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Averages over Surfaces with Infinitely Flat Points

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Let S be a hypersurface in $\mathbf{R}^n$, $n \geqslant 2$, and let $d\mu = \psi\, d\sigma$, where $\psi \in C_0^\infty(\mathbf{R}^n)$ and σ denotes the surface area measure on S. Define the maximal function $\mathcal{M}$ associated to S and μ by

$$\mathcal{M}f(x) = \sup_{t>0} \left| \int_S f(x - t\xi)\, d\mu(\xi) \right|, \qquad f \in \mathcal{S}(\mathbf{R}^n).$$

It was shown by Stein that when S is the sphere in $\mathbf{R}^n$, $n \geqslant 3$, $\mathcal{M}$ (the spherical maximal function) is bounded on $L^p(\mathbf{R}^n)$ if and only if $p > n/(n-1)$. It has also been shown that if S is of finite type, i.e., the curvature vanishes to at most a finite order m at every point of S, then there exists some number $p_m < \infty$ such that $\mathcal{M}$ is bounded on $L^p(\mathbf{R}^n)$ ($n \geqslant 3$) for all $p \in (p_m, \infty]$. On the other hand it is well known that if S is *flat*, that is, S contains a point at which the curvature vanishes to infinite order, then $\mathcal{M}$ may not be bounded on any $L^p(\mathbf{R}^n)$, $p < \infty$. We show that under some hypotheses the maximal functions $\mathcal{M}$ associated to flat surfaces $S \subset \mathbf{R}^3$ are bounded on certain Orlicz spaces $L^\Phi(\mathbf{R}^3)$ defined naturally in terms of S.

0. INTRODUCTION

Let S be a hypersurface in $\mathbf{R}^n$, $n \geqslant 2$, and let σ denote the surface area measure on S. Let $d\mu = \psi\, d\sigma$, where $\psi \in C_0^\infty(\mathbf{R}^n)$. The maximal function $\mathcal{M}$ associated to S and μ is defined by

$$\mathcal{M}f(x) = \sup_{t>0} |(f * \mu_t)(x)| = \sup_{t>0} \left| \int_S f(x - t\xi)\, d\mu(\xi) \right|, \qquad (0.1)$$

for suitable functions f, say $f \in \mathcal{S}(\mathbf{R}^n)$, the Schwartz class of functions.

It was shown by Stein that $\mathcal{M}$ is bounded on $L^p(\mathbf{R}^n)$ if and only if $p > n/(n-1)$ in the case that S is the sphere in $\mathbf{R}^n$, $n \geqslant 3$ [S1, SW]. ($\mathcal{M}$ is called the spherical maximal function in this case. The corresponding result

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for the circular maximal function on $\mathbf{R}^2$ was later proved by Bourgain [Bo].)

Recall that S is said to be of *finite type* if the Gaussian curvature vanishes to at most a finite order m at every point of S (the smallest such number m is called the type of S). Stein's result has been extended to all finite type surfaces S: it has been proved that if S is of type m then $\mathscr{M}$ is bounded on $L^p(\mathbf{R}^n)$ $(n \geqslant 3)$ for all sufficiently large p, namely all $p \in (p_m, \infty]$, where $p_m = p_m(n)$ is some number (not necessarily optimal) such that $p_m \to \infty$ as $m \to \infty$ (see [G, SS, CM, NSW]).

On the other hand if S is (infinitely) *flat*, that is, S contains a point at which the curvature vanishes to infinite order, then $\mathscr{M}$ may not be bounded on any $L^p(\mathbf{R}^n)$, $p < \infty$ (see [SS] or [S3, p. 512]). The purpose of this paper is to prove *Orlicz space* estimates for $\mathscr{M}$ associated to some flat surfaces (in $\mathbf{R}^3$), which may be regarded as natural substitutes for the L^p space estimates. For example take $\gamma(t) = \exp(-1/t^b)$ for some $b > 0$ if $t > 0$ and $\gamma(0) = 0$. And consider the radial surface $S = \{(y, 1 + \gamma(|y|)) : y \in \mathbf{R}^2\}$ (with a flat point at $y = 0$) and the measure $d\mu(y, 1 + \gamma(|y|)) = \psi(|y|)\, dy$, where $\psi \in C_0^\infty(\mathbf{R})$ is a cutoff function with $\psi(0) > 0$. (That is, μ acts on functions f by $\langle \mu, f \rangle = \int f(y, 1 + \gamma(|y|))\, \psi(|y|)\, dy$.) Let $L^\Phi(\mathbf{R}^3)$ denote the Orlicz space associated to a Young's function Φ given by $\Phi(t) = \exp(t^r)$ for large t (see Example 1.3.a). It turns out that the estimate

$$\|\mathscr{M}f\|_{L^\Phi(\mathbf{R}^3)} \leqslant C\, \|f\|_{L^\Phi(\mathbf{R}^3)}$$

holds if and only if $r > b/2$. In Section 3 we prove a result on $\mathscr{M}$ (Theorem 3.1), which includes this as an example (see Example 3.3(a)). We wish to point out that this problem is in the spirit of the problems discussed by Wainger [W] (see also [B] and [BMO] for related phenomena).

The proof of Theorem 3.1 is based on an interpolation lemma for Orlicz spaces and certain uniform estimates on the Fourier transforms of measures on $S \subset \mathbf{R}^3$ obtained by decomposing μ radially, combined with the standard methods in [S1] and [SS]. We prove our interpolation result in Section 1 and the Fourier transform estimates in Section 2. If these Fourier transform estimates could be extended to higher dimensions $(n \geqslant 4)$, then one would immediately obtain an extension of our result on $\mathscr{M}$ to higher dimensions.

1. An Interpolation Lemma

Let $(X, \mathscr{N}_1, \mu)$ and $(Y, \mathscr{N}_2, \nu)$ be measure spaces, where μ, ν are positive σ-finite measures, and let T be a sublinear operator defined on a suitable linear space of functions f on X such that Tf is a measurable function on Y. The Orlicz space $L^{\Phi}(d\mu)$ associated to a Young's function Φ is equipped with the (Luxemburg) norm

$$\|f\|_{\Phi} = \|f\|_{L^{\Phi}(d\mu)} \equiv \inf\left\{s > 0 : \int \Phi(|f(x)|/s)\, d\mu \leqslant 1\right\}. \tag{1.0}$$

The (generalized) inverse of Φ is defined for $t \in [0, \infty)$ by

$$\Phi^{-1}(t) \equiv \inf\{s > 0 : \Phi(s) > t\}.$$

The following lemma is an extension of (a special case of) the Marcinkiewicz interpolation theorem. For related results see [GP] and [To]. Throughout this paper the letter C will denote a constant which may not be the same at each occurrence.

LEMMA 1.1. *Let $r \in [1, \infty)$. Suppose that T is simultaneously of weak types (r, r) and (∞, ∞), namely there exist constants A and $B > 0$ such that*

$$\nu(\{x : |Tf(x)| > t\}) \leqslant \left(\frac{A\,\|f\|_r}{t}\right)^r \qquad \forall t > 0, \tag{1.1}$$

$$\|Tf\|_{\infty} \leqslant B\,\|f\|_{\infty}. \tag{1.2}$$

Assume that a Young's function Φ is given by $\Phi(s) = \int_0^s \phi(t)\, dt$, where $\phi: [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\phi(t) = 0$ for $0 \leqslant t \leqslant 1$, and $\phi(t) > 0$ for $t > 1$. Also assume that there exist constants $c > 1$, C_0, and C_1 such that

$$\int_1^u \frac{\phi(t)}{t^r}\, dt \leqslant C_0 \frac{\phi(u)}{u^{r-1}} \qquad \textit{for} \quad u > 1, \tag{1.3}$$

and for every $\lambda > 1$

$$C_1 \frac{\phi(\lambda t)}{\phi(t)} \geqslant \phi(\lambda) \qquad \textit{for} \quad t \geqslant c. \tag{1.4}$$

Then there exists a constant $C = C(\Phi, r)$ depending only on Φ and r such that

$$\|Tf\|_{L^{\Phi}(d\nu)} \leqslant CB\Phi^{-1}((A/B)^r)\, \|f\|_{L^{\Phi}(d\mu)}.$$

Remark 1.2. (a) The condition (1.3) will be verified, for instance, if there exist some $\varepsilon > 0$ and $c_0 > 1$ such that

$$\frac{\phi(t)}{t^{r-1+\varepsilon}} \quad \text{is nondecreasing for} \quad t \geqslant c_0, \tag{1.3'}$$

and (1.4) will follow if for every $\lambda > 1$ the quotient function

$$\frac{\phi(\lambda t)}{\phi(t)} \quad \text{is nondecreasing for} \quad t \geqslant c. \tag{1.4'}$$

(b) Our main interest in this lemma lies in the operator norm $CB\Phi^{-1}((A/B)^r)$, which makes it useful when Φ grows exponentially at ∞ and the ratio A/B is large. See Example 1.3, Corollary 1.4, and Theorem 3.1.

EXAMPLE 1.3. In the following we define $\phi(t) = \Phi'(t)$ by the given expressions if $t \geqslant 2$ and let $\phi(t) = 0$ if $0 \leqslant t \leqslant 1$. We also require it to be non-decreasing for $t \geqslant 0$. Assume the ratio A/B is sufficiently large.

(a) $\phi(t) = e^{t^s}$, $s > 0$. In this case $B\Phi^{-1}((A/B)^r) \leqslant CB(\log(A/B))^{1/s}$.

(b) $\phi(t) = \exp \cdots \exp(t^s)$, $s > 0$. Here $B\Phi^{-1}((A/B)^r) \leqslant CB(\log \cdots \log(A/B))^{1/s}$.

(c) $\phi(t) = e^{(\log t)^s}$, $s > 1$. Then $B\Phi^{-1}((A/B)^r) \leqslant CB\exp((r\log(A/B))^{1/s})$.

(d) $\phi(t) = t^{p-1}(\log t)^s$, with $p \in (r, \infty)$ and $s \in \mathbf{R}$. In this case $B\Phi^{-1}((A/B)^r) \leqslant CA^{r/p}B^{1-r/p}(\log(A/B))^{-s/p}$, and when $s = 0$ the lemma essentially reduces to (a special case of) the Marcinkiewicz interpolation theorem.

Proof of Lemma 1.1. The proof relies on the method of proof of the Marcinkiewicz interpolation theorem (see [Z]). We may assume $B = 1$, since otherwise we may replace T by $\tilde{T} = B^{-1}T$. It is then enough to show

$$I \equiv \int \Phi(|Tf(x)|/2D)\, d\nu \leqslant 1$$

with some constant $D \leqslant C\Phi^{-1}(A^r)$, assuming $\|f\|_\Phi = 1$ (see (1.0)). By a well known representation we have

$$I = \int_0^\infty \phi(t)\, \nu_{Tf}(2Dt)\, dt,$$

where $\nu_g(t) \equiv \nu(\{x: |g(x)| > t\})$ denotes the distribution function of g. For $z > 0$ let $f_z(x) = f(x)$ if $|f(x)| \leqslant z$, and $f_z(x) = ze^{i\arg f(x)}$ if $|f(x)| > z$, and let $f^z = f - f_z$. Since T is sublinear and $\phi(t) = 0$ when $t \leqslant 1$,

$$I \leqslant \int_1^\infty \phi(t)\, \nu_{Tf_z}(Dt)\, dt + \int_1^\infty \phi(t)\, \nu_{Tf^z}(Dt)\, dt \equiv I_1 + I_2.$$

By (1.2) $\|Tf_z\|_\infty \leqslant \|f_z\|_\infty \leqslant z$. Now choose $z = Dt$. Then $\nu_{Tf_z}(Dt) = 0$, so $I_1 = 0$. Hence by (1.1)

$$I \leqslant I_2 \leqslant \int_1^{\infty} \phi(t) \left(\frac{A \, \|f^z\|_r}{Dt}\right)^r dt$$

$$\leqslant rA^r D^{-r} \int_1^{\infty} \frac{\phi(t)}{t^r} \int_z^{\infty} s^{r-1} \mu_f(s) \, ds \, dt$$

$$= rA^r D^{-r} \int_D^{\infty} \left(\int_1^{s/D} \frac{\phi(t)}{t^r} dt\right) s^{r-1} \mu_f(s) \, ds$$

since $\|f^z\|_r^r = r \int_0^{\infty} s^{r-1} \mu_{f^z}(s) \, ds = r \int_z^{\infty} (s-z)^{r-1} \mu_f(s) \, ds \leqslant r \int_z^{\infty} s^{r-1} \mu_f(s) \, ds$. Now (1.3) implies

$$\int_1^{s/D} \frac{\phi(t)}{t^r} dt \leqslant C_0 \phi(s/D)(s/D)^{1-r},$$

and (1.4) (with $\lambda = D/2$ and $t = s/\lambda$) implies

$$\phi(s/D) \leqslant \phi(2s/D) \leqslant C_1 \phi(s)/\phi(D/2) \qquad \text{if} \quad s \geqslant D > 2. \tag{1.5}$$

Thus

$$I \leqslant rC_0 C_1 \frac{A^r}{D\phi(D/2)} \int_D^{\infty} \phi(s) \, \mu_f(s) \, ds.$$

We have $\int_D^{\infty} \phi(s) \, \mu_f(s) \, ds \leqslant \int_0^{\infty} \phi(s) \, \mu_f(s) \, ds = \int \Phi(|f|) \, d\mu \leqslant 1$, if $\|f\|_{\Phi} = 1$. So $I \leqslant rC_0 C_1 A^r/[D\phi(D/2)] \leqslant rC_0 C_1 A^r/[2\Phi(D/2)]$, because $\Phi(D/2) = \int_0^{D/2} \phi(t) \, dt \leqslant \phi(D/2) \, D/2$. Thus we get $I \leqslant 1$, if we choose $D = 2\Phi^{-1}(rC_0 C_1 A^r/2)$. Since $\Phi^{-1}(t) > 1$ for $t > 0$, we have $D > 2$, as was required in (1.5). Therefore we conclude that

$$\|Tf\|_{\Phi} \leqslant 2D \, \|f\|_{\Phi}$$

with $D \leqslant C\Phi^{-1}(A^r)$, where $C = \max\{2, rC_0 C_1\}$, since the inequality $\Phi^{-1}(\lambda u) \leqslant \max\{1, \lambda\} \Phi^{-1}(u)$ holds by the convexity of Φ. ▮

The following is an immediate corollary of Lemma 1.1. We state it to indicate the way Lemma 1.1 is intended to be applied.

COROLLARY 1.4. *Let T and T_k be sublinear operators such that $|Tf(x)| \leqslant \sum_{k=1}^{\infty} |T_k f(x)|$ a.e. Suppose that for some $r \in [1, \infty)$ and $k = 1, 2, \ldots$*

$$\nu\{x : |T_k f(x)| > t\} \leqslant \left(\frac{A_k \, \|f\|_r}{t}\right)^r \qquad \forall t > 0,$$

$$\|T_k f\|_{\infty} \leqslant B_k \, \|f\|_{\infty}.$$

If Φ is a Young's function as in Lemma 1.1 *such that*

$$C \equiv \sum_{k=1}^{\infty} B_k \Phi^{-1}((A_k/B_k)^r) < \infty,$$

then

$$\|Tf\|_{\Phi} \leqslant C \|f\|_{\Phi}.$$

2. Uniform Estimates for Fourier Transforms of Measures Carried on a Surface

Let $\gamma \in C^2([0, \infty))$ be a nonnegative strictly convex function such that $\gamma(0) = \gamma'(0) = 0$. For simplicity assume $\gamma'(1) = 1$. Assume in addition that

$$\gamma'(t)/t \quad \text{is nondecreasing for} \quad t > 0. \tag{2.1}$$

Let S be a hypersurface in $\mathbf{R}^n$ given by $S = \{(y, b + \gamma(|y|)) : y \in \mathbf{R}^{n-1}\}$ for some $b \in \mathbf{R}$. Write $\tilde{\gamma}(t) = b + \gamma(t)$ and $\tilde{y} = (y, \tilde{\gamma}(|y|))$.

If $K(\tilde{y})$ denotes the Gaussian curvature of S at $\tilde{y}$, then $K(\tilde{y}) \approx \gamma''(|y|)[\gamma'(|y|)/|y|]^{n-2}$ (on any compact set). (Here the symbol $\approx$ means the ratio of the expressions on either side is bounded between two positive absolute constants.) Since (2.1) implies $\gamma''(t) \geqslant \gamma'(t)/t > 0$ for $t > 0$, the Gaussian curvature of S can only vanish at the origin.

The following theorem is proved by using the method of proof of Theorem 2.2 in [BMVW]. Therefore it may be generalized to include surfaces in $\mathbf{R}^3$ whose horizontal cross sections are dilates of a single smooth convex curve with nonvanishing curvature (see [BMVW]). For simplicity we state and prove it only in the radial case (where the cross sections are concentric circles).

Theorem 2.1. *Let $\chi \in C_0^1([0, \infty))$ be a nonnegative function that is compactly supported in the interval (a, ∞), where $a > 0$. Let $n = 3$ and let γ, $\tilde{\gamma}$, and S be as above. Let ν be the measure on the surface S such that $d\nu(y, \tilde{\gamma}(|y|)) = \chi(|y|)\, dy$.*

Then for every multiindex α with $|\alpha| \leqslant 1$ there exists a constant C independent of a, ξ, and χ such that

$$|(\partial/\partial\xi)^{\alpha}\, \hat{\nu}(\xi)| \leqslant CC_{\chi} \frac{a}{\sqrt{\gamma'(a)\, \gamma'(a/2)}} (1 + |\xi|)^{-1}, \tag{2.2}$$

where $C_{\chi} \leqslant \|\chi\|_{\infty} + \|\chi'\|_1$ if $|\alpha| = 0$, and $C_{\chi} \leqslant \|\chi\|_{\infty} + \|\chi\|_1 + \|\chi'\|_1$ if $|\alpha| = 1$.

Remark 2.2. For each fixed $a>0$ the fact that (2.2) holds with *some* constant C is a trivial consequence of the classical result, since the curvature of S does not vanish on the support of χ (see, e.g., Theorem 1 in [S2] or Theorem 7.7.14 in [H]).

An analog of Theorem 2.1 in $\mathbf{R}^2$ is well known (see Section 6 in [BNW]). Therefore the following corollary holds.

COROLLARY 2.3. *Let $n=2$ or 3. Let v, χ, and a be as in Theorem 2.1. If* $|\alpha|\leqslant 1$

$$|(\partial/\partial\xi)^\alpha \hat{v}(\xi)| \leqslant CC_\chi \left(\frac{a}{\gamma'(a/2)}\right)^{(n-1)/2} (1+|\xi|)^{-(n-1)/2}. \tag{2.2'}$$

If $n=3$ assume in addition that $\gamma'(t)\geqslant t\gamma''(t/2)$ for $t>0$. Then

$$|(\partial/\partial\xi)^\alpha \hat{v}(\xi)| \leqslant CC_\chi[\kappa(a/2)]^{-1/2}(1+|\xi|)^{-(n-1)/2},$$

where $\kappa(t)=\inf\{|K(y,\tilde{\gamma}(|y|))| : |y|\geqslant t\}$.

Proof of Theorem 2.1. We first prove the case $\alpha=0$ and comment on the other cases at the end. It is enough to take $\xi=\lambda\zeta$ with $\lambda>0$, $\zeta=(0,-\varepsilon,1)$, and $0<\varepsilon<1$. Then $|\xi|\approx\lambda$. We may assume $b=0$. Using polar coordinates we obtain

$$\begin{aligned}\hat{v}(\xi)=\hat{v}(\lambda\zeta)&=\int_S e^{i\lambda\zeta\cdot x}\,dv(x)=\int_{\mathbf{R}^2} e^{i\lambda\zeta\cdot(y,\gamma(|y|))}\chi(|y|)\,dy\\ &=\int_0^\infty e^{i\lambda\gamma(r)}r\chi(r)\int_0^{2\pi} e^{-i\lambda\varepsilon r\sin\theta}\,d\theta\,dr\\ &=\int_0^\infty e^{i\lambda\gamma(r)}r\chi(r)\,J_0(\lambda\varepsilon r)\,dr.\end{aligned}$$

Here $J_0(r)=e^{-ir}F_1(r)+e^{ir}F_2(r)$ is the Bessel function of order 0, where F_l satisfy the following estimates (see, e.g., [BNW]).

$$|F_l^{(j)}(r)|\leqslant C_j, \tag{2.3a}$$

$$|F_l^{(j)}(r)|\leqslant C_j r^{-1/2-j} \qquad \text{for } j\geqslant 0 \text{ and } l=1,2. \tag{2.3b}$$

Thus $\hat{v}(\xi)=I_1+I_2$, where

$$I_l=\int_0^\infty e^{i\lambda\gamma(r)}r\chi(r)\exp((-1)^l\, i\lambda\varepsilon r)\,F_l(\lambda\varepsilon r)\,dr.$$

Since $|I_l| \leqslant C$, it suffices to prove

$$|I_l| \leqslant C \frac{a}{\lambda \sqrt{\gamma'(a)\,\gamma'(a/2)}}. \tag{2.4}$$

Define the number $\rho > 0$ by $\gamma'(\rho) = \varepsilon$. We may also assume $\|\chi\|_\infty + \|\chi'\|_1 \leqslant 1$.

We first prove (2.4) for I_1. Write $F(r) = F_1(r)$. Let $\psi(r) = \gamma(r) - \gamma(\rho) - \gamma'(\rho)(r-\rho)$. Then $\psi(\rho) = \psi'(\rho) = 0$, and $\psi'(r) > 0$ for $r > \rho$. And (2.1) implies that

$$\frac{\psi'(r)}{r} \quad \text{is strictly increasing for} \quad r > \rho. \tag{2.5}$$

We also need the following consequence of (2.1): If $r \geqslant 2\rho$, then

$$\frac{r}{\psi'(r)} = \frac{r}{\gamma'(r) - \gamma'(\rho)} \leqslant \frac{2r}{\gamma'(r)}, \tag{2.6}$$

since $\gamma'(r) \geqslant \gamma'(\rho)\, r/\rho \geqslant 2\gamma'(\rho)$.

Write

$$e^{i\lambda[\gamma'(\rho)\rho - \gamma(\rho)]} I_1 = \int_0^\infty e^{i\lambda\psi(r)} F(\lambda\varepsilon r)\, r\chi(r)\, dr = \int_0^{2\rho} + \int_{2\rho}^\infty .$$

We estimate the second term first. Put $\beta = \max\{a, 2\rho\}$. By integration by parts

$$\begin{aligned}\int_{2\rho}^\infty = \int_\beta^\infty &= \frac{1}{i\lambda} \int_\beta^\infty \frac{d}{dr}\left(e^{i\lambda\psi(r)}\right) F(\lambda\varepsilon r) \frac{r}{\psi'(r)} \chi(r)\, dr \\ &= \frac{1}{i\lambda} e^{i\lambda\psi(r)} F(\lambda\varepsilon r) \frac{r}{\psi'(r)} \chi(r) \Big]_\beta^\infty \\ &\quad - \frac{1}{i\lambda} \int_\beta^\infty e^{i\lambda\psi(r)} \frac{d}{dr}\left[F(\lambda\varepsilon r) \frac{r}{\psi'(r)} \chi(r)\right] dr.\end{aligned}$$

Using (2.3a), (2.6), and (2.1) gives the following estimate for the boundary term BT.

$$|BT| \leqslant C \frac{\beta}{\lambda\psi'(\beta)} \leqslant C \frac{\beta}{\lambda\gamma'(\beta)} \leqslant C \frac{a}{\lambda\gamma'(a)},$$

since $\beta \geqslant 2\rho$ and $\beta \geqslant a$. Likewise, the integrated term IT is estimated by using (2.3), (2.6), and (2.1).

$$|IT| \leqslant C \frac{1}{\lambda} \int_{\beta}^{\infty} \lambda\varepsilon \, |F'(\lambda\varepsilon r)| \frac{r}{\psi'(r)} \, dr + C \frac{1}{\lambda} \int_{\beta}^{\infty} \left| \frac{d}{dr} \left[\frac{r}{\psi'(r)} \right] \right| |F(\lambda\varepsilon r)| \, dr$$

$$+ C \frac{1}{\lambda} \int_{\beta}^{\infty} |F(\lambda\varepsilon r)| \frac{r}{\psi'(r)} \, |\chi'(r)| \, dr.$$

Observe that by (2.3)

$$\int_{\beta}^{\infty} \lambda\varepsilon \, |F'(\lambda\varepsilon r)| \, dr \leqslant C \int_{0}^{\infty} (1+r)^{-3/2} \, dr \leqslant C.$$

Hence the first term is bounded by $Ca/[\lambda\gamma'(a)]$ just like the boundary term. For the second we use (2.3a), (2.6), and the fact that the monotonicity (2.5) of $\psi'(t)/t$ implies

$$\int_{\beta}^{\infty} \left| \frac{d}{dr} \frac{r}{\psi'(r)} \right| dr = \left| \int_{\beta}^{\infty} \frac{d}{dr} \frac{r}{\psi'(r)} \, dr \right|.$$

For the third term we again use (2.3a) and (2.6). Combining these we obtain

$$|IT| \leqslant C \frac{a}{\lambda\gamma'(a)}.$$

Therefore

$$\left| \int_{2\rho}^{\infty} e^{i\lambda\psi(r)} F(\lambda\varepsilon r) \, r\chi(r) \, dr \right| \leqslant C \frac{a}{\lambda\gamma'(a)}.$$

Next we estimate the term $\int_0^{2\rho}$. Note that we may assume $a < 2\rho$, since otherwise the integrand vanishes identically. Put

$$H(r) = \int_{a}^{r} e^{i\lambda\psi(s)} \, ds.$$

Since (2.1) implies $\psi''(r) = \gamma''(r) \geqslant \gamma'(r)/r \geqslant \gamma'(a)/a$ for $r \geqslant a$, it follows from van der Corput's lemma that

$$|H(r)| \leqslant C \sqrt{\frac{a}{\lambda\gamma'(a)}} \tag{2.7}$$

if $r \geqslant a$ (see, e.g., [BNW, Section 6]). Now we integrate $\int_0^{2\rho}$ by parts as follows:

$$\int_0^{2\rho} = \int_a^{2\rho} = H(r)\, F(\lambda\varepsilon r)\, r\chi(r)]_a^{2\rho} - \int_a^{2\rho} H(r) \frac{d}{dr}[F(\lambda\varepsilon r)\, r\chi(r)]\, dr.$$

The boundary term is estimated using (2.3b), (2.7), and (2.1).

$$|BT| \leqslant C \sqrt{\frac{a}{\lambda\gamma'(a)}} \sqrt{\frac{\rho}{\lambda\varepsilon}} = C \sqrt{\frac{a}{\lambda\gamma'(a)}} \sqrt{\frac{\rho}{\lambda\gamma'(\rho)}} \leqslant C \frac{a}{\lambda\sqrt{\gamma'(a)\,\gamma'(a/2)}}.$$

We estimate the integrated term also by using (2.1), (2.3b), and (2.7).

$$\begin{aligned}
|IT| &\leqslant \int_a^{2\rho} |H(r)\, F(\lambda\varepsilon r)|\, \chi(r)\, dr + \int_a^{2\rho} |H(r)\, F(\lambda\varepsilon r)\, r\chi'(r)|\, dr \\
&\quad + \int_a^{2\rho} |H(r)\, \lambda\varepsilon F'(\lambda\varepsilon r)|\, r\chi(r)\, dr \\
&\leqslant C \sqrt{\frac{a}{\lambda\gamma'(a)}} \left(\int_0^{2\rho} (\lambda\varepsilon r)^{-1/2}\, dr + \sup_{r\in[0,2\rho]} [(\lambda\varepsilon r)^{-1/2} r] \cdot \int_0^{2\rho} |\chi'(r)|\, dr \right. \\
&\quad \left. + \int_0^{2\rho} \lambda\varepsilon(\lambda\varepsilon r)^{-3/2}\, r\, dr \right) \\
&\leqslant C \sqrt{\frac{a}{\lambda\gamma'(a)}} \sqrt{\frac{\rho}{\lambda\varepsilon}} \leqslant C \frac{a}{\lambda\sqrt{\gamma'(a)\,\gamma'(a/2)}}.
\end{aligned}$$

Hence (2.4) holds for I_1. This completes the estimation of the term I_1.

We now give an outline of the estimation of the term I_2, since it is analogous to I_1. We write

$$I_2 = \int_0^{\infty} e^{i\lambda(\gamma(r) + \varepsilon r)} F(\lambda\varepsilon r)\, r\chi(r)\, dr = \int_0^{2\rho} + \int_{2\rho}^{\infty}.$$

As before we integrate the second term $\int_{2\rho}^{\infty}$ by parts. Recall the notation $\beta = \max\{a, 2\rho\}$.

$$\begin{aligned}
\int_{2\rho}^{\infty} = \int_{\beta}^{\infty} &= \frac{1}{i\lambda} \int_{\beta}^{\infty} \frac{d}{dr}(e^{i\lambda(\gamma(r)+\varepsilon r)})\, F(\lambda\varepsilon r) \frac{r}{\gamma'(r)+\varepsilon} \chi(r)\, dr \\
&= \frac{1}{i\lambda} e^{i\lambda(\gamma(r)+\varepsilon r)} F(\lambda\varepsilon r) \frac{r}{\gamma'(r)+\varepsilon} \chi(r)]_{\beta}^{\infty} \\
&\quad - \frac{1}{i\lambda} \int_{\beta}^{\infty} e^{i\lambda(\gamma(r)+\varepsilon r)} \frac{d}{dr}\left[F(\lambda\varepsilon r) \frac{r}{\gamma'(r)+\varepsilon} \chi(r) \right] dr.
\end{aligned}$$

Here unlike the quotient $r/\psi'(r)$ that arises in the case of I_1, the quotient $r/(\gamma'(r)+\varepsilon)$ is not monotone, but we may write it as the product of the monotone functions $r/\psi'(r)$ and $\psi'(r)/(\gamma'(r)+\varepsilon) = 1-[2\varepsilon/(\gamma'(r)+\varepsilon)]$. The rest of the argument is parallel to that for I_1. To estimate the first term $\int_0^{2\rho}$ put

$$H(r) = \int_a^r e^{i\lambda(\gamma(s)+\varepsilon s)}\, ds$$

and proceed as in I_1. This completes the proof of the theorem in the case $\alpha = 0$.

Finally, suppose $|\alpha| = 1$. If $\alpha = (1, 0, 0)$ or $(0, 1, 0)$ we get

$$(\partial/\partial\xi)^\alpha\, \hat{\nu}(\xi) = (\partial/\partial\xi_j)\, \hat{\nu}(\xi) = c_0 \int_0^\infty e^{i\lambda\gamma(r)} r[r\chi(r)]\, J_1(\lambda\varepsilon r)\, dr,$$

where $\xi = \lambda\zeta$, $\zeta = (\zeta_1, \zeta_2, 1)$, $\varepsilon = |(\zeta_1, \zeta_2)|$, $|c_0| \leqslant 1$, and $J_1(r)$ denotes the Bessel function of order 1. If $\alpha = (0, 0, 1)$ we get

$$(\partial/\partial\xi)^\alpha\, \hat{\nu}(\xi) = (\partial/\partial\xi_3)\, \hat{\nu}(\xi) = i \int_0^\infty e^{i\lambda\gamma(r)} r[\gamma(r)\, \chi(r)]\, J_0(\lambda\varepsilon r)\, dr.$$

Thus, $(\partial/\partial\xi)^\alpha\, \hat{\nu}(\xi)$ involves only a new cutoff function and possibly a Bessel function of different order, and hence we obtain similar estimates. ∎

3. The Maximal Functions Associated to Flat Surfaces in $\mathbf{R}^3$

Let γ be as in the first paragraph in Section 2. Recall the notation $\tilde{\gamma}(t) = 1 + \gamma(t)$ (with $b = 1$), $\tilde{y} = (y, \tilde{\gamma}(|y|))$, and $S = \{(y, 1+\gamma(|y|)) : y \in \mathbf{R}^{n-1}\}$. Let $d\mu(\tilde{y}) = \psi(|y|)\, dy$, where $\psi \in C_0^\infty(\mathbf{R})$ is a nonnegative function with $\psi(t) = 1$ if $|t| \leqslant 1$ and $\psi(t) = 0$ if $|t| \geqslant 2$. Let the maximal function $\mathcal{M}$ be defined by (0.1). We obtain the following result for $\mathcal{M}$.

THEOREM 3.1. *Let S be given as above with $n = 3$. Assume that for each $\lambda > 1$*

$$\gamma'(\lambda t)/\gamma'(t) \quad \textit{is nondecreasing for} \quad t > 0. \tag{3.1}$$

Put $G(t) = t^2\gamma'(t)$. For $\beta > 1$ and $d > 0$ let $\phi\colon [0, \infty) \to [0, \infty)$ be a nondecreasing function such that $\phi(t) = t^{-1}[G(t^{-d})]^{-\beta}$ if t is sufficiently large, $\phi(t) > 0$ if $t > 1$, and $\phi(t) = 0$ if $0 \leqslant t \leqslant 1$. Let $\Phi(u) = \int_0^u \phi(t)\, dt$. Then for every $d > \frac{1}{2}$ there exists a constant C such that

$$\|\mathcal{M}f\|_{L^\Phi(\mathbf{R}^3)} \leqslant C\, \|f\|_{L^\Phi(\mathbf{R}^3)}, \qquad f \in \mathcal{S}(\mathbf{R}^3). \tag{3.2}$$

For some necessary conditions see Example 3.3. To prove this result we need the following lemma, whose proof closely follows that of Theorem 3 in [SS].

LEMMA 3.2. *Let $n \geqslant 3$. Let S be as above and let $d\nu(\bar{y}) = \chi(|y|)\, dy$, where $\chi \in C_0^\infty(\mathbf{R})$, and assume that there exists a constant A such that for all $\xi \in \mathbf{R}^n$*

$$|(\partial/\partial\xi)^\alpha \hat{\nu}(\xi)| \leqslant A(1 + |\xi|)^{-(n-1)/2} \tag{3.3}$$

whenever $|\alpha| \leqslant 1$. Let

$$\mathcal{M}f(x) = \sup_{t>0} |(f * \nu_t)(x)| = \sup_{t>0} \left| \int_S f(x - t\xi)\, d\nu(\xi) \right|.$$

Then for every small $\delta > 0$ and r with $(n-1)/n > 1/r > (n-1)/n - \delta/2$ there exists a constant $C = C(n, \delta, r)$ independent of A such that

$$\|\mathcal{M}f\|_{L^r(\mathbf{R}^n)} \leqslant CA^b \|f\|_{L^r(\mathbf{R}^n)}, \tag{3.4}$$

where $b = (n-1)/n + \delta$.

Proof. Write $x = (x', x_n)$ and $x' = (x_1, ..., x_{n-1})$. Fix a function $\eta \in C_0^\infty(\mathbf{R})$ which equals 1 near the origin. Define an analytic family of operators by

$$(f * \nu_z)(x) = \frac{e^{z^2}}{\Gamma(z/2)} \int_{\mathbf{R}^n} f(x - \xi)\, |\xi_n - \bar{\gamma}(|\xi'|)|^{z-1}\, \chi(|\xi'|)\, \eta(\xi_n - \bar{\gamma}(|\xi'|))\, d\xi,$$

for $f \in \mathcal{S}(\mathbf{R}^n)$ (initially for $\operatorname{Re} z > 0$, then analytically continued to all $z \in \mathbf{C}$). Let

$$\mathcal{M}_z f(x) = \sup_{t>0} |(f * \nu_{z,t})(x)|.$$

We have

$$\hat{\nu}_z(\xi) = \frac{e^{z^2}}{\Gamma(z/2)} \int_{\mathbf{R}^n} e^{ix\cdot\xi}\, |x_n - \bar{\gamma}(|x'|)|^{z-1}\, \chi(|x'|)\, \eta(x_n - \bar{\gamma}(|x'|))\, dx.$$

By the change of variables $x_n \to x_n + \bar{\gamma}(|x'|)$

$$\begin{aligned}\hat{\nu}_z(\xi) &= \int_{\mathbf{R}^{n-1}} e^{i(x'\cdot\xi' + \bar{\gamma}(|x'|)\xi_n)} \chi(|x'|)\, dx' \cdot \frac{e^{z^2}}{\Gamma(z/2)} \int_{-\infty}^{\infty} e^{ix_n\xi_n}\, |x_n|^{z-1}\, \eta(x_n)\, dx_n \\ &= \hat{\nu}(\xi) \cdot \zeta_z(\xi_n).\end{aligned}$$

The first factor is bounded by $A(1 + |\xi|)^{-(n-1)/2}$ by the hypothesis (3.3). And by analytic continuation ζ_z is an entire function in z and $\zeta_0 \equiv 1$, so

$\tilde{\mathcal{M}}_0 f = \tilde{\mathcal{M}} f$. Also $|\zeta_z(\xi_n)| \leq C(1+|\xi|)^{-\operatorname{Re} z}$, if $\operatorname{Re} z$ is bounded (see p. 327 in [S2]). Hence it follows that

$$|\hat{v}_z(\xi)| \leq CA(1+|\xi|)^{-1/2-\varepsilon}$$

if $\operatorname{Re} z = -n/2 + 1 + \varepsilon$ for some $\varepsilon > 0$. Similar estimates hold for $(\partial/\partial\xi_j)\, \hat{v}_z(\xi)$. Hence by Proposition 1 in [SS]

$$\|\tilde{\mathcal{M}}_z f\|_{L^2(\mathbf{R}^n)} \leq CA\, \|f\|_{L^2(\mathbf{R}^n)}, \qquad \text{if} \quad \operatorname{Re} z > -\frac{n}{2} + 1.$$

If $\operatorname{Re} z = 1$ then $\tilde{\mathcal{M}}_z f(x) \leq CM(|f|)(x)$, where M is the Hardy–Littlewood maximal function, and hence

$$\|\tilde{\mathcal{M}}_z f\|_p \leq C_p\, \|f\|_p$$

for $1 < p < \infty$. Interpolating the last two estimates (with p sufficiently close to 1 and ε small enough) using Stein's analytic interpolation theorem (see [SWe]) we obtain

$$\|\tilde{\mathcal{M}} f\|_r \leq CA^b\, \|f\|_r,$$

where r and b are as above. ∎

Proof of Theorem 3.1. Fix a nonnegative function $\chi_0 \in C_0^\infty((2, 8))$ such that $\sum_{k=-\infty}^{\infty} \chi_0(2^k t) \equiv 1$ for $t > 0$. Put $\chi_k(t) = \chi_0(2^k t)$, and $d\mu_k(\tilde{y}) = \chi_k(|y|)\, d\mu(\tilde{y})$. Set

$$\mathcal{M}_k f(x) = \sup_{t>0} |(f * \mu_{k,t})(x)| = \sup_{t>0} \left| \int_S f(x - t\xi)\, d\mu_k(\xi) \right|.$$

Since $d\mu = \sum_{k=1}^{\infty} d\mu_k$, we have $\mathcal{M} f(x) \leq \sum_{k=1}^{\infty} \mathcal{M}_k(f)(x)$.

Note that $d\mu_k(\tilde{y}) = \chi_k(|y|)\, \psi(|y|)\, dy = \chi_k(|y|)\, dy$ for $k \geq 3$, and χ_k is supported in the interval $(a_k, 4a_k)$, where $a_k = 2^{-k+1}$. Therefore it follows from Theorem 2.1 that

$$|(\partial/\partial\xi)^\alpha \widehat{\mu_k}(\xi)| \leq A_k (1 + |\xi|)^{-1} \tag{3.5}$$

if $|\alpha| \leq 1$, where

$$A_k \leq C \frac{a_k}{\gamma'(a_k/2)}$$

for some constant C independent of k. (Note that $\|(d/dt)\, \chi_k\|_1 = \|(d/dt)\, \chi_0\|_1$ for all k.) It then follows from (3.5) and Lemma 3.2 that

$$\|\mathcal{M}_k f\|_{L^r(\mathbf{R}^3)} \leq CA_k^b\, \|f\|_{L^r(\mathbf{R}^3)}, \tag{3.6}$$

where $b = \frac{2}{3} + \delta$ and $r > \frac{3}{2}$ may be chosen such that $br/\beta \leqslant 1$. We also have the trivial estimate

$$\|\mathcal{M}_k f\|_\infty \leqslant C a_k^2 \|f\|_\infty. \tag{3.7}$$

It is easy to see that the function ϕ as defined above satisfies the hypotheses of Lemma 1.1 if r is chosen close enough to $\frac{3}{2}$. In fact the conditions (1.3′) and (1.4′) in Remark 1.2(a) are direct consequences of (2.1) (if $\beta \geqslant 1$ and $d > \frac{1}{2}$) and (3.1), respectively. Now $\tilde{\Phi}(u/2) \leqslant \Phi(u) \leqslant \tilde{\Phi}(u)$, where $\tilde{\Phi}(u) \equiv u\phi(u) = [G(u^{-d})]^{-\beta}$ for large u. Hence $\Phi^{-1}(u) \approx \tilde{\Phi}^{-1}(u) = [G^{-1}(u^{-1/\beta})]^{-1/d}$ for large u. Put $G_\delta(t) = t^{2/b-1}\gamma'(t)$. Then $G_\delta(t)^{br/\beta} \geqslant G_\delta(t) \geqslant G(t)$ if $t \leqslant 1$. Interpolating the estimates (3.6) and (3.7) using Lemma 1.1 we obtain

$$\begin{aligned}\|\mathcal{M}_k f\|_\Phi &\leqslant C a_k^2 \Phi^{-1}(C[A_k^b/a_k^2]^r)\,\|f\|_\Phi \\ &\leqslant C a_k^2 [G^{-1}([G_\delta(a_k/2)]^{br/\beta})]^{-1/d}\,\|f\|_\Phi \\ &\leqslant C a_k^2 [G^{-1}(G(a_k/2))]^{-1/d}\,\|f\|_\Phi \leqslant C a_k^{2-1/d}\,\|f\|_\Phi.\end{aligned}$$

Therefore

$$\|\mathcal{M} f\|_\Phi \leqslant \sum_{k=1}^{\infty} \|\mathcal{M}_k(f)\|_\Phi \leqslant C \sum_{k=1}^{\infty} 2^{-k(2-1/d)}\,\|f\|_\Phi \leqslant C\,\|f\|_\Phi$$

if $d > \frac{1}{2}$. ∎

Example 3.3. Put $\gamma(0) = 0$ and define γ by the given expressions for sufficiently small $t > 0$. Note that each γ in (a)–(c) gives rise to a flat surface S (with curvature vanishing to infinite order at the origin), since $\gamma^{(j)}(0) = 0$ for all $j \geqslant 1$.

(a) $\gamma(t) = e^{-1/t^b}$, $b > 0$. By Theorem 3.1 (3.2) holds for every $d > \frac{1}{2}$. (Actually here we may take $\Phi(u) \approx [\gamma(u^{-d})]^{-\beta} = \exp(\beta u^{bd})$, or simply $\Phi(u) \approx \exp(u^{bd})$ (for large u), since these functions define the same Orlicz space.) This is sharp in the sense that (3.2) fails whenever $d \leqslant \frac{1}{2}$. To see this take $f(x) = f(x_1, x_2, x_3) = \eta(x)\,\Phi^{-1}(|x_3|^{-1+\varepsilon})$, for small $\varepsilon > 0$, where $\eta \in C_0^\infty(\mathbf{R}^3)$ satisfies $0 \leqslant \eta(x) \leqslant 1$ for all x, $\eta(x) = 1$ if $|x| \leqslant \frac{1}{2}$ and $\eta(x) = 0$ if $|x| \geqslant 1$. Then $\int \Phi(|f|)\,dx \leqslant C \int_0^1 x_3^{-1+\varepsilon}\,dx_3 < \infty$, so $\|f\|_\Phi < \infty$. On the other hand for $|x|$ small with $x_3 > 0$,

$$\begin{aligned}\mathcal{M}f(x) &= \sup_{t>0} \left| \int_S f(x - t\xi)\,d\mu(\xi) \right| \geqslant \int_S f(x - x_3\xi)\,d\mu(\xi) \\ &\geqslant \int_{\{y \in \mathbf{R}^2 : |y| < c\}} \Phi^{-1}([x_3\gamma(|y|)]^{-1+\varepsilon})\,dy\end{aligned}$$

$$\geqslant \int_{|y| < c} \Phi^{-1}(\gamma(|y|)^{-1+\varepsilon})\, dy$$

$$\geqslant c \int_{|y| < c} |y|^{-1/d}\, dy = \infty$$

if $d \leqslant \frac{1}{2}$.

(b) $\gamma(t) = 1/[\exp \cdots \exp(1/t^b)]$, $b > 0$. As in (a) the estimate (3.2) holds if and only if $d > \frac{1}{2}$.

(c) $\gamma(t) = e^{-(\log(1/t))^b}$, $b > 1$. Note that this γ is "less flat" (at the origin) than the preceding two examples. We can show that (3.2) actually holds with $\beta = 1$ and $d > \frac{1}{2}$. And the example $f(x) = \eta(x)\, \Phi^{-1}(|x_3|^{-1} [\log(1/|x_3|)]^{-1+\varepsilon})$ shows that (3.2) fails if $\beta \leqslant 1$ and $d \leqslant \frac{1}{2}$.

(d) $\gamma(t) = t^m$, $m \geqslant 2$. The conclusion that (3.2) holds when $d > \frac{1}{2}$ may be restated as $\|\mathcal{M}f\|_p \leqslant C\, \|f\|_p$ for $p > (m+1)/2$. This range of p is not sharp if $m > 2$ (see [NSW]). For best known results to date for finite type hypersurfaces see [NSW].

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