.** A network $(G, U)$ has a nowhere-zero integral $k$-flow if and only if $(G, U)$ has a nowhere-zero $A$-flow for any abelian group $A$ of order $k \geq 2$. 
THEOREM 2.5. If \((G, U)\) is a network, then \(\chi_A(G, U) = \chi_A(G, U)\) for any abelian group \(A\) of order \(k \geq 2\).

PROOF. Follows directly from Theorem 2.4 and Proposition 2.3.

We have shown that the study of nowhere-zero integral \(k\)-flows is, in certain sense, equivalent to the study of nowhere-zero \(A\)-flows where \(A\) is an abelian group of order \(k\). But flows with values from finite groups are easier to handle than integral flows. Accordingly, we define a (nowhere-zero) \(k\)-flow and \(k\)-chain in a network \((G, U)\) to be any (nowhere-zero) \(A\)-flow and \(A\)-chain in \((G, U)\), respectively, where \(A\) is an abelian group of order \(k\). Indeed, \(\chi_A(G, U)\) is, by Theorem 2.5, independent of \(A\) and equal to \(\chi_A(G, U)\). In fact we need integral flows only in the proofs of the next two statements.

PROPOSITION 2.6. If a network \((G, U)\) admits a nowhere-zero \(k\)-flow, then it admits a nowhere-zero \((k + 1)\)-flow.

PROOF. Every nowhere-zero integral \(k\)-flow is a nowhere-zero integral \((k + 1)\)-flow. Thus the statement follows from Theorem 2.4.

PROPOSITION 2.7. If \((G, U)\) is a network, then \(\chi_k(G, U) \subseteq \chi_{k+1}(G, U)\).

PROOF. Follows directly from Propositions 2.3 and 2.6.

From now on integral flows will not be used.

3. \(k\)-SNARKS

DEFINITION 3.1. By a \(k\)-snark we mean every network without a nowhere-zero \(k\)-flow \((k \geq 2)\). We say that a graph \(G\) is a \(k\)-snark if \((G, \emptyset)\) is a \(k\)-snark.

It is well known that a cubic graph is a 4-snark if and only if it is not edge-3-colorable (see, e.g., [17]). Cubic graphs without edge-3-colorings, with girth at least 5, and cyclical edge-connectivity at least 4 (i.e., deleting fewer that 4 edges does not disconnect them into components each containing a circuit) are called snarks (see, e.g., [8, 41, 42]). Thus snarks do not admit nowhere-zero 4-flows, and this fact has led us to defining \(k\)-snarks as networks without nowhere-zero \(k\)-flows.

By Proposition 2.7, every \(k\)-snark is also a \(k'\)-snark for every integer \(2 \leq k' \leq k\). It is well known (see [17, 44]) that a graph is a 2-snark if and only if it has no vertex of odd valency and that a graph containing a bridge is a \(k\)-snark for every \(k \geq 2\). By Seymour [35], there are no bridgeless \(k\)-snarks for any \(k \geq 6\). The 5-flow conjecture of Tutte is that there are no bridgeless \(5\)-snarks.

Two networks are homeomorphic if they arise from the same network after applying finitely many subdivisions (subdivision vertices are always assumed to be inner). The following statement contains a trivial but useful observation.

PROPOSITION 3.2. Let \((G, U)\) and \((G', U)\) be two homeomorphic networks. Then \(\chi_k(G, U) = \chi_k(G', U)\) for every \(k \geq 2\). In particular, \((G, U)\) is a \(k\)-snark if and only if \((G', U)\) is.
4. SUPERPOSITION

Suppose that we get a network \((G', U')\), \(U' = \{u'_1, \ldots, u'_n\}\), from a network \((G, U)\), \(U = \{u_1, \ldots, u_n\}\), by the following process. Take a vertex \(w\) of \(G\) and replace it by a graph \(H\) disjoint from \(G\) so that each edge of \(G\) with one end (or two ends) \(w\) gets a new end (or two new ends) from \(V(H)\). Moreover, assume that \(u'_1 = u_i\) if \(u_i \neq w\) and \(u'_i \in V(H)\) if \(u_i = w\).

Then \((G', U')\) is called a \(w\)-superposition (or a vertex superposition) of \((G, U)\). Now \(D(G)\) is a subset of \(D(G')\). Thus if \(\phi\) is a \(k\)-flow in \((G', U')\), then its restriction to \(D(G)\), denoted by \(\phi_{(G, U)}\), is a \(k\)-chain in \((G, U)\).

**Proposition 4.1.** If \(\phi\) is a (nowhere-zero) \(k\)-flow in \((G', U')\), then \(\phi_{(G, U)}\) is a (nowhere-zero) \(k\)-flow in \((G, U)\) and \(\partial\phi_{(G, U)}(u_i) = \partial\phi(u'_i)\) for \(i = 1, \ldots, n\). In particular, \(\chi(\phi_{(G, U)}) = \chi(\phi)\).

**Proof.** For every vertex \(v\) of \(G\), \(v \neq w\), \(\omega^+_G(v) = \omega^+_G(v)\) and \(\partial\phi_{(G, U)}(v) = \partial\phi(v)\). Since \(\omega^+_G(w) = \omega^+_G(V(H))\), then, by Proposition 2.1, \(\partial\phi_{(G, U)}(w) = \sum_{e \in \omega^+_G(w)} \phi_{(G, U)}(x) = \sum_{e \in \omega^+_G(V(H))} \phi(x) = \sum_{v \in U' \cap V(H)} \partial\phi(v)\). If \(w \notin U\), then \(U' \cap V(H) = \emptyset\) and \(\partial\phi_{(G, U)}(w) = 0\). If \(w = u_j\) for some index \(j\), then \(U' \cap V(H) = \{u'_j\}\) and \(\partial\phi_{(G, U)}(w) = \partial\phi(u'_j)\). Thus \(\phi_{(G, U)}\) is a (nowhere-zero) \(k\)-flow in \((G, U)\) and \(\partial\phi_{(G, U)}(u_i) = \partial\phi(u'_i)\) for \(i = 1, \ldots, n\).

For example, the networks \((G_1, (u_1, u_2))\) and \((G'_1, (u_1', u_2'))\), indicated in Figure 4.1, are \(w_1\) and \(u_2\)-superpositions of \((P, (u_1, u_2))\), respectively.

Suppose that we get a network \((G', U')\), \(U' = \{u'_1, \ldots, u'_n\}\), from a network \((G, U)\), \(U = \{u_1, \ldots, u_n\}\), by the following process. Take a network \((H, (v_1, v_2))\) disjoint from \((G, U)\), delete from \(G\) an edge \(e\) with ends \(w_1, w_2\) and identify the sets of vertices \(\{w_1, v_1\}\) and \(\{w_2, v_2\}\) to new vertices \(w'_1\) and \(w'_2\), respectively. Furthermore, let \(u'_i = u_i\) if \(u_i \neq w_1, w_2\), and \(u'_i = w'_i\) (or \(u'_i = w'_i\)) if \(u_i = w_1\) (or \(w_2\)). Then \((G', U')\) is called an \(e\)-superposition (or an edge superposition) of \((G, U)\). Now \(D(G - e)\) and \(D(H)\) are subsets of \(D(G')\). Suppose that \(x_1\) and \(x_2\) are the distinct arcs on \(e\) directed from \(w_1\) and \(w_2\), respectively. Let \(\phi\) be a \(k\)-flow in \((G', U')\). Then denote by \(\phi'\) its restriction to \(D(H)\). Clearly, \(\phi'\) is a \(k\)-flow in \((H, (v_1, v_2))\) and, by Proposition 2.2, \(\partial\phi'(v_1) = -\partial\phi'(v_2)\). Then define a \(k\)-chain \(\phi_{(G, U)}\) in \((G, U)\) so that

\[
\begin{align*}
\phi_{(G, U)}(x_j) &= \partial\phi'(v_j), \quad (j = 1, 2), \\
\phi_{(G, U)}(x) &= \phi(x) \quad \text{for } x \in D(G - e).
\end{align*}
\]

We always assume that \(v_1 \neq v_2\). If \(e\) is a loop (i.e., \(v_1 = v_2\)) then \(v_1, v_2\) and \(w_1\) are identified to one vertex \(w'_1 = w'_2\). But \(\phi_{(G, U)}\) is well defined also in this case.

**Proposition 4.2.** If \(\phi\) is a \(k\)-flow in \((G', U')\), then \(\phi_{(G, U)}\) is a \(k\)-flow in \((G, U)\) and \(\partial\phi_{(G, U)}(u_i) = \partial\phi(u'_i)\) for \(i = 1, \ldots, n\). In particular, \(\chi(\phi_{(G, U)}) = \chi(\phi)\).
Now we study edge superpositions.

For every $v \in V(G) \setminus \{w_1, w_2\}$, $\omega_{G,v}^+(v) = \omega_{G,v}^-(v)$ and $\partial \phi(G,U)(v) = \partial \phi(v)$. If $w_1 \neq w_2$, then $\omega_{G,v}^+(w) = \omega_{G,v}^-(w)$ is disjoint union of $\omega_{G,v_1}^+(w_1)$ and $\omega_{G,v_2}^-(v_2)$, whence, by (4.1) and (2.1), $\partial \phi(G,U)(w_j) = \partial \phi(w_j)$, for $j = 1, 2$. If $w_1 = w_2$, then $w_1 = w_2$ and $\omega_{G,v}^+(w_1)$ is disjoint union of $\omega_{G,v_1}^+(w_1)$, $\omega_{G,v_2}^+(v_2)$, and $\omega_{G,v_2}^-(v_2)$, whence, by (4.1), (2.1), and since $\partial \phi(v_1) = -\partial \phi(v_2)$, we have $\partial \phi(G,U)(w_1) = \partial \phi(w_1)$. Thus $\phi(G,U)$ is a $k$-flow in $(G, U)$ and $\partial \phi(G,U)(u_i) = \partial \phi(u_i)$ for $i = 1, \ldots, n$.

For example $(G_2, (u_1, u_2))$ is an $e$-superposition of $(P, (u_1, u_2))$ (see Figure 4.2).

A network $(G', U')$, $U' = \langle u'_1, \ldots, u'_n \rangle$, is a superposition of $(G, U)$, $U = \langle u_1, \ldots, u_n \rangle$, if there exists a sequence $(G_1, U_1) = (G, U), (G_2, U_2), \ldots, (G_r, U_r) = (G', U')$ such that $(G_{j+1}, U_{j+1})$ is a vertex or edge superposition of $(G_j, U_j)$ for $j = 1, \ldots, r - 1$. If $\phi$ is a $k$-flow in $(G', U')$, then $\phi(G,U)$ denotes the $k$-flow $(\cdots (\phi(G_{r-1}, U_{r-1}))(G_{r-2}, U_{r-2}) \cdots (G_1, U_1))$ in $(G, U)$. By Propositions 4.1 and 4.2, $\phi(G,U)$ is well defined and $\partial \phi(u'_i) = \partial \phi(G,U)(u_i)$ for $i = 1, \ldots, n$, whence $\chi(\phi(G,U)) = \chi(\phi)$. We write $\phi(G)$ instead of $\phi(G,U)$.

**Definition 4.3.** A superposition $(G', U')$ of $(G, U)$ is called $A$-strong if $\phi(G,U)$ is a nowhere-zero $A$-flow in $(G, U)$ for every nowhere-zero $A$-flow $\phi$ in $(G', U')$. A superposition is called $k$-strong if it is $A'$-strong for every abelian group $|A'| \leq k$.

In view of Theorems 2.4 and 2.5, one would expect that if $A$ is an abelian group of order $k$, then the notions of $A$-strong and $k$-strong superpositions coincide. In Example 12.3 we show that this is not true.

The following statement is a cornerstone of our constructions.

**Lemma 4.4.** Let $(G', U')$ be an $A$-strong ($k$-strong) superposition of $(G, U)$ where $A$ is an abelian group of order $k \geq 2$. Then $\chi_k(G', U') \subseteq \chi_k(G, U)$. In particular, if $(G, U)$ is a $k$-snark, then so is $(G', U')$.

**Proof.** If $\phi$ is a nowhere-zero $A$-flow in $(G', U')$, then $\phi(G,U)$ is a nowhere-zero $A$-flow in $(G, U)$ and, by Propositions 4.1 and 4.2, $\chi(\phi(G,U)) = \chi(\phi)$. Clearly, if $(G', U')$ is an $A$-strong ($k$-strong) superposition of $(G, U)$ and $(G'', U'')$ is an $A$-strong ($k$-strong) superposition of $(G', U')$, then $(G'', U'')$ is an $A$-strong ($k$-strong) superposition of $(G, U)$. By Proposition 4.1, a vertex superposition is $k$-strong for every $k \geq 2$. Now we study edge superpositions.

**Proposition 4.5.** Every $A$-strong edge superposition is $|A|$-strong.

**Proof.** Let $(G', U')$ be an $e$-superposition of $(G, U)$ that is $A$-strong but not $A'$-strong where $|A'| \leq |A|$. Thus there exists a nowhere-zero $A'$-flow $\phi$ in $(G', U')$ such that $\phi(G,U)$ is
not nowhere-zero. Let \((H, (v_1, v_2))\), \(\psi', x_1, x_2\), have the same meaning as in the text preceding (4.1). Then \(\psi(G, U')\) can have zero values only on the arcs \(x_1, x_2\), i.e., \(\partial \psi'(v_1) = \partial \psi'(v_2) = 0\), whence \(\psi'\) is a nowhere-zero \(A'\)-flow in \(H\). Thus \((G - e, U)\) and \(H\) admit nowhere-zero \(A'\)-flows and, by Proposition 2.6, also nowhere-zero \(A\)-flows \(\psi_1\) and \(\psi_2\), respectively. Take a nowhere-zero \(A\)-flow \(\psi\) in \((G', U')\) so that its restrictions to \(D(G - e)\) and \(D(H)\) are \(\psi_1\) and \(\psi_2\), respectively. Then \(\psi(G, U)(x_1) = -\psi(G, U)(x_2) = 0\), and the superposition is not \(A\)-strong—a contradiction.

**PROPOSITION 4.6.** Let \((G', U')\) be an \(e\)-superposition of \((G, U)\) replacing an edge \(e\) of \(G\) by a network \((H, (v_1, v_2))\) so that \(H\) is a \(k\)-snark. Then this superposition is \(k\)-strong.

**PROOF.** If the superposition is not \(k\)-strong, then there exists a nowhere-zero \(A'\)-flow \(\psi\) in \((G', U')\) so that \(|A'| \leq k\) and \(\psi(G, U)\) is not nowhere-zero. Now similarly as in the proof of Proposition 4.5, we get a nowhere-zero \(A'\)-flow in \(H\), which contradicts the fact that \(H\) is a \(k\)-snark.

The edge superposition indicated in Figure 4.2 is 4-strong, because it replaces an edge by a snark. As pointed out before, the vertex superpositions indicated in Figure 4.1 are \(k\)-strong for any \(k \geq 2\).

Every superposition arising so that edges are replaced by \(k\)-snarks and vertices by arbitrary graphs is \(k\)-strong. (Note that in Example 12.2 we show that not every \(k\)-strong superposition arise by this process.) Anyway, by Lemma 4.4, this technique produces an infinite class of bridgeless \(k\)-snarks if we have at least one such a \(k\)-snark. We show that this approach also generalizes some known constructions.

**Dot product.** Take two snarks \(G_1\) and \(G_2\) and delete two adjacent vertices \(v_1, v_2\) from \(G_1\) and two edges \((x, y), (z, t)\) from \(G_2\) so that the vertices \(x, y, z, t\) are pairwise distinct. Then joining the vertices of degree two by four new edges as indicated in Figure 4.3 we get a new graph \(G_1 \cdot G_2\), which is known to be a snark. It is called a dot product of \(G_1\) and \(G_2\). This construction was introduced independently by Isaacs [15] and Adelson-Velskij and Titov [1]. It can be obtained using the techniques of superposition as well. First take the 4-snark \(G'_2\), homeomorphic with \(G_2\) (see Figure 4.4). Replacing in \(G_1\) the edge \((v_1, v_2)\) by \((G'_2, (u'_1, u'_2))\) we get \(G'\). Replacing the vertices of \(G'\) of valency 4 by subgraphs consisting of two isolated
vertices we get \( G'' \), which is a 4-snark by Propositions 4.1 and 4.6, and Lemma 4.4. Thus \( G_1 \cdot G_2 \) is a snark by Proposition 3.2.

Let \( H_1 \) be a \( k \)-snark and \( H_2 \) be a graph. Suppose that \( v_1 \) and \( v_2 \) are vertices of \( H_1 \) and \( H_2 \), respectively, having the same valency. Then consider the graph \( H_3 \) indicated in Figure 4.5. \( H_3 \) is a \( k \)-snark, because it is homeomorphic with the graph \( H_4 \) which is a \( v_1 \)-superposition of \( H_1 \). This operation is well known and it has often been used to construct cyclically 3-edge-connected cubic 4-snarks (see, e.g., [8, 42, 44]).

5. JOIN AND IMMERSION

Suppose that \((G, U)\) is a network, \( U = \{u_1, \ldots, u_n\} \), and \( r \) is an integer, \( 0 \leq r \leq n/2 \). Then the \( r \)-join (or simply a join) of \((G, U)\) is the network \((G_1, \{u_{2r+1}, \ldots, u_n\})\) where \( G_1 \) is the graph obtained from \( G \) after identifying the sets of vertices \( \{u_1, u_2, \ldots, u_{2r-1}, u_{2r}\} \) to new vertices \( v_1, \ldots, v_r \), respectively. Furthermore, if \( n = 2r \), then we speak about the complete join of \((G, U)\).

Suppose that \((G, U)\) and \((G', U')\) are disjoint networks, \( U = \{u_1, \ldots, u_n\}, U' = \{u'_1, \ldots, u'_m\} \), and \( r \) is an integer, \( 0 \leq r \leq n, m \). Then the \( r \)-join (or simply a join) of \((G, U)\) and \((G', U')\) is the \( r \)-join of the network \((G \cup G', \{u_1, u'_1, \ldots, u_r, u'_r, u_{r+1}, \ldots, u_n, u'_{r+1}, \ldots, u'_m\})\). If \( n = m = r \), then we speak about the complete join of \((G, U)\) and \((G', U')\).

Suppose that \((G, U)\), \((G', U')\) are disjoint networks, \( U = \{u_1, \ldots, u_n\}, \) and \( U' = \{u'_1, \ldots, u'_m\} \). By an immersion of \((G', U')\) into \((G, U)\) we mean a network \((G'', U)\) arising by the following process: choose pairwise different vertices \( v_1, \ldots, v_m \) from \( G \) and identify the sets of vertices \( \{v_1, u'_1\}, \ldots, \{v_m, u'_m\} \) to vertices which will be denoted by \( v_1, \ldots, v_m \), respectively (in other words, every vertex \( u'_i \) is deleted and the incident edges are joined with \( v_i \), \( i = 1, \ldots, m \)).

For instance, let \((G, \{u_1, u_2, u_3\})\) and \((G', \{u'_1, u'_2\})\) be the networks from Figure 5.1. Then \((G'', \{u_1, u_2, u_3\})\) is an immersion of \((G', \{u'_1, u'_2\})\) into \((G, \{u_1, u_2, u_3\})\). We shall use this notion in the following section (Lemma 6.4).
6. PROPER AND IMPROPER NETWORKS

A network \((G, U)\) is called \(k\)-proper (resp. \(k\)-improper, weakly \(k\)-proper, weakly \(k\)-improper) if \(\chi_k(G, U) \subseteq \{[1, \ldots, 1]\}\) (resp. \(\chi_k(G, U) \subseteq \{0, \ldots, 0\}\), \((0, \ldots, 0) \notin \chi_k(G, U)\), \((1, \ldots, 1) \notin \chi_k(G, U)\)).

For example, if \(G\) is a graph and \(u_1, \ldots, u_n\) are vertices of \(G\) of valency 1, then the network \((G, \langle u_1, \ldots, u_n \rangle)\) is \(k\)-proper for every \(k \geq 2\). By Proposition 2.2, a network \((G, \langle u \rangle)\) is \(k\)-improper for every vertex \(u \in V\) of \(G\) and \(k \geq 2\).

**Proposition 6.1.** Let \(G\) be a \(k\)-snark and \(u_1, \ldots, u_n\) be pairwise distinct vertices of \(G\). Then \((G, \langle u_1, \ldots, u_n \rangle)\) is weakly \(k\)-proper. Furthermore, if \(u_1, \ldots, u_n\) are adjacent to \(u\), \(u \neq u_1, \ldots, u_n\), and \(H\) is the graph arising from \(G\) after deleting edges \(\langle u, u_1 \rangle, \ldots, \langle u, u_n \rangle\), then \((H, \langle u_1, \ldots, u_n \rangle)\) is weakly \(k\)-improper.

**Proof.** The first part follows from the fact that if \((G, \langle u_1, \ldots, u_n \rangle)\) has a nowhere-zero \(k\)-flow \(\varphi\) with \(\chi(\varphi) = \{0, \ldots, 0\}\), then \(\varphi\) is a nowhere-zero \(k\)-flow in \(G\). If \((H, \langle u_1, \ldots, u_n \rangle)\) has a nowhere-zero \(k\)-flow \(\varphi\) so that \(\chi(\varphi) = \{1, \ldots, 1\}\), then, by Proposition 2.3, the graph \(H' = (H, \langle u_1, \ldots, u_n \rangle) / \{1, \ldots, 1\}\) has a nowhere-zero \(k\)-flow. But \(H'\) is a \(u\)-superposition of \(G\), thus, by Proposition 4.1, it is a \(k\)-snark. This contradiction implies the second part.

**Proposition 6.2.** Let \(G\) be a \(k\)-snark and \(u_1, u_2\) be two distinct vertices of \(G\). Then \((G, \langle u_1, u_2 \rangle)\) is \(k\)-proper. Furthermore, if \(u_1\) and \(u_2\) are joined by an edge \(e\), then \((G - e, \langle u_1, u_2 \rangle)\) is \(k\)-improper.

**Proof.** By Proposition 2.2, no network can have a vector in its \(k\)-characteristic set with exactly one nonzero coordinate. Thus \((G, \langle u_1, u_2 \rangle)\) is \(k\)-proper by Proposition 6.1. Similarly subdividing \(e\) by a new vertex \(u\) and deleting the edges \(\langle u, u_1 \rangle, \langle u, u_2 \rangle\), we get that \((G - e, \langle u_1, u_2 \rangle)\) is \(k\)-improper by the second part of Proposition 6.1.

Clearly, every \(k\)-proper or \(k\)-improper network with exactly two outer vertices can be constructed using Proposition 6.2. Similarly \((G, U)\) is weakly \(k\)-proper or weakly \(k\)-improper if and only if \(G\) or \((G, U) / \{1, \ldots, 1\}\) are \(k\)-snarks, respectively.

The following statement contains useful observations. Its proof is trivial.

**Proposition 6.3.**
(a) The complete join of a \(k\)-proper and a weakly \(k\)-improper (a \(k\)-improper and a weakly \(k\)-improper) networks is a \(k\)-snark.
(b) An \(r\)-join \((r \geq 1)\) of a \(k\)-proper and a \(k\)-improper networks is a \(k\)-snark.

The following statement presents another cornerstone of our techniques.

**Lemma 6.4.** Let \((G''', U)\) be an immersion of \((G', U')\) into \((G, U)\) and \((G', U')\) be \(k\)-improper. Then \(\chi_k(G'', U) \subseteq \chi_k(G, U)\). In particular, if \((G, U)\) is a \(k\)-snark, then so is \((G'', U)\).

**Proof.** Let \(\varphi''\) be a nowhere-zero \(k\)-flow in \((G'', U)\) and \(\varphi'\) be its restriction to \(D(G')\), \(D(G)\), respectively. Since \((G', U')\) is \(k\)-improper, then \(\varphi'\) is a nowhere-zero \(k\)-flow in \((G', \emptyset)\), whence \(\varphi\) is a nowhere-zero \(k\)-flow in \((G, U)\), and, furthermore, \(\chi(\varphi') = \chi(\varphi)\) holds.

The dot product (see Figures 4.3 and 4.4) can be also expressed as an application of the techniques from this section. By Proposition 6.2, networks \((G'_2, \langle u'_1, u'_2 \rangle)\) and \((G_1 - \langle v_1, v_2 \rangle, \langle v_1, v_2 \rangle)\) are \(4\)-proper and \(4\)-improper, respectively. Their complete join is \(G'\), which is a \(4\)-snark by Proposition 6.3. We can also apply Lemma 6.4, because \(G'\) is an immersion of \((G_1 - \langle v_1, v_2 \rangle, \langle v_1, v_2 \rangle)\) into the \(4\)-snark \(G'_2\).
In this section we generalize some ideas from Goldberg [10].

A network \((G, U)\) is called \(k\)-even (\(k\)-odd) if every vector from \(\chi_k(G, U)\) has an even (odd) number of nonzero coordinates.

**Proposition 7.1.** Every weakly \(k\)-improper network with precisely three outer vertices is \(k\)-even.

**Proof.** Let \((G, U)\) be weakly \(k\)-improper network, \(U = \langle u_1, u_2, u_3 \rangle\), and \(z \in \chi_k(G, U)\). Then the number of nonzero coordinates in \(z\) is at most 2, but not 1, by Proposition 2.2. Thus \((G, U)\) is \(k\)-even.

The proof of the following statement is also trivial and thus left to the reader.

**Proposition 7.2.**
(a) A join of a \(k\)-even and a \(k\)-odd networks is a \(k\)-odd network. In particular, the complete join of a \(k\)-even and a \(k\)-odd networks is a \(k\)-snark.
(b) A join of a \(k\)-even (\(k\)-odd) network is \(k\)-even (\(k\)-odd). In particular, the complete join of a \(k\)-odd network is a \(k\)-snark.

Let \(v_1, v_2, v_3\) be the vertices of valency 2 in \(P - v\) (see Figure 7.1). Thus by Propositions 6.1 and 7.1, \((P - v, \langle v_1, v_2, v_3 \rangle)\) is 4-even and so is \((G_3, \langle u_1, u_2, u_3 \rangle)\) by Proposition 7.2. Thus the graph \(G_4\) from Figure 7.2 is a 4-snark.
Consider the network \((G_5, \langle w_1, w_2, w_3, w_4 \rangle)\) depicted in Figure 7.3. It is 4-improper, because it arises from four copies of 4-even network \((P - v, \langle v_1, v_2, v_3 \rangle)\) after identifying 8 of their outer vertices with 8 outer vertices of four copies of 4-proper network \((P, \langle u_1, u_2 \rangle)\). In a similar way we can construct 4-improper networks with arbitrary many outer vertices.

8. NETWORKS WITH LARGE RELUCTANCE

The \(k\)-reluctance of a network \((G, U)\), denoted by \(\rho_k(G, U)\), is the smallest number of inner vertices of \((G, U)\) that can be added to \(U\) so that the resulting network admits a nowhere-zero \(k\)-flow.

By Proposition 2.6, \(\rho_k(G, U) \geq \rho_{k+1}(G, U)\). A network \((G, U)\) is a \(k\)-snark if and only if \(\rho_k(G, U) \geq 1\).

**Proposition 8.1.**

(a) \(\rho_k(G', U') \geq \rho_k(G, U)\) if \((G', U')\) is a join of \((G, U)\).

(b) \(\rho_k(G', U') \geq \rho_k(G, U)\) if \((G', U')\) is a vertex superposition of \((G, U)\).

(c) \(\rho_k(G', U') \geq \rho_k(G, U)\) if \((G', U')\) is a \(k\)-strong edge superposition of \((G, U)\).

(d) \(\rho_k(G, U) = \rho_k(G_1, U_1) + \rho_k(G_2, U_2)\) if \((G, U)\) is \(0\)-join of \((G_1, U_1)\) and \((G_2, U_2)\).

(e) \(\rho_k(G, U) = \rho_k(G', U)\) if \((G, U)\) and \((G', U)\) are homeomorphic.

**Proof.** Let \((G', U')\) be a join of \((G, U)\). Take \(r = \rho_k(G', U')\) inner vertices \(v_1, \ldots, v_r\) of \((G', U')\) that can be added to \(U'\) so that the resulting network is not a \(k\)-snark. Then adding to \(U\) the vertices from \(v_1, \ldots, v_r\) which are inner also in \((G, U)\), we get from \((G, U)\) a new network with a nowhere-zero \(k\)-flow. This implies (a).

Let \((G', U')\) be a vertex superposition of \((G, U)\) so that a vertex \(w\) is replaced by a graph \(H\). Take \(r = \rho_k(G', U')\) inner vertices \(v_1, \ldots, v_r\) of \((G', U')\) that can be added to \(U'\) so that
the resulting network is not a $k$-snark. Then adding to $U$ the vertices from $v_1, \ldots, v_r$ which are not in $H$ but also $w$ if $H$ contains some $v_i$, we get from $(G, U)$ a new network with a nowhere-zero $k$-flow. This implies (b).

Suppose that $(G', U')$, $U' = \langle u'_1, \ldots, u'_n \rangle$, is an edge superposition of $(G, U)$, $U = \langle u_1, \ldots, u_n \rangle$, so that an edge $e$ is replaced by a network $(H, \langle v_1, v_2 \rangle)$ where $H$ is a $k$-snark. Let $x_1, x_2, w_1, w_2, w'_1$, and $w'_2$ have the same meaning as in the definition of $e$-superposition in Section 4. Take $r = \rho_k(G', U')$ inner vertices $v'_1, \ldots, v'_r$ of $(G', U')$ so that $(G', \langle u'_1, \ldots, u'_n, v'_1, \ldots, v'_r \rangle)$ has a nowhere-zero $k$-flow. Let $Q = \{v'_1, \ldots, v'_r\} \cap (V(H) \cup \{w'_1, w'_2\})$. If $Q = \emptyset$, then $(G, \langle u_1, \ldots, u_n, v'_1, \ldots, v'_r \rangle)$ has a nowhere-zero $k$-flow, because $H$ is a $k$-snark. If $|Q| \geq 2$ (resp. $Q = \{w'_j\}$, $j \in \{1, 2\}$), then adding to $U$ the vertices from $v'_1, \ldots, v'_r$ which are not in $V(H) \cup \{w'_1, w'_2\}$ together with $w_1, w_2$ (resp. $w_j$) we get from $(G, U)$ a network with a nowhere-zero $k$-flow. Suppose that $Q = \{v'_i\}$, $i \in \{1, \ldots, r\}$, $v'_i \neq w'_1, w'_2$. Let $\phi$ be a nowhere-zero $k$-flow in $(G', \langle u'_1, \ldots, u'_n, v'_1, \ldots, v'_r \rangle)$ and $\phi'$ be its restriction to $D(H)$. Then, by Proposition 2.1, $\phi'(v'_1) + \phi'(v_2) + \phi'(v'_i) = 0$, whence, since $H$ is a $k$-snark, at least one of $\phi'(v'_1), \phi'(v'_i)$ is nonzero. Without loss of generality we can assume that $\phi'(v'_1) \neq 0$ and $i = 1$. Then take a nowhere-zero $k$-flow $\phi''$ in $(G, \langle u_1, \ldots, u_n, w_2, v'_2, \ldots, v'_r \rangle)$ (or in $(G, \langle u_1, \ldots, u_n, v'_2, \ldots, v'_r \rangle)$ if $w_2 \in U$) so that $\phi''(x) = \phi(x)$ if $x \in D(G - e)$ and $\phi''(x_1) = -\phi''(x_2) = \phi(v'_1)$. Thus $\rho_k(G, U) \leq r$ and (c) holds by Propositions 4.5 and 4.6.

Item (d) is trivial. Let $(G_1, U)$ be a network and $G_2$ be the graph obtained from $G_1$ after subdividing an edge. Then $G_2$ is a $k$-proper edge superposition of $G_1$, whence, by (c), $\rho_k(G_1, U) \leq \rho_k(G_2, U)$. The opposite inequality is trivial. This implies (e).}

Clearly, if $\rho_k(G) \geq r$ and $v_1, \ldots, v_r$ are pairwise different vertices of $G$, then $(G, \langle v_1, \ldots, v_r \rangle)$ is a $k$-proper network and $(G, \langle v_1, \ldots, v_{r-1} \rangle)$ is a $k$-snark. For example, by Proposition 6.2, $(P, \langle u_1, u_2 \rangle)$ and $(P - \langle u'_1, u'_2 \rangle, \langle u'_1, u'_2 \rangle)$ (see Figure 8.1) are 4-proper and 4-improper networks, respectively. Applying 1-join to them we get a 4-snark $(H, \langle u_2, u'_2 \rangle)$, i.e., $\rho_4(H, \langle u_2, u'_2 \rangle) \geq 1$. Applying joins to four copies of $(H, \langle u_2, u'_2 \rangle)$, we get, by Proposition 8.1, a 4-snark $H'$ so that $\rho_4(H') \geq 4$ (see Figure 8.2). Then $(H', \langle v_1, v_2, v_3, v_4 \rangle)$ is 4-proper and $(H', \langle v_1, v_2, v_3 \rangle)$ is a 4-snark.
9. 3-SNARKS

It is well known that a cubic graph is a 3-snark if and only if it is not bipartite (see, e.g., [17]). The 3-flow conjecture of Tutte is that every graph without 1- and 3-edge cuts has a nowhere-zero 3-flow. An equivalent form of this conjecture is that there does not exist a 4-edge-connected 3-snark (see, e.g., [17]). This conjecture is true for planar graphs as follows from the Theorem of Grötzsch [11] (see also Thomassen [37]) and the result of Tutte [38] which says that a planar graph admits a nowhere-zero k-flow if and only if it is face-k-colorable. We show that the 3-flow conjecture is equivalent to formally stronger statements.

**Theorem 9.1.** The following statements are pairwise equivalent.

1. Every graph without 1- and 3-edge cuts has a nowhere-zero 3-flow.
2. There is no bridgeless 3-snark with at most three edge cuts of cardinality 3.
3. There is no bridgeless 3-snark $G$ with vertices $v_1, v_2, v_3$ such that for every 3-edge cut $C$ of $G$ each component of $G - C$ has at least one vertex from $v_1, v_2, v_3$.

**Proof.** Clearly, (2) implies (1). Let $G$ be the smallest counterexample to (2). Then every 3-edge cut in $G$ must be trivial, i.e., it is the set of edges incident to one vertex. Otherwise we can apply standard reductions (see, e.g., [17, Chapter 9]) to obtain a smaller counterexample. Thus there exist vertices $v_1, v_2, v_3$ such that every 3-edge cut in $G$ is the set of edges incident to one of them and $G$ is also a counterexample to (3). Hence (3) implies (2).

Suppose that $G$ is a counterexample to (3). Let edges $e_1, e_2$ form a perfect matching in $K_4$ and $G_1$ be a 3-strong superposition of $K_4$ so that $e_1$ and $e_2$ are replaced by two distinct copies of $(G, (v_1, v_2))$. $G_1$ has two copies $v'_1, v''_1$ of the vertex $v_1$. Let $G_2$ be a 3-strong superposition of $K_4$ so that $e_1$ and $e_2$ are replaced by two distinct copies of $(G_1, (v'_1, v''_1))$. By Lemma 4.4, $G_2$ is a 3-snark. But $G_2$ has no 1- and 3-edge cuts, thus it is a counterexample to (1). Hence (1) implies (3).

By a theorem of Grünbaum [12] (see also [2] and [5]), item (2) from Theorem 9.1 is satisfied for planar graphs. Then the same holds for item (3) (because the smallest planar counterexample to (3) is a counterexample to (2)).

Theorem 9.1 is, in certain sense, the strongest possible, because the following holds.

**Theorem 9.2.** There exists an infinite family of (planar) 3-snarks having exactly four trivial edge cuts of cardinality 3 and no other edge cut of cardinality $\leq 3$. 
PROOF. Take the minimal class of graphs containing $K_4$ and closed under the operation of replacing an edge between two vertices of valency 3 by a copy of $K_4$. This class has the required property.

In [29] is proved that the 3-flow conjecture suffices to verify for 5-edge-connected graphs. The weak 3-flow conjecture of Jaeger [17] is that there exists $k \geq 4$ such that every $k$-edge-connected graph has a nowhere-zero 3-flow. Now we study what happens if this (or the 3-flow) conjecture fails.

**Theorem 9.3.** If there exists a $k$-edge-connected 3-snark, $k \geq 4$, then it is an NP-complete problem to determine whether a $k$-edge-connected graph admits a nowhere-zero 3-flow.

**Proof.** By Tutte [38], a planar graph admits a nowhere-zero 3-flow if and only if its dual is vertex-3-colorable. By Garey et al. [9], it is an NP-complete problem to decide whether a planar graph is vertex-3-colorable. Thus, it is an NP-complete problem to decide whether a graph admits a nowhere-zero 3-flow.

If the weak 3-flow conjecture is not true, then there exists a $k$-edge-connected 3-snark $G$ with the smallest possible order ($k \geq 4$). Let $u, v$ be two distinct vertices of $G$. Then $(G, (u, v))$ is not a 3-snark (otherwise the 1-join of $(G, (u, v))$ is a 3-snark, which contradicts the minimality of $G$). Thus, by Proposition 6.2, $\chi_k(G, (u, v)) = \lceil 1, 1 \rceil$ and there exist nowhere-zero 3-flows $\varphi$ and $\psi$ in $(G, (u, v))$ so that $\varphi(u) = -\varphi(v) = -\psi(u) = \psi(v) = 1$. Let $H$ be a graph and $H'$ be a 3-strong superposition of $H$ so that every edge of $H$ is replaced by a copy of the network $(G, (u, v))$. If $H$ is a 3-snark, then so is $H'$, by Lemma 4.4. If $H$ has a nowhere-zero 3-flow, then this flow together with copies of $\varphi$ and $\psi$ induce a nowhere-zero 3-flow in $H'$. Hence $H$ is a 3-snark if and only if $H'$ is. Furthermore $H'$ is $k$-edge-connected if $H$ is connected. Thus we get a polynomial reduction to the previous problem.

**Theorem 9.4.** If there exists a $k$-edge-connected 3-snark $G$ of order $p$, then for every $n \geq 2$ there exists a $k$-edge-connected graph $G_n$ of order $n$ satisfying $\rho_3(G_n) \geq \lfloor n/p \rfloor$.

**Proof.** Let $n \geq 2$ and $r = \lfloor n/p \rfloor$. Take a $k$-edge-connected graph $H$ so that $V(H) = \{u_1, \ldots, u_r\}$. For $i = 1, \ldots, r - 1$, let $(H_r, (v_r))$ be a copy of a network $(G, (v))$ where $v \in V(G)$. Take $(H_r, (v_r)) = (G, (v))$ if $r = \lfloor n/p \rfloor$, otherwise take $(H_r, (v_r))$ so that $H_r$ is a $k$-edge-connected graph of order $n - p(r - 1)$ and $v_r \in V(H_r)$. Identify the sets of vertices $\{u_1, v_1\}, \ldots, \{u_r, v_r\}$ to $r$ new vertices. We get a $k$-edge-connected graph $G_n$ of order $n$. By Proposition 2.2, $\rho_3(G, (v)) \geq 1$, whence, by Proposition 8.1, $\rho_3(G_n) \geq \lfloor n/p \rfloor$.

Let $\rho_3^{(k)}(n)$ denote the maximal 3-reluctance of a $k$-edge-connected graph of order $n$. If there is no $k$-edge-connected 3-snark, then $\rho_3^{(k)}(n)/n = 0$ for every $n > 0$. Otherwise, by Theorem 9.4, $\liminf_{n \to \infty} \rho_3^{(k)}(n)/n \geq 1/p > 0$. Therefore the (weak) 3-flow conjecture is equivalent to the conjecture that $\liminf_{n \to \infty} \rho_3^{(4)}(n)/n = 0$ (lim inf $\rho_3^{(k)}(n)/n = 0$ for some $k \geq 4$). See [27] for more details.

10. **Snarks**

From our everyday experience we might concede that snarks are not common among cubic graphs. Indeed a result of Robinson and Wormald [34] states that almost every cubic graph is hamiltonian, hence also edge-3-colorable.

The general methods from Sections 2–8 can be directly used for constructions of 4-snarks. From them we can easily obtain snarks after splitting vertices of valency greater than four to...
vertices of valencies two and three and subsequently suppressing the vertices of valency two. In this section we present several examples.

An edge cut of a graph is called cyclic if after deleting its edges we get at least two components having cycles. A graph is called cyclically $k$-edge-connected if it does not have a cyclic edge cut of cardinality smaller than $k$.

10.1. Cyclically 6-edge-connected snarks. In Figure 10.1 are depicted flower snarks $I_3$, $I_5$, and $I_7$. Similarly, we can construct graph $I_{2k+1}$ of order $4(2k + 1)$ for every $k \geq 1$. These graphs were introduced by Isaacs [15] (as pointed out in [10], flower snarks have been also constructed independently by Grinberg). The flower snarks $I_7$, $I_9$, ... were the only cyclically 6-edge-connected snarks known previous to the introduction of superposition techniques.

Using superposition we can construct new infinite families of cyclically 6-edge-connected snarks. For instance, the following construction was announced in [24]. Let $C$ be a circuit of length 6 in the Petersen graph $P$. Replace the edges of $C$ by copies of $I_5$ and each vertex of valency 7 replace by vertices of valencies 3, 2, and 2 as indicated in Figure 10.2. The resulting graph is a 4-snark by Propositions 4.6 and 4.1. Suppressing the vertices of valency 2 we get a cyclically 6-edge-connected snark $G_{118}$ of order 118 depicted in Figure 10.3. We can similarly
construct a snark $G_{120}$ of order 120 depicted in Figure 10.4 (it suffices to replace one vertex of $C$ by $K_4$) and snarks of orders 122 or 124. Furthermore, using $I_{2k+1}$ ($k \geq 3$) instead of $I_5$ we can obtain cyclically 6-edge-connected snarks of any even order $\geq 118$. Thus the following statement holds.

**Theorem 10.1.** For every even $n \geq 118$, there exists a cyclically 6-edge-connected snark of order $n$.

In fact, we can replace every edge of $P$ by a copy of flower snark and then split the vertices of valency 9 to three vertices of valency 3. We again get a cyclically 6-edge-connected snark. Furthermore, we can use any snark or 2-edge-connected (cubic) 4-snark instead of $P$.

10.2. Simple 5-cut snarks. A cyclic $k$-edge cut is called simple, if after removing its edges we get at least one component consisting of a circuit. If a cubic graph $G$ is cyclically 5-edge-connected and every cyclic 5-edge cut of $G$ is simple, then we say that $G$ is simple 5-cut cubic graph. For example, the Petersen graph and the flower snark $I_5$ are simple 5-cut snarks. These are the only such snarks known until recently. Simple 5-cut snarks are of some interest, because by Birkhoff [3], the smallest planar snark must belong to them (see also Remark 12.1).

If we use a circuit $C$ of length 5 in the construction preceding Theorem 10.1, we get a simple 5-cut snark of order 100. Furthermore, if we use the Blanuša snark $B_{18}$ (see [4]) instead of $I_5$, we get a simple 5-cut snark $G_{90}$ of order 90 depicted in Figure 10.5. Clearly, using in this construction copies of $I_{2k+1}$ ($k \geq 2$) instead of $B_{18}$, we obtain simple 5-cut snark of any even order $\geq 90$. 
10.3. Snarks with large girths. Jaeger and Swart [18] conjectured that every snark has girth at most 6. This conjecture was very interesting because, if it had been true, it would imply positive solutions of the cycle double cover and the 5-flow conjectures (see, e.g., [17, 23]). Now we sketch a construction of snarks with arbitrary large girth and refer to [23] for more details.

Let $G$ be a cyclically 5-edge-connected cubic graph with girth $c \geq 7$. Remove vertex sets of paths of lengths 2 and 4 from $G$ obtaining graphs $A$ and $B$, respectively, as indicated in Figure 10.6. The graph $J$ from Figure 10.7 is a snark because it is a 4-strong superposition of $P$. Every circuit in $J$ of length less than $c$ contains either $w$ or $w'$. Then the graph $S$ depicted in Figure 10.8 is a cyclically 5-edge-connected snark (it is a 4-strong superposition of $P$) and has girth at least $c$. Thus, taking for $G$ known cubic graphs with large girth, we get 5-edge-connected snarks with arbitrarily large girth.

10.4. A 4-improper cubic network. By Proposition 6.2, $(P - (u'_1, u'_2), (u'_1, u'_2))$ is a 4-improper network (see Figure 8.1). The network $(G_{102}, (u_1, u_2))$ depicted in Figure 10.9 is
also 4-improper, because it is homeomorphic with a 4-strong superposition of \((P - (u'_1, u'_2), (u'_1, u'_2))\).

Applying immersion of three copies of \((G_{102}, (u_1, u_2))\) into the disjoint union of two copies of \(I_7\) we get the graph \(G_{356}\) indicated in Figure 10.10. By Lemma 6.4, \(G_{356}\) is a 4-snark.

Splitting the vertices of \(G_{356}\) of valency 6 to three vertices of valency 2 and suppressing them we get the cyclically 6-edge-connected snark \(G_{350}\) of order 350 from Figure 10.10. In this way we can insert two snarks to obtain a new snark.

Take \(K_{1,3}\) with vertices \(w, w_1, w_2, w_3\) so that \(w\) has valency 3. The network \((K_{1,3}, (w_1, w_2, w_3))\) is k-proper and k-odd for any \(k \geq 2\). Applying immersion of 4-improper network \((I_5 - (v_1, v_3), (v_1, v_3))\) (see Proposition 6.2) into \((K_{1,3}, (w_1, w_2, w_3))\) we get, by Lemma 6.4, a 4-improper network \((G_{22}, (w_1, w_2, w_3))\) (see Figure 10.11). Applying immersion of \((G_{102}, (u_1, u_2))\) into \((G_{22}, (w_1, w_2, w_3))\) we get a 4-proper network \((G_{122}, (w_1, w_2, w_3))\). Furthermore, replacing the vertices \(w_1, w_2, v,\) and \(w_3\) by two isolated vertices, \(K_2,\) three isolated vertices, and \(K_2,\) respectively, and suppressing the inner vertices of valency 2, we get a network \((G_{123}, (w'_1, w'_2, w'_3))\) indicated in Figure 10.11. This network is 4-proper by Propositions 3.2 and 4.1.

10.5. Improving a construction of Goldberg. Insert \(n\) \((n\) is odd) copies of the 4-even network \((P - v, (v_1, v_2, v_3))\) (see Figure 7.1) and one 4-odd network \((K_{1,3}, (w_1, w_2, w_3))\).

Applying complete join to these networks we get a 4-snark by Proposition 7.2. (For example, if \(n = 3\), we can get the 4-snark \(G_3\) depicted in Figure 7.2.) Now splitting the vertices of valency 4 to vertices of valency 2 and suppressing them we get a cyclically 5-edge-connected snark of order \(6n + 4\). (For example, from \(G_3\) we get the snark \(G'_3\) depicted in Figure 10.12.)

A similar method can be found in Goldberg [10].

Using \((G_{123}, (w'_1, w'_2, w'_3))\) instead of \((K_{1,3}, (w_1, w_2, w_3))\) in this construction we can get a cyclically 6-edge-connected snark of order \(6n + 120\). (For example, for \(n = 3\) we get the
snark $G_4''$ of order 138 depicted in Figure 10.13.) The smallest cyclically 6-edge connected snark constructed by this method has 126 vertices.

10.6. Snarks with large reluctance. Applying 1-join to the networks $(G_{102}, \langle u_1, u_2 \rangle)$ and $(I_5, \langle v_1, v_2 \rangle)$ we get a 4-snark $(G_{121}, \langle u_2, v_2 \rangle)$. Taking $r$ copies of $(G_{121}, \langle u_2, v_2 \rangle)$ and identifying pairs of their outer vertices we get a graph $H_r$ from Figure 10.14. By items (a), (d) from Proposition 8.1, $\rho_4(H_r) \geq r$. Splitting the vertices of valency 6 to vertices of valency 2 and suppressing them we get a cyclically 6-edge-connected cubic graph $H'_r$ of order 118$r$ indicated in Figure 10.15. By items (b), (e) from Proposition 8.1, $\rho_4(H'_r) \geq \rho_4(H_r) \geq r$. Thus the following statement holds.

**Theorem 10.3.** For every $r > 0$, there exists a cyclically 6-edge-connected snark of order $118r$ and 4-reluctance at least $r$.

10.7. Snarks with large oddness. By oddness of a cubic graph $G$ we mean the smallest $r$ such that the vertices of $G$ can be covered by a family of vertex disjoint circuits containing $r$ odd circuits (an isolated vertex is also an odd circuit). This parameter was introduced in [22] and [14]. Clearly, the oddness of a cubic graph is always an even number. Furthermore, a cubic graph has oddness zero if and only if it has an edge-3-coloring (i.e., a nowhere-zero
a symbolic representation of $I_5$

Figure 10.11.

$G_i$

Figure 10.12.

$G'_4$

Figure 10.13.

Figure 10.14.
4-flow). Note that the 5-flow and cycle double cover conjectures are verified for snarks with oddness 2 (see [14, 17, 22]).

**Lemma 10.4.** The oddness of a cubic graph is greater than or equal to its 4-reluctance.

**Proof.** Let $G$ be a cubic graph with oddness $r$ and $C_1, \ldots, C_n$ be vertex disjoint circuits covering the vertices of $G$ such that $C_1, \ldots, C_r$ are odd. Choose $v_i \in V(C_i)$ for $i = 1, \ldots, r$ and assign the edges of $G$ nonzero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ so that the edges incident with any inner vertex of $(G, (v_1, \ldots, v_r))$ are assigned different elements (i.e., the edges of $C_1, \ldots, C_n$ receive $(0, 1), (1, 0)$ and the rest of the edges receive $(1, 1)$). Take $\psi : D(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ so that if $x, x^{-1}$ are the arcs on an edge $e$, then $\psi(x), \psi(x^{-1})$ equal the element assigning $e$. From the arithmetic in the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ it follows that $\psi$ is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow in $(G, (v_1, \ldots, v_r))$. Thus $\rho_4(G) \leq r$.

**Theorem 10.5.** For every $r > 0$, there exists a cyclically 6-edge-connected snark of order $118(2r - 1)$ and oddness at least $2r$.

**Proof.** Follows from Theorem 10.3, Lemma 10.4 and the fact that oddness of a cubic graph is always even.

10.8. Avoiding the flower snarks. With respect to the constructions used until now, it seems that without flower snarks we would not be able to construct cyclically 6-edge-connected snarks. Now we show that this is not true. Replace in the Petersen graph $P$ two edges by copies of $P$. We obtain the 4-snark $G_{26}'$ from Figure 10.16. Splitting the vertices of valency 5 to vertices of valencies 3 and 2 and suppressing the latter ones we can get the snark $G_{26}$...
from Figure 10.16. Using \((G_{26}, (v_1, v_2))\) instead of \((I_5, (v_1, v_2))\) in the construction from Section 10.1, we get a new cyclically 6-edge-connected snark of order 154. It was constructed only from copies of the Petersen graph.

11. 5-SNARKS

If there exists a bridgeless 5-snark, then, by [25], it is an NP-complete problem to decide whether a (cubic) graph admits a nowhere-zero 5-flow. Now we study a potential growth of 5-reluctance.

**Theorem 11.1.** If the 5-flow conjecture is not true, then there exist integers \(p, n_0 > 0\) and \(s \geq 0\) such that for every integer \(n \geq n_0\) there exists a cyclically 5-edge-connected cubic 5-snark \(G_n\) of order \(2n\), with girth at least 7, and \(\rho_5(G_n) \geq \lceil n/p \rceil - s\).

**Proof.** If the 5-flow conjecture is not true, then, by Celmins [6] (see also [17]), there exists a cyclically 5-edge-connected cubic 5-snark \(G\) with girth at least 7. Let \(u_1, u_2\) be vertices of \(G\) so that their distance is at least 3. Consider a circuit \(C\) in \(G\) and an edge \(e = (v_1, v_2)\) of \(C\). Let \(C - e\) consists of the edges \(e_1, \ldots, e_m\). Take a 5-strong superposition of \(G - e\) so that the edges \(e_1, \ldots, e_m\) are replaced by \(m\) copies of \((G, (u_1, u_2))\). Applying reductions similarly as in the beginning of Section 10.4 we get a 5-improper network \((G', (w_1, w_2))\). Symbolic representations of \((G, (u_1, u_2)), (G - e, (v_1, v_2)), \) and \((G', (w_1, w_2))\) are indicated in Figure 11.1. The 1-join of \((G', (w_1, w_2))\) and \((G, (u_1, u_2))\) is a 5-snark \((G'', (w_2, u_2))\).

Assume that \(G''\) has order \(2p + 3\). Then \(p > 0\), because the smallest cubic graph with girth 7 has order 24 (see [30]). Since the McGee’s graph is also cyclically 7-edge-connected, using the techniques from Section 10.1 we can construct a cyclically 6-edge-connected graph \(H_k\) of order \(2k\) and with girth at least 7 for all \(k\) larger than some \(s > 0\). Take \(n \geq n_0 = p(s+1), r = \lceil n/p \rceil - s, \) and \(q = n - rp + 1\). Then \(r > 0\) and \(q > s\). Take \(r\) disjoint copies of \((G'', (w_2, u_2))\) and a network \((H_q, (v'_1, v'_2))\) (where \(v'_1, v'_2\) are vertices of \(H_q\) so that their distance is at least 3) and join the pairs of outer vertices similarly as in Figure 10.14. Splitting the vertices of valency 6 to vertices of valency 2 and suppressing them we get a cyclically 5-edge-connected
Let $\overline{\rho}_5(k)$ denote the maximal 5-reluctance of a cyclically 5-edge-connected cyclic graph of order $2k$ and with girth at least 7. Then, similarly as in the end of Section 9, we can show that, by Theorem 11.1, the 5-flow conjecture is equivalent to the conjecture that $\lim \inf_{n \to \infty} \overline{\rho}_5(n)/n = 0$.

12. CONCLUDING REMARKS

Remark 12.1. If the four-color theorem had been false, then there would exist a planar snark and, by [3], also a simple 5-cut cubic planar snark (see Section 10.2). Furthermore, using the same ideas as in Theorem 11.1 we can show that then there exist integers $p, n_0 > 0$ and $s \geq 0$ so that for every integer $n \geq n_0$ we have a simple 5-cut planar cubic graph $G_n$ of order $2n$ satisfying $\rho_4(G_n) \geq [n/p] - s$. Thus, denoting by $\overline{\rho}_4(n)$ the maximal 4-reluctance of a simple 5-cut cubic planar graph of order $2n$ we get that the four-color theorem is equivalent to the statement that $\lim \inf_{n \to \infty} \overline{\rho}_4(n)/n = 0$.

Example 12.2. We construct 4-strong superpositions of the Petersen graph $P$ without replacing edges by 4-snarks. Take a circuit $C$ of length 5 in $P$ and a superposition $P'$ of $P$ so that the vertices of $C$ are replaced by graphs $H_1, \ldots, H_5$ which are copies of $P - e$ ($e \in E(P)$) and the edges of $C$ are replaced by two parallel edges as indicated in Figure 12.1. Let $\varphi$ be a nowhere-zero 4-flow in $P'$ and $\varphi_i$ be its restriction to $D(H_i), i = 1, \ldots, 5$. Then $\varphi_1$ is a 4-flow in $(H_1, (v_1, v_2, v_3))$ satisfying $\chi(\varphi_1) = (1, 1, 1)$ (because $\partial \varphi_1(v_1) \neq 0$, $\chi(\varphi_1) \neq (1, 0, 0), (1, 1, 0)$ by Propositions 2.2 and 6.2, respectively, and if $\chi(\varphi_1) = (1, 0, 1)$, we get the forbidden cases $\chi(\varphi_2) = (1, 0, 0)$ or $(1, 1, 0)$ for the 4-flow $\varphi_2$ in $(H_2, (v_4, v_5, v_6))$. Since similar properties have also $\varphi_2, \ldots, \varphi_5$, then $\varphi_P$ is a nowhere-zero 4-flow in $P$. Thus the superposition is 4-strong and $P'$ is a 4-snark.

Furthermore, let $P''$ be a superposition of $P$ so that the edges of $P$ are replaced by two parallel edges and the vertices of $P$ are replaced by copies of $G_{123}$ (see Figure 10.11) in such
a way that all copies of vertices \( w_1', w_2', w_3' \) are adjacent with parallel edges in \( P'' \) (i.e., have valency 4 in \( P'' \)). This superposition is 4-strong, because the network \((G_{123}, \langle w_1', w_2', w_3' \rangle)\) is 4-proper (see Section 10.4).

**Example 12.3.** We construct \( A \)-strong superpositions which are not \(|A|\)-strong.

Let \( G \) and \( G' \) be the graphs with orientations \( X \) and \( X' \), respectively, indicated in Figure 12.2 and \( e \) denote the edge of \( G \) associated with arc \( a\). \( G' \) arises from \( G \) after replacing \( e \) by three parallel edges and the vertices \( u \) and \( v \) by graphs consisting of three isolated vertices \( u_1, u_2, u_3 \) and \( v_1, v_2, v_3 \), respectively. Thus \( G' \) is a superposition of \( G \) and if \( \varphi \) is a \( k \)-flow in \( G' \), then \( \varphi_G(a) = \varphi_x(a_1) + \varphi_x(a_2) + \varphi_x(a_3) \) and \( \varphi_G(x) = \varphi(x_1) \) for \( x = b, c, d, i \). We claim that \( G' \) is \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-strong but not \( \mathbb{Z}_4 \)-strong superposition of \( G \). Let \( \varphi \) be a nowhere-zero \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-flow in \( G' \). Then \( \varphi(a_1) = \varphi(a_2) \), whence \( \varphi_G(a) = \varphi(a_3) \neq 0 \). Thus \( \varphi_G \) is nowhere-zero. On the other hand take a nowhere-zero \( \mathbb{Z}_4 \)-flow \( \psi \) on \( G' \) so that \( \psi(a_1) = \psi(c_1) = 1 \) and \( \psi(d_1) = 2 \). Then \( \psi_G(a) = 1 + 1 + 2 = 0 \).

Let \( G \) and \( G'' \) be the graphs with orientations \( X \) and \( X'' \), respectively, indicated in Figure 12.2. \( G'' \) arises from \( G \) after replacing \( e \) by two parallel edges and the vertices \( u \) and \( v \) by graphs consisting of three isolated vertices \( u_1, u_2, u_3 \) and \( v_1, v_2, v_3 \), respectively. Thus \( G'' \) is a superposition of \( G \) and if \( \varphi \) is a \( k \)-flow in \( G'' \), then \( \varphi_G(a) = \varphi(x_1) + \varphi(x_2) \) and \( \varphi_G(x) = \varphi(x_3) \) for \( x = b, c, d, i \). We claim that \( G'' \) is \( A \)-strong superposition of \( G \) if and only if the order of \( A \) is odd. Let \(|A| \) be odd and \( \varphi \) be a nowhere-zero \( A \)-flow in \( G'' \).

Then \( \varphi_G(a) = \varphi(a_1) + \varphi(a_3) = 2\varphi(a_1) \neq 0 \). Thus \( \varphi_G \) is nowhere-zero. If \(|A| \) is even, then there exists \( \alpha \in A \) satisfying \( 2\alpha = 0 \) and a nowhere-zero \( \alpha \)-flow \( \psi \) in \( G'' \) so that \( \psi(a_1) = \psi(d_1) = \alpha \). Thus \( \psi_G(a) = 2\alpha = 0 \).

**Remark 12.4.** By a (nowhere-zero) free \( k \)-flow \( \varphi \) in a network \((G, U), U = \{u_1, \ldots, u_n\}\), we mean a (nowhere-zero) \( \mathbb{Z}_k \)-flow in \((G, U)\) such that \( |\varphi(x)| < k \) for every \( x \in D(G) \). The characteristic vector \( \chi(\varphi) \) of \( \varphi \) is the vector \( z = (z_1, \ldots, z_k) \) such that \( z_i = 0 \) if \( \varphi(u_i) \equiv 0 \) mod \( k \) and \( z_i = 1 \) otherwise. Let \( \chi_k(G, U) \) be the set of all vectors \( \chi(\varphi) \) where \( \varphi \) is a nowhere-zero free \( k \)-flow in \((G, U)\). We claim that \( \chi_k(G, U) = \chi_k(G, U) \) and that \((G, U)\) admits a nowhere-zero free \( k \)-flow if and only if \((G, U)\) admits a nowhere-zero \( k \)-flow. If \((G, U)\) has a nowhere-zero free \( k \)-flow, then taking its values mod \( k \) we get a nowhere-zero \( \mathbb{Z}_k \)-flow. The converse holds by Theorem 2.4 since every integral \( k \)-flow is also a free \( k \)-flow.

The latter fact also implies that \( \chi_k(G, U) \subseteq \chi_k(G, U) \) and if \( z \in \chi_k(G, U) \) and \( \phi \) is a nowhere-zero free \( k \)-flow in \((G, U)\) such that \( \chi'(\varphi) = z \), then taking the values of \( \varphi \) mod \( k \) we get a nowhere-zero \( \mathbb{Z}_k \)-flow \( \psi \) so that \( \chi(\psi) = \chi'(\varphi) \), whence, by Theorem 2.5, \( z \in \chi_k(G, U) \). Thus \( \chi_k(G, U) \subseteq \chi_k(G, U) \). This proves the claim.

**Remark 12.5.** Propositions 2.3, 2.7 and the theorem of Seymour [35] imply that \( \chi_2(G, U) \subseteq \chi_3(G, U) \subseteq \chi_4(G, U) \subseteq \chi_5(G, U) \subseteq \chi_6(G, U) = \chi_7(G, U) = \cdots \) holds. Furthermore, the 5-flow conjecture is equivalent to the statement that also \( \chi_5(G, U) = \chi_6(G, U) \) holds for every network \((G, U)\). On the other hand the sequence of \( k \)-characteristic sets can increase
for parameters $k = 2, 3, 4$. For instance, we claim that if $P$ is the Petersen graph and $U$ is an ordered set containing all vertices of $P$, then $\chi_2(P, U) \subset \chi_3(P, U) \subset \chi_4(P, U) \subset \chi_5(P, U) = \chi_6(P, U) = \cdots$. Really, $\chi_2(P, U)$ contains only the vector with all coordinates equal to 1. On the other hand $\chi_3(P, U)$ contains also vectors with some zero coordinates, but no vector with exactly two nonzero coordinates, and such vectors are in $\chi_4(P, U)$. We can check that $\chi_5(P, U)$ also has the zero vector (which is not in $\chi_4(P, U)$) and that $\chi_5(P, U) = \chi_6(P, U)$. This proves the claim.

Methods discussed in this paper have two common features. Primarily, they are recursive, that means they create new $k$-snarks from known $k$-snarks. Furthermore, they are based on properties of $k$-characteristic sets of networks, that means on 0–1 laws. Let us note that majority of the constructions of snarks known until recently were also based on 0–1 laws (see, e.g., [1, 7, 8, 10, 13, 15, 41, 42]).

We have avoided sophisticated constructions in order to preserve the general character of this paper. More applications of superposition techniques can be found in [21, 23, 25–29] and also in [32, 33]. Further details about nowhere-zero flows in graphs and related problems can be found in [17, 38–40, 43, 44].

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