The intersection of the spectra of operator completions

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Abstract

Let $A \in B(H)$, $B \in B(K)$, $C \in B(K, H)$, $X \in B(H, K)$ and $M_X = \left( \begin{array}{cc} A & C \\ X & B \end{array} \right)$ be an operator completion of the partial operator matrix $Q = \left( \begin{array}{cc} A \\ X & B \end{array} \right)$. In this note, we consider the intersection of the spectra of $M_X$ when $X$ runs over $B(H, K)$. Denote by $\sum(A, B, C)$ the set of scalar $\lambda \in \mathbb{C}$ such that either $(A - \lambda, C)$ or $(B^* - \bar{\lambda}, C^*)$ is not right invertible. We prove that

$$\bigcap_{X \in B(H, K)} \sigma(M_X) = \begin{cases} \sum(A, B, C) & \text{if } \dim(R(C)) = \infty, \\ \sum(A, B, C) \cup \Delta(A, B, C) & \text{if } \dim(R(C)) < \infty, \end{cases}$$

where $\Delta(A, B, C)$ is the set of scalars $\lambda \in \mathbb{C}$ such that $R((A - \lambda, C)) = H$, $R((B^* - \bar{\lambda}, C^*)) = K$, and $\text{ind}(A - \lambda) + \text{ind}(B - \lambda) \neq 0$. We also prove that the intersection is empty if and only if $(A, C)$ and $(B^*, C^*)$ are controllable.

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null-space of $A$, respectively. When $A \in B(H)$, $B \in B(K)$, $X \in B(H, K)$ and $C \in B(K, H)$ are given, put

$$M_X = \begin{pmatrix} A & C \\ X & B \end{pmatrix}, \quad Q = \begin{pmatrix} A & C \\ \ast & B \end{pmatrix}.$$ 

The operator $M_X$ on $H \oplus K$ can be viewed as an operator completion of the partial operator matrix $Q$. The some properties of the spectrum of $M_X$ were discussed in [1]. Takahashi discussed in [2] the invertible completion of $Q$. The relationship between operator completion problem and spectrum assignment can be found in [3,4]. In this paper, we discuss the intersection of the spectra of $M_X$ when $X$ runs over $B(H, K)$.

To do this, we need some notations and definitions.

For given $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$, let

$$\sum(A, B, C) := \{ \lambda \in \mathbb{C} : (A - \lambda, C) \text{ or } (B\ast - \bar{\lambda}, C\ast) \text{ is not right invertible}. \}$$

Clearly, for any $X \in B(H, K)$ we have $\sum(A, B, C) \subset \sigma(M_X)$. Thus

$$\sum(A, B, C) \subset \bigcap_{X \in B(H, K)} \sigma(M_X).$$

**Definition 1.** Let $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$. A pair of operators $(A, C)$ is called controllable if there exists a positive integer $p$ with $\sum_{i=1}^{p} R(A^{i-1}C) = H$; a triple of operators $(A, B, C)$ is called controllable if $(A, C)$ and $(B\ast, C\ast)$ are controllable.

**Definition 2.** Let $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$. For a pair of operators $A \in B(H)$ and $C \in B(K, H)$ denote $R_p(A, C) = R(C, AC, \ldots, A^{p-1}C)$. The pair of operators $(A, C)$ is called admissible if for some positive integer $p$, $R_p(A, C) = R_{p+1}(A, C)$ and the linear set $R_p(A, C)$ is closed. If $p$ is the minimal positive integer with these properties, the pair $(A, C)$ is $p$-admissible; the triple of operators $(A, B, C)$ is called admissible if $(A, C)$ and $(B\ast, C\ast)$ are admissible.

For two operators $S \in B(H)$ and $R \in B(K, H)$, let

$$N(S \mid R) = \{ G \in B(K, H) : R(SG) \subset R(R) \}. \quad (1)$$

As well known (see [2]), an operator $G$ belongs to $N(S \mid R)$ if and only if there exists an $D \in B(K)$ such that $SG = -RD$.

The following results are in [2] which we state as lemmas.

**Lemma 1** (Theorem 1 in [2]). Let $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$. Assume that the operator $(A, C) : H \oplus K \to H$ and $(B\ast, C\ast) : K \oplus H \to K$ are right invertible. Then the following conditions are equivalent:

(1) There exists $X \in B(H, K)$ such that $M_X$ is invertible.

(2) There exists $X \in B(H, K)$ such that $M_X$ is Fredholm with $\text{ind } M_X = 0$. 

null-space of $A$, respectively. When $A \in B(H)$, $B \in B(K)$, $X \in B(H, K)$ and $C \in B(K, H)$ are given, put

$$M_X = \begin{pmatrix} A & C \\ X & B \end{pmatrix}, \quad Q = \begin{pmatrix} A & C \\ \ast & B \end{pmatrix}.$$
The operator $M_0(= M(A, B, C, 0))$ is Fredholm with $\text{ind } M_0 = 0$ or both $N(A | C)$ and $N(B^* | C^*)$ contain non-compact operators.

Lemma 2 (Theorem 2 in [2]). Let $S \in B(H)$ and $R \in B(K, H)$. Assume that $(S, R): H \oplus K \rightarrow H$ is right invertible.

1. When $R$ is compact, there exists $F \in B(H, K)$ such that $S + RF$ is invertible if and only if $S$ is Fredholm with $\text{ind } S = 0$.
2. When $R$ is not compact, there exists $F \in B(H, K)$ such that $S + RF$ is invertible if and only if $N(S | R)$ contains a non-compact operator.

Our main results are the followings.

Theorem 1. Let $M_X \in B(H \oplus K)$, then $igcap_{X \in B(H,K)} \sigma(M_X) = \emptyset$ if and only if both pairs $(A, C)$ and $(B^*, C^*)$ of operators are controllable.

Proof. Suppose that $igcap_{X \in B(H,K)} \sigma(M_X) = \emptyset$, then for each $\lambda \in \mathbb{C}$, there exists $X_\lambda \in B(H,K)$ such that

$$
M_{X_\lambda} = 
\begin{pmatrix}
A - \lambda & C \\
X_\lambda & B - \lambda
\end{pmatrix}
$$

is invertible, this implies that $R(A - \lambda) + R(C) = H$ and $R(B^* - \tilde{\lambda}) + R(C^*) = K$. Thus, the pairs of operators $(A, C)$ and $(B^*, C^*)$ are controllable.

Conversely, assume that $(A, B)$ and $(B^*, C^*)$ are controllable. Then for each $\lambda \in \mathbb{C}$, $(A - \lambda, C)$ and $(B^* - \tilde{\lambda}, C^*)$ are also controllable. Therefore there exist operators $F_\lambda \in B(H, K)$ and $G_\lambda \in B(H, K)$ such that $A - \lambda + CF_\lambda$ and $B - \lambda + G_\lambda C$ are invertible. Now we construct $X_\lambda$ such that

$$
M_{X_\lambda} = 
\begin{pmatrix}
A - \lambda & C \\
X_\lambda & B - \lambda
\end{pmatrix}
$$

is invertible. Let $X_\lambda = -G_\lambda(A - \lambda + CF_\lambda) - (B - \lambda)F_\lambda$, then $G_\lambda = -(X_\lambda + (B - \lambda)F_\lambda)(A - \lambda + CF_\lambda)^{-1}$. Note that

$$
\begin{pmatrix}
A - \lambda & C \\
X_\lambda & B - \lambda
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
F_\lambda & I
\end{pmatrix}
= 
\begin{pmatrix}
A - \lambda + CF_\lambda & C \\
X_\lambda + (B - \lambda)F_\lambda & B - \lambda
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
A - \lambda + CF_\lambda & C \\
X_\lambda + (B - \lambda)F_\lambda & B - \lambda
\end{pmatrix}
\begin{pmatrix}
I & -(A - \lambda + CF_\lambda)^{-1}C \\
0 & I
\end{pmatrix}
= 
\begin{pmatrix}
A - \lambda + CF_\lambda & 0 \\
X_\lambda + (B - \lambda)F_\lambda & B - \lambda - (X_\lambda + (B - \lambda)F_\lambda)(A - \lambda + CF_\lambda)^{-1}C
\end{pmatrix}
= 
\begin{pmatrix}
A - \lambda + CF_\lambda & 0 \\
X_\lambda + (B - \lambda)F_\lambda & B - \lambda + G_\lambda C
\end{pmatrix}.
$$
Consequently,
\[
M_{X_\lambda} = \begin{pmatrix}
A - \lambda & C \\
X_\lambda & B - \lambda
\end{pmatrix}
\]
is invertible. This shows that \( \lambda \not\in \bigcap_{X \in B(H, K)} \sigma(M_X) \). Since \( \lambda \) is arbitrary, \( \bigcap_{X \in B(H, K)} \sigma(M_X) = \emptyset \). The proof is completed. \( \square \)

**Theorem 2.** Let \((A, B, C)\) be an admissible triple of operators.

1. If \( R(C) \) is infinite dimensional, then
   \[
   \bigcap_{X \in B(H, K)} \sigma(M_X) = \sum (A, B, C).
   \]
2. If \( R(C) \) is finite dimensional, then
   \[
   \bigcap_{X \in B(H, K)} \sigma(M_X) = \sum (A, B, C) \bigcup \Lambda,
   \]
   where \( \Lambda = \{ \lambda : R((A - \lambda, C)) = H, R((B^* - \bar{\lambda}, C^*)) = K, \text{ and } \text{ind}(A - \lambda) + \text{ind}(B - \lambda) \neq 0 \} \).

**Proof.** (1) The inclusion \( \sum (A, B, C) \subseteq \bigcap_{X \in B(H, K)} \sigma(M_X) \) is clear. For each \( \lambda \not\in \sum (A, B, C) \), by Lemma 1, to prove that there exists \( X_\lambda \in B(H, K) \) such that \( M_{X_\lambda} \) is invertible. It suffices to prove that both \( N((A - \lambda) | C) \) and \( N((B^* - \bar{\lambda}) | C^*) \) contain non-compact operators. Since that \((A, B, C)\) is an admissible triple of operators and \( R(C) \) is infinite dimensional, it is easy to know that \( C \) is not compact. Moreover, by (2) of Lemma 2, it suffices to prove that there exist \( F \in B(H, K) \) and \( G \in B(K, H) \) such that \( A - \lambda + CF \) and \( B^* - \bar{\lambda} + C^*G \) are invertible. At first, we prove that there exists an operator \( F \in B(H, K) \) such that \( A - \lambda + CF \) is invertible. Without loss of generality, we assume that \( \lambda = 0 \). Suppose that \((A, C)\) is \( p_1 \)-admissible and \((B^*, C^*)\) is \( p_2 \)-admissible. Denote \( H_2 = \sum_{i=1}^{p_1} R(A^{i-1}C) \) and \( K_1 = \sum_{j=1}^{p_2} R(B^{*p_2-j}C^*) \), then it is easy to know that \( H_2 \) and \( K_1 \) are invariant subspaces under \( A \) and \( B^* \), respectively. Moreover, let \( H_1 = H \oplus H_2 \) and \( K_2 = K \oplus K_1 \), then \( K = K_1 \oplus K_2 \) and \( H = H_1 \oplus H_2 \). Thus, \( A \) and \( C \) have the following operator matrix forms
   \[
   A = \begin{pmatrix}
   A_{11} & 0 \\
   A_{21} & A_{22}
   \end{pmatrix}, \quad
   C = \begin{pmatrix}
   0 & 0 \\
   C_0 & 0
   \end{pmatrix}.
   \]
   For each \( F \in B(H_1 \oplus H_2, K_1 \oplus K_2) \), \( F \) has the operator matrix form
   \[
   F = \begin{pmatrix}
   F_{11} & F_{12} \\
   F_{21} & F_{22}
   \end{pmatrix},
   \]
   Then,
   \[
   A + CF = \begin{pmatrix}
   A_{11} & 0 \\
   A_{21} + C_0 F_{11} & A_{22} + C_0 F_{12}
   \end{pmatrix}.
   \]
We will construct an operator \( F \) such that \( A + CF \) is invertible. For convenience, we divide it into two cases.

**Case 1.** Suppose that \( N(A_{11}) = \{0\} \). Since \((A, C)\) is right invertible, \( A_{11} \) is invertible. On the other hand, since the \((A, B, C)\) is an admissible triple of operators, it is easy to know that \((A_{22}, C_0)\) is controllable. Therefore there exists an operator \( F_{12} \in \mathcal{B}(K_1, H_2) \) such that \( A_{22} + C_0F_{12} \) is an invertible operator on \( H_2 \). Thus, for such an \( F_{12} \) and arbitrary operators \( F_{11}, F_{12}, \) and \( F_{22} \), we obtain that \( A + CF \) is invertible.

**Case 2.** Assume that \( \dim N(A_{11}) \neq 0 \). Let

\[
C_0^*C_0 = \int_0^\infty t \, dE_t
\]

be the spectral decomposition of the positive operator \( C_0^*C_0 \). Because \( C \) is not compact, there exists sufficiently small \( \delta > 0 \) such that the subspace \( K_{11} = \int_0^\infty dE_t K_1 \) is infinite dimensional and \((A_{22}, C_0(\delta))\) is controllable, where \( C_0(\delta) = C_0 E([\delta, \infty)) \) and \( E([\delta, \infty)) = \int_\delta^\infty dE_t \) (see the theorem of [4]). Let \( K_{12} = K_1 \oplus K_{11}, H_{21} = R(C_0(\delta)) \) and \( H_{22} = H_2 \ominus H_{21} \), then \( C_0(\delta) \), as an operator from \( K_{11} = K_{11} \oplus K_{12} \) to \( H_2 = H_{21} \oplus H_{22} \), has the following operator matrix form

\[
C_0(\delta) = \begin{pmatrix}
C_{11}^{11}(\delta) & 0 \\
0 & 0
\end{pmatrix},
\]

where the operator \( C_{11}^{11}(\delta) \) is an invertible operator from \( K_{11} \) onto \( H_{21} \) and \( \dim H_{21} \) is infinite. In this case, the operator \( A_{22} \) has the following operator matrix form

\[
\begin{pmatrix}
A_{22}^{11} & A_{22}^{12} \\
A_{22}^{21} & A_{22}^{22}
\end{pmatrix}
\]

with respect to the decomposition \( H_2 = H_{21} \oplus H_{22} \). It is easy to known that the pair of operators \((A_{22}^{11}, A_{22}^{22})\) is controllable, so we can choose a suitable operator \( D \in \mathcal{B}(H_{22}, H_{21}) \) such that \( A_{22}^{22} + A_{22}^{22}D \) is invertible. Let \( N \in \mathcal{B}(H_{21}) \) is an isometry on \( H_{21} \) with codim \( R(N) = \dim N(A_{11}) \). Now, we define an operator \( C_0(\delta)^+ \) from \( H_2 = H_{21} \oplus H_{22} \) to \( K_1 = K_{11} \oplus K_{12} \) by

\[
C_0(\delta)^+ = \begin{pmatrix}
(C_0^{11}(\delta))^{-1} & 0 \\
0 & 0
\end{pmatrix},
\]

Put

\[
F_{12} = C_0(\delta)^+ \begin{pmatrix}
DA_{22}^{11} + N - A_{22}^{12} & DA_{22}^{22} - ND - A_{22}^{12} \\
0 & 0
\end{pmatrix}.
\]

Then

\[
A_{22} + C_0F_{12} = A_{22} + C_0(\delta)C_0(\delta)^+ \begin{pmatrix}
DA_{22}^{11} + N - A_{22}^{12} & DA_{22}^{22} - ND - A_{22}^{12} \\
0 & 0
\end{pmatrix}.
\]
\[
= \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
+ \begin{pmatrix}
DA_{21} + N - A_{11} & DA_{22} - ND - A_{12} \\
0 & 0
\end{pmatrix}
\]
\[
= \begin{pmatrix}
DA_{21} + N & DA_{22} - ND \\
A_{21} & A_{22}
\end{pmatrix}
\]
\[
= \begin{pmatrix}
I & D \\
0 & I
\end{pmatrix}
\begin{pmatrix}
N & 0 \\
A_{21} & A_{22} + A_{21}D
\end{pmatrix}
\begin{pmatrix}
I & D \\
0 & I
\end{pmatrix}^{-1}.
\]

On the other hand, since \(C_{0}^{11}(\delta)\) is invertible, we can choose \(F_{11}^{11} \in B(H_1, K_{11})\) such that \(F_{11}^{11}\) is bounded below and \(R(C_{0}^{11}(\delta)F_{11}^{11}) = H_1 \oplus R(N)\). Then \(F_{11} = (F_{11}^{11})\) is an operator from \(H_1\) to \(K_1 = K_{11} \oplus K_{12}\). Note that
\[
\begin{pmatrix}
I & D \\
0 & I
\end{pmatrix}^{-1}(A_{21} + C_0 F_{11}) = \begin{pmatrix}
I & -D \\
0 & I
\end{pmatrix} \begin{pmatrix}
A_{21} + \left(C_{0}^{11}(\delta) 0 \right) \\
0 & 0 \end{pmatrix}
\begin{pmatrix}
F_{11}^{11} \\
0
\end{pmatrix}
\]
\[
= \begin{pmatrix}
I & -D \\
0 & I
\end{pmatrix} A_{21} + \left(C_{0}^{11}(\delta)F_{11}^{11} \right) 0 \end{pmatrix}.
\]

Consequently, for such \(F_{11}, F_{12}\) and any \(F_{21}, F_{22}\), it is clear that \(A + CF\) is invertible. Similarly, we can show that there exists an operator \(G \in B(K, H)\) such that \(B^* + CG\) is invertible.

(2) Since that the range \(R(C)\) of \(C\) is finite dimensional, the inclusion
\[
\bigcap_{X \in B(H, K)} \sigma(M_X) \supseteq \sum (A, B, C) \bigcup \Delta
\]
is clear. Let \(\lambda \notin \sum (A, B, C) \bigcup \Delta\). Without loss of generality, assume \(\lambda = 0\). Now we will construct an operator \(X \in B(H, K)\) such that \(M_X\) is invertible. We may assume that \(M_X\) has the following operator matrix form
\[
M_X = \begin{pmatrix}
A_{11} & 0 & 0 & 0 \\
A_{21} & A_{22} & C_0 & 0 \\
X_{11} & X_{12} & B_{11} & 0 \\
X_{21} & X_{22} & B_{21} & B_{22}
\end{pmatrix},
\]
with respect to the decomposition \(H \oplus K = H_1 \oplus H_2 \oplus K_1 \oplus K_2\). Since \((A, B, C)\) is an admissible triple of operators, it is easy to verify that the triple \((A_{22}, B_{11}, C_0)\) is controllable. By Theorem 1, there exists an operator \(X_{12}\) such that
\[
\tilde{M}_X = \begin{pmatrix}
A_{22} & C_0 \\
X_{12} & B_{11}
\end{pmatrix}
\]
is invertible. Since \(\dim R(C)\) is finite, we have \(\text{ind} A + \text{ind} B = \text{ind} A_{11} + \text{ind} B_{22}\). Moreover, since \(0 \notin \sum (A, B, C) \bigcup \Delta\), \(\dim N(A_{11}) = \dim N(B_{22}^*)\). Take \(X_{21}\) as an isometry from \(N(A_{11})\) into \(N(B_{22}^*)\). Put \(X = \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix}\), then
\[ \begin{pmatrix}
A_{11} & 0 & 0 & 0 \\
A_{21} & A_{22} & C_0 & 0 \\
0 & X_{12} & B_{11} & 0 \\
X_{21} & 0 & B_{21} & B_{22}
\end{pmatrix}. \]  
(6)

Since \( R(B_{22}) \oplus R(X_{21}) = K_2 \), \( R(M_X) = R(A_{11}) \oplus R(\tilde{M}_X) \oplus R(B_{22}) \oplus R(X_{21}) = H \oplus K \), \( N(A_{11}) \cap N(X_{21}) = \{0\} \), \( N(\tilde{M}_X) = \{0\} \) and \( N(B_{22}) = \{0\} \). Clearly, \( N(M_X) = \{0\} \). So \( M_X \) is invertible. This shows \( 0 \not\in \bigcap_{X \in B(H,K)} \sigma(M_X) \).

The proof is completed. \( \square \)

**Remark.** For a particular case \( B = I \), Theorems 1 and 2 can be obtained from [3] directly.

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**References**


