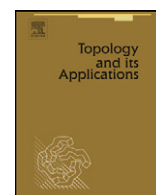




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## On the quasi-component of pseudocompact abelian groups

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## ABSTRACT

In this paper, we describe the relationship between the quasi-component  $q(G)$  of a (perfectly) minimal pseudocompact abelian group  $G$  and the quasi-component  $q(\tilde{G})$  of its completion. Specifically, we characterize the pairs  $(C, A)$  of compact connected abelian groups  $C$  and subgroups  $A$  such that  $A \cong q(G)$  and  $C \cong q(\tilde{G})$ . As a consequence, we show that for every positive integer  $n$  or  $n = \omega$ , there exist plenty of abelian pseudocompact perfectly minimal  $n$ -dimensional groups  $G$  such that the quasi-component of  $G$  is not dense in the quasi-component of the completion of  $G$ .

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## 1. Introduction

A Tychonoff space  $X$  is *pseudocompact* if every continuous real-valued map on  $X$  is bounded. If a topological group  $G$  is pseudocompact, then its completion  $\tilde{G}$  is compact, that is,  $G$  is *precompact* (cf. [4, 1.1]), which allows for the following characterization of pseudocompact groups.

**Theorem 1.1.** ([4]) *A topological group  $G$  is pseudocompact if and only if  $G$  is precompact and  $G_\delta$ -dense in  $\tilde{G}$ , in which case  $\tilde{G} = \beta G$ .*

A Tychonoff space is *zero-dimensional* if it has a base consisting of *clopen* (open-and-closed) sets. For a topological group  $G$ , let  $G_0$  and  $q(G)$  denote the connected component and the quasi-component of the identity, respectively. For every group  $G$ , the quotient  $G/G_0$  is *hereditarily disconnected* (i.e.,  $(G/G_0)_0$  is trivial), and  $G/q(G)$  is *totally disconnected* (i.e.,  $q(G/q(G))$  is trivial). While in the class of locally compact groups (or spaces) these notions of disconnectedness coincide

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with zero-dimensionality, this is not the case in general. Indeed, both  $G/G_0$  and  $G/q(G)$  may fail to be zero-dimensional, or to admit a coarser zero-dimensional group topology, even in the presence of additional compactness-like properties (cf. [3, 7.7], [17], [8, Theorem D]). There is a close relationship between connectedness and disconnectedness properties of pseudocompact groups and those of their completions, which is summarized in the next theorem.

**Theorem 1.2.** *Let  $G$  be a pseudocompact group. Then:*

- (a) ([5, 1.4])  $q(G) = q(\tilde{G}) \cap G = (\tilde{G})_0 \cap G$ ;
- (b) ([26])  $G$  is zero-dimensional if and only if  $\tilde{G}$  is zero-dimensional;
- (c) ([7, 1.7])  $G/q(G)$  is zero-dimensional if and only if  $q(G)$  is dense in  $(\tilde{G})_0 = q(\tilde{G})$ ;
- (d) ([2, 4.11(b)])  $G/G_0$  is zero-dimensional if and only if  $G_0$  is dense in  $(\tilde{G})_0 = q(\tilde{G})$ , in which case  $G_0 = q(G)$ .

We say that a pair  $(C, A)$  of a group  $C$  and its subgroup  $A$  is realized by a group  $G$  if  $A \cong q(G)$  and  $C \cong q(\tilde{G})$ . In a previous paper, the authors obtained sufficient conditions for a pair to be realizable by pseudocompact abelian groups that satisfy various degrees of so-called minimality properties (cf. [8, Theorem D']). In this paper, we obtain a complete characterization (i.e., necessary and sufficient conditions) for the same.

Recall that a (Hausdorff) topological group  $G$  is minimal if there is no coarser (Hausdorff) group topology on  $G$  (cf. [23] and [11]). For a pseudocompact group  $G$ , the quotient  $G/q(G)$  is totally disconnected and pseudocompact, and thus, by an unpublished result of Shakhmatov, admits a coarser zero-dimensional group topology (cf. [8, Theorem B]). Therefore, Shakhmatov's result implies that for a pseudocompact group  $G$ , if  $G/q(G)$  is minimal, then it is zero-dimensional. If in addition  $G$  is minimal and abelian, then the converse is also true.

**Theorem 1.3.** ([6, Theorem 3], [7, 1.7]) *Let  $G$  be a minimal pseudocompact abelian group. Then the following statements are equivalent:*

- (i)  $G/q(G)$  is zero-dimensional;
- (ii)  $q(G)$  is dense in  $(\tilde{G})_0 = q(\tilde{G})$ ;
- (iii)  $G/q(G)$  is minimal.

A group  $G$  is totally minimal if every (Hausdorff) quotient of  $G$  is totally minimal (cf. [9]);  $G$  is perfectly (totally) minimal if the product  $G \times H$  is (totally) minimal for every (totally) minimal group  $H$  (cf. [25]). In an earlier paper, the authors proved the following:

**Theorem 1.4.** ([8, Theorem D']) *Let  $C$  be a connected compact abelian group, and  $A$  a subgroup of  $C$ . Then there exists a pseudocompact abelian group  $G$  such that  $A \cong q(G)$  and  $C \cong (\tilde{G})_0 = q(\tilde{G})$ , and in particular,  $\dim G = \dim C$ . Furthermore, if  $A$  is dense in  $C$  and*

- (a)  $A$  is minimal, then  $G$  may be chosen to be minimal;
- (b)  $A$  is totally minimal, then  $G$  may be chosen to be totally minimal;
- (c)  $A$  is perfectly minimal, then  $G$  may be chosen to be perfectly minimal;
- (d)  $A$  is perfectly totally minimal, then  $G$  may be chosen to be perfectly totally minimal.

By Theorem 1.3, if  $G$  is a totally minimal pseudocompact abelian group, then  $q(G)$  is dense in  $(\tilde{G})_0 = q(\tilde{G})$ . Thus, the condition that  $A$  is dense in  $C$  is not only sufficient, but also necessary in parts (b) and (d) of Theorem 1.4.

In this paper, we show that in order to realize  $(C, A)$  by a (perfectly) minimal pseudocompact abelian group,  $A$  need not be dense in  $C$ , but a milder condition is both sufficient and necessary. A subgroup  $E$  of a topological group  $G$  is essential, and we put  $E \leq_e G$ , if for every non-trivial closed normal subgroup  $N$  of  $G$ , the intersection  $E \cap N$  is non-trivial.

**Theorem A.** *Let  $C$  be a compact connected abelian group, and  $A$  a subgroup. Then:*

- (a)  $(C, A)$  can be realized by a minimal pseudocompact abelian group if and only if  $A \leq_e C$ ;
- (b)  $(C, A)$  can be realized by a perfectly minimal pseudocompact abelian group if and only if  $A$  is perfectly minimal and  $A \leq_e C$ ;
- (c)  $(C, A)$  can be realized by a (perfectly) totally minimal pseudocompact abelian group if and only if  $A$  is (perfectly) totally minimal and dense in  $C$ .

**Remark 1.5.** If  $A$  is an essential subgroup of a compact group  $C$ , then in particular, it is essential in its completion  $\tilde{A} \subseteq C$ , and thus  $A$  is minimal (see Lemma 2.4(a) and Theorem 2.1). Therefore, the condition  $A \leq_e C$  in Theorem A(a) implies that  $A$  is minimal.

By applying Theorem A to the character group  $\text{hom}(\mathbb{Q}, \mathbb{R}/\mathbb{Z})$  equipped with the topology of pointwise convergence and the annihilator of  $\mathbb{Z}$  in it, we also obtain a family of “pathological” examples.

**Theorem B.** *For every positive integer  $n$  or  $n = \omega$ , there exists an abelian pseudocompact perfectly minimal group  $G_n$  with  $\dim G_n = n$  such that  $q(G_n)$  is not dense in  $q(\tilde{G}_n) = (\tilde{G}_n)_0$ , or equivalently,  $G_n/q(G_n)$  is not minimal.*

The proofs of Theorems A and B are presented in §3; they are based on preservation properties of essential subgroups established in §2.

## 2. Preliminaries: Essential and minimal subgroups

While preservation of minimality under formation of closed subgroups and products has been thoroughly studied (cf. [20, Proposition 2.3, Lemma 3.1], [11, Théorème 1-2], [24, 9], and [12, (6), (3)]), it appears that preservation of essentiality has not been well investigated in the context of topological groups. Our aim in this section is to remedy this state of affairs in the realm of abelian groups.

The relationship between minimality and essential subgroups was discovered independently by Stephenson and Prodanov, and generalized by Banaschewski (cf. [23, Theorem 2], [19], and [1, Propositions 1 and 2]).

**Theorem 2.1.** ([10, 2.5.1], [16, 3.21]) *Let  $G$  be a topological group, and  $D$  a dense subgroup. Then  $D$  is minimal if and only if  $G$  is minimal and  $D \leq_e G$ .*

The celebrated Prodanov–Stoyanov Theorem states that every minimal abelian group is precompact (cf. [21] and [22]), and allows for a complete characterization of minimality of abelian groups using the notion of an essential subgroup.

**Theorem 2.2.** ([10, 2.5.2], [16, 3.31]) *An abelian topological group  $G$  is minimal if and only if its completion  $\tilde{G}$  is compact and  $G \leq_e \tilde{G}$ .*

The next two easy lemmas, whose proofs have been omitted, describe elementary properties of the essentiality relation.

**Lemma 2.3.** *Let  $G$  be a topological group, and  $E$  and  $H$  subgroups of  $G$  such that  $E \subseteq H$ .*

- (a) *If  $E \leq_e H$  and  $H \leq_e G$ , then  $E \leq_e G$ .*
- (b) *If  $H$  is dense in  $G$ , then  $E \leq_e G$  if and only if  $E \leq_e H$  and  $H \leq_e G$ .*

**Lemma 2.4.** *Let  $G$  be an abelian topological group, and  $E$  and  $H$  subgroups.*

- (a) *If  $E \subseteq H$ , then  $E \leq_e G$  if and only if  $E \leq_e H$  and  $H \leq_e G$ .*
- (b) *If  $H$  is closed in  $G$  and  $E \leq_e G$ , then  $E \cap H \leq_e H$ .*

**Remark 2.5.** Every closed subgroup of a minimal abelian group is minimal (cf. [20, Proposition 2.3]). Indeed, let  $G$  be a minimal abelian group, and  $M$  a closed subgroup. Then, by Theorem 2.2,  $\tilde{G}$  is compact and  $G \leq_e \tilde{G}$ . So, by Lemma 2.4(b),  $M = G \cap (\text{cl}_{\tilde{G}} M) \leq_e \text{cl}_{\tilde{G}} M = \tilde{M}$ . Thus, by Theorem 2.2,  $M$  is minimal. (In fact, every closed central subgroup of a minimal group is minimal; cf. [10, 7.2.5] and [16, 3.26].) It follows that every closed subgroup of a totally minimal abelian group is totally minimal, and in fact, every closed central subgroup of a totally minimal group is totally minimal (cf. [20, Lemma 3.1], [10, 7.2.5], and [16, 3.27]).

We turn now to preservation of essentiality under formation of products. A topological group  $G$  is *elementwise compact* if every  $g \in G$  is contained in a compact subgroup of  $G$ , or equivalently,  $\text{cl}_G \langle g \rangle$  is compact for every  $g \in G$  (cf. [15, 5.4]). Stephenson showed that every elementwise compact minimal group is perfectly minimal (cf. [24, 9]). The next lemma is a natural extension of Stephenson’s result.

**Lemma 2.6.** *Let  $G$  be a topological group, and  $E \leq_e G$ . If  $L$  is an elementwise compact group, then  $L \times E \leq_e L \times G$ .*

**Proof.** Let  $\pi_2 : L \times G \rightarrow G$  denote the canonical projection, and let  $N$  be a non-trivial closed normal subgroup of  $L \times G$ . If  $\pi_2(N)$  is trivial, then  $N \subseteq L \times \{e\}$ , and consequently  $N \cap (L \times E)$  is non-trivial. Thus, we may assume that  $\pi_2(N)$  is non-trivial. We prove the statement in two steps.

*Step 1.* Suppose that  $L$  is compact. Then  $\pi_2$  is a closed map (cf. [13, 3.1.16]), and so  $\pi_2(N)$  is a closed normal subgroup of  $G$ . Thus,  $\pi_2(N) \cap E$  is non-trivial, because  $E \leq_e G$ . Therefore, the intersection  $N \cap (L \times E)$  is non-trivial, as required.

*Step 2.* In the general case, let  $x = (l, g) \in N$  be such that  $g \neq e$ . Since  $L$  is elementwise compact, there is a compact subgroup  $S$  of  $L$  such that  $l \in S$ . Put  $N' := N \cap (S \times G)$ . Then  $N'$  is a closed normal subgroup of  $S \times G$ , and it is non-trivial, because  $x \in N'$ . By what we have shown so far,  $N' \cap (S \times E)$  is non-trivial, and in particular,  $N \cap (L \times E)$  is non-trivial, as desired.  $\square$

The next example shows that elementwise compactness cannot be replaced with precompactness, minimality, or completeness in Lemma 2.6.

**Example 2.7.** Let  $p$  be a prime,  $\mathbb{Z}_p$  the group of  $p$ -adic integers, and  $(\mathbb{Z}, \tau_p)$  the integers equipped with the  $p$ -adic topology. Since  $(\mathbb{Z}, \tau_p)$  is a minimal group whose completion is  $\mathbb{Z}_p$  (cf. [10, 2.5.6]), by Theorem 2.2,  $\mathbb{Z} \leq_e \mathbb{Z}_p$ . However,  $\mathbb{Z} \times \mathbb{Z}$  is not essential in  $(\mathbb{Z}, \tau_p) \times \mathbb{Z}_p$  or  $\mathbb{Z} \times \mathbb{Z}_p$  (in the latter, the first component is equipped with the discrete topology). Indeed, if  $\xi \in \mathbb{Z}_p \setminus \mathbb{Z}$ , then  $F := \langle (1, \xi) \rangle$  is a non-trivial closed subgroup of  $(\mathbb{Z}, \tau_p) \times \mathbb{Z}_p$  (and thus of  $\mathbb{Z} \times \mathbb{Z}_p$ ) such that  $F \cap (\mathbb{Z} \times \mathbb{Z})$  is trivial.

It is also worth noting that preservation of essentiality does not imply precompactness, minimality, nor completeness, as the next example demonstrates.

**Example 2.8.** Let  $p$  be a prime. Let  $L_1$  denote the direct sum  $\mathbb{Z}_p^{(\omega)}$  equipped with the subgroup topology induced by the direct product  $\mathbb{Z}_p^\omega$ , let  $L_2$  denote the direct sum  $(\mathbb{R}/\mathbb{Z})^{(\omega)}$  equipped with the box topology, and put  $L := L_1 \times L_2$ . Then  $L$  is elementwise compact, because elementwise compactness is preserved under formation of products, coproducts, sums, and  $\Sigma$ -products. Thus, by Lemma 2.6,  $L$  preserves essentiality, that is,  $E \leq_e G$  implies  $L \times E \leq_e L \times G$ . Nevertheless,  $L$  is neither complete nor precompact, and in particular, by Theorem 2.2,  $L$  is not minimal.

These examples provide a natural motivation for the following problem.

**Problem 2.9.** Characterize the topological groups  $L$  with the property that

- (a) for every topological group  $G$ , if  $E \leq_e G$ , then  $L \times E \leq_e L \times G$ ;
- (b) for every abelian topological group  $G$ , if  $E \leq_e G$ , then  $L \times E \leq_e L \times G$ ;
- (c) for every topological group  $G$ , if  $E \leq_e G$  and  $E$  is minimal, then  $L \times E \leq_e L \times G$ .

We provide an answer to Problem 2.9(c) in the special case where  $L$  is minimal and abelian.

**Lemma 2.10.** Let  $M$  be a minimal abelian group,  $G$  a topological group, and  $E \leq_e G$ . If  $M \times E$  is minimal, then:

- (a)  $M \times E \leq_e \tilde{M} \times E$ ;
- (b)  $M \times E \leq_e \tilde{M} \times G$ ;
- (c)  $M \times E \leq_e M \times G$ .

**Proof.** (a) By Theorem 2.1,  $M \times E \leq_e \tilde{M} \times \tilde{E}$ , because  $M \times E$  is minimal. Consequently, by Lemma 2.3(b),  $M \times E \leq_e \tilde{M} \times E$ , since  $\tilde{M} \times E$  is dense in  $\tilde{M} \times \tilde{E}$ .

(b) By Theorem 2.2,  $\tilde{M}$  is compact, and so, by Lemma 2.6,  $\tilde{M} \times E \leq_e \tilde{M} \times G$ . By Lemma 2.3(a), combining this with what has been shown in part (a) yields  $M \times E \leq_e \tilde{M} \times G$ .

(c) As  $M \times G$  is dense in  $\tilde{M} \times G$ , by Lemma 2.3(b), part (b) implies that  $M \times E \leq_e M \times G$ .  $\square$

**Theorem 2.11.** Let  $M$  be a minimal abelian group. The following statements are equivalent:

- (i)  $M$  is perfectly minimal;
- (ii) for every topological group  $G$ , if  $E \leq_e G$  and  $E$  is minimal, then  $M \times E \leq_e M \times G$ ;
- (iii) for every abelian topological group  $G$ , if  $E \leq_e G$  and  $E$  is minimal, then  $M \times E \leq_e M \times G$ ;
- (iv) for every minimal abelian group  $M'$ , the product  $M \times M'$  is minimal.

**Proof.** The implication (i)  $\Rightarrow$  (ii) follows by Lemma 2.10(c), and the implication (ii)  $\Rightarrow$  (iii) is trivial. The equivalence (i)  $\Leftrightarrow$  (iv) is an immediate consequence of the following theorem of Stoyanov: *A topological group  $G$  is perfectly minimal if and only if  $G \times (\mathbb{Z}, \tau_p)$  is minimal for every prime  $p$  (cf. [25]).* We turn now to the remaining implication.

(iii)  $\Rightarrow$  (iv): Let  $M'$  be a minimal abelian group. By Theorem 2.2,  $\tilde{M}$  and  $\tilde{M}'$  are compact, and  $M \leq_e \tilde{M}$  and  $M' \leq_e \tilde{M}'$ . So, by (iii),  $M \times M' \leq_e M \times \tilde{M}'$ , and by Lemma 2.6,  $M \times \tilde{M}' \leq_e \tilde{M} \times \tilde{M}'$ . Therefore, by Lemma 2.3(a),  $M \times M' \leq_e \tilde{M} \times \tilde{M}'$ . Hence, by Theorem 2.2,  $M \times M'$  is minimal.  $\square$

**Remark 2.12.** It follows from Remark 2.5 and Theorem 2.11 that every closed subgroup of a perfectly minimal abelian group is perfectly minimal.

### 3. Proof of Theorems A and B

We prove Theorem A by establishing the following more elaborate statement.

**Theorem A'.** *Let  $A$  be an essential subgroup of a connected compact abelian group  $C$ . Then there exists an abelian group  $G$  such that*

- (a)  $G$  is pseudocompact;
- (b)  $A \cong q(G)$ ;
- (c)  $C \cong q(\tilde{G}) = (\tilde{G})_0$ , and in particular,  $\dim G = \dim C$ ;
- (d)  $G$  is minimal.

Furthermore, if  $A$  is perfectly minimal, then  $G$  may be chosen to be perfectly minimal.

We first show how Theorem A follows from Theorem A'.

**Proof of Theorem A.** Sufficiency of the conditions in (a) and (b) follows from Theorem A', while sufficiency of (c) was already shown in [8, Theorem D']. Thus, we may turn to the necessity of the conditions.

(a) Let  $G$  be a minimal pseudocompact abelian group. Then, by Theorem 2.2,  $G \leq_e \tilde{G}$ , and thus, by Lemma 2.4(b),  $G \cap q(\tilde{G}) \leq_e q(\tilde{G})$ . By Theorem 1.2,  $q(G) = q(\tilde{G}) \cap G$ , and therefore  $q(G) \leq_e q(\tilde{G})$ .

(b) If  $G$  is a perfectly minimal pseudocompact abelian group, then by Remark 2.12,  $q(G)$  is also perfectly minimal, and by part (a),  $q(G) \leq_e q(\tilde{G})$ .

(c) Let  $G$  be a (perfectly) totally minimal pseudocompact abelian group. Then  $G/q(G)$  is minimal, and by Theorem 1.3,  $q(G)$  is dense in  $q(\tilde{G})$ . It follows from Remarks 2.5 and 2.12 that  $q(G)$  is (perfectly) totally minimal, being a closed subgroup of a (perfectly) totally minimal abelian group.  $\square$

We turn now to the proof of Theorem A', and to that end, we recall two technical lemmas.

**Lemma 3.1.** ([8, 5.2]) *For every infinite cardinal  $\lambda$ , there exists a pseudocompact zero-dimensional group  $H$  such that:*

- (i)  $H$  is perfectly totally minimal;
- (ii)  $r_0(\tilde{H}/H) \geq 2^\lambda$ .

**Lemma 3.2.** ([8, 5.3]) *Let  $K_1$  and  $K_2$  be compact topological groups, and let  $h : K_1 \rightarrow K_2$  be a surjective homomorphism such that  $\ker h$  is  $G_\delta$ -dense in  $K_1$ . Then the graph  $\Gamma_h$  of  $h$  is a  $G_\delta$ -dense subgroup of the product  $K_1 \times K_2$ , and in particular,  $\Gamma_h$  is pseudocompact.*

**Proof of Theorem A'.** Put  $\lambda = w(C)$ , and let  $H$  be the group provided by Lemma 3.1. Since  $r_0(\tilde{H}/H) \geq 2^\lambda$ , the quotient  $\tilde{H}/H$  contains a free abelian group  $F$  of rank  $2^\lambda$ . As  $|C| \leq 2^\lambda$ , one may pick a surjective homomorphism  $h_1 : F \rightarrow C$ . The group  $C$  is divisible, because it is compact and connected (cf. [14, 24.25]). Thus,  $h_1$  can be extended to a surjective homomorphism  $h_2 : \tilde{H}/H \rightarrow C$ .

Let  $h : \tilde{H} \rightarrow C$  denote the composition of  $h_2$  with the canonical projection  $\tilde{H} \rightarrow \tilde{H}/H$ . By Theorem 1.1,  $H$  is  $G_\delta$ -dense in  $\tilde{H}$ , because  $H$  is pseudocompact. Thus,  $\ker h$  is  $G_\delta$ -dense in  $\tilde{H}$ , because  $H \subseteq \ker h$ . Clearly,  $h$  is surjective. Therefore, by Lemma 3.2, the graph  $\Gamma_h$  of  $h$  is  $G_\delta$ -dense in the product  $\tilde{H} \times C$ .

Put  $G := \Gamma_h + (\{0\} \times A)$ . Since  $\Gamma_h$  is  $G_\delta$ -dense in  $\tilde{H} \times C$  and contained in  $G$ , the group  $G$  is  $G_\delta$ -dense too. Thus,  $\tilde{G} = \tilde{H} \times C$ , and by Theorem 1.1,  $G$  is pseudocompact. As  $H$  is zero-dimensional,  $q(\tilde{G}) = (\tilde{G})_0 = \{0\} \times C$ , and by Theorem 1.2(a),  $q(G) = q(\tilde{G}) \cap G = \{0\} \times A$ .

We check now that  $\dim G = \dim C$ . Since  $G$  is pseudocompact, by Theorem 1.1,  $\tilde{G} = \beta G$ , and so

$$\dim G = \dim \beta G = \dim \tilde{G}$$

(cf. [13, 7.1.17]). As  $H$  is zero-dimensional and pseudocompact, by Theorem 1.2(b),  $\dim \tilde{H} = 0$ . Thus, by Yamanoshita's Theorem,  $\dim \tilde{G} = \dim \tilde{H} + \dim C = \dim C$  (cf. [27], [18, Corollary 2], and [10, 3.3.12]). Therefore,  $\dim G = \dim C$ .

We turn now to minimality properties of  $G$ . The group  $G$  always contains the product  $H \times A$ . Since  $C$  is compact, so is  $\tilde{A} = \text{cl}_C A$ , and by Lemma 2.4(a),  $A \leq_e \tilde{A}$ , because  $A \leq_e C$ . Thus, by Theorem 2.2,  $A$  is minimal, and consequently,  $H \times A$  is minimal, as  $H$  is perfectly minimal. Therefore, by Lemma 2.10(b),  $H \times A \leq_e \tilde{H} \times C = \tilde{G}$ . Hence, by Lemma 2.4(a),  $G \leq_e \tilde{G}$ , and by Theorem 2.2,  $G$  is minimal.

Suppose now that  $A$  is perfectly minimal, and let  $M'$  be a minimal abelian group. Then  $H \times A$  is perfectly minimal, and so  $H \times A \times M'$  is minimal. Thus, by Lemma 2.10(b),  $H \times A \times M' \leq_e \tilde{G} \times \tilde{M}'$ . Therefore, by Lemma 2.4(a),  $G \times M' \leq_e \tilde{G} \times \tilde{M}'$ , because  $G \times M'$  contains  $H \times A \times M'$ . Hence, by Theorem 2.2,  $G \times M'$  is minimal. Consequently, by Theorem 2.11,  $G$  is perfectly minimal, as desired.  $\square$

In preparation for the proof of Theorem B, we recall some remarkable properties of the Pontryagin dual of the discrete group of the rationals. Let  $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ , and put  $\widehat{\mathbb{Q}} := \text{hom}(\mathbb{Q}, \mathbb{T})$ , equipped with the topology of pointwise convergence. The group  $\widehat{\mathbb{Q}}$  is compact, and it is connected, because  $\mathbb{Q}$  is torsion-free (cf. [10, 3.3.8]).

**Lemma 3.3.** ([10, 3.6.2, 3.6.5]) *Let  $\mathbb{Z}^\perp := \{\chi \in \widehat{\mathbb{Q}} \mid \chi(\mathbb{Z}) = 0\}$  denote the annihilator of  $\mathbb{Z}$  in  $\widehat{\mathbb{Q}}$ .*

- (a)  $\mathbb{Z}^\perp$  is a compact essential subgroup of  $\widehat{\mathbb{Q}}$ ;
- (b)  $\mathbb{Z}^\perp \cong \prod_{p \in \mathbb{P}} \mathbb{Z}_p$ , and in particular,  $\dim \mathbb{Z}^\perp = 0$ ;
- (c)  $\widehat{\mathbb{Q}}/\mathbb{Z}^\perp \cong \mathbb{T}$ , and in particular,  $\dim \widehat{\mathbb{Q}} = 1$ .

**Proof of Theorem B.** Suppose that  $n < \omega$ . Put  $C_n := (\widehat{\mathbb{Q}})^n$  and  $A_n := (\mathbb{Z}^\perp)^n$ . By Lemma 3.3,  $\mathbb{Z}^\perp$  is a compact essential subgroup of  $\widehat{\mathbb{Q}}$ . Thus, by applying Lemmas 2.6 and 2.3(a) repeatedly, one obtains that  $A_n$  is a compact essential subgroup of  $C_n$ . It follows from Lemma 3.3 and Yamanoshita's Theorem that  $\dim C_n = n$  (cf. [27], [18, Corollary 2], and [10, 3.3.12]). By Theorem A', there exists a perfectly minimal pseudocompact group  $G_n$  such that  $A_n \cong q(G_n)$ ,  $C_n \cong q(\widetilde{G}_n)$ , and  $\dim G_n = \dim C_n = n$ . Clearly,  $q(G_n)$  is not only not dense, but in fact it is closed in  $q(\widetilde{G}_n)$ .

For  $n = \omega$ , put  $G_\omega = G_1 \times \mathbb{T}^\omega$ . Then  $\dim G_\omega \geq \omega$ , and  $q(G_\omega) = q(G_1) \times \mathbb{T}^\omega \cong \mathbb{Z}^\perp \times \mathbb{T}^\omega$ , while

$$q(\widetilde{G}_\omega) = (\widetilde{G}_\omega)_0 = (\widetilde{G}_1 \times \mathbb{T}^\omega)_0 = (\widetilde{G}_1)_0 \times \mathbb{T}^\omega \cong \widehat{\mathbb{Q}} \times \mathbb{T}^\omega.$$

Thus,  $q(G_\omega)$  is not dense in  $q(\widetilde{G}_\omega)$ , as required. The last statement follows from Theorem 1.3.  $\square$

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