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# Radiation equations for black holes

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In memory of Bryce DeWitt

**Abstract**

It has been shown in [G.A. Vilkovisky, hep-th/0511182] that the metric in the semiclassical region of the collapse spacetime is expressed purely kinematically through the Bondi charges. Here the Bondi charges are expressed through this metric by calculating the vacuum radiation against its background. The result is closed equations for the metric and the Bondi charges. Notably, there is a nonvanishing flux of the vacuum-induced matter charge.



In Ref. [1], a new approach is proposed to the problem of backreaction of the Hawking radiation. For the spherically symmetric collapse of a compact matter source it has been shown that, in the semiclassical region [1] of the expectation-value spacetime, the gravity equations close purely kinematically leaving the arbitrariness only in the data functions. The data functions are two Bondi charges appearing as coefficients of the expansion of the metric at the future null infinity $\mathcal{I}^+$

$$(\nabla r)^2\big|_{\mathcal{I}^+} = 1 - \frac{2\mathcal{M}(u)}{r} + \frac{Q^2(u)}{r^2} + O\left(\frac{1}{r^3}\right), \tag{1}$$

$$(\nabla r, \nabla u)|_{\mathcal{I}^+} = -1 - \frac{c_1}{r} + O\left(\frac{1}{r^2}\right). \tag{2}$$

Here $c_1$ is a constant [1], and, for the model of the vacuum considered below, $c_1 = 0$. The notation in Eqs. (1), (2) and below is conventional for spherical symmetry and is the same as in Ref. [1] with the following exception. The retarded time $u^+$ normalized at the future null infinity is in the present notation just $u$, and the retarded time $u^-$ counted out by an early falling observer [1] will here be denoted as

$$u^- = U(u), \qquad \frac{du^-}{du^+} = \dot{U}(u). \tag{3}$$

The advanced time $v$ that also figures below remains normalized at the past null infinity $\mathcal{I}^-$. The partial derivative $\partial_u$ is the derivative at fixed $v$.

As shown in Ref. [1], the metric in the semiclassical region depends only on the Bondi charges $\mathcal{M}(u)$ and $Q^2(u)$. On the other hand, the Bondi charges depend only on the metric in the semiclassical region, this dependence being already a subject of the quantum dynamics. Having both dependences obtained, one gets self-consistent equations for the Bondi charges and, thereby, for the expectation value of the metric in the semiclassical region.

The calculation thus falls into three stages. The first stage: solving the kinematical equations for the metric in terms of the Bondi charges is accomplished in Ref. [1]. The purpose of the present work is the second stage: calculation of the vacuum radiation against

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the thus obtained gravitational background. This should express the Bondi charges through themselves, and there will remain only the last stage: solving the resultant self-consistent equations.

Both Bondi charges $\mathcal{M}(u)$ and $Q^2(u)$ appear in the expansion of the flux component of the energy–momentum tensor at $\mathcal{I}^+$:

$$\int d^2\mathcal{S}\, r^2 T^{\mu\nu} \frac{\nabla_\mu v \nabla_\nu v}{(\nabla v, \nabla u)^2}\bigg|_{\mathcal{I}^+} = -\partial_u \mathcal{M} + \frac{1}{2}\partial_u Q^2 \frac{1}{r} + O\left(\frac{1}{r^2}\right). \tag{4}$$

Here the integration is over the unit 2-sphere $\mathcal{S}$ entering the product $\mathcal{I}^+ = (u\text{-axis}) \times \mathcal{S}$. Since the collapsing matter source is assumed having a compact spatial support [1], the $T^{\mu\nu}$ in Eq. (4) is the energy–momentum tensor of the in-vacuum, $T^{\mu\nu}_{\text{vac}}$. The quantity (4) with $T^{\mu\nu} = T^{\mu\nu}_{\text{vac}}$ is the only one that needs to be calculated. There is more than one way to do this calculation in semiclassical theory. The present calculation uses the WKB technique along the lines of Refs. [2,3].

To model the vacuum, the simplest quantum field will be chosen: a massless scalar field satisfying the equation

$$(\Box - \xi R)\boldsymbol{\Phi} = 0, \quad \xi = \text{const}, \tag{5}$$

and having the energy–momentum tensor

$$\mathbf{T}^{\mu\nu} = (1-2\xi)\nabla^\mu \boldsymbol{\Phi} \nabla^\nu \boldsymbol{\Phi} - 2\xi \boldsymbol{\Phi} \nabla^\mu \nabla^\nu \boldsymbol{\Phi} + \xi R^{\mu\nu} \boldsymbol{\Phi}^2 + g^{\mu\nu}\left[\left(2\xi - \frac{1}{2}\right)(\nabla \boldsymbol{\Phi})^2 + 2\xi \boldsymbol{\Phi} \Box \boldsymbol{\Phi} - \frac{1}{2}\xi R \boldsymbol{\Phi}^2\right]. \tag{6}$$

The calculation will be carried out with an arbitrary $\xi$ but the case of interest is $\xi = 1/6$. The latter is important because the locality of the trace of $T^{\mu\nu}_{\text{vac}}$ is assumed in Ref. [1].

The $T^{\mu\nu}_{\text{vac}}$ at $\mathcal{I}^+$ can be obtained by a direct averaging of the operator (6):

$$T^{\mu\nu}_{\text{vac}}\big|_{\mathcal{I}^+} = \big[\langle \text{in vac}|\mathbf{T}^{\mu\nu}|\text{in vac}\rangle - \langle \text{out vac}|\mathbf{T}^{\mu\nu}|\text{out vac}\rangle\big]\big|_{\mathcal{I}^+}, \tag{7}$$

and here one of the main points emerges. The subtraction in Eq. (7) is necessary because there is a noise at $\mathcal{I}^+$ even when the background field is absent. The principal requirement that the subtraction term must satisfy is its locality in the background field. In the effective-action technique, this requirement is secured by the locality of the counterterms. In the present technique, it should be secured by the choice of the quantum state in the subtraction term. The definition of this state should not involve the quantum-field operators in the future or past of the observation point of $T^{\mu\nu}_{\text{vac}}$. With the observation point at $\mathcal{I}^+$, the normalization state is the out-vacuum. Below, the notation is introduced

$$\langle \mathbf{X} \rangle = \langle \text{in vac}|\mathbf{X}|\text{in vac}\rangle - \langle \text{out vac}|\mathbf{X}|\text{out vac}\rangle. \tag{8}$$

From Eq. (6) one obtains

$$T^{\mu\nu}_{\text{vac}} \frac{\nabla_\mu v \nabla_\nu v}{(\nabla v, \nabla u)^2}\bigg|_{\mathcal{I}^+} = \big\langle (\partial_u \boldsymbol{\Phi})^2 \big\rangle - \xi \partial^2_{uu} \big\langle \boldsymbol{\Phi}^2 \big\rangle + O\left(\frac{1}{r^4}\right). \tag{9}$$

Expanding $\boldsymbol{\Phi}$ in spherical harmonics, one can replace the quantum field in 4 dimensions with a sequence of quantum fields $\boldsymbol{\Phi}_l$ in 2 dimensions. The fields $\boldsymbol{\Phi}_l$, $l = 0, 1, \ldots$, are defined on the Lorentzian subspace of a spherically symmetric spacetime and, there, satisfy the equation

$$\boldsymbol{\Phi}_l = \frac{1}{r}\boldsymbol{\Psi}_l, \tag{10}$$

$$\left(\triangle - \frac{\triangle r}{r} - \xi R - \frac{l(l+1)}{r^2}\right)\boldsymbol{\Psi}_l = 0 \tag{11}$$

with $\triangle$ the D'Alembert operator in this subspace. In terms of $\boldsymbol{\Psi}_l$ one obtains from Eq. (9)

$$\int d^2\mathcal{S}\, r^2 T^{\mu\nu}_{\text{vac}} \frac{\nabla_\mu v \nabla_\nu v}{(\nabla v, \nabla u)^2}\bigg|_{\mathcal{I}^+} = \sum_{l=0}^{\infty}(2l+1)\left[\big\langle (\partial_u \boldsymbol{\Psi}_l)^2 \big\rangle - \xi \partial^2_{uu}\big\langle {\boldsymbol{\Psi}_l}^2 \big\rangle + \frac{1}{r}\left(\frac{1}{2} - 2\xi\right)\partial_u \big\langle {\boldsymbol{\Psi}_l}^2 \big\rangle + O\left(\frac{1}{r^2}\right)\right]. \tag{12}$$

The expansion of $\boldsymbol{\Psi}_l$ at $\mathcal{I}^+$ is readily obtained from Eq. (11). One needs the retarded solution which is

$$\boldsymbol{\Psi}_l|_{\mathcal{I}^+} = \boldsymbol{\Psi}_l(u) + \frac{1}{r}\frac{l(l+1)}{2}\int_{-\infty}^{u} du\, \boldsymbol{\Psi}_l(u) + O\left(\frac{1}{r^2}\right). \tag{13}$$

Here $\boldsymbol{\Psi}_l(u)$ are the data for $\boldsymbol{\Psi}_l$ at $\mathcal{I}^+$. Inserting this expansion in Eq. (12) and using Eq. (4) one obtains finally

$$-\partial_u \mathcal{M} = \sum_{l=0}^{\infty}(2l+1)\big[\big\langle \big(\partial_u \boldsymbol{\Psi}_l(u)\big)^2 \big\rangle - \xi \partial^2_{uu}\big\langle {\boldsymbol{\Psi}_l}^2(u) \big\rangle\big], \tag{14}$$

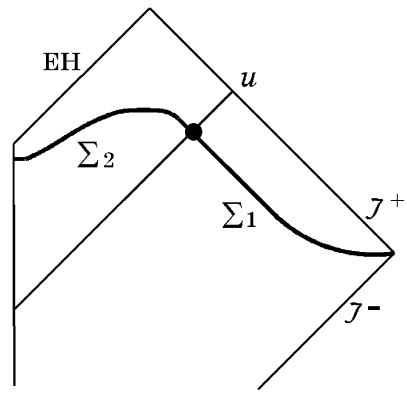


Fig. 1. Choice of the Cauchy surface $\Sigma$ for the inner product, $\Sigma = \Sigma_1 \cup \Sigma_2$. The division into $\Sigma_1$ and $\Sigma_2$ is specified by the location of the observation point $u$ at $\mathcal{I}^+$. The event horizon EH if any is in the future of $\Sigma$.

$$\partial_u Q^2 = \sum_{l=0}^{\infty}(2l+1)\left[(1-4\xi)\partial_u\langle\boldsymbol{\Psi}_l{}^2(u)\rangle + l(l+1)\partial_u\langle\boldsymbol{\Psi}_l{}^2(u)\rangle - \xi l(l+1)\partial^3_{uuu}\left\langle\left(\int_{-\infty}^{u} du\,\boldsymbol{\Psi}_l(u)\right)^2\right\rangle\right]. \tag{15}$$

The flux of $Q^2$ in Eq. (15) appears as a total derivative (candidate for discarding) but appearances may deceive.

Eqs. (14), (15) leave one with the expectation values of the form

$$\langle(\mathcal{D}\boldsymbol{\Psi}_l(u))^2\rangle = \langle(\mathcal{D}\boldsymbol{\Psi}_l(x))^2\rangle|_{x\in\mathcal{I}^+}, \tag{16}$$

where $\mathcal{D} = \mathcal{D}(\partial_u)$ is some (retarded) linear operator. Let $\psi^{\text{in}}_{la}(x)$, $a = \varepsilon_{\text{in}}$, be the solution of Eq. (11) that asymptotically at $\mathcal{I}^-$ becomes the eigenfunction of the energy operator with the eigenvalue $\varepsilon_{\text{in}}$. Let $\psi^{\text{out}}_{lA}(x)$, $A = \varepsilon_{\text{out}}$, be the solution defined by a similar condition at $\mathcal{I}^+$. Normalized with the aid of the inner product, the $\psi^{\text{in}}_{la}(x)$ and $\psi^{\text{out}}_{lA}(x)$ make two bases of solutions of Eq. (11) (for each $l$), related by the Bogolyubov transformation

$$\psi^{\text{out}}_{lA}(x) = \alpha^l_{Aa}\psi^{\text{in}}_{la}(x) + \beta^l_{Aa}\psi^{\dagger\,\text{in}}_{la}(x). \tag{17}$$

(Summation is assumed over the repeating $a$ or/and $A$ but not $l$.) The dagger designates complex conjugation, and "the complex conjugate quantity" will be abbreviated as c.c. Expanding the quantum field $\boldsymbol{\Psi}_l(x)$ in any of the bases and calculating the difference in Eq. (8) one obtains

$$\langle(\mathcal{D}\boldsymbol{\Psi}_l(x))^2\rangle = -\big(\mathcal{D}\psi^{\dagger\,\text{out}}_{lA}(x)\big)\big(\mathcal{D}\psi^{\dagger\,\text{in}}_{la}(x)\big)\beta^l_{Aa} + \text{c.c.}, \tag{18}$$

$$\beta^l_{Aa} = \mathrm{i}\int d^2x\,{}^{(2)}g^{1/2}\delta(\sigma)(\nabla_\mu\sigma)\big(\psi^{\text{in}}_{la}\nabla^\mu\,\psi^{\text{out}}_{lA}(x) - \psi^{\text{out}}_{lA}\nabla^\mu\,\psi^{\text{in}}_{la}(x)\big). \tag{19}$$

Here $d^2x\,{}^{(2)}g^{1/2}$ is the volume element of the Lorentzian subspace, and the equation

$$\Sigma:\quad \sigma(x) = 0 \tag{20}$$

with $\nabla\sigma$ past-directed describes an arbitrary Cauchy surface $\Sigma$ in this subspace.

To see that the expectation value (18) with $x \in \mathcal{I}^+$ is causal, i.e., does not contain the background field in the future of the observation point, note that it can be expressed through the commutator function of the field $\boldsymbol{\Psi}_l$

$$G_l(x,x') = \frac{1}{\mathrm{i}}\big[\boldsymbol{\Psi}_l(x),\boldsymbol{\Psi}_l(x')\big] = -\mathrm{i}\psi^{\text{in}}_{la}(x)\psi^{\dagger\,\text{in}}_{la}(x') + \text{c.c.} = -\mathrm{i}\psi^{\text{out}}_{lA}(x)\psi^{\dagger\,\text{out}}_{lA}(x') + \text{c.c.} \tag{21}$$

Namely,

$$\begin{aligned}\langle(\mathcal{D}\boldsymbol{\Psi}_l(x))^2\rangle = \int d^2x'\,{}^{(2)}g'^{1/2}\delta(\sigma')(\nabla'_\mu\sigma')\big[&\big(\mathcal{D}\psi^{\dagger\,\text{out}}_{lA}\psi'^{\,\text{out}}_{lA} - \mathcal{D}\psi^{\dagger\,\text{in}}_{la}\psi'^{\,\text{in}}_{la}\big)\mathcal{D}\nabla'^\mu G_l(x,x') \\ &+ \big(\mathcal{D}\psi^{\text{in}}_{la}\nabla'^\mu\,\psi'^{\dagger\,\text{in}}_{la} - \mathcal{D}\psi^{\text{out}}_{lA}\nabla'^\mu\,\psi'^{\dagger\text{out}}_{lA}\big)\mathcal{D}G_l(x,x')\big].\end{aligned} \tag{22}$$

With the observation point $x$ at $\mathcal{I}^+$ at a given value of $u$, choose the Cauchy surface $\Sigma$ as shown in Fig. 1, $\Sigma = \Sigma_1 \cup \Sigma_2$. The contribution of $\Sigma_2$ drops out of (22) because the commutator function vanishes when $x'$ is outside the light cone of $x$. Then shift $\Sigma_1$ to $\mathcal{I}^+$. The result is

$$\langle(\mathcal{D}\boldsymbol{\Psi}_l(u))^2\rangle = -\mathrm{i}\big(\mathcal{D}\psi^{\dagger\,\text{out}}_{lA}(u)\big)\big(\mathcal{D}\psi^{\dagger\,\text{in}}_{la}(u)\big)\int_{-\infty}^{u+0} du'\,\big[\psi^{\text{out}}_{lA}(u')\,\partial_{u'}\,\psi^{\text{in}}_{la}(u') - \psi^{\text{in}}_{la}(u')\partial_{u'}\psi^{\text{out}}_{lA}(u')\big] + \text{c.c.} \tag{23}$$

Here $\psi^{\rm out}_{lA}(u)$ and $\psi^{\rm in}_{la}(u)$ are the data at $\mathcal{I}^+$ for the basis functions $\psi^{\rm out}_{lA}(x)$ and $\psi^{\rm in}_{la}(x)$. Only these data and only in the past of the observation point $u$ are needed to calculate the expectation value (23).

The WKB solution for $\psi^{\rm in}_{la}(x)$ boils down to the solution for the geodesics in the background metric. The background metric is the sought for expectation value of the metric—the final goal of the calculation—but kinematics gives it in terms of arbitrary Bondi charges [1]. One needs to solve for the congruence of the null geodesics that start at $\mathcal{I}^-$ with one and the same energy $\varepsilon_{\rm in}$ and one and the same angular momentum $L=\hbar l$. It suffices to know which of these geodesics come to the point $u$ of $\mathcal{I}^+$, and what then is their energy with respect to the Killing vector at $\mathcal{I}^+$. Denote this energy as

$$\varepsilon_+ = \varepsilon_+(u, \varepsilon_{\rm in}, L). \tag{24}$$

One can put the question differently. Consider the geodesic that comes to the point $u$ of $\mathcal{I}^+$ with the energy $\varepsilon_+$ and angular momentum $L$, and trace it back to $\mathcal{I}^-$. Then what is its energy with respect to the Killing vector at $\mathcal{I}^-$? The answer is given by Eq. (24) solved with respect to $\varepsilon_{\rm in}$. The result is

$$\varepsilon_+ = \begin{cases} \varepsilon_{\rm in}, & L > \varepsilon_{\rm in} H(u), \\ \varepsilon_{\rm in}\frac{dU(u)}{du}, & L \leqslant \varepsilon_{\rm in} H(u)\frac{dU(u)}{du}. \end{cases} \tag{25}$$

Two functions of the background metric govern this behaviour. One is the $U(u)$ of Eq. (3), and the other one, $H(u)$, may be interpreted as defining the variable height of the centrifugal barrier. Both are expressed through the Bondi charges:

$$H^{-2}(u) = 2\frac{\mathcal{M}(u)+\sqrt{9\mathcal{M}^2(u)-8Q^2(u)}}{(3\mathcal{M}(u)+\sqrt{9\mathcal{M}^2(u)-8Q^2(u)}\,)^3}, \tag{26}$$

and [1]

$$\frac{d}{du}\ln\frac{dU(u)}{du} = -\kappa(u), \tag{27}$$

$$\kappa(u) = \frac{\sqrt{\mathcal{M}^2(u)-Q^2(u)}}{(\mathcal{M}(u)+\sqrt{\mathcal{M}^2(u)-Q^2(u)}\,)^2}, \qquad u > u_0. \tag{28}$$

Here $u_0$ labels the radial light ray tangent to the apparent horizon [1]. Expression (28) is invalid for early $u$, and the range of $u$ for which Eq. (25) holds is $u > u_0 + O(\mathcal{M})$ but this is the range in which one needs to calculate the radiation. At earlier $u$, the radiation is negligible. The geodesics whose $\varepsilon_{\rm in}$ and $L$ do not satisfy either of the inequalities in Eq. (25) do not come to the point $u$ of $\mathcal{I}^+$.

The derivation of the result above will here be omitted. It will only be mentioned that the geodesics corresponding to the second line of Eq. (25) turn (in $r$) three times. The respective particles start at $\mathcal{I}^-$ as incoming and turn the first time before the black hole has formed. They are outgoing already when the collapsing mass comes and turns them back. They cross the apparent horizon and continue falling down but, before they reach small $r$ and even before they cross a possible event horizon [1], gravity weakens and lets them go. They cross the apparent horizon the second time, next turn the third time, and go out to $\mathcal{I}^+$. The third turn is a quantum effect.

The congruence of geodesics considered is hypersurface orthogonal, and the function (24) determines the phase of $\psi^{\rm in}_{la}(x)$ at $\mathcal{I}^+$. Hence one obtains the data at $\mathcal{I}^+$ for the basis functions:

$$\psi^{\rm in}_{la}(u) = \frac{1}{\sqrt{4\pi\varepsilon_{\rm in}}}\big[\theta(\varepsilon_{\rm in}H\dot{U}-l)\exp(-{\rm i}\varepsilon_{\rm in}U) + \theta(l-\varepsilon_{\rm in}H)\exp(-{\rm i}\varepsilon_{\rm in}u)\big], \tag{29}$$

$$\psi^{\rm out}_{lA}(u) = \frac{1}{\sqrt{4\pi\varepsilon_{\rm out}}}\exp(-{\rm i}\varepsilon_{\rm out}u). \tag{30}$$

The expectation values (23) will thus be expressed entirely through the functions $H(u)$ and $U(u)$. It is important that, even kinematically, the Bondi charges are not completely arbitrary [1]. By their properties, $H(u)$ and $\kappa(u)$ in Eq. (27) are macroscopic quantities whereas the derivatives of these functions in $u$ are microscopic quantities. In the notation of Ref. [1],

$$|\mathcal{O}| < H, \frac{1}{\kappa} < \frac{1}{|\mathcal{O}|}, \qquad \frac{d}{du}H = \mathcal{O}, \qquad \frac{d}{du}\frac{1}{\kappa} = \mathcal{O}, \tag{31}$$

$$\dot{U} = \mathcal{O}, \qquad u > u_0, \tag{32}$$

where $\mathcal{O}$ vanishes when the quantum parameter tends to zero.

In the upper limit of the integral in Eq. (23) one may exploit condition (32):

$$\dot{U}(u+0) < \dot{U}(u) = \mathcal{O} \tag{33}$$

and set $\dot{U}(u+0)=0$. In view of Eqs. (27) and (31), this is equivalent to setting $u+0=\infty$, and then the integration by parts proves that the two terms of the integral are equal. In the product of two functions (29) that figures in Eq. (23) only one diagonal term survives and takes the form

$$\psi^{\dagger\,\mathrm{in}}_{la}(u)\psi^{\mathrm{in}}_{la}(u') = \int_0^\infty \frac{d\varepsilon_{\mathrm{in}}}{4\pi\,\varepsilon_{\mathrm{in}}}\theta(\varepsilon_{\mathrm{in}} H\dot U - l)\exp\bigl(\mathrm{i}\varepsilon_{\mathrm{in}}(U - U')\bigr) \tag{34}$$

because, for $u > u'$, $H\dot U < H'\dot U'$. The contribution of the other diagonal term to the expectation values vanishes because the contribution of the term (34) vanishes when $\kappa \equiv 0$, and expression (23) vanishes when $\psi^{\mathrm{in}} = \psi^{\mathrm{out}}$. The cross terms vanish because they imply $\dot U > |\mathcal{O}|$ or $\dot U' > |\mathcal{O}|$.

It follows from Eqs. (23) and (34) that

$$\sum_{l=0}^\infty p_1(l)\bigl\langle\bigl(\mathcal{D}\boldsymbol{\Psi}_l(u)\bigr)^2\bigr\rangle = p_1(0)\bigl\langle\bigl(\mathcal{D}\boldsymbol{\Psi}_l(u)\bigr)^2\bigr\rangle\big|_{l=0} + \int_0^\infty dl\; p_2(l)\bigl\langle\bigl(\mathcal{D}\boldsymbol{\Psi}_l(u)\bigr)^2\bigr\rangle, \tag{35}$$

where $p_1(l)$ is any given polynomial, and $p_2(l)$ is some other polynomial. Doing the sum over $l$ first and the sum over $\varepsilon_{\mathrm{out}}$ last, one obtains each contribution to (35) as a spectral integral over the energy $\varepsilon_{\mathrm{out}}$, the spectral function being a combination of the functions

$$I_n(\varepsilon_{\mathrm{out}}, u) = \int_0^\infty d\varepsilon_{\mathrm{in}}\,\dot U(\mathrm{i}\varepsilon_{\mathrm{in}}\dot U)^n \int_{-\infty}^\infty du'\,\varepsilon_{\mathrm{in}}\dot U'\exp\bigl(\mathrm{i}(\Omega - \Omega')\bigr), \tag{36}$$

$$\Omega - \Omega' = \varepsilon_{\mathrm{in}}(U - U') + \varepsilon_{\mathrm{out}}(u - u'). \tag{37}$$

Specifically,

$$\bigl\langle\bigl(\partial_u\boldsymbol{\Psi}_l(u)\bigr)^2\bigr\rangle\big|_{l=0} = \frac{2}{(4\pi)^2}\int_0^\infty d\varepsilon_{\mathrm{out}}\,I_0(\varepsilon_{\mathrm{out}}, u) + \text{c.c.}, \tag{38}$$

$$\bigl\langle\boldsymbol{\Psi}_l{}^2(u)\bigr\rangle\big|_{l=0} = \frac{2}{(4\pi)^2}\int_0^\infty d\varepsilon_{\mathrm{out}}\,\frac{1}{\mathrm{i}\varepsilon_{\mathrm{out}}}I_{-1}(\varepsilon_{\mathrm{out}}, u) + \text{c.c.} \tag{39}$$

In Eq. (36) introduce the new integration variables

$$y = \varepsilon_{\mathrm{in}}\dot U\frac{1}{\kappa}, \qquad x = \varepsilon_{\mathrm{in}}\dot U'\frac{1}{\kappa'} \tag{40}$$

to obtain

$$I_n(\varepsilon_{\mathrm{out}}, u) = \kappa^{n+1}\int_0^\infty dy\,(\mathrm{i}y)^n\int_0^\infty \frac{dx}{w}\exp\bigl(\mathrm{i}(\Omega - \Omega')\bigr), \tag{41}$$

$$w = 1 - \frac{d}{du'}\frac{1}{\kappa'}. \tag{42}$$

Next use Eq. (27) to write

$$\dot U = \exp\left(-\int_{-\infty}^u d\bar u\,\bar\kappa\right), \tag{43}$$

$$U' - U = \int_u^{u'} du''\exp\left(-\int_{-\infty}^{u''} d\bar u\,\bar\kappa\right) \tag{44}$$

and integrate in $u''$ by parts:

$$U' - U = \left(1 + \frac{d}{du}\frac{1}{\kappa} + \frac{d}{du}\left(\frac{1}{\kappa}\frac{d}{du}\frac{1}{\kappa}\right) + \cdots\right)\frac{1}{\kappa}\dot U - \left(1 + \frac{d}{du'}\frac{1}{\kappa'} + \frac{d}{du'}\left(\frac{1}{\kappa'}\frac{d}{du'}\frac{1}{\kappa'}\right) + \cdots\right)\frac{1}{\kappa'}\dot U'. \tag{45}$$

Owing to condition (31), all the corrections with the derivatives of $\kappa$ are negligible both in (42) and (45):

$$w = 1, \qquad \varepsilon_{\mathrm{in}}(U - U') = x - y. \tag{46}$$

There remains to be expressed through $x$ and $y$ the difference $(u-u')$ in Eq. (37). For that one has the equation

$$\ln\frac{y}{x}=\int_u^{u'}du''\,\kappa''\left(1-\frac{d}{du''}\frac{1}{\kappa''}\right) \tag{47}$$

in which the last term is negligible. This equation can be solved by expanding in the derivatives of $\kappa$:

$$\kappa(u'-u)=\ln\frac{y}{x}+\frac{1}{2}\left(\frac{d}{du}\frac{1}{\kappa}\right)\ln^2\frac{y}{x}+\frac{1}{6}\left(\frac{d}{du}\left(\frac{1}{\kappa}\frac{d}{du}\frac{1}{\kappa}\right)\right)\ln^3\frac{y}{x}+\cdots \tag{48}$$

but here the corrections with the derivatives are not unconditionally negligible as they are in Eqs. (42) and (45). The point here is that the integrals in $x$ and $y$ are cut off by oscillations at both the upper and lower limits so that the integration regions for $\ln x$ and $\ln y$ are effectively compact. Then $\ln(y/x)$ is bounded, and, in Eq. (48), all the terms with the derivatives of $\kappa$ are negligible. As a consequence, the spectral function (41) is calculable. Another important consequence is

$$\kappa(u'-u)<\frac{1}{|\mathcal{O}|}. \tag{49}$$

It was tacitly assumed in the derivations above that conditions (31), (32) valid at the observation point $u$ are valid also at the integration point $u'$. Eq. (49) proves that this is the case.

Now one comes to the central point of the present consideration. The argument above about the effective compactness of the integration regions for $\ln x$ and $\ln y$ needs a reserve. At the upper limits, the integrals in $x$ and $y$ are cut off by oscillations always whereas at the lower limits only when $\varepsilon_{\text{out}}\neq 0$. Therefore, the argument may break down for the low-energy part of the spectrum: $\varepsilon_{\text{out}}\to 0$. This can be checked. Introduce the notation

$$z=\frac{\varepsilon_{\text{out}}}{\kappa(u)}. \tag{50}$$

If the corrections in Eq. (48) are small indeed, their contributions to the spectral function can be calculated by expanding the exponential

$$\exp\bigl(\mathrm{i}\varepsilon_{\text{out}}(u-u')\bigr)=\left[1+P\left(\mathrm{i}z,\ln\frac{x}{y}\right)\right]\exp\left(\mathrm{i}z\ln\frac{x}{y}\right). \tag{51}$$

Here $P$ is a series of the form

$$P\left(\mathrm{i}z,\ln\frac{x}{y}\right)=\sum\mathcal{O}(\mathrm{i}z)^k\left(\ln\frac{x}{y}\right)^s,\quad k\geqslant 1,\ s\geqslant k+1. \tag{52}$$

Insertion of the expansion (51) in Eq. (41) yields the following result:

$$I_n(\varepsilon_{\text{out}},u)=\kappa^{n+1}(u)\left[1+P\left(\mathrm{i}z,\frac{d}{d\mathrm{i}z}\right)\right]F_n(z), \tag{53}$$

$$F_n(z)=\mathrm{e}^{-\pi z}\Gamma(n+1-\mathrm{i}z)\Gamma(1+\mathrm{i}z), \tag{54}$$

where $\Gamma$ is the Euler's function. When $n\geqslant 0$, the function $F_n(z)$ and all its derivatives are bounded including at $z\to 0$. Then the entire contribution of $P$ in Eq. (53) is $\mathcal{O}$ and is negligible. However, when $n<0$, the function $F_n(z)$ behaves as $1/z$ at $z\to 0$. Then, because, in the series $P$, the power of $(d/d\mathrm{i}z)$ exceeds the power of $(\mathrm{i}z)$ at least by one, the corrections due to $P$ are *even more singular*, and, therefore, at $\varepsilon_{\text{out}}\to 0$ they are not small.

Thus, for $n\geqslant 0$ the spectral function (36) is successfully calculated. One has

$$n\geqslant 0:\quad I_n(\varepsilon_{\text{out}},u)=\kappa^{n+1}(u)\mathrm{e}^{-\pi z}\Gamma(n+1-\mathrm{i}z)\Gamma(1+\mathrm{i}z) \tag{55}$$

with $z$ in Eq. (50). Hence one obtains, in particular, the expectation value (38)

$$\bigl\langle\bigl(\partial_u\boldsymbol{\Psi}_l(u)\bigr)^2\bigr\rangle\big|_{l=0}=\frac{4}{(4\pi)^2}\int_0^\infty d\varepsilon_{\text{out}}\frac{2\pi\varepsilon_{\text{out}}}{\mathrm{e}^{2\pi z}-1} \tag{56}$$

but does not obtain (39).

For $I_n$ with $n<0$ one has a difficulty. In order that the function $\kappa(u)$ could be regarded as slowly varying, the operator $\mathcal{D}$ acting on $\psi^{\dagger\,\text{in}}(u)$ in Eq. (23) should be $\partial_u$ to the power 1 or higher. This suggests the way of overcoming the difficulty. For $\mathcal{D}={\partial_u}^0$ and $\mathcal{D}={\partial_u}^{-1}$ write in Eq. (23)

$$\psi^{\dagger\,\text{in}}(u)=\int_{-\infty}^{u}d\bar{u}\,\partial_{\bar{u}}\,\bar{\psi}^{\dagger\,\text{in}}, \tag{57}$$

$$\int_{-\infty}^{u} du\, \psi^{\dagger\,\mathrm{in}}(u) = \int_{-\infty}^{u} d\bar u\, (u-\bar u)\partial_{\bar u}\, \bar\psi^{\dagger\,\mathrm{in}}. \tag{58}$$

Alternatively, write in Eq. (36)

$$\exp(\mathrm{i}\varepsilon_{\mathrm{in}} U) = \int_{-\infty}^{u} d\bar u\, (\mathrm{i}\varepsilon_{\mathrm{in}}\dot{\bar U})\exp(\mathrm{i}\varepsilon_{\mathrm{in}}\bar U) \tag{59}$$

to obtain

$$I_{-1}(\varepsilon_{\mathrm{out}}, u) = \int_{-\infty}^{u} d\bar u\, I_0(\varepsilon_{\mathrm{out}}, \bar u)\exp\bigl(\mathrm{i}\varepsilon_{\mathrm{out}}(u-\bar u)\bigr). \tag{60}$$

This calculates $I_{-1}$ through $I_0$, and for $I_0$ one has the result (55) but the new obstacle is that the integral (60) involves $u$ down to $u=-\infty$ whereas the result (55) is valid only for $u>u_0$. Indeed, only at late $u$ is $\kappa(u)$ slowly varying by virtue of condition (31). The obstacle is not big, however. Expression (55) is inaccurate at early time but, since the radiation at early time is negligible altogether, this inaccuracy is unessential. The specific form of $\kappa(u)$ at early time is also unessential. It is only important that $\kappa(u)$ falls off at $u\to-\infty$ so that the integral in Eq. (43) converges. For the calculational purposes, one may just set $\kappa(u)=0$ for $u<u_0$, this being equivalent to switching off the background curvature at early time.

Using Eqs. (60) and (55), one obtains the expectation value (39):

$$\bigl\langle \boldsymbol{\Psi}_l{}^2(u)\bigr\rangle\bigr|_{l=0} = \frac{4}{(4\pi)^2}\int_0^\infty d\varepsilon_{\mathrm{out}}\int_{-\infty}^{u} d\bar u\,\frac{2\pi\,\sin(\varepsilon_{\mathrm{out}}(u-\bar u))}{\mathrm{e}^{2\pi\bar z}-1}, \tag{61}$$

where $\bar z=\varepsilon_{\mathrm{out}}/\bar\kappa$, and $\bar\kappa=\kappa(\bar u)$. The integral over $\varepsilon_{\mathrm{out}}$ can be done:

$$\bigl\langle \boldsymbol{\Psi}_l{}^2(u)\bigr\rangle\bigr|_{l=0} = \frac{4}{(4\pi)^2}\int_{-\infty}^{u} d\bar u\,\bar\kappa\left[\frac{\pi}{2}-\frac{\pi}{\bar\kappa(u-\bar u)}+\frac{\pi}{\mathrm{e}^{\bar\kappa(u-\bar u)}-1}\right], \tag{62}$$

and finally one obtains

$$\bigl\langle \boldsymbol{\Psi}_l{}^2(u)\bigr\rangle\bigr|_{l=0} = \frac{4}{(4\pi)^2}\left[\frac{\pi}{2}\left(\int_{-\infty}^{u} d\bar u\,\bar\kappa\right)-\pi\ln\left(\int_{-\infty}^{u} d\bar u\,\bar\kappa\right)+O(1)\right],\qquad \left(\int_{-\infty}^{u} d\bar u\,\bar\kappa\right)\to\infty. \tag{63}$$

This is the message of the present Letter. The operator $\boldsymbol{\Phi}^2|_{\mathcal{I}^+}$ averaged over the in-vacuum or out-vacuum is infrared-divergent. The expectation value $\langle\boldsymbol{\Phi}^2\rangle|_{\mathcal{I}^+}$ obtained as the difference in Eq. (8) is finite but *growing at late time*. Its first derivative contributes significantly to the fluxes (14), (15):

$$\partial_u\bigl\langle \boldsymbol{\Psi}_l{}^2(u)\bigr\rangle\bigr|_{l=0} = \frac{4}{(4\pi)^2}\frac{\pi}{2}\kappa(u)(1+\mathcal{O}) \tag{64}$$

whereas the second derivative is already negligible:

$$\partial^2_{uu}\bigl\langle \boldsymbol{\Psi}_l{}^2(u)\bigr\rangle\bigr|_{l=0} = O\left(\frac{d\kappa}{du}\right)=\kappa^2\mathcal{O}. \tag{65}$$

This mechanism of emergence of the vacuum fluxes is familiar. In the effective-action technique they *all* emerge as total derivatives of growing vertex functions [4].

An alternative way of calculating expression (61) is introducing the integration variables

$$\gamma=\varepsilon_{\mathrm{out}}(u-\bar u),\qquad \sigma=\bar\kappa(u-\bar u),\quad \bar u>u_0 \tag{66}$$

and doing the integral over $\sigma$ first. It works also when calculating the $l>0$ contributions to the expectation values. Using Eqs. (57), (58) one obtains

$$\int_0^\infty dl\, l^n\bigl\langle \boldsymbol{\Psi}_l{}^2(u)\bigr\rangle = \frac{4}{(4\pi)^2}\frac{n!}{n+1}H^{n+1}(u)\kappa^{n+1}(u)\begin{cases}-(-1)^{\frac{n}{2}}\ln\bigl(\int_{-\infty}^{u} d\bar u\,\bar\kappa\bigr)+O(1), & n\ \text{even},\\ O(1), & n\ \text{odd},\end{cases} \tag{67}$$

$$\int_0^\infty dl\, l^n \left\langle \left( \int_{-\infty}^u du\, \boldsymbol{\Psi}_l(u) \right)^2 \right\rangle = \frac{1}{(4\pi)^2} \frac{n!}{n+1} H^{n+1}(u)\kappa^{n-1}(u) \begin{cases} (-1)^{\frac{n}{2}} \left( \int_{-\infty}^u d\bar u\, \bar\kappa \right)^2 + O\left( \int_{-\infty}^u d\bar u\, \bar\kappa \right), & n \text{ even}, \\ (-1)^{\frac{n+1}{2}} 2\pi \left( \int_{-\infty}^u d\bar u\, \bar\kappa \right) + O\left( \ln\left( \int_{-\infty}^u d\bar u\, \bar\kappa \right)\right), & n \text{ odd}. \end{cases} \tag{68}$$

Here the powers of growth are different for even and odd powers of $l$ but in all cases this growth is insufficient. Since, in Eq. (15), on the terms (67) there acts $\partial_u$ and on the terms (68) $\partial^3_{uuu}$, the contributions of all of these terms are negligible. *Only the s-mode contributes to the flux of the charge $Q^2$.*

The $l > 0$ modes contribute only to the flux of the gravitational charge $\mathcal{M}$ via

$$\int_0^\infty dl\, l^n \left\langle \left( \partial_u \boldsymbol{\Psi}_l(u) \right)^2 \right\rangle = \frac{2}{(4\pi)^2} \frac{1}{n+1} H^{n+1}(u)\kappa^{n+1}(u) \int_0^\infty d\varepsilon_{\text{out}}\, \varepsilon_{\text{out}} \left[ -(-\mathrm{i})^n \mathrm{e}^{-\pi z} \Gamma(n+1-\mathrm{i}z)\Gamma(1+\mathrm{i}z) + \text{c.c.} \right]. \tag{69}$$

The WKB technique is inaccurate for $l$ of order 1 but the contribution of the s-mode is unambiguous and so is the contribution of the $l \gg 1$ modes. The latter is given by the highest power of $l$ in the polynomial $p_2(l)$ in Eq. (35). Retaining only these two contributions one obtains

$$-\partial_u \mathcal{M} = \frac{4}{(4\pi)^2} 2\pi \int_0^\infty d\varepsilon_{\text{out}} \frac{\varepsilon_{\text{out}} + H^2(u)\varepsilon^3_{\text{out}}}{\mathrm{e}^{2\pi z} - 1}. \tag{70}$$

The contribution of the s-mode describes correctly the low-energy behaviour of the spectral function, and the contribution of the $l \gg 1$ modes describes correctly the high-energy behaviour. The inaccuracy at intermediate energies is a question of the grey-body factor.

The final result is the following set of equations for the Bondi charges:

$$-\partial_u \mathcal{M} = \frac{1}{48\pi} \kappa^2(u) \left( 1 + \frac{1}{10} \left( H(u)\kappa(u) \right)^2 \right), \tag{71}$$

$$\partial_u Q^2 = \frac{1}{8\pi} \kappa(u)(1 - 4\xi). \tag{72}$$

The contribution of the $l \gg 1$ modes to the total energy flux (71) is by an order of magnitude less than the contribution of the s-mode because the high-energy part of the Planckian spectrum is exponentially suppressed. The flux of the charge $Q^2$ depends on the value of $\xi$, and there are values for which it is zero or negative but, if $\xi \neq 1/6$, the present calculation is inconsistent because the background metric of Ref. [1] is invalid. The nonvanishing flux of the charge $Q^2$ is a surprise.

In conclusion, a failure of the 2-dimensional effective action is worth mentioning. At $\xi = 0$, to the s-mode $\boldsymbol{\Psi}_0$ there corresponds the effective action in the Lorentzian subspace:

$${}^{(2)}S_{\text{vac}} = -\frac{1}{96\pi} \int d^2x\, {}^{(2)}g^{1/2}\, {}^{(2)}R \frac{1}{\triangle} {}^{(2)}R \tag{73}$$

so that

$$\frac{2}{{}^{(2)}g^{1/2}} \frac{\delta\, {}^{(2)}S_{\text{vac}}}{\delta g_{\mu\nu}} \frac{\nabla_\mu v \nabla_\nu v}{(\nabla v, \nabla u)^2} = \left\langle (\partial_u \boldsymbol{\Psi}_0)^2 \right\rangle \tag{74}$$

(with the retarded current on the left-hand side). Eq. (74) is to be compared with Eq. (12) at $\xi = 0$ and $l = 0$. Only to the leading order in $1/r$ do these expressions coincide. The effective action (73) should, therefore, give the correct energy flux for the s-mode, and it does. But the flux of the charge $Q^2$ is not contained in this action even for the s-mode and even at $\xi = 0$. This may explain why the 2-dimensional models of the effective equations miss the backreaction of radiation. The 4-dimensional effective action should, of course, reproduce the present results in full but here it will not be considered.

The equations above for the Bondi charges close. Thereby, the expectation-value equations for the metric close already at the level of functions of one variable [1]. The solution will be reported.

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