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The intersection of the spectra of operator completions ☆

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Abstract

Let $A \in B(H)$, $B \in B(K)$, $C \in B(K, H)$, $X \in B(H, K)$ and $M_X = \begin{pmatrix} A & C \\ X & B \end{pmatrix}$ be an operator completion of the partial operator matrix $Q = \begin{pmatrix} A & C \\ ? & B \end{pmatrix}$. In this note, we consider the intersection of the spectra of M_X when X runs over $B(H, K)$. Denote by $\sum(A, B, C)$ the set of scalar $\lambda \in \mathbb{C}$ such that either $(A - \lambda, C)$ or $(B^* - \bar{\lambda}, C^*)$ is not right invertible. We prove that

$$\bigcap_{X \in B(H,K)} \sigma(M_X) = \begin{cases} \sum(A, B, C) & \text{if } \dim R(C) = \infty, \\ \sum(A, B, C) \cup \Delta(A, B, C) & \text{if } \dim R(C) < \infty, \end{cases}$$

where $\Delta(A, B, C)$ is the set of scalars $\lambda \in \mathbb{C}$ such that $R((A - \lambda, C)) = H$, $R((B^* - \bar{\lambda}, C^*)) = K$, and $\operatorname{ind}(A - \lambda) + \operatorname{ind}(B - \lambda) \neq 0$. We also prove that the intersection is empty if and only if (A, C) and (B^*, C^*) are controllable.

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Let H and K be separable Hilbert spaces. Let $B(H, K)$ denote the space of all bounded linear operators from H to K and abbreviate $B(H, H)$ to $B(H)$. For an operator A, $\sigma(A)$, $R(A)$, and $N(A)$ denote the spectrum, the range, and the

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null-space of A, respectively. When $A \in B(H)$, $B \in B(K)$, $X \in B(H, K)$ and $C \in B(K, H)$ are given, put

$$M_X = \begin{pmatrix} A & C \\ X & B \end{pmatrix}, \quad Q = \begin{pmatrix} A & C \\ ? & B \end{pmatrix}.$$

The operator M_X on $H \oplus K$ can be viewed as an operator completion of the partial operator matrix Q. The some properties of the spectrum of M_X were discussed in [1]. Takahashi discussed in [2] the invertible completion of Q. The relationship between operator completion problem and spectrum assignment can be found in [3,4]. In this paper, we discuss the intersection of the spectra of M_X when X runs over $B(H, K)$. To do this, we need some notations and definitions.

For given $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$, let

$$\sum(A, B, C) := \{\lambda \in \mathbf{C} : (A - \lambda, C) \text{ or } (B^* - \bar{\lambda}, C^*) \text{ is not right invertible}\}.$$

Clearly, for any $X \in B(H, K)$ we have $\sum(A, B, C) \subset \sigma(M_X)$. Thus

$$\sum(A, B, C) \subset \bigcap_{X \in B(H,K)} \sigma(M_X).$$

Definition 1. Let $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$. A pair of operators (A, C) is called controllable if there exists a positive integer p with $\sum_{i=1}^{p} R(A^{i-1}C) = H$; a triple of operators (A, B, C) is called controllable if (A, C) and (B^*, C^*) are controllable.

Definition 2. Let $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$. For a pair of operators $A \in B(H)$ and $C \in B(K, H)$ denote $R_p(A, C) = R(C, AC, \ldots, A^{p-1}C)$. The pair of operators (A, C) is called admissible if for some positive integer p, $R_p(A, C) = R_{p+1}(A, C)$ and the linear set $R_p(A, C)$ is closed. If p is the minimal positive integer with these properties, the pair (A, C) is p-admissible; the triple of operators (A, B, C) is called admissible if (A, C) and (B^*, C^*) are admissible.

For two operators $S \in B(H)$ and $R \in B(K, H)$, let

$$N(S \mid R) = \{G \in B(K, H) : R(SG) \subset R(R)\}. \tag{1}$$

As well known (see [2]), an operator G belongs to $N(S \mid R)$ if and only if there exists an $D \in B(K)$ such that $SG = -RD$.

The following results are in [2] which we state as lemmas.

Lemma 1 (Theorem 1 in [2]). *Let $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$. Assume that the operator $(A, C) : H \oplus K \to H$ and $(B^*, C^*) : K \oplus H \to K$ are right invertible. Then the following conditions are equivalent*:

(1) *There exists $X \in B(H, K)$ such that M_X is invertible.*
(2) *There exists $X \in B(H, K)$ such that M_X is Fredholm with* $\operatorname{ind} M_X = 0$.

(3) *The operator* $M_0 (= M(A, B, C, 0))$ *is Fredholm with* $\operatorname{ind} M_0 = 0$ *or both* $N(A \mid C)$ *and* $N(B^* \mid C^*)$ *contain non-compact operators.*

Lemma 2 (Theorem 2 in [2]). *Let* $S \in B(H)$ *and* $R \in B(K, H)$. *Assume that* $(S, R) : H \oplus K \to H$ *is right invertible.*

(1) *When R is compact, there exists* $F \in B(H, K)$ *such that* $S + RF$ *is invertible if and only if S is Fredholm with* $\operatorname{ind} S = 0$.
(2) *When R is not compact, there exists* $F \in B(H, K)$ *such that* $S + RF$ *is invertible if and only if* $N(S \mid R)$ *contains a non-compact operator.*

Our main results are the followings.

Theorem 1. *Let* $M_X \in B(H \oplus K)$, *then* $\bigcap_{X \in B(H,K)} \sigma(M_X) = \emptyset$ *if and only if both pairs* (A, C) *and* (B^*, C^*) *of operators are controllable.*

Proof. Suppose that $\bigcap_{X \in B(H,K)} \sigma(M_X) = \emptyset$, then for each $\lambda \in \mathbb{C}$, there exists $X_\lambda \in B(H, K)$ such that

$$M_{X_\lambda} = \begin{pmatrix} A - \lambda & C \\ X_\lambda & B - \lambda \end{pmatrix}$$

is invertible, this implies that $R(A - \lambda) + R(C) = H$ and $R(B^* - \bar{\lambda}) + R(C^*) = K$. Thus, the pairs of operators (A, C) and (B^*, C^*) are controllable.

Conversely, assume that (A, B) and (B^*, C^*) are controllable. Then for each $\lambda \in \mathbb{C}$, $(A - \lambda, C)$ and $(B^* - \bar{\lambda}, C^*)$ are also controllable. Therefore there exist operators $F_\lambda \in B(H, K)$ and $G_\lambda \in B(H, K)$ such that $A - \lambda + CF_\lambda$ and $B - \lambda + G_\lambda C$ are invertible. Now we construct X_λ such that

$$M_{X_\lambda} = \begin{pmatrix} A - \lambda & C \\ X_\lambda & B - \lambda \end{pmatrix}$$

is invertible. Let $X_\lambda = -G_\lambda(A - \lambda + CF_\lambda) - (B - \lambda)F_\lambda$, then $G_\lambda = -(X_\lambda + (B - \lambda)F_\lambda)(A - \lambda + CF_\lambda)^{-1}$. Note that

$$\begin{pmatrix} A - \lambda & C \\ X_\lambda & B - \lambda \end{pmatrix} \begin{pmatrix} I & 0 \\ F_\lambda & I \end{pmatrix} = \begin{pmatrix} A - \lambda + CF_\lambda & C \\ X_\lambda + (B - \lambda)F_\lambda & B - \lambda \end{pmatrix}$$

and

$$\begin{aligned} &\begin{pmatrix} A - \lambda + CF_\lambda & C \\ X_\lambda + (B - \lambda)F_\lambda & B - \lambda \end{pmatrix} \begin{pmatrix} I & -(A - \lambda + CF_\lambda)^{-1}C \\ 0 & I \end{pmatrix} \\ &\quad = \begin{pmatrix} A - \lambda + CF_\lambda & 0 \\ X_\lambda + (B - \lambda)F_\lambda & B - \lambda - (X_\lambda + (B - \lambda)F_\lambda)(A - \lambda + CF_\lambda)^{-1}C \end{pmatrix} \\ &\quad = \begin{pmatrix} A - \lambda + CF_\lambda & 0 \\ X_\lambda + (B - \lambda)F_\lambda & B - \lambda + G_\lambda C \end{pmatrix}. \end{aligned}$$

Consequently,

$$M_{X_\lambda} = \begin{pmatrix} A-\lambda & C \\ X_\lambda & B-\lambda \end{pmatrix}$$

is invertible. This shows that $\lambda \notin \bigcap_{X\in B(H,K)} \sigma(M_X)$. Since λ is arbitrary, $\bigcap_{X\in B(H,K)} \sigma(M_X) = \emptyset$. The proof is completed. □

Theorem 2. *Let (A, B, C) be an admissible triple of operators.*

(1) *If $R(C)$ is infinite dimensional, then*

$$\bigcap_{X\in B(H,K)} \sigma(M_X) = \sum(A, B, C). \tag{2}$$

(2) *If $R(C)$ is finite dimensional, then*

$$\bigcap_{X\in B(H,K)} \sigma(M_X) = \sum(A, B, C) \bigcup \Delta, \tag{3}$$

where $\Delta = \{\lambda : R((A-\lambda, C)) = H,\ \ R((B^* - \bar{\lambda}, C^*)) = K,\ \ \text{and}\ \ \text{ind}(A-\lambda) + \text{ind}(B-\lambda) \neq 0\}$.

Proof. (1) The inclusion $\sum(A, B, C) \subset \bigcap_{X\in B(H,K)} \sigma(M_X)$ is clear. For each $\lambda \notin \sum(A, B, C)$, by Lemma 1, to prove that there exists $X_\lambda \in B(H, K)$ such that M_{X_λ} is invertible. It suffices to prove that both $N((A-\lambda) \mid C)$ and $N((B^* - \bar{\lambda}) \mid C^*)$ contain non-compact operators. Since that (A, B, C) is an admissible triple of operators and $R(C)$ is infinite dimensional, it is easy to know that C is not compact. Moreover, by (2) of Lemma 2, it suffices to prove that there exist $F \in B(H, K)$ and $G \in B(K, H)$ such that $A - \lambda + CF$ and $B^* - \bar{\lambda} + C^*G$ are invertible. At first, we prove that there exists an operator $F \in B(H, K)$ such that $A - \lambda + CF$ is invertible. Without loss of generality, we assume that $\lambda = 0$. Suppose that (A, C) is p_1-admissible and (B^*, C^*) is p_2-admissible. Denote $H_2 = \sum_{i=1}^{p_1} R(A^{i-1}C)$ and $K_1 = \sum_{j=1}^{p_2} R(B^{*^{p_2-1}} C^*)$, then it is easy to know that H_2 and K_1 are invariant subspaces under A and B^*, respectively. Moreover, let $H_1 = H \ominus H_2$ and $K_2 = K \ominus K_1$, then $K = K_1 \oplus K_2$ and $H = H_1 \oplus H_2$. Thus, A and C have the following operator matrix forms

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ C_0 & 0 \end{pmatrix}.$$

For each $F \in B(H_1 \oplus H_2, K_1 \oplus K_2)$, F has the operator matrix form

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}.$$

Then,

$$A + CF = \begin{pmatrix} A_{11} & 0 \\ A_{21} + C_0F_{11} & A_{22} + C_0F_{12} \end{pmatrix}.$$

We will construct an operator F such that $A + CF$ is invertible. For convenience, we divide it into two cases.

Case 1. Suppose that $N(A_{11}) = \{0\}$. Since (A, C) is right invertible, A_{11} is invertible. On the other hand, since the (A, B, C) is an admissible triple of operators, it is easy to know that (A_{22}, C_0) is controllable. Therefore there exists an operator $F_{12} \in B(K_1, H_2)$ such that $A_{22} + C_0F_{12}$ is an invertible operator on H_2. Thus, for such an F_{12} and arbitrary operators F_{11}, F_{12}, and F_{22}, we obtain that $A + CF$ is invertible.

Case 2. Assume that $\dim N(A_{11}) \neq 0$. Let

$$C_0^*C_0 = \int_0^\infty t\, \mathrm{d}E_t$$

be the spectral decomposition of the positive operator $C_0^*C_0$. Because C is not compact, there exists sufficiently small $\delta > 0$ such that the subspace $K_{11} = \int_\delta^\infty \mathrm{d}E_t K_1$ is infinite dimensional and $(A_{22}, C_0(\delta))$ is controllable, where $C_0(\delta) = C_0E([\delta, \infty))$ and $E([\delta, \infty)) = \int_\delta^\infty \mathrm{d}E_t$ (see the theorem of [4]). Let $K_{12} = K_1 \ominus K_{11}$, $H_{21} = R(C_0(\delta))$ and $H_{22} = H_2 \ominus H_{21}$, then $C_0(\delta)$, as an operator from $K_1 = K_{11} \oplus K_{12}$ to $H_2 = H_{21} \oplus H_{22}$, has the following operator matrix form

$$C_0(\delta) = \begin{pmatrix} C_0^{11}(\delta) & 0 \\ 0 & 0 \end{pmatrix},$$

where the operator $C_0^{11}(\delta)$ is an invertible operator from K_{11} onto H_{21} and $\dim H_{21}$ is infinite. In this case, the operator A_{22} has the following operator matrix form

$$\begin{pmatrix} A_{22}^{11} & A_{22}^{12} \\ A_{22}^{21} & A_{22}^{22} \end{pmatrix}$$

with respect to the decomposition $H_2 = H_{21} \oplus H_{22}$. It is easy to known that the pair of operators $(A_{22}^{22}, A_{22}^{21})$ is controllable, so we can choose a suitable operator $D \in B(H_{22}, H_{21})$ such that $A_{22}^{22} + A_{22}^{21}D$ is invertible. Let $N \in B(H_{21})$ is an isometry on H_{21} with $\operatorname{codim} R(N) = \dim N(A_{11})$. Now, we define an operator $C_0(\delta)^+$ from $H_2 = H_{21} \oplus H_{22}$ to $K_1 = K_{11} \oplus K_{12}$ by

$$C_0(\delta)^+ = \begin{pmatrix} (C_0^{11}(\delta))^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Put

$$F_{12} = C_0(\delta)^+ \begin{pmatrix} DA_{22}^{21} + N - A_{22}^{12} & DA_{22}^{22} - ND - A_{22}^{12} \\ 0 & 0 \end{pmatrix}.$$

Then

$$A_{22} + C_0F_{12} = A_{22} + C_0(\delta)C_0(\delta)^+ \begin{pmatrix} DA_{22}^{21} + N - A_{22}^{11} & DA_{22}^{22} - ND - A_{22}^{12} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} A_{22}^{11} & A_{22}^{12} \\ A_{22}^{21} & A_{22}^{22} \end{pmatrix} + \begin{pmatrix} DA_{22}^{21} + N - A_{22}^{11} & DA_{22}^{22} - ND - A_{22}^{12} \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} DA_{22}^{21} + N & DA_{22}^{22} - ND \\ A_{22}^{21} & A_{22}^{22} \end{pmatrix}$$
$$= \begin{pmatrix} I & D \\ 0 & I \end{pmatrix} \begin{pmatrix} N & 0 \\ A_{22}^{21} & A_{22}^{22} + A_{22}^{21}D \end{pmatrix} \begin{pmatrix} I & D \\ 0 & I \end{pmatrix}^{-1}.$$

On the other hand, since $C_0^{11}(\delta)$ is invertible, we can choose $F_{11}^{11} \in B(H_1, K_{11})$ such that F_{11}^{11} is bounded below and $R(C_0^{11}(\delta)F_{11}^{11}) = H_1 \ominus R(N)$. Then $F_{11} = \binom{F_{11}^{11}}{0}$ is an operator from H_1 to $K_1 = K_{11} \oplus K_{12}$. Note that

$$\begin{pmatrix} I & D \\ 0 & I \end{pmatrix}^{-1} (A_{21} + C_0F_{11}) = \begin{pmatrix} I & -D \\ 0 & I \end{pmatrix} \left(A_{21} + \begin{pmatrix} C_0^{11}(\delta) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} F_{11}^{11} \\ 0 \end{pmatrix} \right)$$
$$= \begin{pmatrix} I & -D \\ 0 & I \end{pmatrix} A_{21} + \begin{pmatrix} C_0^{11}(\delta)F_{11}^{11} \\ 0 \end{pmatrix}.$$

Consequently, for such F_{11}, F_{12} and any F_{21}, F_{22}, it is clear that $A + CF$ is invertible. Similarly, we can show that there exists an operator $G \in B(K, H)$ such that $B^* + C^*G$ is invertible.

(2) Since that the range $R(C)$ of C is finite dimensional, the inclusion

$$\bigcap_{X \in B(H,K)} \sigma(M_X) \supseteq \sum(A, B, C) \bigcup \Delta$$

is clear. Let $\lambda \notin \sum(A, B, C) \bigcup \Delta$. Without loss of generality, assume $\lambda = 0$. Now we will construct an operator $X \in B(H, K)$ such that M_X is invertible. We may assume that M_X has the following operator matrix form

$$M_X = \begin{pmatrix} A_{11} & 0 & 0 & 0 \\ A_{21} & A_{22} & C_0 & 0 \\ X_{11} & X_{12} & B_{11} & 0 \\ X_{21} & X_{22} & B_{21} & B_{22} \end{pmatrix}, \tag{4}$$

with respect to the decomposition $H \oplus K = H_1 \oplus H_2 \oplus K_1 \oplus K_2$. Since (A, B, C) is an admissible triple of operators, it is easy to verify that the triple (A_{22}, B_{11}, C_0) is controllable. By Theorem 1, there exists an operator X_{12} such that

$$\widetilde{M}_X = \begin{pmatrix} A_{22} & C_0 \\ X_{12} & B_{11} \end{pmatrix} \tag{5}$$

is invertible. Since $\dim R(C)$ is finite, we have $\operatorname{ind} A + \operatorname{ind} B = \operatorname{ind} A_{11} + \operatorname{ind} B_{22}$. Moreover, since $0 \notin \sum(A, B, C) \bigcup \Delta$, $\dim N(A_{11}) = \dim N(B_{22}^*)$. Take X_{21} as an isometry from $N(A_{11})$ into $N(B_{22}^*)$. Put $X = \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix}$, then

$$M_X = \begin{pmatrix} A_{11} & 0 & 0 & 0 \\ A_{21} & A_{22} & C_0 & 0 \\ 0 & X_{12} & B_{11} & 0 \\ X_{21} & 0 & B_{21} & B_{22} \end{pmatrix}. \tag{6}$$

Since $R(B_{22}) \oplus R(X_{21}) = K_2$, $R(M_X) = R(A_{11}) \oplus R(\widetilde{M}_X) \oplus R(B_{22}) \oplus R(X_{21}) = H \oplus K$, $N(A_{11}) \cap N(X_{21}) = \{0\}$, $N(\tilde{M}_X) = \{0\}$ and $N(B_{22}) = \{0\}$. Clearly, $N(M_X) = \{0\}$. So M_X is invertible. This shows $0 \notin \bigcap_{X \in B(H,K)} \sigma(M_X)$.

The proof is completed. □

Remark. For a particular case $B = I$, Theorems 1 and 2 can be obtained from [3] directly.

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References

[1] H. Du, C. Gu, On the spectra of operator completion problems, in: I. Gohberg (Ed.), New aspects in Interpolation and Completion Theories, Birkhäuser, Basel, 1993, pp. 103–117.
[2] K. Takahashi, Invertible completions of operator matricis, Integr. Equat. Oper. Theory 21 (1995) 355–361.
[3] L. Gurvits, L. Rodman, I. Spitkovsky, Spectral assignment for Hilbert space operators, Houston J. Math. 17 (1991) 501–523.
[4] K. Takahashi, Exact controllability and spectrum assignment, J. Math. Anal. Appl. 104 (1984) 537–545.