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Global stability for the multi-channel Gel'fand–Calderón inverse problem in two dimensions

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Abstract

We prove a global logarithmic stability estimate for the multi-channel Gel'fand–Calderón inverse problem on a two-dimensional bounded domain, i.e., the inverse boundary value problem for the equation $-\Delta\psi + v\psi = 0$ on D, where v is a smooth matrix-valued potential defined on a bounded planar domain D.

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1. Introduction

The Schrödinger equation at zero energy,

$$-\Delta\psi + v(x)\psi = 0 \quad \text{on } D \subset \mathbb{R}^2, \tag{1.1}$$

arises in quantum mechanics, acoustics and electrodynamics. The reconstruction of the complex-valued potential v in Eq. (1.1) through the Dirichlet-to-Neumann operator is one of the most studied inverse problems (see [11,10,4,12–14] and references therein).

In this article we consider the multi-channel two-dimensional Schrödinger equation, i.e., Eq. (1.1) with matrix-valued potentials and solutions; this case was already studied in [15,14]. One of the motivations for studying the multi-channel equation is that it comes up as a 2D-approximation for the 3D equation (see [14, Section 2]).

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The main purpose of this paper is to give a global stability estimate for this inverse problem in the multi-channel case.

Let D be an open bounded domain in $\mathbb{R}^2$ with C^2 boundary and $v \in C^1(\bar{D}, M_n(\mathbb{C}))$, where $M_n(\mathbb{C})$ is the set of the $n \times n$ complex-valued matrices. The Dirichlet-to-Neumann map associated with v is the operator $\Phi : C^1(\partial D, M_n(\mathbb{C})) \to L^p(\partial D, M_n(\mathbb{C}))$, $p < \infty$, defined by

$$\Phi(f) = \frac{\partial \psi}{\partial \nu}\bigg|_{\partial D}, \tag{1.2}$$

where $f \in C^1(\partial D, M_n(\mathbb{C}))$, ν is the outer normal of ∂D and ψ is the $H^1(\bar{D}, M_n(\mathbb{C}))$-solution of the Dirichlet problem

$$-\Delta \psi + v(x)\psi = 0 \quad \text{on } D, \qquad \psi|_{\partial D} = f; \tag{1.3}$$

here we assume that

$$0 \text{ is not a Dirichlet eigenvalue of the operator } -\Delta + v \text{ in } D. \tag{1.4}$$

This construction gives rise to the following inverse boundary value problem: given Φ, find v.

This problem can be considered as the Gel'fand inverse boundary value problem for the multichannel Schrödinger equation at zero energy (see [8,11]) and can also be seen as a generalization of the Calderón problem for the electrical impedance tomography (see [5,11]). Note also that we can think of this problem as a model for monochromatic ocean tomography (e.g., see [2] for similar problems arising in this type of tomography).

In the case of complex-valued potentials the global injectivity of the map $v \to \Phi$ was first proved for $D \subset \mathbb{R}^d$ with $d \geqslant 3$ in [11] and for $d = 2$ with $v \in L^p$ in [4]: in particular, these results were obtained by the use of global reconstructions developed in the same papers. The first global uniqueness result (along with an exact reconstruction method) for matrix-valued potentials was given in [14], which deals with C^1 matrix-valued potentials defined on a domain in $\mathbb{R}^2$. A global stability estimate for the Gel'fand–Calderón problem with $d \geqslant 3$ was first found by Alessandrini in [1]; this result was recently improved in [12]. In the two-dimensional case the first global stability estimate was given in [13].

In this paper we extend the results of [13] to the matrix-valued case. We do not discuss global results for special real-valued potentials arising from conductivities: for this case the reader is referred to the references given in [1,4,10–13].

Our main result is the following:

Theorem 1.1. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with a C^2 boundary, $v_1, v_2 \in C^2(\bar{D}, M_n(\mathbb{C}))$ two matrix-valued potentials which satisfy (1.4), with $\|v_j\|_{C^2(\bar{D})} \leqslant N$ for $j = 1, 2$, and Φ_1, Φ_2 the corresponding Dirichlet-to-Neumann operators. For simplicity we also assume that $v_1|_{\partial D} = v_2|_{\partial D}$ and $\frac{\partial}{\partial \nu} v_1|_{\partial D} = \frac{\partial}{\partial \nu} v_2|_{\partial D}$. Then there exists a constant $C = C(D, N, n)$ such that*

$$\begin{aligned} &\|v_2 - v_1\|_{L^\infty(D)} \\ &\quad \leqslant C\big(\log\big(3 + \|\Phi_2 - \Phi_1\|^{-1}\big)\big)^{-\frac{3}{4}}\big(\log\big(3\log\big(3 + \|\Phi_2 - \Phi_1\|^{-1}\big)\big)\big)^2, \end{aligned} \tag{1.5}$$

where $\|\cdot\|$ is the induced operator norm on $L^\infty(\partial D, M_n(\mathbb{C}))$ and $\|v\|_{L^\infty(D)} = \max_{1 \leqslant i,j \leqslant n} \|v_{i,j}\|_{L^\infty(D)}$ (likewise for $\|v\|_{C^2(\bar{D})}$) for a matrix-valued potential v.

This is the first global stability result for the multi-channel ($n \geqslant 2$) Gel'fand–Calderón inverse problem in two dimensions. In addition, Theorem 1.1 is new also for the scalar case, as the estimate obtained in [13] is weaker. We remark, in particular, that this result is true in the special case when $v_1 \equiv v_2 \equiv \Lambda \in M_n(\mathbb{C})$ in a neighborhood of ∂D (situation which appears in the approximation of the 3D equation, see [14, Remark 3 and Section 2]).

Instability estimates complementing the stability estimates of [1,12,13] and of the present work are given in [10,9].

The proof of Theorem 1.1 is based on results obtained in [13,14], which take inspiration mostly from [4] and [1]. In particular, for $z_0 \in D$ we use the existence and uniqueness of a family of solutions $\psi_{z_0}(z,\lambda)$ of Eq. (1.1) where in particular $\psi_{z_0} \to e^{\lambda(z-z_0)^2} I$, for $\lambda \to \infty$ (where I is the identity matrix). Then, using an appropriate matrix-valued version of Alessandrini's identity along with stationary phase techniques, we obtain the result. Note that this matrix-valued identity is one of the new results of this paper.

A generalizations of Theorem 1.1 in the case where we do not assume that $v_1|_{\partial D} = v_2|_{\partial D}$ and $\frac{\partial}{\partial \nu} v_1|_{\partial D} = \frac{\partial}{\partial \nu} v_2|_{\partial D}$, is given in Section 5.

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2. Preliminaries

In this section we introduce and give details on the above-mentioned family of solutions of Eq. (1.1), which will be used throughout the paper.

We identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the coordinates $z = x_1 + ix_2$, $\bar z = x_1 - ix_2$ where $(x_1, x_2) \in \mathbb{R}^2$. Let us define the function spaces $C^1_{\bar z}(\bar D) = \{u\colon u, \frac{\partial u}{\partial \bar z} \in C(\bar D, M_n(\mathbb{C}))\}$ with the norm $\|u\|_{C^1_{\bar z}(\bar D)} = \max(\|u\|_{C(\bar D)}, \|\frac{\partial u}{\partial \bar z}\|_{C(\bar D)})$, where $\|u\|_{C(\bar D)} = \sup_{z\in \bar D} |u|$ and $|u| = \max_{1\leqslant i,j\leqslant n} |u_{i,j}|$; we also define $C^1_z(\bar D) = \{u\colon u, \frac{\partial u}{\partial z} \in C(\bar D, M_n(\mathbb{C}))\}$ with an analogous norm. Following [13,14], we consider the functions:

$$G_{z_0}(z,\zeta,\lambda) = e^{\lambda(z-z_0)^2} g_{z_0}(z,\zeta,\lambda) e^{-\lambda(\zeta-z_0)^2}, \tag{2.1}$$

$$g_{z_0}(z,\zeta,\lambda) = \frac{e^{\lambda(\zeta-z_0)^2 - \bar\lambda(\bar\zeta-\bar z_0)^2}}{4\pi^2} \int_D \frac{e^{-\lambda(\eta-z_0)^2+\bar\lambda(\bar\eta-\bar z_0)^2}}{(z-\eta)(\bar\eta-\bar\zeta)}\, d\operatorname{Re}\eta\, d\operatorname{Im}\eta, \tag{2.2}$$

$$\psi_{z_0}(z,\lambda) = e^{\lambda(z-z_0)^2}\mu_{z_0}(z,\lambda), \tag{2.3}$$

$$\mu_{z_0}(z,\lambda) = I + \int_D g_{z_0}(z,\zeta,\lambda) v(\zeta)\mu_{z_0}(\zeta,\lambda)\, d\operatorname{Re}\zeta\, d\operatorname{Im}\zeta, \tag{2.4}$$

$$h_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z-\bar z_0)^2} v(z)\mu_{z_0}(z,\lambda)\, d\operatorname{Re} z\, d\operatorname{Im} z, \tag{2.5}$$

where $z, z_0, \zeta \in D$, $\lambda \in \mathbb{C}$ and I is the identity matrix. In addition, Eq. (2.4) at fixed z_0 and λ, is considered as a linear integral equation for $\mu_{z_0}(\cdot,\lambda) \in C^1_{\bar z}(\bar D)$. The functions $G_{z_0}(z,\zeta,\lambda)$, $g_{z_0}(z,\zeta,\lambda)$, $\psi_{z_0}(z,\lambda)$, $\mu_{z_0}(z,\lambda)$ defined above, satisfy the following equations (see [13,14]):

$$4\frac{\partial^2}{\partial z \partial \bar z} G_{z_0}(z,\zeta,\lambda) = \delta(z-\zeta), \tag{2.6}$$

$$4\frac{\partial^2}{\partial\zeta\partial\bar\zeta}G_{z_0}(z,\zeta,\lambda)=\delta(\zeta-z), \tag{2.7}$$

$$4\left(\frac{\partial}{\partial z}+2\lambda(z-z_0)\right)\frac{\partial}{\partial\bar z}g_{z_0}(z,\zeta,\lambda)=\delta(z-\zeta), \tag{2.8}$$

$$4\frac{\partial}{\partial\bar\zeta}\left(\frac{\partial}{\partial\zeta}-2\lambda(\zeta-z_0)\right)g_{z_0}(z,\zeta,\lambda)=\delta(\zeta-z), \tag{2.9}$$

$$-4\frac{\partial^2}{\partial z\partial\bar z}\psi_{z_0}(z,\lambda)+v(z)\psi_{z_0}(z,\lambda)=0, \tag{2.10}$$

$$-4\left(\frac{\partial}{\partial z}+2\lambda(z-z_0)\right)\frac{\partial}{\partial\bar z}\mu_{z_0}(z,\lambda)+v(z)\mu_{z_0}(z,\lambda)=0, \tag{2.11}$$

where $z, z_0, \zeta \in D$, $\lambda \in \mathbb{C}$, δ is the Dirac delta. (In addition, we assume that (2.4) is uniquely solvable for $\mu_{z_0}(\cdot,\lambda) \in C^1_{\bar z}(\bar D)$ at fixed z_0 and λ.)

We say that the functions G_{z_0}, g_{z_0}, ψ_{z_0}, μ_{z_0}, h_{z_0} are the Bukhgeim-type analogues of the Faddeev functions (see [14]). We recall that the history of these functions goes back to [7,3].

Now we state some fundamental lemmata. Let

$$g_{z_0,\lambda}u(z)=\int_D g_{z_0}(z,\zeta,\lambda)u(\zeta)\,d\operatorname{Re}\zeta\,d\operatorname{Im}\zeta, \quad z\in\bar D,\ z_0,\lambda\in\mathbb{C}, \tag{2.12}$$

where $g_{z_0}(z,\zeta,\lambda)$ is defined by (2.2) and u is a test function.

Lemma 2.1. *(See [13].) Let $g_{z_0,\lambda}u$ be defined by* (2.12)*. Then, for $z_0,\lambda\in\mathbb{C}$, the following estimates hold*

$$g_{z_0,\lambda}u \in C^1_{\bar z}(\bar D), \quad \textit{for } u\in C(\bar D), \tag{2.13}$$

$$\|g_{z_0,\lambda}u\|_{C^1(\bar D)} \leqslant c_1(D,\lambda)\|u\|_{C(\bar D)}, \quad \textit{for } u\in C(\bar D), \tag{2.14}$$

$$\|g_{z_0,\lambda}u\|_{C^1_{\bar z}(\bar D)} \leqslant \frac{c_2(D)}{|\lambda|^{\frac12}}\|u\|_{C^1_{\bar z}(\bar D)}, \quad \textit{for } u\in C^1_{\bar z}(\bar D),\ |\lambda|\geqslant 1. \tag{2.15}$$

Given a potential $v \in C^1_{\bar z}(\bar D)$ we define the operator $g_{z_0,\lambda}v$ simply as $(g_{z_0,\lambda}v)u(z) = g_{z_0,\lambda}w(z)$, $w = vu$, for a test function u. If $u \in C^1_{\bar z}(\bar D)$, by Lemma 2.1 we have that $g_{z_0,\lambda}v : C^1_{\bar z}(\bar D) \to C^1_{\bar z}(\bar D)$,

$$\|g_{z_0,\lambda}v\|^{op}_{C^1_{\bar z}(\bar D)} \leqslant 2n\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar z}(\bar D)}\|v\|_{C^1_{\bar z}(\bar D)}, \tag{2.16}$$

where $\|\cdot\|^{op}_{C^1_{\bar z}(\bar D)}$ denotes the operator norm in $C^1_{\bar z}(\bar D)$, $z_0,\lambda\in\mathbb{C}$. In addition, $\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar z}(\bar D)}$ is estimated in Lemma 2.1. Inequality (2.16) and Lemma 2.1 imply the existence and uniqueness of $\mu_{z_0}(z,\lambda)$ (and thus also of $\psi_{z_0}(z,\lambda)$) for $|\lambda| > \rho(D,K,n)$, where $\|v\|_{C^1_{\bar z}(\bar D)} < K$.

Let

$$\mu^{(k)}_{z_0}(z,\lambda)=\sum_{j=0}^{k}(g_{z_0,\lambda}v)^j I,$$

$$h^{(k)}_{z_0}(\lambda)=\int_D e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}v(z)\mu^{(k)}_{z_0}(z,\lambda)\,d\operatorname{Re}z\,d\operatorname{Im}z,$$

where $z, z_0 \in D$, $\lambda\in\mathbb{C}$, $k\in\mathbb{N}\cup\{0\}$.

Lemma 2.2. *(See [13].) For $v \in C^1_{\bar z}(\bar D)$ such that $v|_{\partial D} = 0$ the following formula holds*

$$v(z_0) = \frac{2}{\pi} \lim_{\lambda\to\infty} |\lambda| h^{(0)}_{z_0}(\lambda), \quad z_0 \in D. \tag{2.17}$$

In addition, if $v \in C^2(\bar D)$, $v|_{\partial D} = 0$ and $\frac{\partial v}{\partial \nu}|_{\partial D} = 0$ then

$$\left| v(z_0) - \frac{2}{\pi} |\lambda| h^{(0)}_{z_0}(\lambda) \right| \leqslant c_3(D,n) \frac{\log(3|\lambda|)}{|\lambda|} \|v\|_{C^2(\bar D)}, \tag{2.18}$$

for $z_0 \in D$, $\lambda \in \mathbb{C}$, $|\lambda| \geqslant 1$.

Let

$$W_{z_0}(\lambda) = \int\limits_D e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} w(z)\, d\operatorname{Re} z\, d\operatorname{Im} z,$$

where $z_0 \in \bar D$, $\lambda \in \mathbb{C}$ and w is some $M_n(\mathbb{C})$-valued function on $\bar D$. (One can see that $W_{z_0} = h^{(0)}_{z_0}$ for $w = v$.)

Lemma 2.3. *(See [13].) For $w \in C^1_{\bar z}(\bar D)$ the following estimate holds*

$$\left| W_{z_0}(\lambda) \right| \leqslant c_4(D) \frac{\log(3|\lambda|)}{|\lambda|} \|w\|_{C^1_{\bar z}(\bar D)}, \quad z_0 \in \bar D,\ |\lambda| \geqslant 1. \tag{2.19}$$

Lemma 2.4. *(See [14].) For $v \in C^1_{\bar z}(\bar D)$ and for $\|g_{z_0,\lambda} v\|^{op}_{C^1_{\bar z}(\bar D)} \leqslant \delta < 1$ we have that*

$$\left\| \mu_{z_0}(\cdot,\lambda) - \mu^{(k)}_{z_0}(\cdot,\lambda) \right\|_{C^1_{\bar z}(\bar D)} \leqslant \frac{\delta^{k+1}}{1-\delta}, \tag{2.20}$$

$$\left| h_{z_0}(\lambda) - h^{(k)}_{z_0}(\lambda) \right| \leqslant c_5(D,n) \frac{\log(3|\lambda|)}{|\lambda|} \frac{\delta^{k+1}}{1-\delta} \|v\|_{C^1_{\bar z}(\bar D)}, \tag{2.21}$$

where $z_0 \in D$, $\lambda \in \mathbb{C}$, $|\lambda| \geqslant 1$, $k \in \mathbb{N} \cup \{0\}$.

The proofs of Lemmata 2.1–2.4 can be found in the references given.
We will also need the following two new lemmata.

Lemma 2.5. *Let $g_{z_0,\lambda} u$ be defined by (2.12), where $u \in C^1_{\bar z}(\bar D)$, $z_0, \lambda \in \mathbb{C}$. Then the following estimate holds*

$$\|g_{z_0,\lambda} u\|_{C(\bar D)} \leqslant c_6(D) \frac{\log(3|\lambda|)}{|\lambda|} \|u\|_{C^1_{\bar z}(\bar D)}, \quad |\lambda| \geqslant 1. \tag{2.22}$$

Lemma 2.6. *The expression*

$$W(u,v)(\lambda) = \int\limits_D e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} u(z) (g_{z_0,\lambda} v)(z)\, d\operatorname{Re} z\, d\operatorname{Im} z, \tag{2.23}$$

defined for $u, v \in C^1_{\bar z}(\bar D)$ with $\|u\|_{C^1_{\bar z}(\bar D)}, \|v\|_{C^1_{\bar z}(\bar D)} \leqslant N_1$, $\lambda \in \mathbb{C}$, $z_0 \in D$, satisfies the estimate

$$\left| W(u,v)(\lambda) \right| \leqslant c_7(D, N_1, n) \frac{(\log(3|\lambda|))^2}{|\lambda|^{1+3/4}}, \quad |\lambda| \geqslant 1. \tag{2.24}$$

The proofs of Lemmata 2.5, 2.6 are given in Section 4.

3. Proof of Theorem 1.1

We begin with a technical lemma, which will prove useful when generalizing Alessandrini's identity.

Lemma 3.1. *Let $v \in C^1(\bar{D}, M_n(\mathbb{C}))$ be a matrix-valued potential which satisfies condition* (1.4) (*i.e.,* 0 *is not a Dirichlet eigenvalue for the operator $-\Delta + v$ in D*). *Then ${}^t v$, the transpose of v, also satisfies condition* (1.4).

The proof of Lemma 3.1 is given in Section 4.

We can now state and prove a matrix-valued version of Alessandrini's identity (see [1] for the scalar case).

Lemma 3.2. *Let $v_1, v_2 \in C^1(\bar{D}, M_n(\mathbb{C}))$ be two matrix-valued potentials which satisfy* (1.4), *Φ_1, Φ_2 their associated Dirichlet-to-Neumann operators, respectively, and $u_1, u_2 \in C^2(\bar{D}, M_n(\mathbb{C}))$ two matrix-valued functions such that*

$$(-\Delta + v_1)u_1 = 0, \qquad \left(-\Delta + {}^t v_2\right)u_2 = 0 \quad \text{on } D,$$

where ${}^t A$ stand for the transpose of A. Then we have the identity

$$\int_{\partial D} {}^t u_2(z)(\Phi_2 - \Phi_1)u_1(z)\,|dz| = \int_D {}^t u_2(z)\big(v_2(z) - v_1(z)\big)u_1(z)\,d\,\mathrm{Re}\,z\,d\,\mathrm{Im}\,z. \tag{3.1}$$

Proof. If $v \in C^1(\bar{D}, M_n(\mathbb{C}))$ is any matrix-valued potential (which satisfies (1.4)) and $f_1, f_2 \in C^1(\partial D, M_n(\mathbb{C}))$ then we have

$$\int_{\partial D} {}^t f_2 \Phi f_1\,|dz| = \int_{\partial D} {}^t\big({}^t f_1 \Phi^* f_2\big)\,|dz|, \tag{3.2}$$

where Φ and Φ^* are the Dirichlet-to-Neumann operators associated with v and ${}^t v$, respectively (these operators are well defined thanks to Lemma 3.1). Indeed, it is sufficient to extend f_1 and f_2 in D as the solutions of the Dirichlet problems $(-\Delta + v)\tilde{f}_1 = 0$, $(-\Delta + {}^t v)\tilde{f}_2 = 0$ on D and $\tilde{f}_j|_{\partial D} = f_j$, for $j = 1, 2$, so that one obtains

$$\begin{aligned}\int_{\partial D} \big({}^t f_2 \Phi f_1 - {}^t\big({}^t f_1 \Phi^* f_2\big)\big)\,|dz| &= \int_{\partial D} \left({}^t f_2 \frac{\partial \tilde{f}_1}{\partial \nu} - {}^t\left(\frac{\partial \tilde{f}_2}{\partial \nu}\right) f_1\right)|dz| \\ &= \int_D \big({}^t \tilde{f}_2 \Delta \tilde{f}_1 - {}^t(\Delta \tilde{f}_2)\tilde{f}_1\big)\,d\,\mathrm{Re}\,z\,d\,\mathrm{Im}\,z \\ &= \int_D \big({}^t \tilde{f}_2 v \tilde{f}_1 - {}^t\big({}^t v \tilde{f}_2\big)\tilde{f}_1\big)\,d\,\mathrm{Re}\,z\,d\,\mathrm{Im}\,z = 0,\end{aligned}$$

where for the second equality we used the following matrix-valued version of the classical scalar Green's formula:

$$\int_{\partial D}\left({}^t\!\left(\frac{\partial f}{\partial \nu}\right)g-{}^tf\frac{\partial g}{\partial \nu}\right)|dz| = \int_D \left({}^t(\Delta f)g-{}^tf\,\Delta g\right) d\,\mathrm{Re}\,z\, d\,\mathrm{Im}\,z, \tag{3.3}$$

for any $f, g \in C^2(D, M_n(\mathbb{C})) \cap C^1(\bar{D}, M_n(\mathbb{C}))$.

Identities (3.2) and (3.3) imply

$$\begin{aligned}
&\int_{\partial D} {}^tu_2(z)(\Phi_2-\Phi_1)u_1(z)\,|dz| \\
&\quad= \int_{\partial D} \left({}^t\!\left({}^tu_1(z)\Phi_2^*u_2(z)\right) - {}^tu_2(z)\Phi_1u_1(z)\right)|dz| \\
&\quad= \int_{\partial D} \left({}^t\!\left(\frac{\partial u_2(z)}{\partial \nu}\right)u_1(z) - {}^tu_2(z)\frac{\partial u_1(z)}{\partial \nu}\right)|dz| \\
&\quad= \int_D \left({}^t\!\left(\Delta u_2(z)\right)u_1(z) - {}^tu_2(z)\Delta u_1(z)\right) d\,\mathrm{Re}\,z\, d\,\mathrm{Im}\,z \\
&\quad= \int_D \left({}^t\!\left({}^tv_2(z)u_2(z)\right)u_1(z) - {}^tu_2(z)v_1(z)u_1(z)\right) d\,\mathrm{Re}\,z\, d\,\mathrm{Im}\,z \\
&\quad= \int_D {}^tu_2(z)\left(v_2(z)-v_1(z)\right)u_1(z)\, d\,\mathrm{Re}\,z\, d\,\mathrm{Im}\,z. \qquad \square
\end{aligned}$$

Now let $\bar{\mu}_{z_0}$ denote the complex conjugate of μ_{z_0} (the solution of (2.4)) for an $M_n(\mathbb{R})$-valued potential v and, more generally, the solution of (2.4) with $g_{z_0}(z,\zeta,\lambda)$ replaced by $\overline{g_{z_0}(z,\zeta,\lambda)}$ for an $M_n(\mathbb{C})$-valued potential v. In order to make use of (3.1) we define

$$\begin{aligned}
u_1(z) &= \psi_{1,z_0}(z,\lambda) = e^{\lambda(z-z_0)^2}\mu_1(z,\lambda), \\
u_2(z) &= \overline{\psi}_{2,z_0}(z,-\lambda) = e^{-\bar{\lambda}(\bar{z}-\bar{z}_0)^2}\bar{\mu}_2(z,-\lambda),
\end{aligned}$$

for $z_0 \in D$, $\lambda \in C$, $|\lambda| > \rho$ (ρ is mentioned in Section 2), where we set $\mu_1 = \mu_{1,z_0}$, $\mu_2 = \mu_{2,z_0}$ for simplicity's sake and μ_{1,z_0}, μ_{2,z_0} are the solutions of (2.4) with v replaced by v_1, ${}^tv_2$, respectively.

Eq. (3.1), with the above-defined u_1, u_2, now reads

$$\begin{aligned}
&\int_{\partial D}\int_{\partial D} e^{-\bar{\lambda}(\bar{z}-\bar{z}_0)^2}\,{}^t\bar{\mu}_2(z,-\lambda)(\Phi_2-\Phi_1)(z,\zeta)e^{\lambda(\zeta-z_0)^2}\mu_1(\zeta,\lambda)\,|d\zeta|\,|dz| \\
&\quad= \int_D e_{\lambda,z_0}(z)\,{}^t\bar{\mu}_2(z,-\lambda)(v_2-v_1)(z)\mu_1(z,\lambda)\, d\,\mathrm{Re}\,z\, d\,\mathrm{Im}\,z
\end{aligned} \tag{3.4}$$

with $e_{\lambda,z_0}(z) = e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{z}-\bar{z}_0)^2}$ and $(\Phi_2-\Phi_1)(z,\zeta)$ is the Schwartz kernel of the operator $\Phi_2-\Phi_1$.

The right side $I(\lambda)$ of (3.4) can be written as the sum of four integrals, namely

$$I_1(\lambda) = \int_D e_{\lambda,z_0}(z)(v_2-v_1)(z)\, d\,\mathrm{Re}\,z\, d\,\mathrm{Im}\,z,$$

$$I_2(\lambda) = \int_D e_{\lambda,z_0}(z)^t(\bar{\mu}_2 - I)(v_2 - v_1)(z)(\mu_1 - I)\, d\operatorname{Re} z\, d\operatorname{Im} z,$$

$$I_3(\lambda) = \int_D e_{\lambda,z_0}(z)^t(\bar{\mu}_2 - I)(v_2 - v_1)(z)\, d\operatorname{Re} z\, d\operatorname{Im} z,$$

$$I_4(\lambda) = \int_D e_{\lambda,z_0}(z)(v_2 - v_1)(z)(\mu_1 - I)\, d\operatorname{Re} z\, d\operatorname{Im} z,$$

for $z_0 \in D$.

Since $(v_2 - v_1)|_{\partial D} = \frac{\partial}{\partial \nu}(v_2 - v_1)|_{\partial D} = 0$, the first term, I_1, can be estimated using Lemma 2.2 as

$$\left| \frac{2}{\pi}|\lambda| I_1 - \big(v_2(z_0) - v_1(z_0)\big) \right| \leqslant c_3(D, n)\frac{\log(3|\lambda|)}{|\lambda|}\|v_2 - v_1\|_{C^2(\bar{D})}, \tag{3.5}$$

for $|\lambda| \geqslant 1$. The other terms, I_2, I_3, I_4, satisfy, by Lemmata 2.1 and 2.4,

$$|I_2| \leqslant \left| \int_D e_{\lambda,z_0}(z)^t\big(\overline{g_{z_0,\lambda}}^t v_2\big)(v_2 - v_1)(z)(g_{z_0,\lambda} v_1)\, d\operatorname{Re} z\, d\operatorname{Im} z \right| + O\left(\frac{\log(3|\lambda|)}{|\lambda|^2}\right) c_8(D, N, n), \tag{3.6}$$

$$|I_3| \leqslant \left| \int_D e_{\lambda,z_0}(z)^t\big(\overline{g_{z_0,\lambda}}^t v_2\big)(v_2 - v_1)(z)\, d\operatorname{Re} z\, d\operatorname{Im} z \right| + O\left(\frac{\log(3|\lambda|)}{|\lambda|^2}\right) c_9(D, N, n), \tag{3.7}$$

$$|I_4| \leqslant \left| \int_D e_{\lambda,z_0}(z)(v_2 - v_1)(z)(g_{z_0,\lambda} v_1)\, d\operatorname{Re} z\, d\operatorname{Im} z \right| + O\left(\frac{\log(3|\lambda|)}{|\lambda|^2}\right) c_{10}(D, N, n), \tag{3.8}$$

where N is the constant in the statement of Theorem 1.1 and $|\lambda|$ is sufficiently large, for example for λ such that

$$2n\frac{c_2(D)}{|\lambda|^{\frac{1}{2}}} \leqslant \frac{1}{2}, \quad |\lambda| \geqslant 1. \tag{3.9}$$

Lemmata 2.5, 2.6, applied to (3.6)–(3.8), give us

$$|I_2| \leqslant c_{11}(D, N, n)\frac{(\log(3|\lambda|))^2}{|\lambda|^2}, \tag{3.10}$$

$$|I_3| \leqslant c_{12}(D, N, n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{1+3/4}}, \tag{3.11}$$

$$|I_4| \leqslant c_{13}(D, N, n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{1+3/4}}. \tag{3.12}$$

The left side $J(\lambda)$ of (3.4) can be estimated as follows

$$|\lambda||J(\lambda)| \leqslant c_{14}(D,n)e^{(2L^2+1)|\lambda|}\|\Phi_2-\Phi_1\|, \tag{3.13}$$

for λ which satisfies (3.9), and $L=\max_{z\in\partial D,\, z_0\in D}|z-z_0|$.

Putting together estimates (3.5)–(3.13) we obtain

$$\begin{aligned}\big|v_2(z_0)-v_1(z_0)\big| &\leqslant c_{15}(D,N,n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{3/4}}\\ &\quad+\frac{2}{\pi}c_{14}(D,n)e^{(2L^2+1)|\lambda|}\|\Phi_2-\Phi_1\|\end{aligned} \tag{3.14}$$

for any $z_0\in D$. We call $\varepsilon=\|\Phi_2-\Phi_1\|$ and impose $|\lambda|=\gamma\log(3+\varepsilon^{-1})$, where $0<\gamma<(2L^2+1)^{-1}$ so that (3.14) reads

$$\begin{aligned}\big|v_2(z_0)-v_1(z_0)\big| &\leqslant c_{15}(D,N,n)\big(\gamma\log\big(3+\varepsilon^{-1}\big)\big)^{-\frac{3}{4}}\big(\log\big(3\gamma\log\big(3+\varepsilon^{-1}\big)\big)\big)^2\\ &\quad+\frac{2}{\pi}c_{14}(D,n)\big(3+\varepsilon^{-1}\big)^{(2L^2+1)\gamma}\varepsilon,\end{aligned} \tag{3.15}$$

for every $z_0\in D$, with

$$0<\varepsilon\leqslant\varepsilon_1(D,N,\gamma,n), \tag{3.16}$$

where ε_1 is sufficiently small or, more precisely, where (3.16) implies that $|\lambda|=\gamma\log(3+\varepsilon^{-1})$ satisfies (3.9).

As $(3+\varepsilon^{-1})^{(2L^2+1)\gamma}\varepsilon\to 0$ for $\varepsilon\to 0$ more rapidly then the other term, we obtain that

$$\|v_2-v_1\|_{L^\infty(D)}\leqslant c_{16}(D,N,\gamma,n)\frac{(\log(3\log(3+\|\Phi_2-\Phi_1\|^{-1})))^2}{(\log(3+\|\Phi_2-\Phi_1\|^{-1}))^{\frac{3}{4}}} \tag{3.17}$$

for any $\varepsilon=\|\Phi_2-\Phi_1\|\leqslant\varepsilon_1(D,N,\gamma,n)$.

Estimate (3.17) for general ε (with modified c_{16}) follows from (3.17) for $\varepsilon\leqslant\varepsilon_1(D,N,\gamma,n)$ and the assumption that $\|v_j\|_{L^\infty(D)}\leqslant N$, $j=1,2$. This completes the proof of Theorem 1.1. □

4. Proofs of Lemmata 2.5, 2.6, 3.1

Proof of Lemma 2.5. We decompose the operator $g_{z_0,\lambda}$, defined in (2.12), as the product $\frac{1}{4}T_{z_0,\lambda}\bar{T}_{z_0,\lambda}$, where

$$T_{z_0,\lambda}u(z)=\frac{1}{\pi}\int_D\frac{e^{-\lambda(\zeta-z_0)^2+\bar\lambda(\bar\zeta-\bar z_0)^2}}{z-\zeta}u(\zeta)\,d\operatorname{Re}\zeta\,d\operatorname{Im}\zeta, \tag{4.1}$$

$$\bar{T}_{z_0,\lambda}u(z)=\frac{1}{\pi}\int_D\frac{e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2}}{\bar z-\bar\zeta}u(\zeta)\,d\operatorname{Re}\zeta\,d\operatorname{Im}\zeta, \tag{4.2}$$

for $z_0,\lambda\in\mathbb{C}$. From the proof of [13, Lemma 3.1] we have the estimate

$$\|\bar{T}_{z_0,\lambda}u\|_{C(\bar D)}\leqslant\frac{\eta_1(D)}{|\lambda|^{1/2}}\|u\|_{C(\bar D)}+\eta_2(D)\frac{\log(3|\lambda|)}{|\lambda|}\left\|\frac{\partial u}{\partial\bar z}\right\|_{C(\bar D)}, \tag{4.3}$$

for $u\in C^1_{\bar z}(\bar D)$, $z_0\in D$, $|\lambda|\geqslant 1$. As the kernels of $T_{z_0,\lambda}$ and $\bar{T}_{z_0,\lambda}$ are conjugates of each other we deduce immediately that

$$\|T_{z_0,\lambda}u\|_{C(\bar D)} \leqslant \frac{\eta_1(D)}{|\lambda|^{1/2}}\|u\|_{C(\bar D)} + \eta_2(D)\frac{\log(3|\lambda|)}{|\lambda|}\left\|\frac{\partial u}{\partial z}\right\|_{C(\bar D)}, \quad |\lambda| \geqslant 1, \tag{4.4}$$

for $u \in C^1_z(\bar D)$. Combining the two estimates we obtain

$$\begin{aligned}
\|g_{\lambda,z_0}u\|_{C(\bar D)} &= \frac{1}{4}\|T_{z_0,\lambda}\bar T_{z_0,\lambda}u\|_{C(\bar D)} \\
&\leqslant \frac{1}{4}\left(\eta_1(D)\frac{\|\bar T_{z_0,\lambda}u\|_{C(\bar D)}}{|\lambda|^{1/2}} + \eta_2(D)\frac{\log(3|\lambda|)}{|\lambda|}\left\|\frac{\partial}{\partial z}\bar T_{z_0,\lambda}u\right\|_{C(\bar D)}\right) \\
&\leqslant \eta_3(D)\left(\frac{\|u\|_{C(\bar D)}}{|\lambda|} + \frac{\log(3|\lambda|)}{|\lambda|^{3/2}}\left\|\frac{\partial u}{\partial \bar z}\right\|_{C(\bar D)} + \frac{\log(3|\lambda|)}{|\lambda|}\|u\|_{C(\bar D)}\right) \\
&\leqslant \eta_4(D)\frac{\log(3|\lambda|)}{|\lambda|}\|u\|_{C^1_{\bar z}(\bar D)}, \quad |\lambda| \geqslant 1,
\end{aligned}$$

where we use the fact that $\|\frac{\partial}{\partial z}\bar T_{z_0,\lambda}u\|_{C(D)} = \|u\|_{C(D)}$. □

Proof of Lemma 2.6. For $0 < \varepsilon \leqslant 1$, $z_0 \in D$, let $B_{z_0,\varepsilon} = \{z \in \mathbb{C}\colon |z - z_0| \leqslant \varepsilon\}$. We write $W(u,v)(\lambda) = W^1(\lambda) + W^2(\lambda)$, where

$$W^1(\lambda) = \int\limits_{D\cap B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}u(z)g_{z_0,\lambda}v(z)\,d\operatorname{Re}z\,d\operatorname{Im}z,$$

$$W^2(\lambda) = \int\limits_{D\setminus B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}u(z)g_{z_0,\lambda}v(z)\,d\operatorname{Re}z\,d\operatorname{Im}z.$$

The first term, W^1, can be estimated as follows

$$\left|W^1(\lambda)\right| \leqslant \sigma_1(D,n)\|u\|_{C(\bar D)}\|v\|_{C^1_{\bar z}(\bar D)}\frac{\varepsilon^2\log(3|\lambda|)}{|\lambda|}, \quad |\lambda| \geqslant 1, \tag{4.5}$$

where we use estimates (2.16) and (2.22).

For the second term, W^2, we proceed using integration by parts, in order to obtain

$$\begin{aligned}
W^2(\lambda) = {} & \frac{1}{4i\bar\lambda}\int\limits_{\partial(D\setminus B_{z_0,\varepsilon})} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\frac{u(z)g_{z_0,\lambda}v(z)}{\bar z-\bar z_0}\,dz \\
& - \frac{1}{2\bar\lambda}\int\limits_{D\setminus B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\frac{\partial}{\partial\bar z}\left(\frac{u(z)g_{z_0,\lambda}v(z)}{\bar z-\bar z_0}\right)d\operatorname{Re}z\,d\operatorname{Im}z.
\end{aligned}$$

This implies that

$$\begin{aligned}
\left|W^2(\lambda)\right| \leqslant {} & \frac{1}{4|\lambda|}\int\limits_{\partial(D\setminus B_{z_0,\varepsilon})}\frac{\|u(z)g_{z_0,\lambda}v(z)\|_{C(\bar D)}}{|\bar z-\bar z_0|}|dz| \\
& + \frac{1}{2|\lambda|}\left|\int\limits_{D\setminus B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\frac{\partial}{\partial\bar z}\left(\frac{u(z)g_{z_0,\lambda}v(z)}{\bar z-\bar z_0}\right)d\operatorname{Re}z\,d\operatorname{Im}z\right|,
\end{aligned} \tag{4.6}$$

for $\lambda \neq 0$. Again by estimates (2.16) and (2.22) we obtain

$$\begin{aligned}\left|W^2(\lambda)\right| \leqslant{}& \sigma_2(D,n)\|u\|_{C^1_{\bar z}(\bar D)}\|v\|_{C^1_{\bar z}(\bar D)}\frac{\log(3\varepsilon^{-1})\log(3|\lambda|)}{|\lambda|^2}\\ &+\frac{1}{8|\lambda|}\left|\int\limits_{D\setminus B_{z_0,\varepsilon}} u(z)\frac{\bar T_{z_0,\lambda}v(z)}{\bar z-\bar z_0}\,d\operatorname{Re}z\,d\operatorname{Im}z\right|,\quad |\lambda|\geqslant 1,\end{aligned}\tag{4.7}$$

where we used the fact that $\frac{\partial}{\partial \bar z}g_{z_0,\lambda}v(z)=\frac14 e^{-\lambda(z-z_0)^2+\bar\lambda(\bar z-\bar z_0)^2}\bar T_{z_0,\lambda}v(z)$, with $\bar T_{z_0,\lambda}$ defined in (4.2).

The last term in (4.7) can be estimated independently of ε by

$$\sigma_3(D,n)\|u\|_{C(\bar D)}\|v\|_{C^1_{\bar z}(\bar D)}\frac{\log(3|\lambda|)}{|\lambda|^{1+3/4}}.\tag{4.8}$$

This is a consequence of (4.3) and of the estimate

$$\left|\bar T_{z_0,\lambda}u(z)\right|\leqslant\frac{\log(3|\lambda|)(1+|z-z_0|)\tau_1(D)}{|\lambda||z-z_0|^2}\|u\|_{C^1_{\bar z}(\bar D)},\quad |\lambda|\geqslant 1,\tag{4.9}$$

for $u\in C^1_{\bar z}(\bar D)$, $z,z_0\in D$ (a proof of (4.9) can be found in the proof of [13, Lemma 3.1]).

Indeed, for $0<\delta\leqslant\frac12$ we have

$$\begin{aligned}&\left|\int\limits_D u(z)\frac{\bar T_{z_0,\lambda}v(z)}{\bar z-\bar z_0}\,d\operatorname{Re}z\,d\operatorname{Im}z\right|\\ &\leqslant\int\limits_{B_{z_0,\delta}\cap D}\left|u(z)\right|\frac{|\bar T_{z_0,\lambda}v(z)|}{|z-z_0|}\,d\operatorname{Re}z\,d\operatorname{Im}z+\int\limits_{D\setminus B_{z_0,\delta}}\left|u(z)\right|\frac{|\bar T_{z_0,\lambda}v(z)|}{|z-z_0|}\,d\operatorname{Re}z\,d\operatorname{Im}z\\ &\leqslant\|u\|_{C(\bar D)}\|v\|_{C^1_{\bar z}(\bar D)}\frac{\tau_2(D,n)}{|\lambda|^{1/2}}\int\limits_{B_{z_0,\delta}\cap D}\frac{d\operatorname{Re}z\,d\operatorname{Im}z}{|z-z_0|}\\ &\quad+\|u\|_{C(\bar D)}\|v\|_{C^1_{\bar z}(\bar D)}\frac{\log(3|\lambda|)}{|\lambda|}\tau_3(D,n)\int\limits_{D\setminus B_{z_0,\delta}}\frac{d\operatorname{Re}z\,d\operatorname{Im}z}{|z-z_0|^3}\\ &\leqslant 2\pi\|u\|_{C(\bar D)}\|v\|_{C^1_{\bar z}(\bar D)}\tau_2(D,n)\frac{\delta}{|\lambda|^{\frac12}}+\|u\|_{C(\bar D)}\|v\|_{C^1_{\bar z}(\bar D)}\tau_4(D,n)\frac{\log(3|\lambda|)}{|\lambda|\delta},\end{aligned}$$

for $|\lambda|\geqslant 1$. Putting $\delta=\frac12|\lambda|^{-1/4}$ in the last inequality gives (4.8).

Finally, defining $\varepsilon=|\lambda|^{-1/2}$ in (4.7), (4.5) and using (4.8), we obtain the main estimate (2.24), which thus finishes the proof of Lemma 2.6. □

Proof of Lemma 3.1. Take $u\in H^1(D,M_n(\mathbb{C}))$ such that $(-\Delta+{}^t v)u=0$ on D and $u|_{\partial D}=0$. We want to prove that $u\equiv 0$ on D.

By our hypothesis, for any $f\in C^1(\partial D,M_n(\mathbb{C}))$ there exists a unique $\tilde f\in H^1(D,M_n(\mathbb{C}))$ such that $(-\Delta+v)\tilde f=0$ on D and $\tilde f|_{\partial D}=f$. Thus we have, using Green's formula (3.3),

$$\int_{\partial D} {}^t\!\left(\frac{\partial u}{\partial \nu}\right) f\,|dz| = \int_D \left({}^t(\Delta u)\tilde{f} - {}^t u \Delta \tilde{f}\right) d\,\mathrm{Re}\, z\, d\,\mathrm{Im}\, z$$
$$= \int_D \left({}^t\left({}^t v u\right)\tilde{f} - {}^t u v \tilde{f}\right) d\,\mathrm{Re}\, z\, d\,\mathrm{Im}\, z = 0,$$

which yields $\frac{\partial u}{\partial \nu}|_{\partial D} = 0$. Now consider the following straightforward generalization of Green's formula (3.3),

$$\int_{\partial D} \left({}^t\!\left(\frac{\partial f}{\partial \nu}\right) g - {}^t f \frac{\partial g}{\partial \nu}\right) |dz| = \int_D {}^t\!\left((\Delta - {}^t v) f\right) g - {}^t f\left((\Delta - v) g\right) d\,\mathrm{Re}\, z\, d\,\mathrm{Im}\, z, \tag{4.10}$$

which holds (weakly) for any $f, g \in H^1(D, M_n(\mathbb{C}))$. If we put $f = u$ we obtain

$$\int_D {}^t u(-\Delta + v) g\, d\,\mathrm{Re}\, z\, d\,\mathrm{Im}\, z = 0, \tag{4.11}$$

for any $g \in H^1(D, M_n(\mathbb{C}))$. By Fredholm alternative (see [6, Section 6.2]), for each $h \in L^2(D, M_n(\mathbb{C}))$ there exists a unique $g \in H_0^1(D, M_n(\mathbb{C})) = \{g \in H^1(D, M_n(\mathbb{C})):\ g|_{\partial D} = 0\}$ such that $(-\Delta + v)g = h$. This yields $u \equiv 0$ on D and thus Lemma 3.1 is proved. □

5. An extensions of Theorem 1.1

As an extension of Theorem 1.1 to the case where we do not assume that $v_1|_{\partial D} = v_2|_{\partial D}$ and $\frac{\partial}{\partial \nu} v_1|_{\partial D} = \frac{\partial}{\partial \nu} v_2|_{\partial D}$, we give the following proposition:

Proposition 5.1. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with a C^2 boundary, $v_1, v_2 \in C^2(\bar{D}, M_n(\mathbb{C}))$ two matrix-valued potentials which satisfy (1.4), with $\|v_j\|_{C^2(\bar{D})} \leqslant N$ for $j = 1, 2$, and Φ_1, Φ_2 the corresponding Dirichlet-to-Neumann operators. Then, for any $0 < \alpha < \frac{1}{5}$, there exists a constant $C = C(D, N, n, \alpha)$ such that*

$$\|v_2 - v_1\|_{L^\infty(D)} \leqslant C\left(\log\left(3 + \|\Phi_2 - \Phi_1\|_1^{-1}\right)\right)^{-\alpha}, \tag{5.1}$$

where, for an operator A which acts on $L^\infty(\partial D, M_n(\mathbb{C}))$ with kernel $A(x, y)$, $\|A\|_1$ is the norm defined as $\|A\|_1 = \sup_{x,y \in \partial D} |A(x, y)|(\log(3 + |x - y|^{-1}))^{-1}$ and $|A(x, y)| = \max_{1 \leqslant i,j \leqslant n} |A_{i,j}(x, y)|$.

The only properties of $\|\cdot\|_1$ we will use are the following:

(i) $\|A\|_{L^\infty(\partial D) \to L^\infty(\partial D)} \leqslant const(D, n)\|A\|_1$;
(ii) In a similar way as in formula (4.9) of [11] one can deduce

$$\|v\|_{L^\infty(\partial D)} \leqslant const(n)\|\Phi_v - \Phi_0\|_1,$$

for a matrix-valued potential v, Φ_v its associated Dirichlet-to-Neumann operator and Φ_0 the Dirichlet-to-Neumann operator of the 0 potential.

We recall a lemma from [13], which generalizes Lemma 2.2 to the case of potentials without boundary conditions. We then define $(\partial D)_\delta = \{z \in \mathbb{C}:\ dist(z, \partial D) < \delta\}$.

Lemma 5.2. *For* $v \in C^2(\bar{D})$ *we have that*

$$\left| v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}^{(0)}(\lambda) \right| \leqslant \kappa_1(D,n)\delta^{-4}\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar{D})} + \kappa_2(D,n)\log\bigl(3+\delta^{-1}\bigr)\|v\|_{C(\partial D)}, \tag{5.2}$$

for $z_0 \in D \setminus (\partial D)_\delta$, $0 < \delta < 1$, $\lambda \in \mathbb{C}$, $|\lambda| \geqslant 1$.

The proof of Lemma 5.2 for the scalar case can be found in [13] and its generalization to the matrix-valued case is straightforward.

Proof of Proposition 5.1. Fix $0 < \alpha < \frac{1}{5}$ and $0 < \delta < 1$. We then have the following chain of inequalities

$$\begin{aligned}
\|v_2 - v_1\|_{L^\infty(D)} &= \max\bigl(\|v_2 - v_1\|_{L^\infty(D\cap(\partial D)_\delta)}, \|v_2 - v_1\|_{L^\infty(D\setminus(\partial D)_\delta)}\bigr) \\
&\leqslant C_1 \max\Biggl(2N\delta + \|\Phi_2 - \Phi_1\|_1, \frac{\log(3\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))}{\delta^4 \log(3 + \|\Phi_2 - \Phi_1\|^{-1})} \\
&\quad + \log\left(3 + \frac{1}{\delta}\right)\|\Phi_2 - \Phi_1\|_1 + \frac{(\log(3\log(3 + \|\Phi_2 - \Phi_1\|^{-1})))^2}{(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{\frac{3}{4}}}\Biggr) \\
&\leqslant C_2 \max\Biggl(2N\delta + \|\Phi_2 - \Phi_1\|_1, \frac{1}{\delta^4}\bigl(\log\bigl(3 + \|\Phi_2 - \Phi_1\|_1^{-1}\bigr)\bigr)^{-5\alpha} \\
&\quad + \log\left(3 + \frac{1}{\delta}\right)\|\Phi_2 - \Phi_1\|_1 + \frac{(\log(3\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})))^2}{(\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}))^{\frac{3}{4}}}\Biggr),
\end{aligned}$$

where we followed the outline of the proof of Theorem 1.1 with the following modifications: we made use of Lemma 5.2 instead of Lemma 2.2 and we also used (i)–(ii); note that $C_1 = C_1(D, N, n)$ and $C_2 = C_2(D, N, n, \alpha)$.

Putting $\delta = (\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}))^{-\alpha}$ we obtain the desired inequality

$$\|v_2 - v_1\|_{L^\infty(D)} \leqslant C_3\bigl(\log\bigl(3 + \|\Phi_2 - \Phi_1\|_1^{-1}\bigr)\bigr)^{-\alpha}, \tag{5.3}$$

with $C_3 = C_3(D, N, n, \alpha)$, $\|\Phi_2 - \Phi_1\|_1 = \varepsilon \leqslant \varepsilon_1(D, N, n, \alpha)$ with ε_1 sufficiently small or, more precisely when $\delta_1 = (\log(3 + \varepsilon_1^{-1}))^{-\alpha}$ satisfies

$$\delta_1 < 1, \qquad \varepsilon_1 \leqslant 2N\delta_1, \qquad \log\left(3 + \frac{1}{\delta_1}\right)\varepsilon_1 \leqslant \delta_1.$$

Estimate (5.3) for general ε (with modified C_3) follows from (5.3) for $\varepsilon \leqslant \varepsilon_1(D, N, n, \alpha)$ and the assumption that $\|v_j\|_{L^\infty(\bar{D})} \leqslant N$ for $j = 1, 2$. This completes the proof of Proposition 5.1. □

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