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Pullback attractors for a semilinear heat equation on time-varying domains ☆

Peter E. Kloeden [a,*], José Real [b], Chunyou Sun [c]

[a] Institut für Mathematik, Johann Wolfgang Goethe-Universität, D-60054 Frankfurt am Main, Germany
[b] Dpto. Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080 Sevilla, Spain
[c] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China

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ABSTRACT

The existence of a pullback attractor is established for the nonautonomous dynamical system generated by the weak solutions of a semilinear heat equation on time-varying domains with homogeneous Dirichlet boundary conditions. It is assumed that the spatial domains $\mathcal{O}_t$ in $\mathbb{R}^N$ are obtained from a bounded base domain $\mathcal{O}$ by a C^2-diffeomorphism, which is continuously differentiable in the time variable, and are contained, in the past, in a common bounded domain.

1. Introduction

A semilinear heat equation on a time-varying domain is intrinsically nonautonomous even if the terms in the equation do not depend explicit on time. Investigations of its attractor thus require the concept of a nonautonomous attractor, specifically, that of a pullback attractor [3–8,10,12]. In a recent paper [13] Kloeden, Marín-Rubio and Real established the existence of a global pullback attractor for a semilinear heat equation with a homogeneous Dirichlet boundary condition in the case that the spatial domains $\mathcal{O}_t$ are bounded and increase with time. Much of the effort required there was to prove

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* Corresponding author.
E-mail addresses: kloeden@math.uni-frankfurt.de (P.E. Kloeden), jreal@us.es (J. Real), sunchy@lzu.edu.cn (C. Sun).

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the existence and uniqueness of a variational solution satisfying an energy equality. It built on the extensive existence and regularity theory of partial differential equations on non-cylindrical domains, especially with nested spatial domains, see for example [1,11,17,16,19] and the bibliography therein.

In the present paper we do not require the domains $\mathcal{O}_t$ to increase in time, but instead assume they are obtained from a bounded base domain $\mathcal{O}$ by a C^2-diffeomorphism, which is continuously differentiable in the time variable, and are all contained, in the past, in a common bounded domain. We follow the ideas and methods sketched in a paper of Límaco, Medeiros and Zuazua [15]. Here much of our effort is also directed at appropriately formulating the problem and in proving the existence and uniqueness of strong and weak solutions in appropriate functions spaces as well as in establishing energy inequalities. In particular, we need to define what we mean, for example, by the continuity and differentiability of a function $t \mapsto u(t) \in L^2(\mathcal{O}_t)$.

We present the basic equations and notation in Section 2 and then, in Section 3, consider in some detail the appropriate function space setting and properties of functions under the time variable coordinate transformation. A compactness result is also presented there. Strong and weak solutions are considered in Sections 4 and 5, respectively, in particular their existence and uniqueness. In Section 6 we show that the weak solutions generate a process, that is a 2-parameter semi-group which defines the nonautonomous dynamical system. This is shown to satisfy an asymptotic compactness condition in Section 7, which is then used to establish the existence of the pullback attractor. Finally, the transformed equations are derived in Section 8 for a spatially linear domain transformation.

2. Equations and notation

Let $\mathcal{O}$ be a nonempty bounded open subset of $\mathbb{R}^N$ with C^2 boundary $\partial\mathcal{O}$, and $r = r(y,t)$ a vector function

$$r \in C^1\big(\overline{\mathcal{O}} \times \mathbb{R}; \mathbb{R}^N\big), \tag{2.1}$$

such that

$$r(\cdot,t) : \mathcal{O} \to \mathcal{O}_t \quad \text{is a } C^2\text{-diffeomorphism for all } t \in \mathbb{R}. \tag{2.2}$$

We define

$$Q_{\tau,T} := \bigcup_{t\in(\tau,T)} \mathcal{O}_t \times \{t\} \quad \text{for any } T > \tau \tag{2.3}$$

and denote

$$Q_\tau := \bigcup_{t\in(\tau,+\infty)} \mathcal{O}_t \times \{t\} \quad \forall \tau \in \mathbb{R},$$

$$\Sigma_{\tau,T} := \bigcup_{t\in(\tau,T)} \partial\mathcal{O}_t \times \{t\}, \qquad \Sigma_\tau := \bigcup_{t\in(\tau,+\infty)} \partial\mathcal{O}_t \times \{t\} \quad \forall \tau < T.$$

For any $T > \tau$, the set $Q_{\tau,T}$ is an open subset of $\mathbb{R}^{N+1}$, with boundary

$$\partial Q_{\tau,T} = \Sigma_{\tau,T} \cup \big(\mathcal{O}_\tau \times \{\tau\}\big) \cup \big(\mathcal{O}_T \times \{T\}\big).$$

We will also assume that the function $\bar r = \bar r(x,t)$, where $\bar r(\cdot,t) = r^{-1}(\cdot,t)$ denotes the inverse of $r(\cdot,t)$, satisfies

$$\bar r \in C^{2,1}\big(\overline{Q}_{\tau,T}; \mathbb{R}^N\big) \quad \text{for all } \tau < T, \tag{2.4}$$

i.e., $\bar r$, $\frac{\partial \bar r}{\partial t}$, $\frac{\partial \bar r}{\partial x_i}$ and $\frac{\partial^2 \bar r}{\partial x_i \partial x_j}$ belong to $C(\overline{Q}_{\tau,T}; \mathbb{R}^N)$, for all $1 \leqslant i, j \leqslant N$, for any $\tau < T$.

We consider the following initial boundary value problem for a semilinear parabolic equation with homogeneous Dirichlet boundary condition,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + g(u) = f(t) \quad \text{in } Q_\tau, \\ u = 0 \quad \text{on } \Sigma_\tau, \\ u(\tau, x) = u_\tau(x), \quad x \in \mathcal{O}_\tau, \end{cases} \tag{2.5}$$

and, for each $T > \tau$, the auxiliary problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + g(u) = f(t) \quad \text{in } Q_{\tau,T}, \\ u = 0 \quad \text{on } \Sigma_{\tau,T}, \\ u(\tau, x) = u_\tau(x), \quad x \in \mathcal{O}_\tau, \end{cases} \tag{2.6}$$

where $\tau \in \mathbb{R}$, $u_\tau : \mathcal{O}_\tau \to \mathbb{R}$ and $f : Q_\tau \to \mathbb{R}$ are given, and $g \in C^1(\mathbb{R}, \mathbb{R})$ is also a given function for which there exist nonnegative constants α_1, α_2, β and l, and $p \geqslant 2$, such that

$$-\beta + \alpha_1 |s|^p \leqslant g(s)s \leqslant \beta + \alpha_2 |s|^p \quad \forall s \in \mathbb{R} \tag{2.7}$$

and

$$g'(s) \geqslant -l \quad \forall s \in \mathbb{R}. \tag{2.8}$$

For later observe that, by (2.7), there then exist nonnegative constants $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\beta}$ such that

$$-\tilde{\beta} + \tilde{\alpha}_1 |s|^p \leqslant G(s) \leqslant \tilde{\beta} + \tilde{\alpha}_2 |s|^p \quad \forall s \in \mathbb{R}, \tag{2.9}$$

where

$$G(s) := \int_0^s g(r)\, dr.$$

3. Preliminaries

3.1. Functional spaces and preliminary lemmas

We consider a fixed finite time interval $[\tau, T]$. Let $(X_t, \|\cdot\|_{X_t})$ $(t \in [\tau, T])$ be a family of Banach spaces such that $X_t \subset L^1_{\mathrm{loc}}(\mathcal{O}_t)$ for all $t \in [\tau, T]$. For any $1 \leqslant q \leqslant \infty$, we denote by $L^q(\tau, T; X_t)$ the vector space of all functions $u \in L^1_{\mathrm{loc}}(Q_{\tau,T})$ such that $u(t) = u(\cdot, t) \in X_t$ a.e. $t \in (\tau, T)$, and the function $\|u(\cdot)\|_{X_\cdot}$ defined by $t \mapsto \|u(t)\|_{X_t}$, belongs to $L^q(\tau, T)$.

By definition, we consider on $L^q(\tau, T; X_t)$ the norm given by

$$\|u\|_{L^q(\tau,T;X_t)} := \big\| \|u(\cdot)\|_{X_\cdot} \big\|_{L^q(\tau,T)}.$$

For each $u \in L^1_{\mathrm{loc}}(Q_{\tau,T})$, we can extend u trivially to $\mathbb{R}^N \times (\tau, T)$ by

$$\hat{u}(x,t) = \begin{cases} u(x,t), & (x,t) \in \mathcal{O}_t \times (\tau, T), \\ 0, & (x,t) \in (\mathbb{R}^N \setminus \mathcal{O}_t) \times (\tau, T). \end{cases}$$

Then, for any $1 \leqslant p, q \leqslant \infty$, we have

$$u \in L^q\big(\tau, T; L^p(\mathcal{O}_t)\big) \quad \Longrightarrow \quad \hat{u} \in L^q\big(\tau, T; L^p\big(\mathbb{R}^N\big)\big),$$

and

$$u \in L^q\big(\tau, T; H_0^1(\mathcal{O}_t)\big) \quad \Longrightarrow \quad \hat{u} \in L^q\big(\tau, T; H_0^1\big(\mathbb{R}^N\big)\big),$$

with

$$\frac{\partial \hat{u}}{\partial x_i} = \widehat{\frac{\partial u}{\partial x_i}} \quad \forall 1 \leqslant i \leqslant N. \tag{3.1}$$

For any $u \in L^1_{\mathrm{loc}}(Q_{\tau,T})$, we will denote $u' = u_t$ the derivative of u with respect to time t in the sense of distributions in $Q_{\tau,T}$, defined by

$$\langle u', \phi \rangle := -\int_\tau^T \int_{\mathcal{O}_t} \phi'(x,t) u(x,t)\,dx\,dt \quad \text{for all functions } \phi \in C_c^\infty(Q_{\tau,T}),$$

where $\phi' = \frac{\partial \phi}{\partial t}$ is the classical partial derivative.

We have the following result.

Lemma 3.1. *If $u \in L^2(\tau, T; H_0^1(\mathcal{O}_t))$ and $u' \in L^2(\tau, T; L^2(\mathcal{O}_t))$, then the trivial extension $\hat{u}$ belongs to $H^1(\mathbb{R}^N \times (\tau, T))$, satisfies* (3.1)*, and its derivative with respect to time is given by*

$$\hat{u}' = \widehat{u'}. \tag{3.2}$$

Proof. Evidently, we must only prove (3.2). Observe that $u \in L^2(\tau, T; H_0^1(\mathcal{O}_t))$ and $u' \in L^2(\tau, T; L^2(\mathcal{O}_t))$, means that $u \in H^1(Q_{\tau,T})$. Applying integration by parts, and using that $u(t) \in H_0^1(\mathcal{O}_t)$ a.e. $t \in (\tau, T)$, for any function $\varphi \in C_c^1(\mathbb{R}^N \times (\tau, T))$ we have

$$\begin{aligned}
\int_{\mathbb{R}^N \times (\tau,T)} \hat{u}(x,t)\varphi'(x,t)\,dx\,dt &= \int_{Q_{\tau,T}} u(x,t)\varphi'(x,t)\,dx\,dt \\
&= -\int_{Q_{\tau,T}} u'(x,t)\varphi(x,t)\,dx\,dt \\
&= -\int_{\mathbb{R}^N \times (\tau,T)} \widehat{u'}(x,t)\varphi(x,t)\,dx\,dt,
\end{aligned}$$

and thus, (3.2) holds. □

Definition 3.2. We say that a function $u \in L^1_{\mathrm{loc}}(Q_{\tau,T})$ belongs to $C([\tau, T]; L^2(\mathcal{O}_t))$ if its trivial extension $\hat{u}$ belongs to $C([\tau, T]; L^2(\mathbb{R}^N))$ and we say that a sequence $\{u_m\}$ converges to u in $C([\tau, T]; L^2(\mathcal{O}_t))$ as $m \to \infty$, if the sequence $\{\hat{u}_m\}$ converges to $\hat{u}$ in $C([\tau, T]; L^2(\mathbb{R}^N))$ as $m \to \infty$.

Definition 3.3. We say that a function $u \in L^1_{\mathrm{loc}}(Q_{\tau,T})$ belongs to $C([\tau, T]; H_0^1(\mathcal{O}_t))$ if its trivial extension $\hat{u}$ belongs to $C([\tau, T]; H^1(\mathbb{R}^N))$ and we say that a sequence $\{u_m\}$ converges to u in $C([\tau, T]; H_0^1(\mathcal{O}_t))$ as $m \to \infty$, if the sequence $\{\hat{u}_m\}$ converges to $\hat{u}$ in $C([\tau, T]; H^1(\mathbb{R}^N))$ as $m \to \infty$.

From now on, we will use $(\cdot,\cdot)_t$ and $|\cdot|_t$ to denote the usual inner product and associated norm in $L^2(\mathcal{O}_t)$ or $(L^2(\mathcal{O}_t))^N$, indistinctly.

As a consequence of Definition 3.2 and Lemma 3.1, we have

Corollary 3.4. *If $u \in L^2(\tau, T; H_0^1(\mathcal{O}_t))$ and $u' \in L^2(\tau, T; L^2(\mathcal{O}_t))$, then u belongs to $C([\tau, T]; L^2(\mathcal{O}_t))$ and satisfies the energy equality*

$$\left|u(t_2)\right|_{t_2}^2 - \left|u(t_1)\right|_{t_1}^2 = 2\int_{t_1}^{t_2} \left(u'(t), u(t)\right)_t \, dt \quad \forall \tau \leqslant t_1 \leqslant t_2 \leqslant T. \tag{3.3}$$

Proof. It is enough to observe that by Lemma 3.1, in particular $\hat{u}$ and $\hat{u}'$ belong to $L^2(\tau, T; L^2(\mathbb{R}^N))$, and as a consequence, $\hat{u}$ belongs to $C([\tau, T]; L^2(\mathbb{R}^N))$, and by (3.2),

$$\left|\hat{u}(t_2)\right|_{L^2(\mathbb{R}^N)}^2 - \left|\hat{u}(t_1)\right|_{L^2(\mathbb{R}^N)}^2 = 2\int_{t_1}^{t_2} \left(\widehat{u'}(t), \hat{u}(t)\right)_{L^2(\mathbb{R}^N)} dt \quad \forall \tau \leqslant t_1 \leqslant t_2 \leqslant T.$$

But this last equality is exactly (3.3). □

3.2. Coordinate transformations

Following [15], we consider a finite time interval $[\tau, T]$, and set

$$v(y, t) = u\left(r(y, t), t\right) \quad \text{for } y \in \mathcal{O},\ t \in [\tau, T], \tag{3.4}$$

or, equivalently,

$$u(x, t) = v\left(\bar{r}(x, t), t\right) \quad \text{for } x \in \mathcal{O}_t,\ t \in [\tau, T]. \tag{3.5}$$

By the assumptions on r and $\bar{r}$, it is immediate to obtain the following result.

Lemma 3.5. *For any $1 \leqslant p, q \leqslant \infty$, $u \in L^q(\tau, T; L^p(\mathcal{O}_t)) \Leftrightarrow v \in L^q(\tau, T; L^p(\mathcal{O}))$. Moreover, there exist two positive constants $C_1(p, q)$ and $C_2(p, q)$ (which depend only on p, q, r and τ, T) such that*

$$C_1(p, q)\|u\|_{L^q(\tau,T;L^p(\mathcal{O}_t))} \leqslant \|v\|_{L^q(\tau,T;L^p(\mathcal{O}))} \leqslant C_2(p, q)\|u\|_{L^q(\tau,T;L^p(\mathcal{O}_t))}, \tag{3.6}$$

for any $u \in L^q(\tau, T; L^p(\mathcal{O}_t))$.

On the other hand, by Proposition IX.6 in [2], one has that if for some $t \in (\tau, T)$ the function $u(t) = u(\cdot, t)$ belongs to $H^1(\mathcal{O}_t)$, then the function $v(\cdot, t) = u(r(\cdot, t), t)$ belongs to $H^1(\mathcal{O})$, and

$$\frac{\partial v}{\partial y_j}(y, t) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}\left(r(y, t), t\right)\frac{\partial r_i(y, t)}{\partial y_j}, \tag{3.7}$$

and analogously, if for some $t \in (\tau, T)$ the function $v(t) = v(\cdot, t)$ belongs to $H^1(\mathcal{O})$, then the function $u(\cdot, t) = v(\bar{r}(\cdot, t), t)$ belongs to $H^1(\mathcal{O}_t)$, and

$$\frac{\partial u}{\partial x_i}(x, t) = \sum_{j=1}^{N} \frac{\partial v}{\partial y_j}\left(\bar{r}(x, t), t\right)\frac{\partial \bar{r}_j}{\partial x_i}(x, t). \tag{3.8}$$

From (3.7), (3.8), the denseness of $C_c^1(\mathcal{O}_t)$ in $H_0^1(\mathcal{O}_t)$, the denseness of $C_c^1(\mathcal{O})$ in $H_0^1(\mathcal{O})$, and the properties of r and $\bar{r}$, one easily obtain the following result:

Lemma 3.6. *$u \in L^2(\tau, T; H_0^1(\mathcal{O}_t)) \Leftrightarrow v \in L^2(\tau, T; H_0^1(\mathcal{O}))$. Moreover, there exist two positive constants C_1 and C_2 (which depend only on r and τ, T) such that*

$$C_1 \|u\|_{L^2(\tau,T;H_0^1(\mathcal{O}_t))} \leqslant \|v\|_{L^2(\tau,T;H_0^1(\mathcal{O}))} \leqslant C_2 \|u\|_{L^2(\tau,T;H_0^1(\mathcal{O}_t))}, \tag{3.9}$$

for any $u \in L^2(\tau, T; H_0^1(\mathcal{O}_t))$.

Analogously, one has:

Lemma 3.7. *$u \in L^2(\tau, T; H^2(\mathcal{O}_t)) \Leftrightarrow v \in L^2(\tau, T; H^2(\mathcal{O}))$. Moreover, there exist two positive constants C_1' and C_2' (which depend only on r and τ, T) such that*

$$C_1' \|u\|_{L^2(\tau,T;H^2(\mathcal{O}_t))} \leqslant \|v\|_{L^2(\tau,T;H^2(\mathcal{O}))} \leqslant C_2' \|u\|_{L^2(\tau,T;H^2(\mathcal{O}_t))}, \tag{3.10}$$

for any $u \in L^2(\tau, T; H^2(\mathcal{O}_t))$.

3.3. Continuity

Now we establish the equivalence of the continuity of u and v.

Lemma 3.8. *Under the assumptions on r, the function u belongs to $C([\tau, T]; L^2(\mathcal{O}_t))$ if and only if the function v given by* (3.4) *belongs to $C([\tau, T]; L^2(\mathcal{O}))$.*

Proof. (a) Assume first that v belongs to $C([\tau, T]; L^2(\mathcal{O}))$. By definition, we must prove that the trivial extension $\hat{u}$ belongs to $C([\tau, T]; L^2(\mathbb{R}^N))$.

For any pair $t_0, t \in [\tau, T]$, we have

$$\int_{\mathbb{R}^N} \left|\hat{u}(x,t) - \hat{u}(x,t_0)\right|^2 dx = \int_{\mathcal{O}_{t_0} \cap \mathcal{O}_t} \left|u(x,t) - u(x,t_0)\right|^2 dx + \int_{\mathcal{O}_{t_0} \setminus \mathcal{O}_t} \left|u(x,t_0)\right|^2 dx + \int_{\mathcal{O}_t \setminus \mathcal{O}_{t_0}} \left|u(x,t)\right|^2 dx. \tag{3.11}$$

In the following we will estimate the right-hand terms one by one.

At first, observe that as a consequence of the uniform continuity of r in $\overline{\mathcal{O}} \times [\tau, T]$,

$$\text{mes}(\mathcal{O}_{t_0} \setminus \mathcal{O}_t) \to 0 \quad \text{as } t \to t_0. \tag{3.12}$$

On the other hand, as $v(y, t_0) \in L^2(\mathcal{O})$, we have that $u(x, t_0) = v(\bar{r}(x, t_0), t_0) \in L^2(\mathcal{O}_{t_0})$, and therefore, by (3.12) we obtain

$$\int_{\mathcal{O}_{t_0} \setminus \mathcal{O}_t} \left|u(x,t_0)\right|^2 dx \to 0 \quad \text{as } t \to t_0. \tag{3.13}$$

Secondly, observe that by (3.12) and the properties of $\bar{r}$ we have $\text{mes}(\bar{r}(\mathcal{O}_t \setminus \mathcal{O}_{t_0}, t)) \to 0$ as $t \to t_0$. Thus, by the continuity of v, we have

$$\int\limits_{\mathcal{O}_t\setminus\mathcal{O}_{t_0}} \left|u(x,t)\right|^2 dx = \int\limits_{\bar{r}(\mathcal{O}_t\setminus\mathcal{O}_{t_0},t)} \left|v(y,t)\right|^2 \mathrm{Jac}(r,y,t)\,dy \leqslant C_r\Bigg(\int\limits_{\mathcal{O}} \left|v(y,t)-v(y,t_0)\right|^2 dy + \int\limits_{\bar{r}(\mathcal{O}_t\setminus\mathcal{O}_{t_0},t)} \left|v(y,t_0)\right|^2 dy\Bigg) \to 0 \quad \text{as } t\to t_0, \tag{3.14}$$

where we have denoted $\mathrm{Jac}(r,y,t)$ the absolute value of the determinant of the Jacobi matrix $(\frac{\partial r_i}{\partial y_j}(y,t))_{N\times N}$.

Finally, we have

$$\int\limits_{\mathcal{O}_{t_0}\cap\mathcal{O}_t} \left|u(x,t)-u(x,t_0)\right|^2 dx = \int\limits_{\bar{r}(\mathcal{O}_t\cap\mathcal{O}_{t_0},t_0)} \left|u\big(r(y,t_0),t\big)-v(y,t_0)\right|^2 \mathrm{Jac}(r,y,t_0)\,dy = \int\limits_{\bar{r}(\mathcal{O}_t\cap\mathcal{O}_{t_0},t_0)} \left|v\big(\bar{r}\big(r(y,t_0),t\big),t\big)-v(y,t_0)\right|^2 \mathrm{Jac}(r,y,t_0)\,dy \leqslant C_r \int\limits_{\bar{r}(\mathcal{O}_t\cap\mathcal{O}_{t_0},t_0)} \Big(\left|v\big(\bar{r}\big(r(y,t_0),t\big),t\big)-v\big(\bar{r}\big(r(y,t_0),t\big),t_0\big)\right|^2 + \left|v\big(\bar{r}\big(r(y,t_0),t\big),t_0\big)-v(y,t_0)\right|^2\Big)\,dy.$$

Note that

$$\int\limits_{\bar{r}(\mathcal{O}_t\cap\mathcal{O}_{t_0},t_0)} \left|v\big(\bar{r}\big(r(y,t_0),t\big),t\big)-v\big(\bar{r}\big(r(y,t_0),t\big),t_0\big)\right|^2 dy \leqslant \int\limits_{U_t} \left|v(z,t)-v(z,t_0)\right|^2 \mathrm{Jac}\big(f^{-1},z,t\big)\,dz \leqslant C_r \int\limits_{U_t} \left|v(z,t)-v(z,t_0)\right|^2 dz \to 0 \quad \text{as } t\to t_0, \tag{3.15}$$

where $z=f(y)=\bar{r}(r(y,t_0),t)$, $(\mathcal{O}\supset)U_t:=\bar{r}(\mathcal{O}_t\cap\mathcal{O}_{t_0},t_0)$ and we used the continuity of v.

On the other hand, for any arbitrary small $\varepsilon\ll 1$, we have

$$\bar{r}\big(r(y,t_0),t\big)\to y \quad \text{uniformly for } y\in\bar{r}\Bigg(\bigcap_{s\in[t_0-\varepsilon,t_0+\varepsilon]}\mathcal{O}_s,t_0\Bigg) \text{ as } t\to t_0,$$

$$(\mathcal{O}_{t_0}\cap\mathcal{O}_t)\setminus\bigcap_{s\in[t_0-\varepsilon,t_0+\varepsilon]}\mathcal{O}_s\subset\mathcal{O}_{t_0}\setminus\bigcap_{s\in[t_0-\varepsilon,t_0+\varepsilon]}\mathcal{O}_s,$$

and (since $\bar r(\cdot,\cdot)$ is Lipschitz)

$$\operatorname{mes}\biggl(\bar r\biggl(\mathcal{O}_{t_0}\setminus\bigcap_{s\in[t_0-\varepsilon,t_0+\varepsilon]}\mathcal{O}_s,t_0\biggr)\biggr)\leqslant C_r\operatorname{mes}\biggl(\mathcal{O}\setminus\bar r\biggl(\bigcap_{s\in[t_0-\varepsilon,t_0+\varepsilon]}\mathcal{O}_s,t_0\biggr)\biggr)\to 0\quad\text{as }\varepsilon\to 0,$$

therefore, as $t\in[t_0-\varepsilon,t_0+\varepsilon]$, from

$$\begin{aligned}&\int_{\bar r(\mathcal{O}_t\cap\mathcal{O}_{t_0},t_0)}\bigl|v\bigl(\bar r\bigl(r(y,t_0),t\bigr),t_0\bigr)-v(y,t_0)\bigr|^2\,dy\\&\quad\leqslant\int_{\bar r(\bigcap_{s\in[t_0-\varepsilon,t_0+\varepsilon]}\mathcal{O}_s,t_0)}\bigl|v\bigl(\bar r\bigl(r(y,t_0),t\bigr),t_0\bigr)-v(y,t_0)\bigr|^2\,dy\\&\qquad+2\int_{\bar r((\mathcal{O}_t\cap\mathcal{O}_{t_0})\setminus\bigcap_{s\in[t_0-\varepsilon,t_0+\varepsilon]}\mathcal{O}_s,t_0)}\bigl(\bigl|v\bigl(\bar r\bigl(r(y,t_0),t\bigr),t_0\bigr)\bigr|^2+\bigl|v(y,t_0)\bigr|^2\bigr)\,dy,\end{aligned}$$

we obtain

$$\int_{\bar r(\mathcal{O}_t\cap\mathcal{O}_{t_0},t_0)}\bigl|v\bigl(\bar r\bigl(r(y,t_0),t\bigr),t_0\bigr)-v(y,t_0)\bigr|^2\,dy\to 0\quad\text{as }t\to t_0.\tag{3.16}$$

Substituting (3.13)–(3.16) into (3.11), we obtain that $\hat u$ belongs to $C([\tau,T];L^2(\mathbb{R}^N))$.

(b) Conversely, assume that $\hat u$ belongs to $C([\tau,T];L^2(\mathbb{R}^N))$. We must prove that then $v\in C([\tau,T];L^2(\mathcal{O}))$.

For any $t_0,t\in[\tau,T]$,

$$\begin{aligned}\int_{\mathcal{O}}\bigl|v(y,t)-v(y,t_0)\bigr|^2\,dy&=\int_{\mathcal{O}_{t_0}}\bigl|v\bigl(\bar r(x,t_0),t\bigr)-u(x,t_0)\bigr|^2\operatorname{Jac}(\bar r,x,t_0)\,dx\\&=\int_{\mathcal{O}_{t_0}}\bigl|u\bigl(r\bigl(\bar r(x,t_0),t\bigr),t\bigr)-u(x,t_0)\bigr|^2\operatorname{Jac}(\bar r,x,t_0)\,dx\\&\leqslant C_r\bigl(I_1(t)+I_2(t)\bigr),\end{aligned}$$

here

$$\begin{aligned}I_1(t)&:=\int_{\mathcal{O}_{t_0}}\bigl|u\bigl(r\bigl(\bar r(x,t_0),t\bigr),t\bigr)-\hat u\bigl(r\bigl(\bar r(x,t_0),t\bigr),t_0\bigr)\bigr|^2\,dx\\&=\int_{\mathcal{O}_t}\bigl|u(z,t)-\hat u(z,t_0)\bigr|^2\operatorname{Jac}\bigl(h^{-1},z,t\bigr)\,dz\to 0\quad\text{as }t\to t_0,\end{aligned}\tag{3.17}$$

where $z=h(x)=r(\bar r(x,t_0),t):\mathcal{O}_{t_0}\mapsto\mathcal{O}_t$ and we used the continuity of $\hat u$; and

$$I_2(t):=\int_{\mathcal{O}_{t_0}}\bigl|\hat u\bigl(r\bigl(\bar r(x,t_0),t\bigr),t_0\bigr)-u(x,t_0)\bigr|^2\,dx$$

$$= \int_{\bigcap_{s\in[t_0-\varepsilon,t_0+\varepsilon]}\mathcal{O}_s} \left|\hat{u}\left(r\left(\bar{r}(x,t_0),t\right),t_0\right) - u(x,t_0)\right|^2 dx$$

$$+ \int_{\mathcal{O}_{t_0}\setminus\bigcap_{s\in[t_0-\varepsilon,t_0+\varepsilon]}\mathcal{O}_s} \left|\hat{u}\left(r\left(\bar{r}(x,t_0),t\right),t_0\right) - u(x,t_0)\right|^2 dx.$$

By a similar argument as that for (3.16) we can get that

$$I_2(t) \to 0 \quad \text{as } t \to t_0. \tag{3.18}$$

The continuity of v follows from (3.17) and (3.18) immediately. □

Similarly using Proposition IX.18 in [2], from (3.7) and (3.8) we can also get the following result:

Lemma 3.9. *Under the assumptions on r, the function u belongs to* $C([\tau,T];H_0^1(\mathcal{O}_t))$ *if and only if the function v given by* (3.4) *belongs to* $C([\tau,T];H_0^1(\mathcal{O}))$.

3.4. A compactness result

From Lemma 3.1 we know that if $u \in L^2(\tau,T;H_0^1(\mathcal{O}_t))$ and $u' \in L^2(\tau,T;L^2(\mathcal{O}_t))$, then u belongs to $H^1(Q_{\tau,T})$, and consequently, by Proposition IX.18 in [2] we deduce that the function v defined by (3.4) belongs to $H^1(\mathcal{O}\times(\tau,T))$, and in particular

$$v'(y,t) = u'\left(r(y,t),t\right) + \left[(\nabla_x u)\left(r(y,t),t\right)\right]\cdot\frac{\partial r}{\partial t}(r,t),$$

where $\cdot$ denotes the inner product in $\mathbb{R}^N$.

From this and Lemma 3.6, we obtain that if $u \in L^2(\tau,T;H_0^1(\mathcal{O}_t))$ and $u' \in L^2(\tau,T;L^2(\mathcal{O}_t))$, then the function v defined by (3.4) belongs to $L^2(\tau,T;H_0^1(\mathcal{O}))$, and its time derivative v' belongs to $L^2(\tau,T;L^2(\mathcal{O}))$, with

$$\|v'\|_{L^2(\tau,T;L^2(\mathcal{O}))} \leqslant C\left(\|u'\|_{L^2(\tau,T;L^2(\mathcal{O}_t))} + \|u\|_{L^2(\tau,T;H_0^1(\mathcal{O}_t))}\right),$$

for some positive constant C independent of u.

We now generalize the above considerations.

Definition 3.10. We define the space $L^2(\tau,T;H^{-1}(\mathcal{O}_t))$ as the vector space of all the distributions w in $Q_{\tau,T}$, of the form

$$w = f_0 - \sum_{i=1}^{N}\frac{\partial f_i}{\partial x_i}, \quad f_i \in L^2(Q_{\tau,T}),\ i = 0,\dots,N, \tag{3.19}$$

i.e.,

$$\langle w,\phi\rangle = \int_{Q_{\tau,T}} f_0\phi\,dx\,dt + \sum_{i=1}^{N}\int_{Q_{\tau,T}} f_i\frac{\partial\phi}{\partial x_i}\,dx\,dt \quad \forall\phi\in C_c^\infty(Q_{\tau,T}).$$

If w is given by (3.19), let us denote

$$w(t) = f_0(t) - \sum_{i=1}^{N} \frac{\partial f_i(t)}{\partial x_i} \quad \text{in the sense of distributions in } \mathcal{O}_t, \text{ a.e. } t \in (\tau, T),$$

where $f_i(t) := f_i(\cdot, t)$.

Let us denote $\langle \cdot,\cdot \rangle_{-1,t}$ the duality product between $H^{-1}(\mathcal{O}_t)$ and $H_0^1(\mathcal{O}_t)$.

Observe that if $\varphi \in C_c^1(\mathcal{O}_t)$, then

$$\langle w(t), \varphi \rangle_{-1,t} = \int_{\mathcal{O}_t} f_0(x,t)\varphi(x)\,dx + \sum_{i=1}^{N} \int_{\mathcal{O}_t} f_i(x,t) \frac{\partial \varphi}{\partial x_i}(x)\,dx,$$

and consequently

$$\|w(t)\|^2_{H^{-1}(\mathcal{O}_t)} \leqslant \sum_{i=0}^{N} \|f_i(t)\|^2_{L^2(\mathcal{O}_t)}.$$

It is well known that, without loss of generality, we can assume that in fact the above inequality is an equality. Thus, we define

$$\|w\|_{L^2(\tau,T;H^{-1}(\mathcal{O}_t))} := \left(\int_{\tau}^{T} \|w(t)\|^2_{H^{-1}(\mathcal{O}_t)}\,dt \right)^{1/2}, \tag{3.20}$$

and we have

$$\|w\|^2_{L^2(\tau,T;H^{-1}(\mathcal{O}_t))} = \sum_{i=0}^{N} \|f_i\|^2_{L^2(Q_{\tau,T})}. \tag{3.21}$$

Also, observe that if $\phi \in C_c^1(Q_{\tau,T})$, then $\phi(t) = \phi(\cdot, t)$ belongs to $C_c^1(\mathcal{O}_t)$ for each $t \in (\tau, T)$, and therefore

$$\begin{aligned} \langle w, \phi \rangle &= \int_{Q_{\tau,T}} f_0 \phi\,dx\,dt + \sum_{i=1}^{N} \int_{Q_{\tau,T}} f_i \frac{\partial \phi}{\partial x_i}\,dx\,dt \\ &= \int_{\tau}^{T} \int_{\mathcal{O}_t} f_0(x,t)\phi(x,t)\,dx\,dt + \sum_{i=1}^{N} \int_{\tau}^{T} \int_{\mathcal{O}_t} f_i(x,t) \frac{\partial \phi}{\partial x_i}(x,t)\,dx\,dt \\ &= \int_{\tau}^{T} \langle w(t), \phi(t) \rangle_{-1,t}\,dt. \end{aligned} \tag{3.22}$$

If $u \in L^2(\tau, T; H_0^1(\mathcal{O}_t))$, in particular $u \in L^2(Q_{\tau,T})$, then there exists the partial derivative u' of u with respect to t, in the sense of distributions on $Q_{\tau,T}$. We say that $u' \in L^2(\tau, T; H^{-1}(\mathcal{O}_t))$, if there exists a function $w \in L^2(\tau, T; H^{-1}(\mathcal{O}_t))$, such that

$$\int_\tau^T \langle w(t), \phi(t)\rangle_{-1,t}\, dt = -\int_{Q_{\tau,T}} u(x,t)\phi'(x,t)\, dx\, dt \quad \forall \phi \in C_c^1(Q_{\tau,T}).$$

In such a case, w is unique. This last assertion is an immediate consequence of (3.22). Thus, we identify u' with w.

Lemma 3.11.

$$\left.\begin{matrix} u \in L^2(\tau, T; H_0^1(\mathcal{O}_t)) \\ \text{and} \quad u' \in L^2(\tau, T; H^{-1}(\mathcal{O}_t)) \end{matrix}\right\} \iff \left\{\begin{matrix} v \in L^2(\tau, T; H_0^1(\mathcal{O})) \\ \text{and} \quad v' \in L^2(\tau, T; H^{-1}(\mathcal{O})). \end{matrix}\right.$$

Moreover, there are two positive constants C_3 and C_4 (which depend only on r and τ, T) such that

$$\|v'\|_{L^2(\tau,T;H^{-1}(\mathcal{O}))} \leqslant C_3\big(\|u'\|_{L^2(\tau,T;H^{-1}(\mathcal{O}_t))} + \|u\|_{L^2(\tau,T;H_0^1(\mathcal{O}_t))}\big) \tag{3.23}$$

and

$$\|u'\|_{L^2(\tau,T;H^{-1}(\mathcal{O}_t))} \leqslant C_4\big(\|v'\|_{L^2(\tau,T;H^{-1}(\mathcal{O}))} + \|v\|_{L^2(\tau,T;H_0^1(\mathcal{O}))}\big). \tag{3.24}$$

Proof. Suppose that $u \in L^2(\tau, T; H_0^1(\mathcal{O}_t))$ and $u' \in L^2(\tau, T; H^{-1}(\mathcal{O}_t))$.

At first, from Lemma 3.6 we know that $v(y,t) = u(r(t,y),t)$ satisfies

$$v \in L^2(\tau, T; H_0^1(\mathcal{O})). \tag{3.25}$$

Secondly, we have that there exist $N+1$ functions $f_i \in L^2(Q_{\tau,T})$, $i = 0, \dots, N$, such that

$$\int_\tau^T \langle u'(t), \phi(t)\rangle_{-1,t}\, dt = \int_\tau^T \int_{\mathcal{O}_t} f_0(x,t)\phi(x,t)\, dx\, dt + \sum_{i=1}^N \int_\tau^T \int_{\mathcal{O}_t} f_i(x,t)\frac{\partial \phi}{\partial x_i}(x,t)\, dx\, dt,$$

for all $\phi(x,t) \in C_c^1(Q_{\tau,T})$, and

$$\|u'\|^2_{L^2(\tau,T;H^{-1}(\mathcal{O}_t))} = \sum_{i=0}^N \|f_i\|^2_{L^2(Q_{\tau,T})}. \tag{3.26}$$

Thus,

$$\int_\tau^T \int_{\mathcal{O}_t} u(x,t)\phi'(x,t)\, dx\, dt = -\int_\tau^T \int_{\mathcal{O}_t} f_0(x,t)\phi(x,t)\, dx\, dt - \sum_{i=1}^N \int_\tau^T \int_{\mathcal{O}_t} f_i(x,t)\frac{\partial \phi}{\partial x_i}(x,t)\, dx\, dt, \tag{3.27}$$

for all $\phi(x,t) \in C_c^1(Q_{\tau,T})$.

Let us denote $\tilde{f}_i(y,t) = f_i(r(y,t),t)$, $\tilde{\phi}(y,t) = \phi(r(y,t),t)$, and $\psi(y,t) = \tilde{\phi}(y,t)\,\mathrm{Jac}(r,y,t)$. Observe that $\psi \in C_c^1(\mathcal{O} \times (\tau,T))$, and from the identity $\phi(x,t) = \tilde{\phi}(\bar{r}(x,t),t)$, we obtain

$$\frac{\partial \phi}{\partial x_i}(x,t) = \sum_{j=1}^{N} \frac{\partial \tilde{\phi}}{\partial y_j}\big(\bar{r}(x,t),t\big)\frac{\partial \bar{r}_j}{\partial x_i}(x,t) \quad \forall (x,t) \in Q_{\tau,T}. \tag{3.28}$$

Thus, from (3.27) we obtain

$$\begin{aligned}
&\int_\tau^T \int_{\mathcal{O}_t} u(x,t)\phi'(x,t)\,dx\,dt \\
&\quad = -\int_\tau^T \int_{\mathcal{O}} \tilde{f}_0(y,t)\psi(y,t)\,dy\,dt \\
&\qquad - \sum_{i,j=1}^{N} \int_\tau^T \int_{\mathcal{O}} \tilde{f}_i(y,t)\frac{\partial \bar{r}_j}{\partial x_i}\big(r(y,t),t\big)\frac{\partial \tilde{\phi}}{\partial y_j}(y,t)\,\mathrm{Jac}(r,y,t)\,dy\,dt.
\end{aligned} \tag{3.29}$$

Taking into account that

$$\begin{aligned}
\frac{\partial \psi}{\partial y_j}(y,t) &= \frac{\partial \tilde{\phi}}{\partial y_j}(y,t)\,\mathrm{Jac}(r,y,t) + \tilde{\phi}(y,t)\frac{\partial}{\partial y_j}\big(\mathrm{Jac}(r,y,t)\big) \\
&= \frac{\partial \tilde{\phi}}{\partial y_j}(y,t)\,\mathrm{Jac}(r,y,t) + \psi(y,t)\frac{1}{\mathrm{Jac}(r,y,t)}\frac{\partial}{\partial y_j}\big(\mathrm{Jac}(r,y,t)\big),
\end{aligned}$$

and denoting

$$g_j(y,t) := \sum_{i=1}^{N} \tilde{f}_i(y,t)\frac{\partial \bar{r}_j}{\partial x_i}\big(r(y,t),t\big), \quad j = 1,\dots,N, \tag{3.30}$$

$$g_0(y,t) := \tilde{f}_0(y,t) - \sum_{j=1}^{N} \tilde{g}_j(y,t)\frac{1}{\mathrm{Jac}(r,y,t)}\frac{\partial}{\partial y_j}\big(\mathrm{Jac}(r,y,t)\big), \tag{3.31}$$

we deduce from (3.29) that

$$\int_\tau^T \int_{\mathcal{O}_t} u(x,t)\phi'(x,t)\,dx\,dt = -\int_\tau^T \int_{\mathcal{O}} g_0(y,t)\psi(y,t)\,dy\,dt - \sum_{j=1}^{N} \int_\tau^T \int_{\mathcal{O}} g_j(y,t)\frac{\partial \psi}{\partial y_j}(y,t)\,dy\,dt. \tag{3.32}$$

On the other hand, we have

$$\int_\tau^T \int_{\mathcal{O}_t} u(x,t)\phi'(x,t)\,dx\,dt = \int_\tau^T \int_{\mathcal{O}} v(y,t)\phi'\big(r(y,t),t\big)\,\mathrm{Jac}(r,y,t)\,dy\,dt. \tag{3.33}$$

From the identity $\tilde{\phi}(y,t) = \phi(r(y,t),t)$, and (3.28), we deduce

$$\tilde{\phi}'(y,t) = \phi'\big(r(y,t),t\big) + \sum_{i=1}^{N} \frac{\partial \phi}{\partial x_i}\big(r(y,t),t\big)\frac{\partial r_i}{\partial t}(y,t)$$
$$= \phi'\big(r(y,t),t\big) + \sum_{i,j=1}^{N} \frac{\partial \tilde{\phi}}{\partial y_j}(y,t)\frac{\partial r_i}{\partial t}(y,t)\frac{\partial \bar{r}_j}{\partial x_i}\big(r(y,t),t\big). \tag{3.34}$$

But the identity $\bar{r}(r(y,t),t) = y$ implies that

$$\frac{\partial \bar{r}_j}{\partial t}\big(r(y,t),t\big) + \sum_{i=1}^{N} \frac{\partial \bar{r}_j}{\partial x_i}\big(r(y,t),t\big)\frac{\partial r_i}{\partial t}(y,t) = 0. \tag{3.35}$$

From (3.34) and (3.35) we deduce that

$$\phi'\big(r(y,t),t\big) = \tilde{\phi}'(y,t) + \sum_{j=1}^{N} \frac{\partial \tilde{\phi}}{\partial y_j}(y,t)\frac{\partial \bar{r}_j}{\partial t}\big(r(y,t),t\big),$$

and therefore (3.33) can be written

$$\int_\tau^T \int_{\mathcal{O}_t} u(x,t)\phi'(x,t)\,dx\,dt$$
$$= \int_\tau^T \int_{\mathcal{O}} v(y,t)\tilde{\phi}'(y,t)\,\mathrm{Jac}(r,y,t)\,dy\,dt$$
$$+ \sum_{j=1}^{N} \int_\tau^T \int_{\mathcal{O}} v(y,t)\frac{\partial \tilde{\phi}}{\partial y_j}(y,t)\frac{\partial \bar{r}_j}{\partial t}\big(r(y,t),t\big)\,\mathrm{Jac}(r,y,t)\,dy\,dt. \tag{3.36}$$

Now, observing that

$$\tilde{\phi}'(y,t)\,\mathrm{Jac}(r,y,t) = \psi'(y,t) - \frac{1}{\mathrm{Jac}(r,y,t)}\frac{\partial}{\partial t}\big(\mathrm{Jac}(r,y,t)\big)\psi(y,t),$$

and

$$\frac{\partial \tilde{\phi}}{\partial y_j}(y,t)\,\mathrm{Jac}(r,y,t) = \frac{\partial \psi}{\partial y_j}(y,t) - \frac{1}{\mathrm{Jac}(r,y,t)}\frac{\partial}{\partial y_j}\big(\mathrm{Jac}(r,y,t)\big)\psi(y,t),$$

we can rewrite (3.36) as

$$\int_\tau^T \int_{\mathcal{O}_t} u(x,t)\phi'(x,t)\,dx\,dt = \int_\tau^T \int_{\mathcal{O}} v(y,t)\psi'(y,t)\,dy\,dt$$
$$- \int_\tau^T \int_{\mathcal{O}} \frac{v(y,t)}{\mathrm{Jac}(r,y,t)}\frac{\partial}{\partial t}\big(\mathrm{Jac}(r,y,t)\big)\psi(y,t)\,dy\,dt$$

$$+\sum_{j=1}^{N}\int_{\tau}^{T}\int_{\mathcal{O}} v(y,t)\frac{\partial \bar{r}_j}{\partial t}\big(r(y,t),t\big)\frac{\partial \psi}{\partial y_j}(y,t)\,dy\,dt$$

$$-\sum_{j=1}^{N}\int_{\tau}^{T}\int_{\mathcal{O}} \frac{v(y,t)}{\mathrm{Jac}(r,y,t)}\frac{\partial \bar{r}_j}{\partial t}\big(r(y,t),t\big)\frac{\partial}{\partial y_j}\big(\mathrm{Jac}(r,y,t)\big)\psi(y,t)\,dy\,dt.$$

From this last equality and (3.32), we deduce

$$\begin{aligned}
&\int_{\tau}^{T}\int_{\mathcal{O}} v(y,t)\psi'(y,t)\,dy\,dt \\
&= \int_{\tau}^{T}\int_{\mathcal{O}} \left\{ \frac{v(y,t)}{\mathrm{Jac}(r,y,t)}\left[\frac{\partial}{\partial t}\big(\mathrm{Jac}(r,y,t)\big) + \sum_{j=1}^{N}\frac{\partial \bar{r}_j}{\partial t}\big(r(y,t),t\big)\frac{\partial}{\partial y_j}\big(\mathrm{Jac}(r,y,t)\big)\right] - g_0(y,t)\right\}\psi(y,t)\,dy\,dt \\
&\quad - \sum_{j=1}^{N}\int_{\tau}^{T}\int_{\mathcal{O}} \left(v(y,t)\frac{\partial \bar{r}_j}{\partial t}\big(r(y,t),t\big) + g_j(y,t)\right)\frac{\partial \psi}{\partial y_j}(y,t)\,dy\,dt,
\end{aligned} \tag{3.37}$$

for any $\psi(y,t) \in C_c^1(\mathcal{O}\times(\tau,T))$ (the arbitrariness of ψ comes from the arbitrariness of ϕ and the fact that $r(\cdot,\cdot)$ is a diffeomorphism). Thus, from (3.37) we have that the derivative of v with respect to time is given by

$$v' = h_0 - \sum_{j=1}^{N}\frac{\partial h_j}{\partial y_j} \tag{3.38}$$

in the sense of distributions on $\mathcal{O}\times(\tau,T)$, where

$$h_0(y,t) := g_0(y,t) - \frac{v(y,t)}{\mathrm{Jac}(r,y,t)}\left[\frac{\partial}{\partial t}\big(\mathrm{Jac}(r,y,t)\big) + \sum_{j=1}^{N}\frac{\partial \bar{r}_j}{\partial t}\big(r(y,t),t\big)\frac{\partial}{\partial y_j}\big(\mathrm{Jac}(r,y,t)\big)\right],$$

and

$$h_j(y,t) := v(y,t)\frac{\partial \bar{r}_j}{\partial t}\big(r(y,t),t\big) + g_j(y,t), \quad j = 1,\dots,N.$$

From (3.38) and the definitions of h_0 and h_j, $j = 1,\dots,N$, we obtain that $v' \in L^2(\tau,T;H^{-1}(\mathcal{O}))$. Moreover, using (3.26), we easily deduce (3.23).

Similarly, using the inverse transformation $\bar{r}(x,t)$ of $r(y,t)$, we can obtain the converse results. □

Let $p \geqslant 2$, and set

$$X = L^2\big(\tau,T;H_0^1(\mathcal{O}_t)\big)\cap L^p\big(\tau,T;L^p(\mathcal{O}_t)\big),$$
$$Y = L^2\big(\tau,T;H_0^1(\mathcal{O})\big)\cap L^p\big(\tau,T;L^p(\mathcal{O})\big),$$

and

$$X^* = L^2\big(\tau, T; H^{-1}(\mathcal{O}_t)\big) + L^q\big(\tau, T; L^q(\mathcal{O}_t)\big),$$
$$Y^* = L^2\big(\tau, T; H^{-1}(\mathcal{O})\big) + L^q\big(\tau, T; L^q(\mathcal{O})\big),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

From the previous results, evidently we have:

Corollary 3.12. $u \in X \Leftrightarrow v \in Y$.

With a slight modification of the proof of Lemma 3.11, we have:

Lemma 3.13.

$$\left.\begin{aligned} u \in X \\ \text{and} \quad u' \in X^* \end{aligned}\right\} \quad \Longleftrightarrow \quad \begin{cases} v \in Y \\ \text{and} \quad v' \in Y^*. \end{cases}$$

Moreover, there are two positive constants C_5 and C_6 (which depend only on r and τ, T) such that

$$\|v'\|_{Y^*} \leqslant C_5\big(\|u'\|_{X^*} + \|u\|_{L^2(\tau,T;H_0^1(\mathcal{O}_t))}\big) \tag{3.39}$$

and

$$\|u'\|_{X^*} \leqslant C_6\big(\|v'\|_{Y^*} + \|v\|_{L^2(\tau,T;H_0^1(\mathcal{O}))}\big). \tag{3.40}$$

As a consequence of the results above, we have the following compactness criterion.

Lemma 3.14. *Assume that $\{u_m\}$ is a bounded sequence in X and $\{u'_m\}$ is bounded in X^*. Then, $\{u_m\}$ is relatively compact in $L^2(\tau, T; L^2(\mathcal{O}_t))$.*

Proof. Let $v_m(y,t) = u_m(r(y,t),t)$, then from Lemma 3.13, we know that $\{v_m\}$ is bounded in Y and $\{v'_m\}$ is bounded in Y^*. Then by the well-known result for fixed domain (e.g., see [17]), we know that $\{v_m\}$ is relatively compact in $L^2(\tau, T; L^2(\mathcal{O}))$. Without loss generality, we assume $v_m \to v$ in $L^2(\tau, T; L^2(\mathcal{O}))$, then from

$$\int_\tau^T \int_{\mathcal{O}_t} \big|u_m(x,t) - u(x,t)\big|^2 \,dx\,dt = \int_\tau^T \int_{\mathcal{O}} \big|v_m(y,t) - v(y,t)\big|^2 \operatorname{Jac}(r, y, t)\,dy\,dt,$$

we deduce that $u_m \to u$ in $L^2(\tau, T; L^2(\mathcal{O}_t))$, where $u(x,t) = v(\bar{r}(x,t),t)$. □

Corollary 3.15. *Assume $\{u_m\}$ is a bounded sequence in $L^2(\tau, T; H_0^1(\mathcal{O}_t))$ and $\{u'_m\}$ is bounded in $L^2(\tau, T; H^{-1}(\mathcal{O}_t))$. Then, $\{u_m\}$ is relatively compact in $L^2(\tau, T; L^2(\mathcal{O}_t))$.*

4. Strong solutions

Definition 4.1 *(Strong solution).* A function $u = u(x,t)$ defined in $Q_{\tau,T}$ is said to be a strong solution for problem (2.6) if

$$u \in L^2\big(\tau, T; H^2(\mathcal{O}_t)\big) \cap C\big([\tau, T]; H_0^1(\mathcal{O}_t)\big) \cap L^\infty\big(\tau, T; L^p(\mathcal{O}_t)\big), \qquad u' \in L^2\big(\tau, T; L^2(\mathcal{O}_t)\big),$$

and the three equations in (2.6) are satisfied almost everywhere in their corresponding domains.

Associated to problem (2.6), we consider the problem

$$\begin{cases} \dfrac{\partial v(y,t)}{\partial t} - \displaystyle\sum_{k,j=1}^{N} \frac{\partial}{\partial y_j}\big(a_{jk}(y,t) v_{y_k}(y,t)\big) + b(y,t)\cdot\nabla_y v(y,t) + g\big(v(y,t)\big) = f\big(r(y,t),t\big) \\ \quad \text{in } \mathcal{O}\times(\tau,T), \\ v = 0 \quad \text{on } \partial\mathcal{O}\times(\tau,T), \\ v(y,\tau) = u_\tau\big(r(y,\tau)\big), \quad y\in\mathcal{O}, \end{cases} \tag{4.1}$$

where

$$a_{jk}(y,t) = \sum_{i=1}^{N} \frac{\partial \bar r_k}{\partial x_i}\big(r(y,t),t\big)\frac{\partial \bar r_j}{\partial x_i}\big(r(y,t),t\big), \quad j,k = 1,\dots,N;$$

and $b(y,t) = (b_1(y,t),\dots,b_N(y,t)) \in \mathbb{R}^N$ is defined by

$$b_k(y,t) = \frac{\partial \bar r_k}{\partial t}\big(r(y,t),t\big) - \Delta_x \bar r_k\big(r(y,t),t\big) + \sum_{j=1}^{N}\frac{\partial a_{jk}}{\partial y_j}(y,t), \quad k = 1,2,\dots,N.$$

A strong solution of (4.1) is a function $v \in L^2(\tau,T;H^2(\mathcal{O})) \cap C([\tau,T];H_0^1(\mathcal{O})) \cap L^\infty(\tau,T;L^p(\mathcal{O}))$, with time derivative $v' \in L^2(\tau,T;L^2(\mathcal{O}))$, such that the three equations in (4.1) are satisfied almost everywhere in their corresponding domains.

Lemma 4.2. *For any* $-\infty < \tau \leqslant T < \infty$, $a_{jk} \in C^1(\overline{\mathcal{O}}\times[\tau,T])$, $b_k \in C^0(\overline{\mathcal{O}}\times[\tau,T])$. *In particular,* $a_{jk}, \frac{\partial a_{jk}}{\partial y_j}, b_k \in L^\infty(\mathcal{O}\times(\tau,T))$, $j,k = 1,2,\dots,N$.

Moreover, there exists a $\delta = \delta(r,\tau,T) > 0$ *such that for any* $(y,t) \in \mathcal{O}\times[\tau,T]$,

$$\sum_{j,k=1}^{N} a_{jk}(y,t)\xi_j\xi_k \geqslant \delta|\xi|^2 \quad \text{for all } \xi\in\mathbb{R}^N.$$

Proof. Let $A(y,t) = (a_{jk}(y,t))_{N\times N}$, then $A(y,t) = T^*(y,t)T(y,t)$, where $T(y,t) := F(r(y,t),t)$ with

$$F(x,t) = \begin{pmatrix} \frac{\partial \bar r_1(x,t)}{\partial x_1} & \frac{\partial \bar r_2(x,t)}{\partial x_1} & \cdots & \frac{\partial \bar r_N(x,t)}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \bar r_1(x,t)}{\partial x_N} & \frac{\partial \bar r_2(x,t)}{\partial x_N} & \cdots & \frac{\partial \bar r_N(x,t)}{\partial x_N} \end{pmatrix},$$

and $T^*(y,t)$ is the transpose of $T(y,t)$.

For any $\xi\in\mathbb{R}^N$, we have

$$\sum_{j,k=1}^{N} a_{jk}(y,t)\xi_j\xi_k = \big(A(y,t)\xi,\xi\big)_{\mathbb{R}^N} = \big(T^*(y,t)T(y,t)\xi,\xi\big)_{\mathbb{R}^N} = (T\xi,T\xi)_{\mathbb{R}^N} = \big\|T(y,t)\xi\big\|^2.$$

Note that $T(y,t)$ is reversible, so

$$\big\|T(y,t)\xi\big\| \geqslant \big\|T^{-1}(y,t)\big\|^{-1}\|\xi\|,$$

where $\|T^{-1}(y,t)\|$ is the operator-norm of $T^{-1}(y,t)$ in $\mathbb{R}^N$; consequently, we have

$$(A\xi,\xi) \geqslant \frac{\|\xi\|^2}{\|T^{-1}(y,t)\|^2}.$$

Finally, due to the continuity of $T^{-1}(y,t)$ and compactness of $\overline{\mathcal{O}} \times [\tau, T]$, we know that $\|T^{-1}(y,t)\|$ is uniformly bounded from below in $\overline{\mathcal{O}} \times [\tau, T]$ by a positive constant. $\square$

From the assumptions on $\partial\mathcal{O}$, r and $\bar{r}$, we have (e.g., see [14,9])

Lemma 4.3. *For any $-\infty < \tau \leqslant T < \infty$, there exist two positive constants δ_0 and c_0 which depend on r, τ, T, such that for any $u \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$, the following estimate holds*

$$\delta_0 \int_{\mathcal{O}} \left|\Delta u(y)\right|^2 dy \leqslant \int_{\mathcal{O}} \sum_{k,j=1}^{N} a_{kj}(y,t) u_{y_k y_j} \Delta u \, dy + c_0 \int_{\mathcal{O}} \left|u(y)\right|^2 dy \quad \textit{for all } t \in [\tau, T].$$

We have the following existence and uniqueness result of strong solution for problem (2.6).

Theorem 4.4. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\mathcal{O}_t))$ and r and $\bar{r}$ satisfy assumptions* (2.1), (2.2) *and* (2.4)*. Assume also that*

$$\partial\mathcal{O} \text{ is } C^2 \text{ and } N \leqslant 2p/(p-2), \quad \text{or} \quad \partial\mathcal{O} \text{ is } C^m \text{ with } m \geqslant 2 \text{ integer such that } m \geqslant N(p-2)/2p. \tag{4.2}$$

Then, for any $u_\tau \in H_0^1(\mathcal{O}_\tau) \cap L^p(\mathcal{O}_\tau)$ and any $-\infty < \tau \leqslant T < \infty$, there exists a unique strong solution u of (2.6)*. Moreover, u satisfies the equality of energy*

$$\begin{aligned} &\left|u(t)\right|_t^2 + 2\int_\tau^t \left|\nabla u(s)\right|_s^2 ds + 2\int_\tau^t \int_{\mathcal{O}_s} g\big(u(x,s)\big)u(x,s)\,dx\,ds \\ &\quad = \left|u(\tau)\right|_\tau^2 + \int_\tau^t \big(f(s), u(s)\big)_s ds \quad \textit{for all } t \in [\tau, T], \end{aligned} \tag{4.3}$$

and the following estimates:

$$\left|u(t)\right|_t^2 \leqslant e^{-\lambda_{\tau t}(t-\tau)}\left|u(\tau)\right|_\tau^2 + e^{-\lambda_{\tau t} t}\int_\tau^t e^{\lambda_{\tau t} s}\|f(s)\|^2_{H^{-1}(\mathcal{O}_s)} ds + 2\beta e^{-\lambda_{\tau t} t}\int_\tau^t e^{\lambda_{\tau t} s}|\mathcal{O}_s|\,ds, \tag{4.4}$$

$$\int_\tau^t \left(\left|\nabla u(s)\right|_s^2 + 2\alpha_1 \int_{\mathcal{O}_s} \left|u(s)\right|^p dx\right) ds \leqslant \int_\tau^t \|f(s)\|^2_{H^{-1}(\mathcal{O}_s)} ds + 2\beta \int_\tau^t |\mathcal{O}_s|\,ds + |u_\tau|_\tau^2, \tag{4.5}$$

for all $t \in [\tau, T]$, where $\lambda_{\tau t}$ is the first eigenvalue of $-\Delta$ on $H_0^1(\bigcup_{\tau \leqslant s \leqslant t} \mathcal{O}_s)$.

Proof. (a) Uniqueness.

Let $u_{\tau i} \in H_0^1(\mathcal{O}_\tau) \cap L^p(\mathcal{O}_\tau)$ and $u_i(t)$ $(i = 1, 2)$ be the corresponding strong solutions. Set $w(t) = u_1(t) - u_2(t)$, then $w(t)$ is a strong solution of the following equation

$$\begin{cases} w_t - \Delta w + g(u_1) - g(u_2) = 0 & \text{in } Q_{\tau,T}, \\ w(\tau) = u_{\tau 1} - u_{\tau 2} & \text{on } \mathcal{O}_\tau, \\ w = 0 & \text{on } \Sigma_{\tau,T}. \end{cases}$$

By Corollary 3.4 and (2.8), we have

$$\frac{1}{2}\frac{d}{ds}\left|w(s)\right|_s^2+\left|\nabla w(s)\right|_s^2 \leqslant -\int_{\mathcal{O}_s}\left(g\big(u_1(s)\big)-g\big(u_2(s)\big)\right)w(s)\,dx$$
$$\leqslant l\left|w(s)\right|_s^2 \quad \text{a.e. } s\in(\tau,T),$$

so, integrating over $[\tau,t]$ we deduce that

$$\left|w(t)\right|_t^2 \leqslant e^{2l(t-\tau)}\left|w(\tau)\right|_\tau^2 = e^{2l(t-\tau)}|u_{\tau 1}-u_{\tau 2}|_\tau^2 \quad \text{for any } t\in[\tau,T], \tag{4.6}$$

which implies the uniqueness immediately.

(b) Equality (4.3) and estimates (4.4)–(4.5).

The energy equality (4.3) is a direct consequence of Corollary 3.4.

As a consequence of (4.3), (2.7), and Hölder inequality, we get

$$\frac{d}{ds}\left|u(s)\right|_s^2+\left|\nabla u(s)\right|_s^2+2\alpha_1|u|_{L^p(\mathcal{O}_s)}^p \leqslant \left\|f(s)\right\|_{H^{-1}(\mathcal{O}_s)}^2+2\beta|\mathcal{O}_s| \quad \text{a.e. } s\in(\tau,T). \tag{4.7}$$

Estimate (4.5) is a direct consequence of (4.7). Also, by (4.7) and the definition of $\lambda_{\tau t}$, observing that $\tau\leqslant s\leqslant t \Rightarrow \lambda_{\tau s}\geqslant\lambda_{\tau t}$, we have in particular

$$\frac{d}{ds}\left|u(s)\right|_s^2+\lambda_{\tau t}\left|u(s)\right|_s^2 \leqslant \left\|f(s)\right\|_{H^{-1}(\mathcal{O}_s)}^2+2\beta|\mathcal{O}_s| \quad \text{a.e. } s\in(\tau,t),$$

and multiplying this inequality by $\exp(\lambda_{\tau t}s)$ and integrating between τ and t, we have (4.4).

(c) Existence.

From the results in the previous sections, and using Proposition IX.18 in [2], one obtain (see [15] for the linear case) that $u=u(x,t)$ is a strong solution for problem (2.6) if and only if the function $v=v(y,t):=u(r(y,t),t)$ is a strong solution of the problem (4.1).

Thus, to prove existence of strong solution of problem (2.6) it is enough to prove existence of strong solution of problem (4.1).

The existence of strong solutions for the transformed equation (4.1) can be obtained by the Galerkin method (see [9,17,18]). We sketch the proof.

Without loss of the generality, we can assume that $g(0)=0$ (if not, then let $\hat{g}(\cdot)=g(\cdot)-g(0)$ and $\hat{f}(t)=f(t)-g(0)$).

As in [9], we define the time-dependent bilinear form

$$B[v,w;t]=\int_{\mathcal{O}}\left(\sum_{k,j=1}^{N}a_{kj}(y,t)\frac{\partial v}{\partial y_k}\frac{\partial w}{\partial y_j}+\sum_{k=1}^{N}b_k(y,t)\frac{\partial v}{\partial y_k}w\right)dy \tag{4.8}$$

for $v,w\in H_0^1(\mathcal{O})$ and $\tau\leqslant t\leqslant T$.

Now, let $\omega_k=\omega_k(y)\in H^2(\mathcal{O})\cap H_0^1(\mathcal{O})$ $(k=1,2,\ldots)$ be the eigenfunctions of $-\Delta$ on $H_0^1(\mathcal{O})$, and let $0<\lambda_1<\lambda_2\leqslant\lambda_3\leqslant\cdots$ be the corresponding eigenvalues. Then,

$$\lambda_n\to\infty \quad \text{as } n\to\infty$$

and we can assume that

$$\{\omega_k\}_{k=1}^\infty \text{ is an orthogonal basis of } H_0^1(\mathcal{O}) \text{ and an orthonormal basis of } L^2(\mathcal{O}).$$

For each fixed positive integer m, set

$$v_m(t) := \sum_{k=1}^{m} d_m^k(t)\omega_k, \tag{4.9}$$

and consider the finite dimensional approximate system

$$\begin{cases} \big(v_m'(t), \omega_k\big) + B\big[v_m(t), \omega_k; t\big] + \big(g\big(v_m(t)\big), \omega_k\big) = \big(\tilde{f}(t), \omega_k\big), \\ \quad k = 1, \dots, m \text{ and } \tau \leqslant t \leqslant T, \\ v_m(\tau) = P_m v_\tau, \end{cases} \tag{A_m}$$

where

$$\tilde{f}(y,t) := f\big(r(y,\tau), t\big), \qquad v_\tau(y) := u_\tau\big(r(y,\tau)\big),$$

$(\cdot,\cdot)$ is the inner product in $L^2(\mathcal{O})$, with associated norm $|\cdot|$, and P_m is the projector from $L^2(\mathcal{O})$ to span$\{\omega_1, \omega_2, \dots, \omega_m\}$. Observe that

$$\tilde{f} \in L^2\big(\tau, T; L^2(\mathcal{O})\big), \qquad v_\tau \in H_0^1(\mathcal{O}) \cap L^p(\mathcal{O}),$$

and thanks to the assumption (4.2),

$$P_m v_\tau \to v_\tau \quad \text{in } H_0^1(\mathcal{O}) \cap L^p(\mathcal{O}) \text{ as } m \to \infty. \tag{4.10}$$

Noticing that $g \in C^1(\mathbb{R})$, then as a direct consequence of the existence and uniqueness result for ODEs, we have that for each integer $m = 1, 2, \dots$ there exists a unique local solution v_m of the form (4.9) satisfying (A_m), defined in an interval $[\tau, T_m]$, with $\tau < T_m \leqslant T$.

Now we obtain several estimates about the functions v_m.

Step 1. Multiplying (A_m) by $d_m^k(t)$ and taking the sum with respect to k from 1 to m, we get that

$$\frac{1}{2}\frac{d}{dt}\big|v_m(t)\big|^2 + B\big[v_m(t), v_m(t); t\big] + \big(g\big(v_m(t)\big), v_m(t)\big) = \big(\tilde{f}(t), v_m(t)\big), \quad t \in [\tau, T_m].$$

Then from Lemma 4.2, (4.8) and (2.7) we know that there is a positive constant δ (which depends only on τ, T and the transform function r) such that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big|v_m(t)\big|^2 + \delta\big|\nabla v_m(t)\big|^2 + \alpha_1\big\|v_m(t)\big\|_{L^p(\mathcal{O})}^p \\ &\quad \leqslant \big\|\tilde{f}(t)\big\|_{H^{-1}(\mathcal{O})} \cdot \big\|v_m(t)\big\|_{H_0^1(\mathcal{O})} + \beta|\mathcal{O}| + N^{1/2} \max_{1 \leqslant k \leqslant N} \|b_k\|_{L^\infty(\mathcal{O}\times[\tau,T])}\big|\nabla v_m(t)\big|\big|v_m(t)\big| \\ &\quad \leqslant \frac{\delta}{2}\big|\nabla v_m(t)\big|^2 + C_{\delta,b}\big(\big\|\tilde{f}(t)\big\|_{H^{-1}(\mathcal{O})}^2 + \big|v_m(t)\big|^2\big) + \beta|\mathcal{O}|, \quad t \in [\tau, T_m], \end{aligned}$$

which, combining with the Gronwall lemma and the fact that $|P_m v_\tau| \leqslant |v_\tau|$, implies that

$$T_m = T \quad \text{for any } m = 1, 2, \dots, \tag{4.11}$$

and

$$\text{the sequence } \{v_m\} \text{ is bounded in } C^0\big([\tau, T]; L^2(\mathcal{O})\big) \cap L^2\big(\tau, T; H_0^1(\mathcal{O})\big) \cap L^p\big(\tau, T; L^p(\mathcal{O})\big). \tag{4.12}$$

Step 2. Multiplying (A_m) by $\lambda_k d_m^k(t)$ and summing over $k = 1, \ldots, m$, also noting that $-\Delta v_m = \sum_{k=1}^m \lambda_k d_m^k(t)\omega_k$ equals to 0 on $\partial\mathcal{O}$ and using Lemma 4.3, we deduce that

$$\frac{1}{2}\frac{d}{dt}\left|\nabla v_m(t)\right|^2 + \delta_0\left|\Delta v_m(t)\right|^2 - \int_{\mathcal{O}} g\big(v_m(y,t)\big)\Delta v_m(y,t)\,dy$$
$$\leqslant \left|\tilde{f}(t)\right|\left|\Delta v_m(t)\right| + c_0\left|v_m(t)\right|^2 + N^{1/2}\max_{1\leqslant k\leqslant N}|\bar{b}_k|_{L^\infty(\mathcal{O}\times(\tau,T))}\left|\nabla v_m(t)\right|\left|\Delta v_m(t)\right|, \quad t\in[\tau,T],$$

where

$$\bar{b}_k(y,t) := b_k(y,t) - \sum_{j=1}^N \frac{\partial a_{jk}}{\partial y_j}(y,t), \quad k = 1,2,\ldots,N.$$

From the above inequality, (2.8) and $g(0) = 0$, we can obtain that

$$\frac{d}{dt}\left|\nabla v_m(t)\right|^2 + \delta_0\left|\Delta v_m(t)\right|^2 \leqslant C_{\delta_0,b,l}\big(\left|\tilde{f}(t)\right|^2 + \left|\nabla v_m(t)\right|^2\big), \quad t\in[\tau,T].$$

From this inequality, (4.12), and the fact that $P_m v_\tau$ is bounded in $H_0^1(\mathcal{O})$, we get that

$$\text{the sequence } \{v_m\} \text{ is bounded in } C^0\big([\tau,T];H_0^1(\mathcal{O})\big)\cap L^2\big(\tau,T;H^2(\mathcal{O})\big). \tag{4.13}$$

Step 3. Similarly, multiplying (A_m) by $d_m^{k\prime}(t)$, summing over $k = 1,\ldots,m$, and using that $a_{k,j} = a_{j,k}$, we can get that

$$\left|v_m'(t)\right|^2 + \frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}}\sum_{k,j=1}^N a_{kj}(y,t)\frac{\partial v_m}{\partial y_j}(y,t)\frac{\partial v_m}{\partial y_k}(y,t)\,dy$$
$$-\frac{1}{2}\int_{\mathcal{O}}\sum_{k,j=1}^N \frac{\partial a_{kj}}{\partial t}(y,t)\frac{\partial v_m}{\partial y_j}(y,t)\frac{\partial v_m}{\partial y_k}(y,t)\,dy + \frac{d}{dt}\int_{\mathcal{O}} G\big(v_m(y,t)\big)\,dy$$
$$\leqslant \left|\tilde{f}(t)\right|\left|v_m'(t)\right| + C_b\left|\nabla v_m(t)\right|\left|v_m'(t)\right|, \quad t\in[\tau,T].$$

Integrating over $[\tau,T]$, and using Lemma 4.2, the Cauchy inequality, (4.12), (2.9), and the facts that $a_{kj}\in C^1(\overline{\mathcal{O}}\times[\tau,T])$ $(k,j = 1,2,\ldots,N)$, and by (4.10) the sequence $P_m v_\tau$ is bounded in $H_0^1(\mathcal{O})\cap L^p(\mathcal{O})$, we obtain that

$$\text{the sequence } \{v_m\} \text{ is bounded in } L^\infty\big(\tau,T;L^p(\mathcal{O})\big), \tag{4.14}$$

and

$$\text{the sequence } \big\{v_m'\big\} \text{ is bounded in } L^2\big(\tau,T;L^2(\mathcal{O})\big). \tag{4.15}$$

Step 4. From (4.10), (4.13), (4.14) and (4.15), it is now a standard matter (e.g., see [9,17,18]) to prove that a subsequence of $\{v_m\}$ (in fact, by uniqueness, all the sequence) converges weakly in $L^2(\tau,T;H^2(\mathcal{O}))$, weakly star in $L^\infty(\tau,T;H_0^1(\mathcal{O}))\cap L^\infty(\tau,T;L^p(\mathcal{O}))$, and strongly in $L^2(\tau,T;H_0^1(\mathcal{O}))$, to a function v that is (the unique) strong solution of (4.1). □

5. Weak solutions

Let us denote

$$\mathcal{U}_{\tau,T} := \{\varphi \in L^2(\tau, T; H_0^1(\mathcal{O}_t)) \cap L^p(\tau, T; L^p(\mathcal{O}_t)):\ \varphi' \in L^2(\tau, T; L^2(\mathcal{O}_t)),\ \varphi(\tau) = \varphi(T) = 0\}.$$

Definition 5.1. Let $u_\tau \in L^2(\mathcal{O}_\tau)$, $f \in L^2(\tau, T; H^{-1}(\mathcal{O}_t))$ and $-\infty < \tau \leqslant T < \infty$ be given. We say that a function u is a weak solution of (2.6) if

(1) $u \in C([\tau, T]; L^2(\mathcal{O}_t)) \cap L^2(\tau, T; H_0^1(\mathcal{O}_t)) \cap L^p(\tau, T; L^p(\mathcal{O}_t))$ with $u(\tau) = u_\tau$;
(2) there exists a sequence of regular data $u_{\tau m} \in H_0^1(\mathcal{O}_\tau) \cap L^p(\mathcal{O}_\tau)$ and $f_m \in L^2(\tau, T; L^2(\mathcal{O}_t))$, $m = 1, 2, \ldots$, such that

$$u_{\tau m} \to u_\tau \quad \text{in } L^2(\mathcal{O}_\tau), \qquad f_m \to f \quad \text{in } L^2(\tau, T; H^{-1}(\mathcal{O}_t)),$$

and

$$u_m \to u \quad \text{in } C([\tau, T]; L^2(\mathcal{O}_t)),$$

where u_m is the unique strong solution of (2.6) corresponding to $(u_{\tau m}, f_m)$;
(3) for all $\varphi \in \mathcal{U}_{\tau,T}$,

$$\int_\tau^T \int_{\mathcal{O}_t} u(x,t)\varphi'(x,t)\,dx\,dt + \int_\tau^T \int_{\mathcal{O}_t} \nabla_x u \cdot \nabla_x \varphi\,dx\,dt$$
$$= -\int_\tau^T \int_{\mathcal{O}_t} g(u(x,t))\varphi(x,t)\,dx\,dt + \int_\tau^T \int_{\mathcal{O}_t} f(x,t)\varphi(x,t)\,dx\,dt. \tag{5.1}$$

It is clear from the definition, that every strong solution of (2.6) is a weak solution of (2.6).

Theorem 5.2. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H^{-1}(\mathcal{O}_t))$, and the functions r and $\bar{r}$ satisfy assumptions* (2.1), (2.2) *and* (2.4)*. Assume also that $\partial\Omega$ is C^2 and $N \leqslant 2p/(p-2)$, or $\partial\Omega$ is C^m with $m \geqslant 2$ integer such that $m \geqslant N(p-2)/2p$. Then for any $u_\tau \in L^2(\mathcal{O}_\tau)$ and any $-\infty < \tau \leqslant T < \infty$, there exists a unique weak solution $u(t)$ for Eq.* (2.6)*. Moreover, $u(t)$ satisfies the following estimates for all $t \in [\tau, T]$,*

$$|u(t)|_t^2 \leqslant e^{-\lambda_{\tau t}(t-\tau)}|u(\tau)|_\tau^2 + e^{-\lambda_{\tau t} t}\int_\tau^t e^{\lambda_{\tau t} s}\|f(s)\|^2_{H^{-1}(\mathcal{O}_s)}\,ds + 2\beta e^{-\lambda_{\tau t} t}\int_\tau^t e^{\lambda_{\tau t} s}|\mathcal{O}_s|\,ds, \tag{5.2}$$

$$\int_\tau^t \Big(|\nabla u(s)|_s^2 + 2\alpha_1 \int_{\mathcal{O}_s} |u(s)|^p\,dx\Big)ds \leqslant \int_\tau^t \|f(s)\|^2_{H^{-1}(\mathcal{O}_s)}\,ds + 2\beta\int_\tau^t |\mathcal{O}_s|\,ds + |u_\tau|_\tau^2. \tag{5.3}$$

Proof. Let $u_{\tau m} \in H_0^1(\mathcal{O}_\tau) \cap L^p(\mathcal{O}_\tau)$ and $f_m \in L^2(\tau, T; L^2(\mathcal{O}_t))$ such that

$$u_{\tau m} \to u_\tau \quad \text{in } L^2(\mathcal{O}_\tau) \quad \text{and} \quad f_m \to f \quad \text{in } L^2(\tau, T; H^{-1}(\mathcal{O}_t)) \text{ as } m \to \infty. \tag{5.4}$$

Then for each $(u_{\tau m}, f_m)$, $m = 1, 2, \ldots$, there exists a unique strong solution $u_m = u_m(t)$ for the following problem:

$$\begin{cases} \dfrac{\partial u_m}{\partial t} - \Delta u_m + g(u_m) = f_m(t) & \text{in } Q_{\tau,T}, \\ u_m = 0 \quad \text{on } \Sigma_{\tau,T}, \\ u_m(\tau) = u_{\tau m} \quad \text{in } \mathcal{O}_\tau. \end{cases} \tag{5.5}$$

Moreover, from (4.4) and (4.5) we have that

$$\text{the sequence } \{u_m\} \text{ is bounded in } L^2\big(\tau, T; H_0^1(\mathcal{O}_t)\big) \cap L^p\big(\tau, T; L^p(\mathcal{O}_t)\big). \tag{5.6}$$

Therefore, taking into account Lemmas 3.5 and 3.6, we can extract a subsequence (denoted also by $\{u_m\}$) such that

$$u_m \rightharpoonup u \quad \text{weakly in } L^2\big(\tau, T; H_0^1(\mathcal{O}_t)\big), \tag{5.7}$$

$$u_m \rightharpoonup u \quad \text{weakly in } L^p\big(\tau, T; L^p(\mathcal{O}_t)\big), \tag{5.8}$$

$$g(u_m) \rightharpoonup \Psi \quad \text{weakly in } L^q\big(\tau, T; L^q(\mathcal{O}_t)\big). \tag{5.9}$$

At the same time, by Corollary 3.4, (2.8), and Gronwall lemma, we have

$$\big|u_m(t) - u_n(t)\big|_t^2 \leqslant e^{2l(t-\tau)}|u_{\tau m} - u_{\tau n}|_\tau^2 + e^{2l(t-\tau)} \int_\tau^t \big\| f_m(s) - f_n(s)\big\|_{H^{-1}(\mathcal{O}_s)}^2 \, ds \quad \text{for any } t \in [\tau, T], \tag{5.10}$$

and therefore $\{u_m\}$ is a Cauchy sequence in $C([\tau, T]; L^2(\mathcal{O}_t))$. So, by the uniqueness of the limit and (5.7), we know that

$$u_m \to u \quad \text{in } C\big([\tau, T]; L^2(\mathcal{O}_t)\big). \tag{5.11}$$

Hence, extracting a subsequence if necessary, we can assume that $g(u_m) \to g(u)$, a.e. in $Q_{\tau,T}$, and then, by (5.9), we have $\Psi = g(u)$.

On the other hand, for any test function $\varphi \in \mathcal{U}_{\tau,T}$, we know that u_m satisfies (5.1). Then, using (5.7), (5.9), and (5.11), by passing to the limit, we obtain that u also satisfies (5.1). So, u is a weak solution of (2.6) with initial data u_τ.

The estimates (5.2) and (5.3) follow from (4.4), (4.5), (5.4) and (5.11) directly. The uniqueness follows easily from (5.10). □

6. Process $U(t, \tau)$ generated by the weak solutions

Definition 6.1. A function $u : \bigcup_{t \in [\tau,\infty)} \mathcal{O}_t \times \{t\} \to \mathbb{R}$ is called a weak solution of (2.5) if for any $T \geqslant \tau$, the restriction of u on $\bigcup_{t \in [\tau,T]} \mathcal{O}_t \times \{t\}$ is a weak solution of (2.6).

By Theorem 5.2, we have:

Theorem 6.2. *Under the assumptions of Theorem* 5.2, *for any* $u_\tau \in L^2(\mathcal{O}_\tau)$ *and* $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H^{-1}(\mathcal{O}_t))$, (2.5) *has a unique weak solution. This weak solution satisfies* (5.2) *and* (5.3) *for all* $t \in [\tau, \infty)$.

Consider a fixed $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H^{-1}(\mathcal{O}_t))$. From Theorem 6.2 above, we can define the operators

$$U(t,\tau): L^2(\mathcal{O}_\tau) \to L^2(\mathcal{O}_t), \quad -\infty < \tau \leqslant t < \infty, \tag{6.1}$$

by

$$U(t,\tau)u_\tau := u(t;\tau,u_\tau) = u(t) \quad \text{for any } u_\tau \in L^2(\mathcal{O}_\tau), \tag{6.2}$$

where $u(\cdot;\tau,u_\tau)$ is the unique weak solution of (2.5). Since $u \in C([\tau,T]; L^2(\mathcal{O}_t))$ for any $T \geqslant \tau$, the inclusion $u(t) \in L^2(\mathcal{O}_t)$ makes sense.

Then, by the existence and uniqueness again, we know that the family operators $\{U(t,\tau)\colon -\infty < \tau \leqslant t < \infty\}$ forms a process, that is:

$$U(\tau,\tau) = \mathrm{Id}\ \big(\text{identity on } L^2(\mathcal{O}_\tau)\big) \quad \forall \tau \in \mathbb{R}, \tag{6.3}$$

$$U(t,s)U(s,\tau) = U(t,\tau) \quad \text{for all } -\infty < \tau \leqslant s \leqslant t < \infty. \tag{6.4}$$

In the following, we will give some properties of the process $\{U(t,\tau)\colon -\infty < \tau \leqslant t < \infty\}$ defined above.

Lemma 6.3. *Let $u^i_\tau \in L^2(\mathcal{O}_\tau)$ and $u^i(s) = U(s,\tau)u^i_\tau$ $(i = 1,2)$. Then, we have*

$$\big|u^1(t) - u^2(t)\big|_t^2 \leqslant e^{2l(t-s)}\big|u^1(s) - u^2(s)\big|_s^2 \quad \textit{for any } \tau \leqslant s \leqslant t. \tag{6.5}$$

Proof. Let us fix $t > \tau$. By definition we know that there are two sequences $\{(u^i_{\tau m}, f^i_m)\}$ $(i = 1,2)$ satisfying $u^i_{\tau m} \in H^1_0(\mathcal{O}_\tau) \cap L^p(\mathcal{O}_\tau)$ and $f^i_m \in L^2(\tau,t; L^2(\mathcal{O}_s))$ such that

$$u^i_{\tau m} \to u^i_\tau \quad \text{in } L^2(\mathcal{O}_s) \quad \text{and} \quad f^i_m \to f \quad \text{in } L^2\big(\tau,t; H^{-1}(\mathcal{O}_s)\big) \text{ as } m \to \infty, \tag{6.6}$$

and

$$u^i_m \to u^i \quad \text{in } C^0\big([\tau,t]; L^2(\mathcal{O}_s)\big),\ i = 1,2, \tag{6.7}$$

where u^i_m is the unique strong solution corresponding to the regular data $(u^i_{\tau m}, f^i_m)$.

Then, similarly to (5.10), we have

$$\big|u^1_m(t) - u^2_m(t)\big|_t^2 \leqslant e^{2l(t-s)}\big|u^1_m(s) - u^2_m(s)\big|_s^2 + e^{2l(t-s)}\int_s^t \big\|f^1_m(\theta) - f^2_m(\theta)\big\|^2_{H^{-1}(\mathcal{O}_\theta)}\,d\theta. \tag{6.8}$$

Therefore, we get (6.5) immediately from (6.6) and (6.7). □

As a direct consequence of (6.5), we have the following continuity result:

Lemma 6.4. *For any $-\infty < \tau \leqslant t < \infty$, the operator $U(t,\tau): L^2(\mathcal{O}_\tau) \to L^2(\mathcal{O}_t)$ is continuous.*

7. Pullback $\mathscr{D}_{\lambda_1}$-attractor

Throughout this section, we assume that

$$\Omega_t := \bigcup_{s \leqslant t} \mathcal{O}_s \quad \text{is bounded, for any } t \in \mathbb{R}. \tag{7.1}$$

For any $t \in \mathbb{R}$, we denote

$$\lambda_{1,t} := \min_{v \in H_0^1(\Omega_t),\, v \neq 0} \frac{|\nabla v|^2_{(L^2(\Omega_t))^N}}{|v|^2_{L^2(\Omega_t)}}, \tag{7.2}$$

the first eigenvalue of $-\Delta$ on $H_0^1(\Omega_t)$. Observe that

$$\lambda_{\tau t} \geqslant \lambda_{1,t} \quad \text{for all } \tau \leqslant t, \tag{7.3}$$

and

$$\lambda_{1,s} \geqslant \lambda_{1,t} \quad \text{for all } s \leqslant t. \tag{7.4}$$

7.1. Pullback $\mathscr{D}_{\lambda_1}$-absorbing set

Let $u_\tau \in L^2(\mathcal{O}_\tau)$ and $u(t) = U(t,\tau)u_\tau$.
From (5.2) and (7.3) we have that

$$\left|u(t)\right|_t^2 \leqslant e^{-\lambda_{1,t}(t-\tau)}|u_\tau|_\tau^2 + e^{-\lambda_{1,t}t}\int_\tau^t e^{\lambda_{1,t}s}\left\|f(s)\right\|^2_{H^{-1}(\mathcal{O}_s)}\,ds + 2\beta|\Omega_t|\lambda_{1,t}^{-1} \quad \forall \tau \leqslant t. \tag{7.5}$$

Let R_{λ_1} be the set of all functions $\rho : \mathbb{R} \to [0,\infty)$ such that

$$e^{\lambda_{1,\tau}\tau}\rho^2(\tau) \to 0 \quad \text{as } \tau \to -\infty,$$

and $\mathscr{D}_{\lambda_1}$ the class of all families $\hat{D} := \{D(t)\colon\ t \in \mathbb{R},\ \ D(t) \subset L^2(\mathcal{O}_t),\ \ D(t) \neq \emptyset\}$, such that $D(t) \subset \{u \in L^2(\mathcal{O}_t)\colon\ |u|_t \leqslant \rho_{\hat{D}}(t)\}$ for some $\rho_{\hat{D}} \in R_{\lambda_1}$.
We assume further that $f \in L^2_{\text{loc}}(\mathbb{R}; H^{-1}(\mathcal{O}_t))$ satisfies

$$\int_{-\infty}^t e^{\lambda_{1,t}s}\left\|f(s)\right\|^2_{H^{-1}(\mathcal{O}_s)}\,ds < \infty \quad \text{for all } t \in \mathbb{R}. \tag{7.6}$$

For each $t \in \mathbb{R}$, we set $R(t)$ the positive number given by

$$R^2(t) = e^{-\lambda_{1,t}t}\int_{-\infty}^t e^{\lambda_{1,t}s}\left\|f(s)\right\|^2_{H^{-1}(\mathcal{O}_s)}\,ds + 2\beta|\Omega_t|\lambda_{1,t}^{-1} + 1.$$

Lemma 7.1. *The family of sets*

$$\hat{D}_0 := \left\{\left\{u \in L^2(\mathcal{O}_t)\colon\ |u|_t \leqslant R(t)\right\}\colon\ t \in \mathbb{R}\right\} \tag{7.7}$$

is a pullback $\mathscr{D}_{\lambda_1}$-absorbing family for the solution process $U(t,\tau)$, that is, for any $\hat{D}=\{D(\tau)\colon \tau\in\mathbb{R}\}\in\mathscr{D}_{\lambda_1}$ and any $t\in\mathbb{R}$, there is a $T(t,\hat{D})>0$ satisfying

$$U(t,\tau)D(\tau)\subset\big\{u\in L^2(\mathcal{O}_t)\colon |u|_t\leqslant R(t)\big\}\quad \text{for all } t-\tau\geqslant T(t,\hat{D}). \tag{7.8}$$

Moreover, $\hat{D}_0$ belongs to $\mathscr{D}_{\lambda_1}$.

Proof. That $\hat{D}_0$ is pullback $\mathscr{D}_{\lambda_1}$-absorbing, is an immediate consequence of (7.5).

To see that $\hat{D}_0$ belongs to $\mathscr{D}_{\lambda_1}$, we need to prove that $e^{\lambda_{1,t}t}R^2(t)\to 0$ as $t\to-\infty$. But this is a consequence of the fact that by (7.4) and assumption (7.6), for any $t\leqslant 0$ we have

$$\begin{aligned}e^{\lambda_{1,t}t}R^2(t)&=\int_{-\infty}^{t} e^{\lambda_{1,t}s}\big\|f(s)\big\|^2_{H^{-1}(\mathcal{O}_s)}\,ds+2\beta|\Omega_t|e^{\lambda_{1,t}t}\lambda_{1,t}^{-1}+e^{\lambda_{1,t}t}\\&\leqslant\int_{-\infty}^{t} e^{\lambda_{1,0}s}\big\|f(s)\big\|^2_{H^{-1}(\mathcal{O}_s)}\,ds+2\beta|\Omega_0|e^{\lambda_{1,0}t}\lambda_{1,0}^{-1}+e^{\lambda_{1,0}t}\to 0\quad\text{as } t\to-\infty.\qquad\square\end{aligned}$$

7.2. Pullback $\mathscr{D}_{\lambda_1}$-asymptotic compactness

We recall the notion of pullback $\mathscr{D}_{\lambda_1}$-asymptotic compactness (see [6,13]).

Definition 7.2. The process $U(t,\tau)$ is said to be pullback $\mathscr{D}_{\lambda_1}$-asymptotically compact if the sequence $\{U(t,\tau_n)u_n\}$ is relatively compact in $L^2(\mathcal{O}_t)$ for any $t\in\mathbb{R}$, any $\hat{D}=\{D(\tau)\colon \tau\in\mathbb{R}\}\in\mathscr{D}_{\lambda_1}$, and any sequences $\{\tau_n\}$ and $\{u_n\}$ with $\tau_n\to-\infty$ and $u_n\in D(\tau_n)$.

Let us denote

$$B_{t-1}:=\big\{u\in L^2(\mathcal{O}_{t-1})\colon |u|_{t-1}\leqslant e^{\lambda_{1,t}/2}R(t)\big\},$$

for each $t\in\mathbb{R}$.

We have the following result:

Lemma 7.3. *The family of sets $\{U(t,t-1)B_{t-1}\colon t\in\mathbb{R}\}$ is pullback $\mathscr{D}_{\lambda_1}$-absorbing for the process $U(t,\tau)$. Moreover, for each $t\in\mathbb{R}$, the set $U(t,t-1)B_{t-1}$ is a relatively compact set of $L^2(\mathcal{O}_t)$.*

Proof. Let us fix $t\in\mathbb{R}$, and denote $u(t)=U(t,\tau)u_\tau$. From (7.5) and (7.3), for any $\tau\leqslant t-1$, we have that

$$\begin{aligned}\big|u(t-1)\big|^2_{t-1}&\leqslant e^{-\lambda_{1,t-1}(t-1-\tau)}|u_\tau|^2_\tau+e^{-\lambda_{1,t-1}(t-1)}\int_\tau^{t-1} e^{\lambda_{1,t-1}s}\big\|f(s)\big\|^2_{H^{-1}(\mathcal{O}_s)}\,ds+2\beta|\Omega_{t-1}|\lambda_{1,t-1}^{-1}\\&\leqslant e^{\lambda_{1,t}}\left(e^{-\lambda_{1,t}(t-\tau)}|u_\tau|^2_\tau+e^{-\lambda_{1,t}t}\int_\tau^{t-1} e^{\lambda_{1,t}s}\big\|f(s)\big\|^2_{H^{-1}(\mathcal{O}_s)}\,ds+2\beta|\Omega_t|\lambda_{1,t}^{-1}\right)\\&<e^{\lambda_{1,t}}\cdot R^2(t)\quad\text{as }\tau\to-\infty,\end{aligned}\tag{7.9}$$

which implies that for any $\hat{D}=\{D(\tau)\colon \tau\in\mathbb{R}\}\in\mathscr{D}_{\lambda_1}$,

$$U(t-1,\tau)D(\tau) \subset B_{t-1} \quad \text{as } \tau \to -\infty,$$

and consequently the family $\{U(t,t-1)B_{t-1}\colon t \in \mathbb{R}\}$ is pullback $\mathscr{D}_{\lambda_1}$-absorbing.

In the following, we will show that, for each $t \in \mathbb{R}$, the set $U(t,t-1)B_{t-1}$ is relatively compact in $L^2(\mathcal{O}_t)$.

Step 1. From (5.3) we have

$$\int_0^1 \left(\left|\nabla u(s)\right|_{t-1+s}^2 + 2\alpha_1 \int_{\mathcal{O}_{t-1+s}} \left|u(s)\right|^p dx \right) ds \leqslant \int_{t-1}^t \left\| f(s)\right\|_{H^{-1}(\mathcal{O}_s)}^2 ds + 2\beta|\Omega_t| + |u_0|_{t-1}^2, \tag{7.10}$$

where $u_0 \in B_{t-1}$ and $u(s) = U(t-1+s,t-1)u_0$.

Set $U(t-1+\cdot,t-1)u_0$ the function $U(t-1+\cdot,t-1)u_0 : s \in [0,1] \to U(t-1+s,t-1)u_0 \in L^2(\mathcal{O}_{t-1+s})$, and denote

$$\mathcal{A}_{[t-1,t]} := \left\{U(t-1+\cdot,t-1)u_0\colon\ u_0 \in B_{t-1}\right\}.$$

Then (7.10) shows that

$$\mathcal{A}_{[t-1,t]} \text{ is bounded in } L^2\left(0,1; H_0^1(\mathcal{O}_{t-1+s})\right) \cap L^p\left(0,1; L^p(\mathcal{O}_{t-1+s})\right). \tag{7.11}$$

Consequently, combining this with the equation

$$u' = \Delta u - g(u) + f(t),$$

we know that the set $\{u'\colon\ u \in \mathcal{A}_{[t-1,t]}\}$ is bounded in $L^2(0,1; H^{-1}(\mathcal{O}_{t-1+s})) + L^q(0,1; L^q(\mathcal{O}_{t-1+s}))$. Then, combining this with (7.11), from Lemma 3.14 we know that

$$\mathcal{A}_{[t-1,t]} \text{ is relatively compact in } L^2\left(0,1; L^2(\mathcal{O}_{t-1+s})\right). \tag{7.12}$$

Step 2. Let $u_{0i} \in B_{t-1}$ and $u_i(s) = U(t-1+s,t-1)u_{0i}$, $i = 1,2$. Set $w(s) = u_1(s) - u_2(s)$, then from Lemma 6.3 we have that

$$\left|w(1)\right|_t^2 \leqslant e^{2l}\left|w(s)\right|_{t-1+s}^2 \quad \text{for any } s \in [0,1]. \tag{7.13}$$

Step 3. From (7.12), for any $\varepsilon > 0$, there exist $u_i \in A_{[t,t-1]}$, $i = 1,2,\dots,m_\varepsilon$, such that for any $u \in A_{[t,t-1]}$, there is some u_i satisfying

$$\int_0^1 \left|u(s) - u_i(s)\right|_{t-1+s}^2 ds < e^{-2l}\varepsilon,$$

consequently, there is a $\theta \in [0,1]$ such that

$$\left|u(\theta) - u_i(\theta)\right|_{t-1+\theta}^2 \leqslant e^{-2l}\varepsilon,$$

which, combining with (7.13), implies that

$$\left|u(1) - u_i(1)\right|_t^2 \leqslant \varepsilon. \tag{7.14}$$

Then, by the arbitrariness of ε we know that $U(t,t-1)B_{t-1}$ is relatively compact in $L^2(\mathcal{O}_t)$ for all $t \in \mathbb{R}$. $\square$

As an immediate consequence of the preceding lemma, we have:

Corollary 7.4. *The process $U(t,\tau)$ is pullback $\mathscr{D}_{\lambda_1}$-asymptotically compact.*

7.3. Pullback $\mathscr{D}$-attractor

For each $t \in \mathbb{R}$, and D_1, D_2 nonempty subsets of $L^2(\mathcal{O}_t)$, let us denote $\operatorname{dist}_t(D_1, D_2)$ the Hausdorff semi-distance defined as

$$\operatorname{dist}_t(D_1, D_2) := \sup_{u \in D_1} \inf_{v \in D_2} |u - v|_t.$$

Definition 7.5. A family $\hat{\mathscr{A}} = \{\mathscr{A}(t)\colon\ \mathscr{A}(t) \subset L^2(\mathcal{O}_t),\ \mathscr{A}(t) \neq \emptyset,\ t \in \mathbb{R}\}$ is said to be a pullback $\mathscr{D}_{\lambda_1}$-attractor for the process $U(t,\tau)$, if:

(1) $\mathscr{A}(t)$ is compact in $L^2(\mathcal{O}_t)$ for all $t \in \mathbb{R}$;
(2) $\hat{\mathscr{A}}$ is pullback $\mathscr{D}_{\lambda_1}$-attracting, i.e.,

$$\lim_{\tau \to -\infty} \operatorname{dist}_t\big(U(t,\tau)D(\tau), \mathscr{A}(t)\big) = 0 \quad \text{for all } \hat{D} \in \mathscr{D}_{\lambda_1} \text{ and all } t \in \mathbb{R};$$

(3) $\hat{\mathscr{A}}$ is invariant, i.e.,

$$U(t,\tau)\mathscr{A}(\tau) = \mathscr{A}(t) \quad \text{for any } -\infty < \tau \leqslant t < \infty.$$

Using the results in [5,6,13], from Lemma 6.4 and Corollary 7.4, and the fact that the sets in $\hat{D}_0$ are closed, and the family $\mathscr{D}_{\lambda_1}$ is inclusion closed, we obtain that $U(t,\tau)$ has a pullback $\mathscr{D}_{\lambda_1}$-attractor, and more exactly:

Theorem 7.6. *Under the assumptions of Theorem* 5.2*, furthermore, assume* (7.1) *and that f satisfies* (7.6)*. Then the family $\hat{\mathscr{A}} = \{\mathscr{A}(t)\colon\ t \in \mathbb{R}\}$ defined by*

$$\mathscr{A}(t) = \Lambda(\hat{D}_0, t), \quad t \in \mathbb{R},$$

where $\hat{D}_0$ is given by (7.7)*, and for any $\hat{D} \in \mathscr{D}_{\lambda_1}$,*

$$\Lambda(\hat{D}, t) := \bigcap_{s \leqslant t}\Big(\overline{\bigcup_{\tau \leqslant s} U(t,\tau)D(\tau)}^{L^2(\mathcal{O}_t)}\Big), \quad t \in \mathbb{R} \quad \big(\textit{closure in } L^2(\mathcal{O}_t)\big),$$

is the unique pullback $\mathscr{D}_{\lambda_1}$-attractor for the process $U(t,\tau)$ belonging to $\mathscr{D}_{\lambda_1}$. In addition, $\hat{\mathscr{A}}$ satisfies

$$\mathscr{A}(t) = \overline{\bigcup_{\hat{D} \in \mathscr{D}_{\lambda_1}} \Lambda(\hat{D}, t)}^{L^2(\mathcal{O}_t)} \quad \forall t \in \mathbb{R}.$$

Furthermore, $\hat{\mathscr{A}}$ is minimal in the sense that if $\hat{C} = \{C(t)\colon\ t \in \mathbb{R}\}$ is a family of nonempty sets such that $C(t)$ is a closed subset of $L^2(\mathcal{O}_t)$ and

$$\lim_{\tau \to -\infty} \operatorname{dist}_t\big(U(t,\tau)D_0(\tau), C(t)\big) = 0$$

for all $t \in \mathbb{R}$, then $\mathscr{A}(t) \subset C(t)$ for any $t \in \mathbb{R}$.

8. An example

Let $r(y,t)=h(t)y$ for any $t\in\mathbb{R}$ and $y\in\overline{\mathcal{O}}\subset\mathbb{R}^N$, where $h\in C^1(\mathbb{R})$, and satisfies

$$h(t)\neq 0 \quad \text{and} \quad \sup_{s\leqslant t}\big|h(s)\big|<\infty \quad \text{for all } t\in\mathbb{R}.$$

So $\mathcal{O}_t=h(t)\mathcal{O}$, and $\bar{r}(x,t)=h(t)^{-1}x$ for $x\in\mathcal{O}_t$ and $t\in\mathbb{R}$. We can apply our results to this example.

In particular, for $u(x,t)=v(\bar{r}(x,t),t)=v(y,t)$ we have

$$\frac{\partial u}{\partial x_i}(x,t)=\sum_{j=1}^{N}\frac{\partial v}{\partial y_j}(y,t)\frac{\partial \bar{r}_j(x,t)}{\partial x_i}=h^{-1}(t)\frac{\partial v}{\partial y_i}(y,t),$$

and

$$\frac{\partial^2 u}{\partial x_i^2}(x,t)=h^{-1}(t)\sum_{k=1}^{N}\frac{\partial^2 v}{\partial y_i\partial y_k}(y,t)\frac{\partial \bar{r}_k}{\partial x_i}(x,t)=h^{-2}(t)\frac{\partial^2 v}{\partial y_i^2}(y,t),$$

where $\bar{r}_j(x,t)=h^{-1}(t)x_j$ for $j=1,\dots,N$. So,

$$\Delta_x u(x,t)=h^{-2}(t)\Delta_y v(y,t).$$

On the other hand,

$$\frac{\partial u}{\partial t}(x,t)=\frac{\partial v}{\partial t}(y,t)+\sum_{i=1}^{N}\frac{\partial v}{\partial y_i}(y,t)\frac{\partial r_i}{\partial t}(y,t)=\frac{\partial v}{\partial t}(y,t)+h'(t)\sum_{i=1}^{N}y_i\frac{\partial v}{\partial y_i}(y,t).$$

Thus, the auxiliary equation can be transformed to

$$\begin{cases}\dfrac{\partial v(y,t)}{\partial t}-h^{-2}(t)\Delta v(y,t)+h'(t)b(v,y,t)+g\big(v(y,t)\big)=f\big(h(t)y,t\big) & \text{in } \mathcal{O}\times(\tau,T),\\ v=0 \quad \text{on } \partial\mathcal{O}\times(\tau,T), & \\ v(y,\tau)=u_\tau\big(h(\tau)y\big), \quad y\in\mathcal{O}, & \end{cases} \tag{8.1}$$

where $b(v,y,t)=\nabla_y v(y,t)\cdot y$.

It would be interesting to investigate the effect of the temporal behaviour of the function h, such as periodicity or almost periodicity, on that of the subsets of the pullback attractor.

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