ACADEMIC PRESS
J. Math. Anal. Appl. 272 (2002) 596–603
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.academicpress.com

B-Weyl spectrum and poles of the resolvent

M. Berkani

Département de Mathématiques, Faculté des Sciences, Université Mohammed I, Oujda, Morocco

Received 25 April 2001
Submitted by T.M. Rassias

Dedicated to Ilham, Soufyane and Imad

Abstract

Let T be a bounded linear operator acting on a Banach space and let $\sigma_{\mathrm{BW}}(T) = \{\lambda \in \mathbf{C}$ such that $T - \lambda I$ is not a B-Fredholm operator of index 0$\}$ be the B-Weyl spectrum of T. Define also $E(T)$ to be the set of all isolated eigenvalues in the spectrum $\sigma(T)$ of T, and $\Pi(T)$ to be the set of the poles of the resolvent of T. In this paper two new generalized versions of the classical Weyl's theorem are considered. More precisely, we seek for conditions under which an operator T satisfies the generalized Weyl's theorem: $\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus E(T)$, or the version II of the generalized Weyl's theorem: $\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus \Pi(T)$.

1. Introduction

This paper is a continuation of our previous works [2–4]. We consider a Banach space X and $L(X)$ the Banach algebra of bounded linear operators acting on X. For $T \in L(X)$ we will denote by $N(T)$ the null space of T, by $\alpha(T)$ the nullity of T, by $R(T)$ the range of T and by $\beta(T)$ its defect. If both of $\alpha(T)$ and $\beta(T)$ are finite then T is called a Fredholm operator and the index of T is defined by $\mathrm{ind}(T) = \alpha(T) - \beta(T)$. In this case it is well known that the range $R(T)$ of T is closed in X.

Now for a bounded linear operator T and for each integer n, define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in

E-mail address: berkani@sciences.univ-oujda.ac.ma.

0022-247X/02/$ – see front matter
PII: S0022-247X(02)00179-8

particular, $T_0 = T$). If for some integer n the range space $R(T^n)$ is closed and T_n is a Fredholm operator, then T is called a B-Fredholm operator. In this case and from [4, Proposition 2.1] T_m is a Fredholm operator and $\operatorname{ind}(T_m) = \operatorname{ind}(T_n)$ for each $m \geqslant n$. This enable us to define the index of a B-Fredholm operator T as the index of the Fredholm operator T_n, where n is any integer such that $R(T^n)$ is closed and such that T_n is a Fredholm operator. Let $\mathrm{BF}(X)$ be the class of all B-Fredholm operators. In [4] we studied this class of operators and we proved [4, Theorem 2.7] that an operator $T \in L(X)$ is a B-Fredholm operator if and only if $T = T_0 \oplus T_1$, where T_0 is a Fredholm operator and T_1 is a nilpotent one.

It appears that the concept of Drazin invertibility plays an important role for the class of B-Fredholm operators. Let A be an algebra with a unit e; following [12] we say that an element x of A is Drazin invertible of degree k if there is an element b of A such that

$$x^k b x = x^k, \qquad bxb = b, \qquad xb = bx. \tag{$*$}$$

Recall that the concept of Drazin invertibility was originally considered by Drazin in [8] where elements satisfying relation $(*)$ are called pseudo-invertible elements. The Drazin spectrum is defined by $\sigma_D(a) = \{\lambda \in \mathbf{C}\colon\ a - \lambda e \text{ is not Drazin invertible}\}$ for every $a \in A$. In the case of a bounded linear operator T acting on a Banach space X, it is well known that T is Drazin invertible if and only if it has a finite ascent and descent (Definition 2.1); which is also equivalent to the fact that $T = T_0 \oplus T_1$, where T_0 is an invertible operator and T_1 is a nilpotent one (see [12, Proposition 6] and [11, Corollary 2.2]). In [2] B-Weyl operators and the B-Weyl spectrum are defined as follows:

Definition 1.1. Let $T \in L(X)$. Then T is called a B-Weyl operator if it is a B-Fredholm operator of index 0. The B-Weyl spectrum $\sigma_{\mathrm{BW}}(T)$ of T is defined by $\sigma_{\mathrm{BW}}(T) = \{\lambda \in \mathbf{C}\colon\ T - \lambda I \text{ is not a B-Weyl operator}\}$.

Now let $F_0(X)$ to be the ideal of finite rank operators in the algebra $L(X)$ of bounded linear operators acting on X, and let $E(T)$ to be the set of all isolated eigenvalues in the spectrum $\sigma(T)$ of T. In [2, Theorem 4.3] we showed that for $T \in L(X)$ we have

$$\sigma_{\mathrm{BW}}(T) = \bigcap_{F \in F_0(X)} \sigma_D(T + F),$$

and in the case of a normal operator T acting on a Hilbert space H, we showed in [2, Theorem 4.5] that

$$\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus E(T),$$

which gives a generalization of the classical Weyl theorem. Recall that the classical Weyl theorem [13] asserts that if T is a normal operator acting on a

Hilbert space H, then the Weyl spectrum $\sigma_W(T)$ is exactly the set of all points in $\sigma(T)$ except the isolated eigenvalues of finite multiplicity; that is

$$\sigma_W(T) = \sigma(T) \setminus E_0(T),$$

where $E_0(T)$ is the set of isolated eigenvalues of finite multiplicity and $\sigma_W(T)$ is the Weyl spectrum of T. In other words, $\sigma_W(T) = \{\lambda \in \mathbf{C}$ such that $T - \lambda I$ is not a Fredholm operator of index $0\}$. It is known from [10, Theorem 6.5.2] that

$$\sigma_W(T) = \bigcap_{F \in F_0(X)} \sigma(T + F).$$

In his paper [1], Barnes considered the version II of the Weyl's theorem (called also the Browder's theorem in [7]): For a bounded linear operator T acting on X, what conditions on T implies that T satisfies

$$\sigma_W(T) = \sigma(T) \setminus \Pi_0(T),$$

where $\Pi_0(T)$ is the set of poles of the resolvent of T of finite rank. It is well known [11, Section 2] that an isolated point λ of the spectrum $\sigma(T)$ of T is a pole of the resolvent of T if $T - \lambda I$ is Drazin invertible. A pole of the resolvent of T is of finite rank if the spectral projection associated to the set $\{\lambda\}$ is of finite rank.

The aim of this paper is to study similar questions as in [1], but instead of isolated eigenvalues of finite multiplicity, we consider the set of all isolated eigenvalues, instead of poles of the resolvent of finite rank, we consider the set of all the poles of the resolvent, and instead of the Weyl's spectrum, we use the B-Weyl spectrum. More precisely, let $E(T)$ be the set of all isolated eigenvalues in the spectrum $\sigma(T)$ of T, and $\Pi(T)$ the set of all the poles of the resolvent of T. Then the two following new generalized versions of the classical Weyl's theorem are considered. Under which conditions an operator T satisfies the generalized Weyl's theorem:

$$\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus E(T),$$

or the version II of the generalized Weyl's theorem

$$\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus \Pi(T).$$

As we will see, many of the results valid for Poles of finite rank obtained in [1] are also valid for all poles.

Finally, we also mention the following book [6], which is useful for the reader in this circle of ideas.

2. Results

The following definition is well known:

Definition 2.1. Let $T \in L(X)$, $n \in \mathbf{N}$ and let $c_n(T) = \dim R(T^n)/R(T^{n+1})$ and $c'_n(T) = \dim N(T^{n+1})/N(T^n)$. Then the descent of T is defined by $\delta(T) = \inf\{n\colon\ c_n(T) = 0\} = \inf\{n\colon\ R(T^n) = R(T^{n+1})\}$, and the ascent $a(T)$ of T is defined by $a(T) = \inf\{n\colon\ c'_n(T) = 0\} = \inf\{n\colon\ N(T^n) = N(T^{n+1})\}$, with $\inf\emptyset = \infty$.

Definition 2.2 [9]. Let $T \in L(X)$ and let $d \in \mathbf{N}$. Then T has a uniform descent for $n \geqslant d$ if $R(T) + N(T^n) = R(T) + N(T^d)$ for all $n \geqslant d$. If, in addition, $R(T) + N(T^d)$ is closed then T is said to have a topological uniform descent for $n \geqslant d$.

Remark. As it has already been observed in [4] a B-Fredholm operator is an operator of topological uniform descent.

Theorem 2.3. *Let $T \in L(X)$ and let $\lambda \in \sigma(T)$ be an isolated point of $\sigma(T)$. Then the following properties are equivalent*:

(1) *λ is pole of the resolvent of T.*
(2) *There exists T-invariant subspaces M and N of X such that $X = M \oplus N$, $(T - \lambda I)_{|M}$ is invertible and $(T - \lambda I)_{|N}$ is nilpotent.*
(3) *$T - \lambda I$ is a B-Fredholm operator of index* 0.

Proof. The equivalence of the two first properties is well known as a characterization of poles of the resolvent. Let us show that (2) is equivalent to (3).

If there exists T-invariant subspaces M and N of X such that $X = M \oplus N$, $(T - \lambda I)_{|M}$ is invertible and $(T - \lambda I)_{|N}$ is nilpotent, then from [2, Theorem 4.2] it follows that $T - \lambda I$ is a B-Fredholm operator of index 0.

Conversely, suppose that $T - \lambda I$ is a B-Fredholm operator of index 0. Since λ is isolated in the spectrum of T, then from the Grabiner's punctured neighborhood theorem [5, Theorem 4.5], if $|\lambda - \beta|$ is small enough and for n large enough we have $c_n(T - \beta I) = c_n(T - \lambda I)$, $c'_n(T - \beta I) = c'_n(T - \lambda I)$. Since λ is isolated in the spectrum of T, then if $|\lambda - \beta|$ is small enough and $\lambda \neq \beta$, $T - \beta I$ is invertible. Hence $c_n(T - \beta I) = c'_n(T - \beta I) = 0$. So $T - \lambda I$ is an operator of finite ascent and descent. Therefore λ is a pole of T. □

Theorem 2.4. *Let H be a Hilbert space, let $T \in L(H)$ and let T^* be its adjoint. Then T satisfies the version II of the generalized Weyl's theorem if and only if T^* does.*

Proof. It is easily seen that $\sigma(T^*) = \overline{\sigma(T)}$ and $\Pi(T^*) = \overline{\Pi(T)}$. Moreover, from [2, Remark B] we know that T is a B-Fredholm operator of index 0 if and only if T^* is B-Fredholm operator of index 0. Thus $\sigma_{\mathrm{BW}}(T^*) = \overline{\sigma_{\mathrm{BW}}(T)}$, where $\overline{\sigma_{\mathrm{BW}}(T)}$ denotes the conjugate set of $\sigma_{\mathrm{BW}}(T)$. From those relations it follows that

T satisfies the version II of the generalized Weyl's theorem if and only if T^* does. □

Theorem 2.5. *Let $T \in L(X)$. Then we have the following properties*:

(1) $\sigma_{\mathrm{BW}}(T) \subseteq \sigma(T) \setminus E(T)$ *if and only if* $E(T) = \Pi(T)$.
(2) $\sigma_{\mathrm{BW}}(T) \supseteq \sigma(T) \setminus E(T)$ *if and only if* $\sigma_{\mathrm{BW}}(T) = \sigma_D(T)$.

Proof. (1) Suppose that $\sigma_{\mathrm{BW}}(T) \subseteq \sigma(T) \setminus E(T)$ and let $\lambda \in E(T)$ be an isolated eigenvalue of T. Then $\lambda \notin \sigma_{\mathrm{BW}}(T)$, so $T - \lambda I$ is a B-Fredholm operator of index 0. From the Theorem 2.3 it follows that λ is a pole of the resolvent of T, and so $\lambda \in \Pi(T)$. As we have always $\Pi(T) \subset E(T)$, then $E(T) = \Pi(T)$.

Conversely, if $E(T) = \Pi(T)$ and $\lambda \in E(T)$, then $T - \lambda I$ is a B-Fredholm operator of index 0. Therefore $\lambda \notin \sigma_{\mathrm{BW}}(T)$ and so $\sigma_{\mathrm{BW}}(T) \subseteq \sigma(T) \setminus E(T)$.

(2) Suppose that $\sigma_{\mathrm{BW}}(T) \supseteq \sigma(T) \setminus E(T)$ and let $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{BW}}(T)$. Then $\lambda \in E(T)$, in particular λ is isolated in the spectrum of T. Moreover, $T - \lambda I$ is a B-Fredholm operator of index 0. From the Theorem 2.3 it follows that $T - \lambda I$ is Drazin invertible and $\sigma_D(T) \subset \sigma_{\mathrm{BW}}(T)$. As it is always true that $\sigma_{\mathrm{BW}}(T) \subset \sigma_D(T)$, then $\sigma_{\mathrm{BW}}(T) = \sigma_D(T)$.

Conversely, suppose that $\sigma_{\mathrm{BW}}(T) = \sigma_D(T)$. Let $\lambda \notin \sigma_{\mathrm{BW}}(T)$; then $\lambda \notin \sigma_D(T)$. So $T - \lambda I$ is Drazin invertible and $\lambda \in E(T)$. Hence $\sigma_{\mathrm{BW}}(T) \supseteq \sigma(T) \setminus E(T)$. □

From this theorem we obtain immediately the following corollary:

Corollary 2.6. *Let $T \in L(X)$. Then T satisfies the generalized Weyl's theorem if and only if $\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus \Pi(T)$ and $E(T) = \Pi(T)$.*

Theorem 2.7. *Let $T \in L(X)$. If $F \in F_0(X)$ and $TF = FT$, then $\sigma_D(T) = \sigma_D(T + F)$.*

Proof. Let us show that $\sigma_D(T) \subset \sigma_D(T + F)$. If $\lambda \notin \sigma_D(T + F)$, then $T + F - \lambda I$ is Drazin invertible. Hence from [2, Proposition 3.3] $T - \lambda I = (T - \lambda I + F) - F$ is a B-Fredholm operator. In particular, the two operators $T - \lambda I$ and $T - \lambda I + F$ are operators of topological uniform descent. From [9, Theorem 5.8], and for n large enough, we have $c_n(T - \beta I) = c_n(T - \lambda I + F), c'_n(T - \beta I) = c'_n(T - \lambda I + F)$. Since $T - \lambda I + F$ is Drazin invertible, then for n large enough we have $c_n(T - \lambda I + F) = c'_n(T - \lambda I + F) = 0$. So for n large enough we have $c_n(T - \lambda I) = c'_n(T - \lambda I) = 0$. So $T - \lambda I$ is Drazin invertible and $\lambda \notin \sigma_D(T)$. Therefore $\sigma_D(T) \subset \sigma_D(T + F)$. Since $T = (T + F) - F$ and $(T + F)F = F(T + F)$, then we have also $\sigma_D(T + F) \subset \sigma_D(T)$. □

From this property of the Drazin spectrum, we obtain the following perturbation theorem:

Theorem 2.8. *If T satisfies the version II of the generalized Weyl's theorem and if F is a finite rank operator such that $TF = FT$, then $T + F$ satisfies the version II of the generalized Weyl's theorem.*

Proof. From the characterization of the B-Weyl spectrum [2, Theorem 4.3], it follows that if F is a finite rank operator, then $\sigma_{\mathrm{BW}}(T + F) = \sigma_{\mathrm{BW}}(T)$. Moreover, if F commutes with T, then from the previous theorem we have $\sigma_D(T + F) = \sigma_D(T)$. If T satisfies the version II of the generalized Weyl's theorem, then $\sigma_{\mathrm{BW}}(T) = \sigma_D(T)$. Hence $\sigma_{\mathrm{BW}}(T + F) = \sigma_D(T + F)$, and so $T + F$ satisfies the version II of the generalized Weyl's theorem. □

Theorem 2.9. *Let Γ be a nonempty connected subset of $\mathbf{C}$ such that $T - \lambda I$ is a B-Fredholm operator for all $\alpha \in \Gamma$. If there is $\alpha \in \Gamma$ such that $T - \alpha I$ is Drazin invertible, then every point of $\sigma(T) \cap \Gamma$ is a pole of T and $\sigma(T) \cap \Gamma$ is a countable discrete set.*

Proof. Since $T - \alpha I$ is Drazin invertible, for n large enough we have $c_n(T - \alpha I) = c'_n(T - \alpha I) = 0$. Let $A = \{\mu \in \Gamma \mid T - \mu I \text{ is Drazin invertible}\}$. Then $\alpha \in A$ and $A \neq \emptyset$. If $\lambda \in A$, since $T - \lambda I$ is Drazin invertible, then there is an open neighborhood $B(\lambda, \epsilon)$ such that $B(\lambda, \epsilon) - \{\lambda\} \subset \rho(T)$, where $\rho(T)$ is the resolvent set of T. Therefore $B(\lambda, \epsilon) \cap \Gamma \subset A$, and A is open in Γ. Now let $\lambda \in \overline{A} \cap \Gamma$, where $\overline{A}$ is the closure of A. In particular, $T - \lambda I$ is a B-Fredholm operator. Hence there is an $\epsilon > 0$ such that if $|\lambda - \mu| < \epsilon$ then for n large enough, we have $c_n(T - \lambda) = c_n(T - \mu I)$, $c'_n(T - \lambda I) = c'_n(T - \mu I)$. Since $\lambda \in \overline{A}$, then $B(\lambda, \epsilon) \cap A \neq \emptyset$. So there is $\mu \in B(\lambda, \epsilon) \cap A$. Hence $c_n(T - \lambda I) = c'_n(T - \lambda I) = 0$, and so $\lambda \in A$. Therefore A is closed in Γ. Since Γ is connected, then $A = \Gamma$. Moreover, if $\lambda \in \sigma(T) \cap \Gamma$, then λ is a pole of the resolvent of T. Therefore it is an isolated point of the spectrum $\sigma(T)$. Since $\sigma(T)$ is a compact set, then $\sigma(T) \cap \Gamma$ is a discrete set. □

Theorem 2.10. *Let $T \in L(X)$ and suppose that $\sigma_{\mathrm{BW}}(T)$ is simply connected. Then $T + F$ satisfies the generalized version II of the Weyl's theorem for every $F \in F_0(X)$.*

Proof. Suppose that $\lambda \in \sigma(T)$ and $T - \lambda I$ is a B-Fredholm operator of index 0. Let $\Gamma = \{\alpha \in \mathbf{C} \mid T - \alpha I \text{ is a B-Fredholm operator of index } 0\}$. Then Γ is connected. Since $\Gamma \cap \rho(T)$ is nonempty, from the previous theorem it follows that $\Gamma \cap \sigma(T)$ consists entirely of Poles of the resolvent of T. So $\lambda \in \Pi(T)$ and T satisfies the version II of the generalized Weyl's theorem.

If $F \in F_0(X)$ is a finite rank operator, then $\sigma_{\mathrm{BW}}(T + F) = \sigma_{\mathrm{BW}}(T)$. Thus $\sigma_{\mathrm{BW}}(T + F)$ is simply connected, and so $T + F$ satisfies the version II of the generalized Weyl's theorem. □

As it is well known, a meromorphic operator T is an operator T such that each $\lambda \neq 0$ is a pole of T. In the following theorem we characterize meromorphic operators in terms of B-Fredholm operators.

Theorem 2.11. *Let $T \in L(X)$. Then T is a meromorphic operator if and only if $T - \lambda I$ is a B-Fredholm operator for all $\lambda \neq 0$.*

Proof. If T is a meromorphic operator, then $T - \lambda I$ is Drazin invertible for each $\lambda \neq 0$. In particular, $T - \lambda I$ is a B-Fredholm operator.

Conversely, suppose that for all $\lambda \neq 0$, $T - \lambda I$ is a B-Fredholm operator. Set $\Gamma = \mathbf{C} \setminus \{0\}$. Then Γ is a connected set of B-Fredholm points of T. Since the spectrum of T is bounded, then there is $\lambda \in \Gamma$ such that $T - \lambda I$ is invertible, and so $T - \lambda I$ is Drazin invertible. From Theorem 2.9 it follows that every point of Γ is a pole of T. □

Remark. It is proved in [3] that if T is a bounded linear operator acting on Banach space X, satisfying the generalized Weyl's theorem (respectively, the version II of the generalized Weyl's theorem), then T satisfies also the Weyl's theorem (respectively, the version II of the Weyl's theorem). However, the converse is not true as shown by the following example:

Example. Let X be an infinite-dimensional Banach space. Let $T \in L(X)$ be any nilpotent operator with nonclosed range, and let $Q \in L(X)$ be a quasi-nilpotent operator which is not nilpotent. Consider the operator $S = T \oplus Q$, defined on the Banach space $X \oplus X$. Then $E(S) = \{0\}$, $\Pi(S) = E_0(S) = \Pi_0(S) = \emptyset$, $\sigma(S) = \{0\}$, $\sigma_W(S) = \{0\}$ and $\sigma_{\mathrm{BW}}(S) = \{0\}$. Hence the Weyl's theorem and its version II are satisfied by S, but the generalized Weyl's theorem is not satisfied by S.

References

[1] B.A. Barnes, Riesz points and Weyl's theorem, Integral Equations Operator Theory 34 (1999) 187–196.

[2] M. Berkani, Index of B-Fredholm operators and generalization of a Weyl theorem, Proc. Amer. Math. Soc. 130 (2002) 1717–1723.

[3] M. Berkani, J.J. Koliha, Weyl type theorems for bounded linear operators, Acta Sci. Math. (Szeged), to appear.

[4] M. Berkani, On a class of quasi-Fredholm operators, Integral Equations Operator Theory 34 (1999) 244–249.

[5] M. Berkani, Restriction of an operator to the range of its powers, Studia Math. 140 (2000) 163–175.

[6] M. Craioveanu, M. Puta, Th.M. Rassias, Old and New Aspects in Spectral Geometry, Kluwer Academic, Dordrecht, 2001.

[7] S.V. Djordjević, Y.M. Han, Browder's theorems and spectral continuity, Glasgow Math. J., to appear.

[8] M.P. Drazin, Pseudoinverse in associative rings and semigroups, Amer. Math. Monthly 65 (1958) 506–514.
[9] S. Grabiner, Uniform ascent and descent of bounded operators, J. Math. Soc. Japan 34 (1982) 317–337.
[10] R. Harte, Invertibility and Singularity for Bounded Linear Operators, Marcel Dekker, New York, 1988.
[11] D.C. Lay, Spectral analysis using ascent, descent, nullity and defect, Math. Ann. 184 (1970) 197–214.
[12] S. Roch, B. Silbermann, Continuity of generalized inverses in Banach algebras, Studia Math. 136 (1999) 197–227.
[13] H. Weyl, Über beschränkte quadratische Formen, deren Differenz vollstetig ist, Rend. Circ. Mat. Palermo 27 (1909) 373–392.